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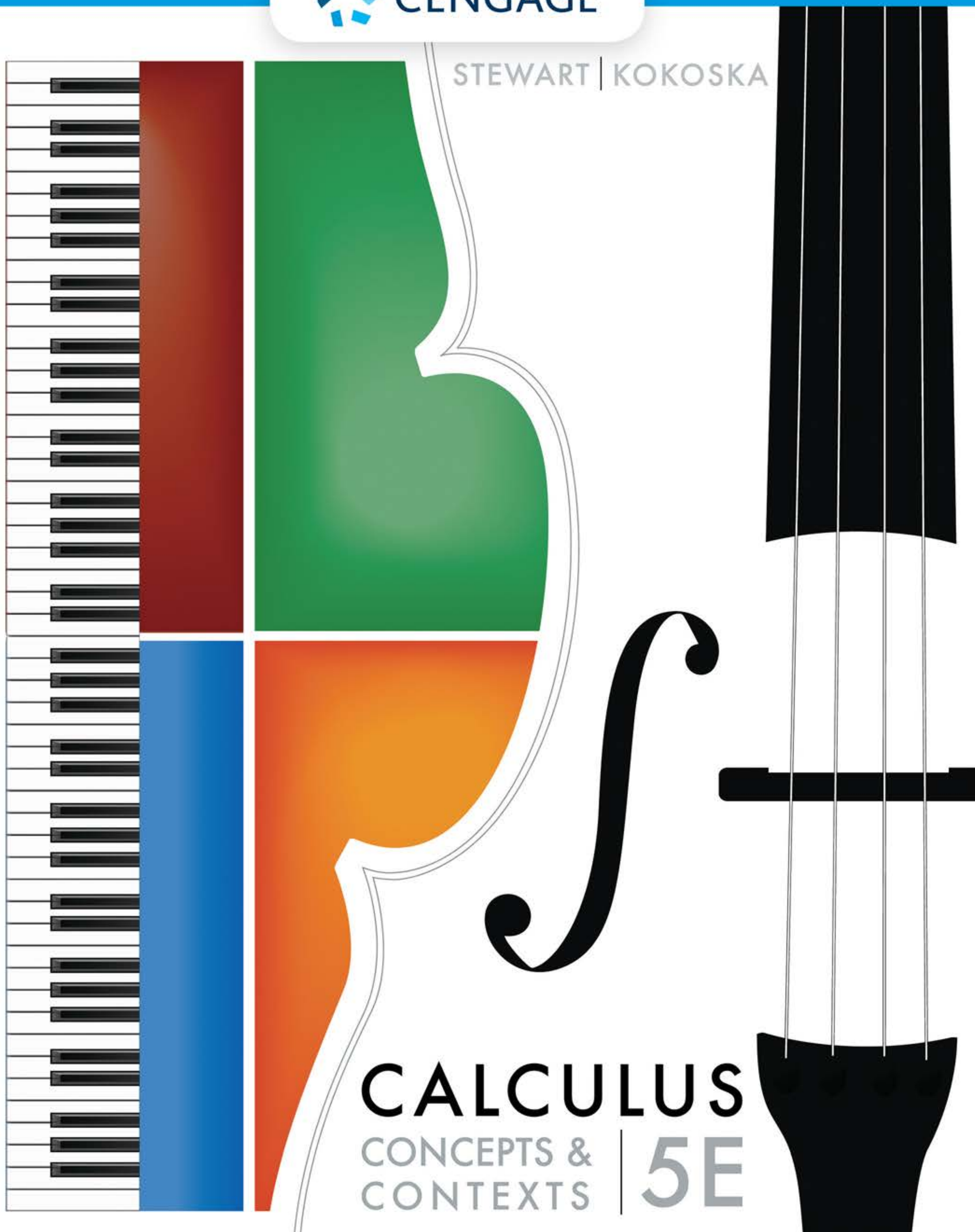

CALCULUS

CONCEPTS & CONTEXTS | 5E

Calculus

Concepts and Contexts

Fifth Edition

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Fifth Edition
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Contents

Preface

When the first edition of this book appeared over 20 years ago, there was a lot of discussion centered on calculus reform. Many mathematics departments were divided on issues including the use of technology, conceptual understanding versus procedural practice, and the role of discovery learning. Since then, the Advanced Placement® Calculus program has embraced calculus reform, and reformers and traditionalists have realized that they have a common goal: to enable students to understand and appreciate calculus.

The first four editions were intended to be a synthesis of reform and traditional approaches to calculus instruction. In the fifth edition, we continue this approach by emphasizing conceptual understanding through graphical, verbal, numerical, and algebraic approaches. We would like students to learn important problem-solving skills, and to see both the practical power of calculus and the intrinsic beauty of the subject.

The principal way in which this book differs from the more traditional calculus textbooks is that it is more streamlined. For example, there is no complete chapter on techniques of integration; we do not prove as many theorems; and the material on transcendental functions and on parametric equations is interwoven throughout the book instead of being treated in separate chapters. Instructors who prefer a more complete coverage of traditional calculus topics should consider *Calculus*, Ninth Edition, and Calculus: *Early Transcendentals*, Ninth Edition.

What's New in the Fifth Edition?

The changes in the fifth edition include a more conversational tone with an uncluttered presentation, all focused on conceptual understanding through the development of problem-solving skills. Here are some of the specific improvements that we have incorporated into this edition:

- *A Closer Look* feature provides straightforward itemized explanations of important concepts. Students will find these easy to read and to connect with the relevant theory.
- Marginal notes titled *Common Error* remind students of common errors and reinforce the proper solution technique.
- More detailed, guided solutions to examples include explanations for most steps (easy to read, in a different color, right justified with the appropriate step). This makes it easier for the student to follow the logical steps to a solution and to apply problem-solving skills to exercises.
- Wherever possible, sections are divided into appropriate subsections, smaller pieces, to accommodate the way students read and learn today.
- All graphs have been redrawn to include more detail and every figure has an appropriate caption to easily link with the appropriate idea.

- Each chapter begins with a real-world situation that introduces the material.
- The data in examples and exercises have been updated to be more timely.
- Section 1.4, "Graphing Calculators and Computers," has been eliminated.
- Former Section 2.8, "What Does f' Say About f," has been incorporated into Section 4.3, "Derivatives and the Shapes of Curves."
- New WebAssign problem types and learning resources build student problem-solving skills and conceptual understanding. These include automatically graded proof problems, Expanded Problems, Explore It interactive learning modules, and an eTextbook with Media Index and Student Solutions Manual.

Features

Conceptual Exercises

The exercises include various types of problems to foster conceptual understanding. Some exercises sets begin with questions that ask for an explanation of some of the basic concepts presented in the section. See, for example, the first few exercises in Sections 2.2, 2.4, 2.5, 5.3, 8.2, 11.2, and 11.3. These problems might be used as a basis for classroom discussions. Similarly, review sections begin with a Concepts and Vocabulary section and a True-False Quiz. Other Exercises test conceptual understanding through graphs and tables. See, for example, Exercises 1.7.22–25, 2.6.19, 2.7.39–42; 45–48, 3.8.5–6, 5.2.65–67, 7.1.12–14, 8.7.2, 10.2.1, 10.3.37–41, 11.1.1–2, 11.1.12–22, 11.3.3–10, 11.6.1–3, 11.7.3–4, 12.1.7–12, 13.1.13–22, 13.2.18–19, and 13.3.1, 2, 13.

Another type of exercise uses verbal description to test conceptual understanding. See, for example, Exercises 2.4.11, 2.7.75, 4.3.80, 4.3.84–85, and 5.10.69. Other exercises combine and compare graphical, numerical, and algebraic approaches; see Exercises 2.5.54–55, 2.5.63, 3.8.27, and 7.5.4.

Graded Exercise Sets

Each exercise set is carefully graded, progressing from basic conceptual exercises and skill-development problems to more challenging problems involving applications and proofs.

Real-World Data

Everyone involved with this writing project has spent a great deal of time looking in libraries, contacting companies and government agencies, and searching the Internet for interesting real-world data to introduce, motivate, and illustrate the concepts of calculus. As a result, many of the examples and exercises are associated with functions defined by numerical data given in a table or graphically. See, for example, Figure 1.1 in Section 1.1 (the rate of water usage in New York City during the 2018 Super Bowl), Exercise 5.1.16 (the velocity of a car racing at the Daytona International Speedway), Exercise 5.1.18 (the velocity of a pod in the SpaceX Hyperloop), Figure 5.40 (San Francisco power consumption), Example 5.9.5 (data traffic on Internet links), and Example 9.6.3 (wave heights).

Functions of two variables are illustrated by a table of values of the wind-chill index as a function of the wind speed and the air temperature (Example 11.1.1). Partial

derivatives are introduced in Section 11.3 by examining a column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity. This example is considered again in connection with linear approximations (Example 11.4.3). Directional derivatives are introduced in Section 11.6 by using a temperature contour map to estimate the rate of change of temperature at Boston in a northwest direction. Double integrals are used to estimate the average snowfall in Colorado during the 2020–2021 winter (Example 12.1.4). Vector fields are introduced in Section 13.1 by depictions of actual velocity vector fields showing wind patterns and ocean currents.

Projects

One way to involve students and to help make them active learners is to have them work (perhaps in groups) on extended projects that lead to a feeling of substantial accomplishment when completed. Applied Projects involve applications that are designed to appeal to the imagination of students. The project after Section 3.1 asks students to design the first ascent and drop for a roller coaster. The project after Section 11.8 uses Lagrange multipliers to determine the masses of the three stages of a rocket so as to minimize the total mass while enabling the rocket to reach a desired velocity. Laboratory Projects involve technology; the project following Section 3.4 shows how to use Bezier curves to design shapes that represent letters for a laser printer. Writing Projects ask students to compare present-day methods with those of the founders of calculus—Fermat's method for finding tangents, for instance. Suggested references are supplied. Discovery Projects anticipate results to be discussed later or cover optional topics (hyperbolic functions) or encourage discovery through pattern recognition (see the project following Section 5.8). Others explore aspects of geometry: tetrahedra (after Section 9.4), hyperspheres (after Section 12.7), and intersections of three cylinders (after Section 12.8). Additional projects can be found in the Instructor's Guide (see, for instance, Group Exercise 5.1: Position from Samples) and also in the CalcLabs supplements.

Rigor

There are fewer proofs included in this text as compared with more traditional calculus books. However, it is still worthwhile to expose students to the idea of proof and to make a clear distinction between a proof and a plausibility argument using, for example, technology (a graph or a table of values). The important thing is to show how to reach a conclusion that seems less obvious from something that seems more obvious. A good example is the use of the Mean Value Theorem to prove the Evaluation Theorem (Part 2 of the Fundamental Theorem of Calculus). Note that we have chosen not to prove the convergence tests but rather present intuitive arguments that they are true.

Problem Solving

Problem solving is perhaps the most difficult concept to teach and learn. Students frequently have difficulty solving problems in which there is no single well-defined procedure or technique for obtaining the final answer. It seems that no one has improved very much on George Polya's four-stage problem-solving strategy and, accordingly, a version of his problem-solving principles is included at the end of Chapter 1. These principles are applied, both explicitly and implicitly, throughout the book. At the end of other chapters, there are sections called Focus on Problem Solving, which feature examples of how to approach challenging calculus problems. The varied problems in

these sections are selected using the following advice from David Hilbert: "A mathematical problem should be difficult in order to entice us, yet not inaccessible lest it mock our efforts." These challenging problems might be used on assignments and tests, but consider grading them in a different way. One might reward a student significantly for presenting ideas toward a solution and for recognizing which problem-solving principles are relevant.

Technology

Graphing calculators and computers are powerful tools that allow us to explore problems, discover concepts, and confirm solutions. However, it is even more important to understand clearly the concepts that underlie the results and images on the screen. We assume that the student has access to either a graphing calculator or a computer algebra system. But technology doesn't make pencil and paper obsolete. Hand calculations and sketches are often preferable to technology for illustrating and reinforcing some concepts. Both instructors and students need to develop the ability to decide where the use of technology is appropriate.

Cengage WebAssign

WebAssign: webassign.com

This Fifth Edition is available with WebAssign, a fully customizable online solution for STEM disciplines from Cengage. WebAssign includes homework, an interactive mobile eBook, videos, tutorials, and Explore It interactive learning modules. Instructors can decide what type of help students can access, and when, while working on assignments. The patented grading engine provides unparalleled answer evaluation, giving students instant feedback, and insightful analytics highlight exactly where students are struggling. For more information, visit webassign.com.

Stewart Website

Visit StewartCalculus.com for these additional materials:

- Homework Hints
- Algebra and Analytic Geometry Review
- Lies My Calculator and Computer Told Me
- History of Mathematics, with links to recommended historical websites
- Additional Topics (complete with exercise sets): Fourier Series, Rotation of Axes, Formulas for the Remainder Theorem in Taylor Series, Second-Order Differential Equations
- Challenge Problems (some from the Problems Plus sections from prior editions)
- Links, for particular topics, to outside Web resources

Content

Diagnostic Tests

The book begins with four diagnostic tests, in Basic Algebra, Analytic Geometry, Functions, and Trigonometry.

A Preview of Calculus

This is an overview of the subject and includes a list of questions to motivate the study of calculus.

1 • Functions and Models

Multiple representations of functions are emphasized throughout the text: verbal, numerical, graphical, and algebraic. A discussion of mathematical models leads to a review of the standard functions, including exponential and logarithmic functions, from these four points of view. Parametric curves are introduced in the first chapter, partly, so that curves can be drawn easily, with technology, whenever needed throughout the text. This early placement also enables tangents to parametric curves to be treated in Section 3.4 and graphing such curves to be covered in Section 4.4.

2 • Limits

The material on limits is motivated by a discussion of the tangent line and instantaneous velocity problems. Limits are treated from descriptive, graphical, numerical, and algebraic points of view. Note that the precise definition of a limit is provided in Appendix D for those who wish to cover this concept. It is important to carefully consider Sections 2.6 and 2.7, which deal with derivatives and rates of change, before the differentiation rules are covered in Chapter 3. The examples and exercises in these sections explore the meanings of derivatives in various contexts.

3 • Differentiation Rules

All of the rules for differentiating basic functions are presented in this chapter. There are many applied examples and exercises in which students are asked to explain the meaning of the derivative in the context of the problem. Optional topics (hyperbolic functions, an early introduction to Taylor polynomials) are explored in Discovery and Laboratory Projects. A full treatment of hyperbolic functions is available to instructors on the website.

4 • Applications of Differentiation

This chapter begins with a section on related rates. Then, the basic facts concerning extreme values and shapes of curves are derived using the Mean Value Theorem as the starting point. The interaction between technology and calculus is discussed and illustrated, and there are a wide variety of optimization problems presented. Indeterminate forms are addressed, Newton's method is presented, and a discussion of antiderivatives prepares students for Chapter 5.

5 • Integrals

The area problem and the distance problem serve to motivate the definite integral. Subintervals of equal width are used in order to make the definition of a definite integral easier to understand. Emphasis is placed on explaining the meanings of integrals in various contexts and on estimating their values from graphs and tables. There is no separate chapter on techniques of integration, but substitution and integration by parts are covered here and other methods are treated briefly. Partial fractions are given full treatment in Appendix G. The use of computer algebra systems is discussed in Section 5.8.

6 • Applications of Integration

General methods, not formulas, are emphasized. The goal is for students to be able to divide a quantity into small pieces, estimate with Riemann sums, and recognize the limit as an integral. There are lots of applications in this chapter, probably too many to cover in any one course. We hope you will select applications that you and your students enjoy. Some instructors like to cover polar coordinates, in Appendix H, here. Others prefer to defer this topic until it is needed in a third semester calculus course, with Section 9.7 or just before Section 12.4.

7 • Differential Equations

Modeling is the theme that unifies this introductory treatment of differential equations. Slope fields and Euler's method are presented before separable equations are solved explicitly, so that qualitative, numerical, and analytic approaches are given equal consideration. These methods are applied to the exponential, logistic, and other models for population growth. Predator-prey models are used to illustrate systems of differential equations.

8 • Infinite Sequences and Series

Tests for the convergence of series are considered briefly, with intuitive rather than formal justifications. Numerical estimates of sums of series are based on the test used to prove convergence. The emphasis is on Taylor series and polynomials, their applications to physics, and error estimates.

9 • Vectors and the Geometry of Space

The dot product and cross product of vectors are given geometric definitions, motivated by work and torque, before the algebraic expressions are derived. To facilitate the discussion of surfaces, functions of two variables and their graphs are introduced here.

10 • Vector Functions

The calculus of vector functions is used to prove Kepler's First Law of planetary motion, with the proofs of the other laws left as a project. Since parametric curves were introduced in Chapter 1, parametric surfaces are introduced as soon as possible, namely, in this chapter. We think an early familiarity with such surfaces is desirable, especially with the capability of computers to produce their graphs. Then tangent planes and areas of parametric surfaces can be discussed in Sections 11.4 and 12.6.

11 • Partial Derivatives

Functions of two or more variables are studied from verbal, numerical, visual, and algebraic points of view. In particular, partial derivatives are introduced by looking at a specific column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity. Directional derivatives are estimated from contour maps of temperature, pressure, and elevation.

12 • Multiple Integrals

Contour maps and the Midpoint Rule are used to estimate the average snowfall and average temperature in given regions. Double and triple integrals are used to compute

probabilities, areas of parametric surfaces, volumes of hyperspheres, and the volume of intersection of three cylinders.

13 • Vector Fields

Vector fields are introduced through pictures of velocity fields showing wind patterns and ocean currents. The similarities among the Fundamental Theorem for line integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem are emphasized.

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James Stewart
Stephen Kokoska

Instructor Resources

Cengage WebAssign

Built by educators, WebAssign from Cengage is a fully customizable online solution for STEM disciplines. WebAssign includes the flexibility, tools, and content you need to create engaging learning experiences for your students. The patented grading engine provides unparalleled answer evaluation, giving students instant feedback, and insightful analytics highlight exactly where students are struggling. For more information, visit *webassign.com.*

Additional instructor resources for this product are available online at the Cengage Instructor Center—an all-in-one resource for class preparation, presentation, and testing. Instructor resources available for download from the Cengage Instructor Center include:

Instructor's Manual

Includes activities and assessments correlated by learning objectives, chapter and section outline, key formulas and terms with definitions, ideas for student collaboration and class discussions, and more.

Solutions and Answer Guide

Provides answers and solutions to all exercises in the text, including exercises in the end-of-chapter Review and Focus on Problem Solving.

Cengage Testing Powered by Cognero®

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PowerPoint® Slides

The PowerPoint® slides are ready to use, visual outlines of each section that can be easily customized for your lectures. Presentations include activities, examples, and opportunities for student engagement and interaction.

Transition Guide

Highlights the content changes from the previous edition to the new edition, including exercise correlations.

Guide to Online Teaching

Provides technological and pedagogical considerations and tips for teaching a Calculus course online.

Educator's Guide

Offers suggested content and activities for Cengage WebAssign—like videos and assignments—that you can integrate into your course to help boost engagement and outcomes.

Sign up or sign in at **www.cengage.com** to search for and access this product and its online resources.

Student Resources

Cengage WebAssign

Prepare for class with confidence using WebAssign from Cengage. This online learning platform, which includes an interactive eBook, encourages practice, so that you truly absorb what you learn and prepare better for tests. Videos and tutorials walk you through concepts and deliver instant feedback and grading, so you always know where you stand in class. Focus your study time and get extra practice where you need it most. Study smarter with WebAssign! Ask your instructor today how you can get access to WebAssign, or learn about self-study options at *webassign.com.*

Student Solutions

Provides step-by-step solutions for all odd-numbered exercises in the text. The Student Solutions are provided in the eBook at no additional cost.

Additional student resources are available online. Sign up or sign in at **www.cengage.com** to search for and access this product and its online resources.

To the Student

Reading a calculus book is different from reading a newspaper or a novel, or even a physics book. Don't be discouraged if you have to read a passage more than once to understand it. It's a good idea to keep a pencil, paper, and graphing calculator handy to check, or try, a calculation or sketch a diagram or graph.

Some students don't read the text at first; they start by trying the homework problems and read the text only when they get stuck on an exercise. We think a better plan is to read and understand a section of the text before attempting the exercises. In particular, you should carefully study the definitions and understand the exact meanings of terms. And before you read an example, we suggest that you cover up the solution and try solving the problem yourself. You will learn a lot more from looking at the solution this way.

This text is designed to teach conceptual understanding through problem-solving skills and pattern recognition, and to train you to think logically. Learn to write solutions to the exercises in a connected, step-by-step fashion, using good communication and proper notation.

The answers to the odd-numbered exercises appear at the back of the book, in Appendix J. Some exercises ask for a verbal explanation or interpretation or description. In such cases there is no single correct way of expressing the answer, so don't worry that you haven't found the definitive answer. In addition, there are often several forms in which to express a numerical or algebraic answer, so if your answer differs from the one in the back of the book, don't immediately assume you are wrong. For example, if the answer given in the back of the book is $\sqrt{2} - 1$ and your answer is $1/(1 + \sqrt{2})$, then you are right, and rationalizing the denominator will show that the answers are equivalent.

The Stewart website is a companion to this text and provides various resources to help you succeed. For example, Homework Hints for representative exercises ask questions that allow you to make progress toward a solution without actually giving you the answer. You will need to pursue each hint in an active manner with pencil and paper to work out the details. If a particular hint doesn't enable you to solve the problem, you can reveal the next hint. There is also an algebra review, some drill exercises to reinforce techniques, and several challenge problems.

We hope that you will keep this book for reference after you finish the course. Because you may forget some of the specific details of calculus, this book will serve as a useful reminder when you need to use calculus in subsequent courses. And, because this book contains more material than can be covered in any one course, it can also serve as a valuable resource for a working scientist or engineer. Calculus is an exciting subject, justly considered to be one of the greatest achievements of the human intellect. We hope that you will discover that calculus is not only useful but also intrinsically beautiful.

James Stewart
Stephen Kokoska

About the Author

Steve received his undergraduate degree from Boston College and his M.S. and Ph.D. from the University of New Hampshire. His initial research interests included the statistical analysis of cancer chemoprevention experiments. He has published a number of research papers in mathematics journals, including *Biometrics, Anticancer Research,* and *Computer Methods and Programs in Biomedicine*; presented results at national conferences; and written several books. He has been awarded grants from the National Science Foundation, the Center for Rural Pennsylvania, and the Ben Franklin Program.

Steve is a longtime consultant for the College Board and has conducted workshops in Brazil, the Dominican Republic, Singapore, and China. He was the AP Calculus Chief Reader for four years, has been involved with calculus reform and the use of technology in the classroom, and recently published an AP Calculus text with James Stewart. He taught at Bloomsburg University for 30 years and has served as Director of the Honors Program.

Steve believes in teaching conceptual understanding through the development of problem-solving skills and pattern recognition. He is still involved with the AP Calculus program and co-hosts the webinar Monday Night Calculus sponsored by Texas Instruments.

Steve's uncle, Fr. Stanley Bezuszka, a Jesuit and professor at Boston College, was one of the original architects of the so-called new math in the 1950s and 1960s. He had a huge influence on Steve's career. Steve helped Fr. B. with text accuracy checks, as a teaching assistant, and even writing projects through high school and college. Steve learned about the precision, order, and elegance of mathematics and developed an unbounded enthusiasm to teach.

Diagnostic Tests

Success in calculus depends to a large extent on knowledge of the mathematics that precedes calculus: algebra, analytic geometry, functions, and trigonometry. The following diagnostic tests will help you assess your proficiency in these subjects. After taking each test, you can check your answers against the given answers and, if necessary, refresh your skills by referring to the review materials that are provided.

A Diagnostic Test: Algebra

1. Evaluate each expression without using a calculator.

(a) $(-3)^4$ (b) -3^4 (c) 3^{-4}

(d) $\dfrac{5^{23}}{5^{21}}$ (e) $\left(\dfrac{2}{3}\right)^{-2}$ (f) $16^{-3/4}$

2. Simplify each expression. Write your answer without negative exponents.

(a) $\sqrt{200} - \sqrt{32}$ (b) $(3a^3b^3)(4ab^2)^2$ (c) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$

3. Expand and simplify.

(a) $3(x + 6) + 4(2x - 5)$ (b) $(x + 3)(4x - 5)$

(c) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$ (d) $(2x + 3)^2$

(e) $(x + 2)^3$

4. Factor each expression.

(a) $4x^2 - 25$ (b) $2x^2 + 5x - 12$

(c) $x^3 - 3x^2 - 4x + 12$ (d) $x^4 + 27x$

(e) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ (f) $x^3y - 4xy$

5. Simplify the rational expression.

(a) $\dfrac{x^2 + 3x + 2}{x^2 - x - 2}$ (b) $\dfrac{2x^2 - x - 1}{x^2 - 9} \cdot \dfrac{x + 3}{2x + 1}$

(c) $\dfrac{x^2}{x^2 - 4} - \dfrac{x + 1}{x + 2}$ (d) $\dfrac{\dfrac{y}{x} - \dfrac{x}{y}}{\dfrac{1}{y} - \dfrac{1}{x}}$

6. Rationalize the expression and simplify.

(a) $\dfrac{\sqrt{10}}{\sqrt{5} - 2}$ (b) $\dfrac{\sqrt{4 + h} - 2}{h}$

7. Rewrite by completing the square.

(a) $x^2 + x + 1$ (b) $2x^2 - 12x + 11$

8. Solve the equation. (Find only the real solutions.)

(a) $x + 5 = 14 - \frac{1}{2}x$ (b) $\dfrac{2x}{x + 1} = \dfrac{2x - 1}{x}$

(c) $x^2 - x - 12 = 0$ (d) $2x^2 + 4x + 1 = 0$

(e) $x^4 - 3x^2 + 2 = 0$ (f) $3|x - 4| = 10$

(g) $2x(4 - x)^{-1/2} - 3\sqrt{4 - x} = 0$

9. Solve each inequality. Write your answer using interval notation.

(a) $-4 < 5 - 3x \leq 17$

(b) $x^2 < 2x + 8$

(c) $x(x-1)(x+2) > 0$

(d) $|x - 4| < 3$

(e) $\dfrac{2x-3}{x+1} \leq 1$

10. State whether each equation is true or false.

(a) $(p+q)^2 = p^2 + q^2$

(b) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(c) $\sqrt{a^2 + b^2} = a + b$

(d) $\dfrac{1 + TC}{C} = 1 + T$

(e) $\dfrac{1}{x-y} = \dfrac{1}{x} - \dfrac{1}{y}$

(f) $\dfrac{1/x}{a/x - b/x} = \dfrac{1}{a-b}$

Answers to Diagnostic Test A: Algebra

1. (a) 81 (b) -81 (c) $\dfrac{1}{81}$

(d) 25 (e) $\dfrac{9}{4}$ (f) $\dfrac{1}{8}$

2. (a) $6\sqrt{2}$ (b) $48a^5b^7$ (c) $\dfrac{x}{9y^7}$

3. (a) $11x - 2$ (b) $4x^2 + 7x - 15$

(c) $a - b$ (d) $4x^2 + 12x + 9$

(e) $x^3 + 6x^2 + 12x + 8$

4. (a) $(2x-5)(2x+5)$ (b) $(2x-3)(x+4)$

(c) $(x-3)(x-2)(x+2)$ (d) $x(x+3)(x^2-3x+9)$

(e) $3x^{-1/2}(x-1)(x-2)$ (f) $xy(x-2)(x+2)$

5. (a) $\dfrac{x+2}{x-2}$ (b) $\dfrac{x-1}{x-3}$

(c) $\dfrac{1}{x-2}$ (d) $-(x+y)$

6. (a) $5\sqrt{2} + 2\sqrt{10}$ (b) $\dfrac{1}{\sqrt{4+h}+2}$

7. (a) $\left(x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$ (b) $2(x-3)^2 - 7$

8. (a) 6 (b) 1 (c) $-3, 4$

(d) $-1 \pm \frac{1}{2}\sqrt{2}$ (e) $\pm 1, \pm\sqrt{2}$ (f) $\dfrac{2}{3}, \dfrac{22}{3}$

(g) $\dfrac{12}{5}$

9. (a) $[-4, 3)$ (b) $(-2, 4)$

(c) $(-2, 0) \cup (1, \infty)$ (d) $(1, 7)$

(e) $(-1, 4)$

10. (a) False (b) True (c) False

(d) False (e) False (f) True

> If you have had difficulty with these problems, you may wish to consult the Review of Algebra on the website www.stewartcalculus.com.

B Diagnostic Test: Analytic Geometry

1. Find an equation for the line that passes through the point $(2, -5)$ and
   (a) has slope -3
   (b) is parallel to the x-axis
   (c) is parallel to the y-axis
   (d) is parallel to the line $2x - 4y = 3$

2. Find an equation for the circle that has center $(-1, 4)$ and passes through the point $(3, -2)$.

3. Find the center and radius of the circle with equation $x^2 + y^2 - 6x + 10y + 9 = 0$.

4. Let $A(-7, 4)$ and $B(5, -12)$ be points in the plane.
   (a) Find the slope of the line that contains A and B.
   (b) Find an equation of the line that passes through A and B. What are the intercepts?
   (c) Find the midpoint of the segment AB.
   (d) Find the length of the segment AB.
   (e) Find an equation of the perpendicular bisector of AB.
   (f) Find an equation of the circle for which AB is a diameter.

5. Sketch the region in the xy-plane defined by the equation or inequalities.

(a) $-1 \le y \le 3$ (b) $|x| < 4$ and $|y| < 2$ (c) $y < 1 - \frac{1}{2}x$

(d) $y \ge x^2 - 1$ (e) $x^2 + y^2 < 4$ (f) $9x^2 + 16y^2 = 144$

Answers to Diagnostic Test B: Analytical Geometry

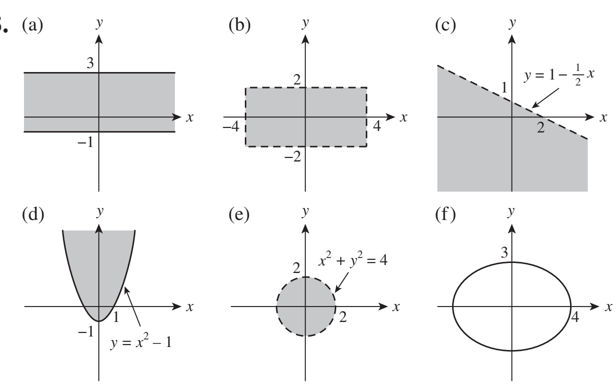
1. (a) $y = -3x + 1$
   (b) $y = -5$
   (c) $x = 2$
   (d) $y = \frac{1}{2}x - 6$

2. $(x + 1)^2 + (y + 4)^2 = 52$

3. Center $(3, -5)$, radius 5

4. (a) $-\frac{4}{3}$
   (b) $4x + 3y + 16 = 0$; x-intercept -4, y-intercept $-\frac{16}{3}$
   (c) $(-1, -4)$
   (d) 20
   (e) $3x - 4y = 13$
   (f) $(x + 1)^2 + (y + 4)^2 = 100$

5.



If you have had difficulty with these problems, you may wish to consult the review of analytic geometry in Appendix B.

C Diagnostic Test: Functions

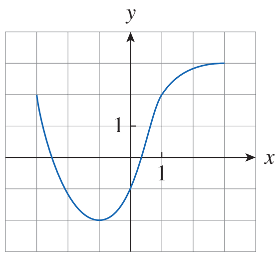


Figure for Problem 1

1. The graph of a function f is given at the left.

(a) State the value of $f(-1)$.
(b) Estimate the value of $f(2)$.
(c) For what values of x is $f(x) = 2$?
(d) Estimate the values of x such that $f(x) = 0$.
(e) State the domain and range of f.

2. If $f(x) = x^3$, evaluate the difference quotient $\dfrac{f(2+h) - f(2)}{h}$ and simplify your answer.

3. Find the domain of the function.

(a) $f(x) = \dfrac{2x+1}{x^2 + x - 2}$ (b) $g(x) = \dfrac{\sqrt[3]{x}}{x^2 + 1}$ (c) $h(x) = \sqrt{4 - x} + \sqrt{x^2 - 1}$

4. How are graphs of the functions obtained from the graph of f?

(a) $y = -f(x)$ (b) $y = 2f(x) - 1$ (c) $y = f(x - 3) + 2$

5. Without using a calculator, make a rough sketch of the graph.

(a) $y = x^3$ (b) $y = (x+1)^3$ (c) $y = (x-2)^3 + 3$
(d) $y = 4 - x^2$ (e) $y = \sqrt{x}$ (f) $y = 2\sqrt{x}$
(g) $y = -2^x$ (h) $y = 1 + x^{-1}$

6. Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \le 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$

(a) Evaluate $f(-2)$ and $f(1)$. (b) Sketch the graph of f.

7. If $f(x) = x^2 + 2x - 1$ and $g(x) = 2x - 3$, find each of the following functions.

(a) $f \circ g$ (b) $g \circ f$ (c) $g \circ g \circ g$

Answers to Diagnostic Test C: Functions

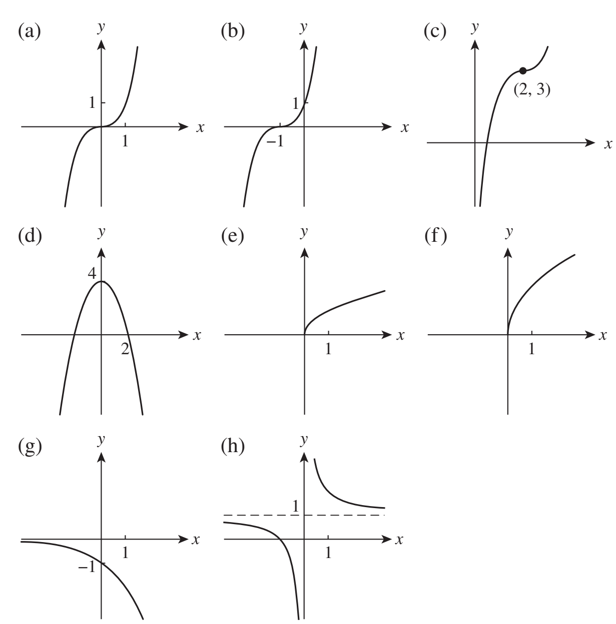
1. (a) -2 (b) 2.8
(c) $-3, 1$ (d) $-2.5, 0.3$
(e) $[-3, 3], [-2, 3]$

2. $12 + 6h + h^2$

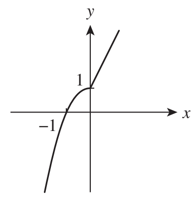
3. (a) $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$
(b) $(-\infty, \infty)$
(c) $(-\infty, -1] \cup [1, 4]$

4. (a) Reflect about the x-axis.
(b) Stretch vertically by a factor of 2, then shift 1 unit downward.
(c) Shift 3 units to the right and 2 units upward.

5.



6. (a) $-3, 3$ (b)



7. (a) $(f \circ g)(x) = 4x^2 - 8x + 2$
(b) $(g \circ f)(x) = 2x^2 + 4x - 5$
(c) $(g \circ g \circ g)(x) = 8x - 21$

If you have had difficulty with these problems, you should look at Sections 1.1–1.3 of this book.

D Diagnostic Test: Trigonometry

1. Convert from degrees to radians.
(a) $300°$ (b) $-18°$

2. Convert from radians to degrees.
(a) $\dfrac{5\pi}{6}$ (b) 2

3. Find the length of an arc of a circle with radius 12 cm if the arc subtends a central angle of $30°$.

4. Find the exact values.
(a) $\tan\left(\dfrac{\pi}{3}\right)$ (b) $\sin\left(\dfrac{7\pi}{6}\right)$ (c) $\sec\left(\dfrac{5\pi}{3}\right)$

5. Express the lengths a and b in the figure in terms of θ.

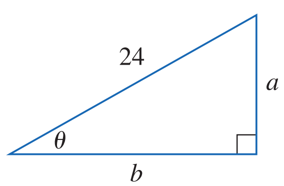


Figure for Problem 5

6. If $\sin x = \dfrac{1}{3}$ and $\sec y = \dfrac{5}{4}$, where x and y lie between 0 and $\dfrac{\pi}{2}$, evaluate $\sin(x + y)$.

7. Prove the identities.
(a) $\tan\theta \sin\theta + \cos\theta = \sec\theta$
(b) $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x$

8. Find all values of x such that $\sin 2x = \sin x$ and $0 \le x \le 2\pi$.

9. Sketch the graph of the function $y = 1 + \sin 2x$ without using a calculator.

Answers to Diagnostic Test D: Trigonometry

1. (a) $\dfrac{5\pi}{3}$ (b) $\dfrac{-\pi}{10}$

2. (a) $150°$ (b) $\dfrac{360°}{\pi} \approx 114.6°$

3. 2π cm

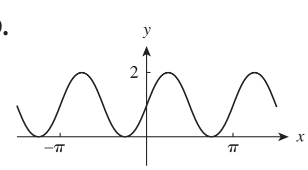
4. (a) $\sqrt{3}$ (b) $-\dfrac{1}{2}$ (c) 2

5. (a) $24\sin\theta$ (b) $24\cos\theta$

6. $\dfrac{1}{15}(4 + 6\sqrt{2})$

8. $0, \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}, 2\pi$

9.



If you have had difficulty with these problems, you should look at Appendix C of this book.

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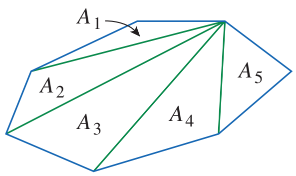
A Preview of Calculus

Mark Twain wrote, "If you don't like the weather in New England now, just wait a few minutes." Indeed, the weather in New England changes quickly, and calculus is about the study of change, instantaneous change. Complex mathematical models can be used to predict changes in weather, and calculus plays an important role in these analyses.

Calculus is fundamentally different from the mathematics that you have studied previously: calculus is less static and more dynamic. It is concerned with change and motion; it deals with the long-run, or limiting, behavior of certain expressions. For that reason it may be useful to have an overview of the subject before beginning its intensive study. This preview provides a glimpse of some of the main ideas of calculus by showing how the concept of a limit arises when we attempt to solve a variety of problems.



$A = A_1 + A_2 + A_3 + A_4 + A_5$

Figure 1
The area of the polygon is the sum of the areas of the triangles.

The Area Problem

The origins of calculus go back at least 2500 years to the ancient Greeks, who found areas using the "method of exhaustion." They knew how to find the area A of any polygon by dividing it into triangles as in Figure 1 and adding the areas of these triangles.

It is a much more difficult problem to find the area of a curved figure. The Greek method of exhaustion was to inscribe polygons in the figure and circumscribe polygons about the figure and then let the number of sides of the polygons increase. Figure 2 illustrates this process for the special case of a circle with inscribed regular polygons.

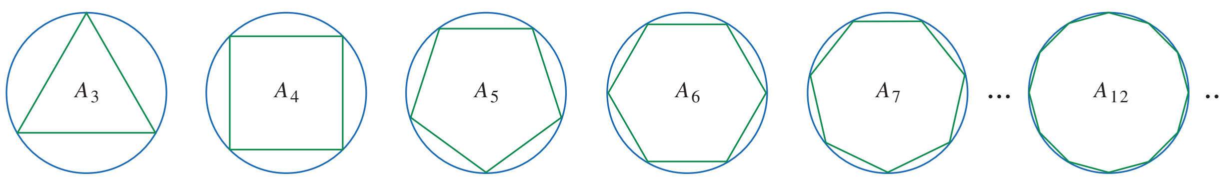


Figure 2
Regular inscribed polygons used to find the area of a circle.

Let A_n be the area of the inscribed polygon with n sides. As n increases, it appears that A_n becomes closer and closer to the area of the circle. We say that the area of the circle is the *limit* of the areas of the inscribed polygons, and we write

$$A = \lim_{n \to \infty} A_n$$

The Greeks themselves did not use limits explicitly. However, by indirect reasoning, Eudoxus (5th century BC) used exhaustion to prove the familiar formula for the area of a circle: $A = \pi r^2$.

We will use a similar idea in Chapter 5 to find areas of regions of the type shown in Figure 3. We will approximate the desired area A by areas of rectangles (as in Figure 4), let the width of the rectangles decrease, and then calculate A as the limit of these sums of areas of rectangles.

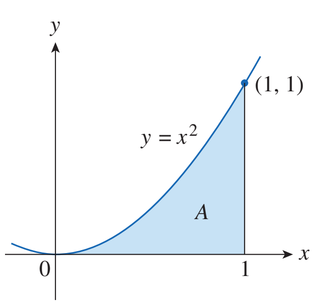


Figure 3
Let A be the area of the shaded region.

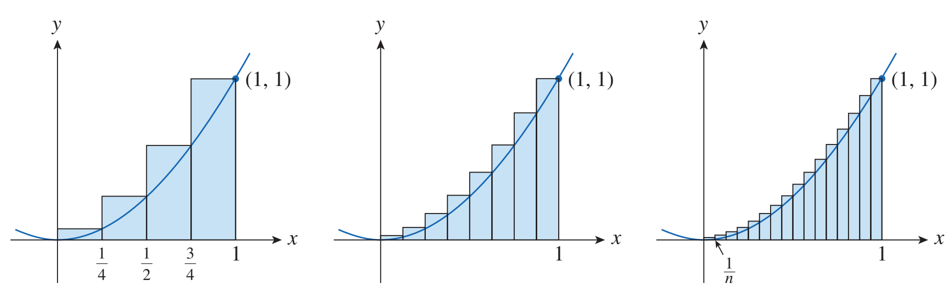


Figure 4
Approximate the area A by areas of rectangles.

The area problem is the central problem in the branch of calculus called *integral calculus*. The techniques that we will develop in Chapter 5 for finding areas will also enable us to compute the volume of a solid, the length of a curve, the force of water against a dam, the mass and center of gravity of a rod, and the work done in pumping water out of a tank.

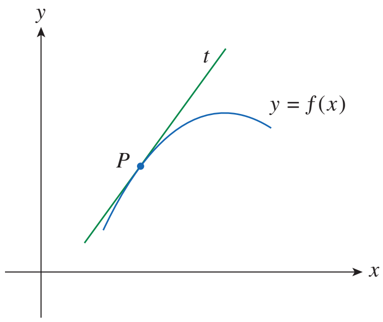


Figure 5
The tangent line to the graph of $y = f(x)$ at the point P.

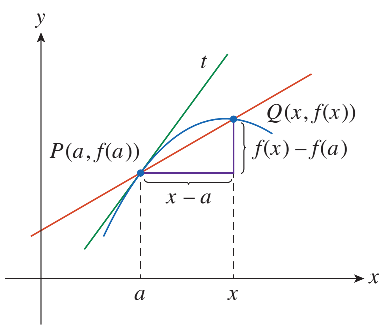


Figure 6
The secant line PQ.

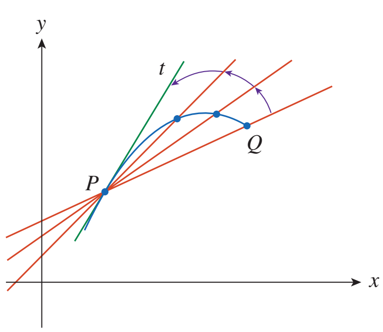


Figure 7
Secant lines approaching the tangent line.

The Tangent Line Problem

Consider the problem of trying to find an equation of the tangent line t to a curve with equation $y = f(x)$ at a given point P. (We will give a precise definition of a tangent line in Chapter 2. For now you can think of it as a line that just touches the curve at P as in Figure 5.) Since we know that the point P lies on the tangent line, we can find the equation of t if we know its slope m. The problem is that we need two points to compute the slope and we know only one point, P, on t. To solve this problem, we first find an approximation to m by taking a nearby point Q on the curve and computing the slope m_{PQ} of the secant line PQ. From Figure 6 we see that

$$m_{PQ} = \frac{f(x) - f(a)}{x - a} \tag{1}$$

Now imagine that Q moves along the curve toward P as in Figure 7. You can see that the secant line rotates and approaches the tangent line as its limiting position. This means that the slope m_{PQ} of the secant line becomes closer and closer to the slope m of the tangent line. We write

$$m = \lim_{Q \to P} m_{PQ}$$

and we say that m is the limit of m_{PQ} as Q approaches P along the curve. Since x approaches a as Q approaches P, we could also use Equation 1 to write

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{2}$$

Specific examples of this procedure will be given in Chapter 2.

The tangent line problem has given rise to the branch of calculus called *differential calculus*, which was not invented until more than 2000 years after integral calculus. The main ideas behind differential calculus are due to the French mathematician Pierre Fermat (1601–1665) and were developed by the English mathematicians John Wallis (1616–1703), Isaac Barrow (1630–1677), and Isaac Newton (1642–1727) and the German mathematician Gottfried Leibniz (1646–1716).

The two branches of calculus and their chief problems, the area problem and the tangent line problem, appear to be very different, but it turns out that there is a very close connection between them. The tangent line problem and the area problem are inverse problems in a sense that will be described in Chapter 5.

Velocity

When we look at the speedometer of a car and read that the car is traveling at 48 mi/h, what does that information really mean? We know that if the velocity remains constant, then after an hour we will have traveled 48 mi. But if the velocity of the car varies, what does it mean to say that the velocity at a given instant is 48 mi/h?

In order to analyze this question, let's examine the motion of a car that travels along a straight road and assume that we can measure the distance traveled by the car (in feet) at 1-second intervals as in the following table:

t = Time elapsed (s)	0	1	2	3	4	5
d = Distance (ft)	0	2	9	24	42	71

As a first step toward finding the velocity after 2 seconds have elapsed, we find the average velocity during the time interval $2 \le t \le 4$:

$$\begin{aligned} \text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} \\ &= \frac{42 - 9}{4 - 2} \\ &= 16.5 \text{ ft/s} \end{aligned}$$

Similarly, the average velocity in the time interval $2 \le t \le 3$ is

$$\text{average velocity} = \frac{24 - 9}{3 - 2} = 15 \text{ ft/s}$$

It seems reasonable that the velocity at the instant $t = 2$ can't be much different from the average velocity during a short time interval starting at $t = 2$. So let's imagine that the distance traveled has been measured at 0.1-second time intervals as in the following table:

t	2.0	2.1	2.2	2.3	2.4	2.5
d	9.00	10.02	11.16	12.45	13.96	15.80

Then we can compute, for instance, the average velocity over the time interval [2, 2.5]:

$$\text{average velocity} = \frac{15.80 - 9.00}{2.5 - 2} = 13.6 \text{ ft/s}$$

The results of such calculations are shown in the following table:

Time interval	[2, 3]	[2, 2.5]	[2, 2.4]	[2, 2.3]	[2, 2.2]	[2, 2.1]
Average velocity (ft/s)	15.0	13.6	12.4	11.5	10.8	10.2

The average velocities over successively smaller intervals appear to be getting closer to a number near 10, and so we expect that the velocity at exactly $t = 2$ is about 10 ft/s. In Chapter 2 we will define the instantaneous velocity of a moving object as the limiting value of the average velocities over smaller and smaller time intervals.

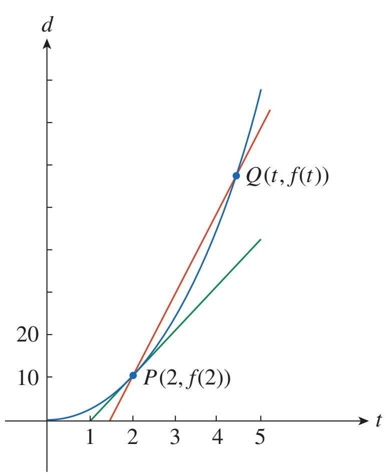


Figure 8
A graphical illustration of the motion of the car.

Figure 8 shows a graphical representation of the motion of the car by plotting the distance traveled as a function of time. If we write $d = f(t)$, then $f(t)$ is the number of feet traveled after t seconds. The average velocity in the time interval $[2, t]$ is

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t) - f(2)}{t - 2}$$

which is the same expression as the slope of the secant line PQ in Figure 8. The velocity v when $t = 2$ is the limiting value of this average velocity as t approaches 2; that is,

$$v = \lim_{t \to 2} \frac{f(t) - f(2)}{t - 2}$$

and we recognize from Equation 2 that this is the same as the slope of the tangent line to the curve at P.

Thus, when we solve the tangent line problem in differential calculus, we are also solving problems concerning velocities. The same techniques also enable us to solve problems involving rates of change in all of the natural and social sciences.

The Limit of a Sequence

In the 5th century BC, the Greek philosopher Zeno of Elea posed four problems, now known as *Zeno's paradoxes*, that were intended to challenge some of the ideas concerning space and time that were held in his day. Zeno's second paradox concerns a race between the Greek hero Achilles and a tortoise that has been given a head start. Zeno argued, as follows, that Achilles could never pass the tortoise: suppose that Achilles starts at position a_1 and the tortoise starts at position t_1. (See Figure 9.) When Achilles reaches the point $a_2 = t_1$, the tortoise is farther ahead at position t_2. When Achilles reaches $a_3 = t_2$, the tortoise is at t_3. This process continues indefinitely and so it appears that the tortoise will always be ahead! But this defies common sense.

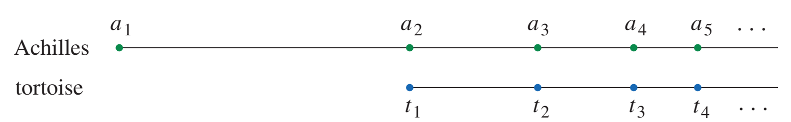


Figure 9
A graphical illustration of the race between Achilles and a tortoise.

One way of explaining this paradox is with the idea of a *sequence*. The successive positions of Achilles $(a_1, a_2, a_3, \ldots)$ or the successive positions of the tortoise $(t_1, t_2, t_3, \ldots)$ form what is known as a sequence.

In general, a sequence, denoted $\{a_n\}$, is a set of numbers written in a definite order. For instance, the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$$

can be described by using the following formula for the nth term:

$$a_n = \frac{1}{n}$$

We can visualize this sequence by plotting its terms on a number line as in Figure 10(a) or by drawing its graph as in Figure 10(b). Observe from either picture that the terms of the sequence $a_n = 1/n$ are becoming closer and closer to 0 as n increases. In fact, we can find terms as small as we please by making n large enough. We say that the limit of the sequence is 0, and we indicate this behavior by writing

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

In general, the notation

$$\lim_{n \to \infty} a_n = L$$

is used if the terms a_n approach the number L as n becomes large. This means that the numbers a_n can be made as close as we like to the number L by taking n sufficiently large.

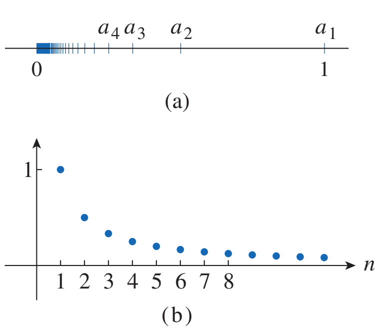


Figure 10
Two ways to visualize a sequence.

The concept of the limit of a sequence occurs whenever we use the decimal representation of a real number. For instance, if

$$\begin{aligned} a_1 &= 3.1 \\ a_2 &= 3.14 \\ a_3 &= 3.141 \\ a_4 &= 3.1415 \\ a_5 &= 3.14159 \\ a_6 &= 3.141592 \\ a_7 &= 3.1415926 \\ &\vdots \end{aligned}$$

then

$$\lim_{n \to \infty} a_n = \pi$$

The terms in this sequence are rational approximations to π.

Let's return to Zeno's paradox. The successive positions of Achilles and the tortoise form sequences $\{a_n\}$ and $\{t_n\}$, where $a_n < t_n$ for all n. It can be shown that both sequences have the same limit:

$$\lim_{n \to \infty} a_n = p = \lim_{n \to \infty} t_n$$

It is precisely at this point p that Achilles overtakes the tortoise.

The Sum of a Series

Another of Zeno's paradoxes involves a person in a room attempting to walk to a wall. As Aristotle indicated, in order to do so, they would first have to go half the distance, then half the remaining distance, and then again half of what still remains. This process can always be continued and can never be ended. (See Figure 11.)

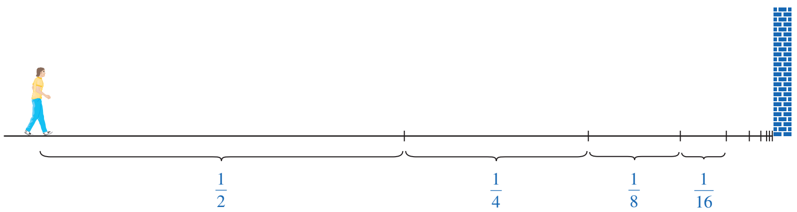


Figure 11
A visualization of a walk to the wall.

Of course, we know that the person can actually reach the wall, so this suggests that perhaps the total distance can be expressed as the sum of infinitely many smaller distances as follows:

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots \tag{3}$$

Zeno was arguing that it doesn't make sense to add infinitely many numbers together. But there are other situations in which we implicitly use infinite sums. For instance, in decimal notation, the symbol $0.\overline{3} = 0.3333\ldots$ means

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots$$

and so, in some sense, it must be true that

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots = \frac{1}{3}$$

More generally, if d_n denotes the nth digit in the decimal representation of a number, then

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_n}{10^n} + \cdots$$

Therefore, some infinite sums, or infinite series as they are called, have a meaning. But we must carefully define the sum of a series.

Returning to the series in Equation 3, the sum of the first n terms of the series is called the partial sum and is denoted by s_n. Thus

$$\begin{aligned}
s_1 &= \frac{1}{2} = 0.5 \\
s_2 &= \frac{1}{2} + \frac{1}{4} = 0.75 \\
s_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875 \\
s_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.9375 \\
s_5 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = 0.96875 \\
s_6 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = 0.984375 \\
s_7 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} = 0.9921875 \\
&\vdots \\
s_{10} &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1024} \approx 0.99902344 \\
&\vdots \\
s_{16} &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{16}} \approx 0.99998474
\end{aligned}$$

Observe that as we add more and more terms, the partial sums become closer and closer to 1. In fact, it can be shown that by taking n large enough (that is, by adding sufficiently many terms of the series), we can make the partial sum s_n as close as we please to the number 1. It therefore seems reasonable to say that the sum of the infinite series is 1 and to write

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1$$

In other words, the reason the sum of the series is 1 is that

$$\lim_{n \to \infty} s_n = 1$$

In Chapter 8 we will discuss these ideas further. We will then use Newton's idea of combining infinite series with differential and integral calculus.

Summary

We have seen that the concept of a limit arises in trying to find the area of a region, the slope of a tangent line to a curve, the velocity of a car, or the sum of an infinite series. In each case the common theme is the calculation of a quantity as the limit of other, easily calculated quantities. It is this basic idea of a limit that sets calculus apart from other areas of mathematics. In fact, we could define calculus as the part of mathematics that deals with limits.

After Sir Isaac Newton invented his version of calculus, he used it to explain the motion of the planets around the sun. Today calculus is used in calculating the orbits of satellites and spacecraft, in predicting population sizes, in estimating how fast oil prices rise or fall, in forecasting weather, in measuring the cardiac output of the heart, in calculating life insurance premiums, and in a wide variety of other areas. We will explore some of these uses of calculus in this book.

In order to convey a sense of the power of the subject, we end this preview with a list of some of the questions that you will be able to answer using calculus:

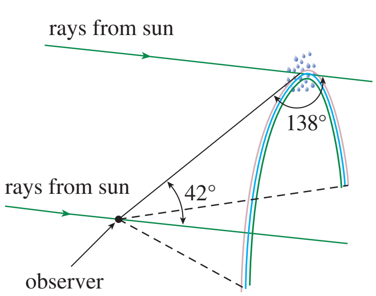
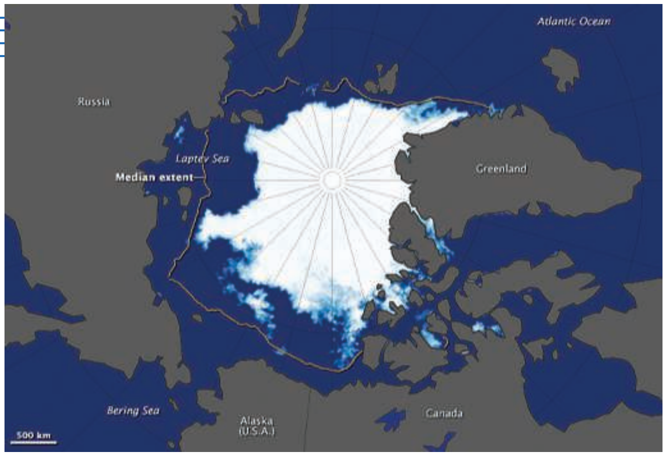
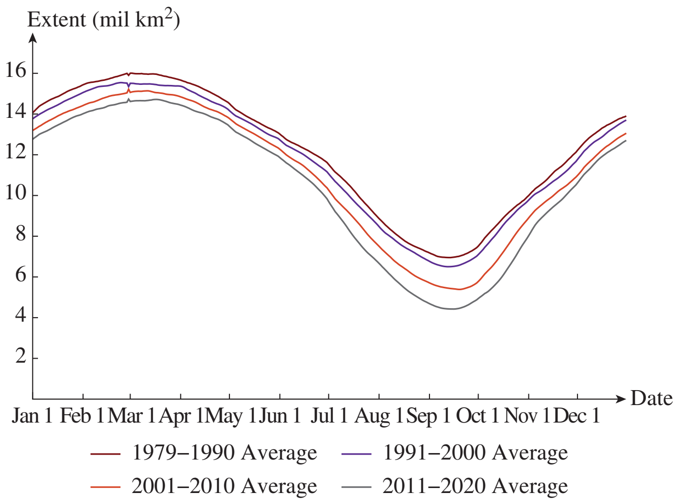


Figure 12
The elevation up to the highest point in a rainbow.

1. How can we explain the fact, illustrated in Figure 12, that the angle of elevation from an observer up to the highest point in a rainbow is 42°? (See page 313.)
2. How can we explain the shapes of cans on supermarket shelves? (See page 372.)
3. Where is the best place to sit in a movie theater? (See page 574.)
4. How far away from an airport should a pilot start descent? (See page 237.)
5. How can we fit curves together to design shapes to represent letters on a laser printer? (See page 236.)
6. Where should an infielder position themself to catch a baseball thrown by an outfielder and relay it to home plate? (See page 653.)
7. Does a ball thrown upward take longer to reach its maximum height or to fall back to its original height? (See page 643.)
8. How can we explain the fact that planets and satellites move in elliptical orbits? (See page 886.)
9. How can we distribute water flow among turbines at a hydroelectric station so as to maximize the total energy production? (See page 994.)
10. If a marble, a squash ball, a steel bar, and a lead pipe roll down a slope, which of them reaches the bottom first? (See page 1070.)



https://earthobservatory.nasa.gov/



Sea ice is simply frozen water, but the extent, or total area, of sea ice is related to the climate in the polar regions and the entire world. Sea ice reflects sunlight. Therefore, a large sea ice extent tends to keep a region cool. A small sea ice extent exposes dark water, which absorbs sunlight. The ocean tends to heat up, which warms the surrounding area. Scientists have studied the change in sea ice extent over the past four decades. Often a graph is the best way to represent this information because it conveys many significant features at a glance. For example, a graph of the average sea ice extent over four different decades is shown in the accompanying figure (National Snow and Ice Data Center). Using this information and some calculus, we will soon be able to estimate the rate of change of sea ice extent at any time of even the time of year at which the sea ice is a maximum or a minimum.

1 Functions and Models

Contents

The fundamental objects that we develop and utilize in calculus are functions. This chapter prepares the way for calculus by presenting and reviewing the basic concepts concerning functions, their graphs, and methods of transforming and combining them. Throughout this text, we stress that a function can be represented in a variety of ways: by an equation, in a table, by a graph, or in words (the Rule of Four). We look at main types of functions that occur in calculus and describe the process of using these functions as mathematical models of real-world phenomena. We will also see that parametric equations provide the best method for graphing certain types of curves.

1.1 Four Ways to Represent a Function

The idea of a function is the most important concept in all of mathematics. Understanding function notation is essential, and you must be able to work with and evaluate a function that is defined algebraically, by a table, in a graph, or even in words. Functions arise in mathematics, especially in calculus, and in real-life whenever one quantity depends on another. Here are a few examples.

Year	Population (in millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870
2020	7755

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r, there is associated one value of A, and we say that A is a *function* of r.

B. The human population of the world P depends on the time t. The table gives estimates of the world population $P(t)$ at time t, for certain years. For example,

$$P(1950) \approx 2{,}560{,}000{,}000$$

Note that for each value of the time t, there is only one corresponding value of P, and we say that P is a function of t.

C. The cost C of riding an Uber from the Fort Myers airport to a location in the area depends on the total distance traveled d. Although there is no simple rule for connecting d and C, Uber does have a formula for determining C when d is known.

D. The rate of water usage r in New York City is a function of elapsed time t. Figure 1.1 is a graph of water usage during the 2018 Super Bowl. For a given value of t, the graph provides a corresponding value of r.

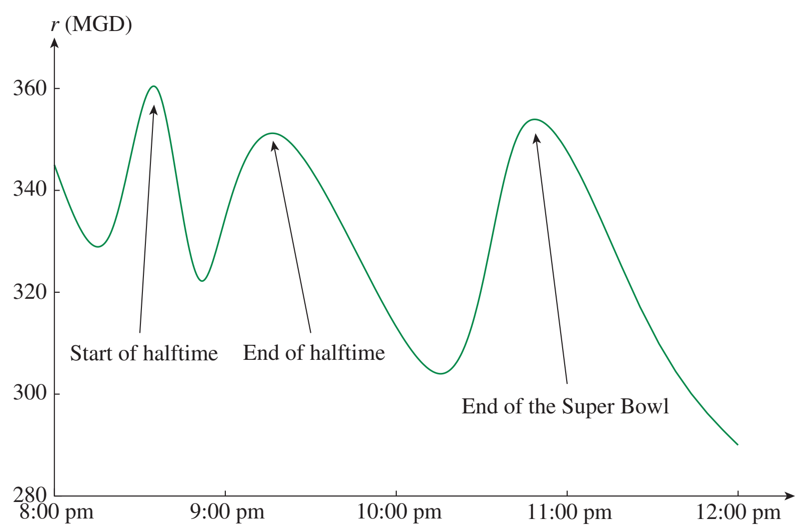


Figure 1.1
Rate of water usage in New York City during the 2018 Super Bowl.

Each of these examples describes a rule such that given a number (r, t, d, or t), another number (A, P, C, or r) is assigned. In each case we say that the second number is a function of the first number.

> A **function** is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The number $f(x)$ is the **value of *f* at *x*** and is read "f of x." The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the **range** of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.

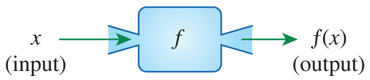


Figure 1.2
Machine diagram for a function f.

It may be helpful to think of a function as a special kind of mathematical **machine** (see Figure 1.2). If x is in the domain of the function f, then when x is placed into the machine, it is accepted as an input and the machine works with x and produces an output $f(x)$ according to the rule of the function. Thus, we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

The acronym CAS stands for Computer Algebra System. For example, the TI-Nspire™ CX CAS calculator is a handheld computer algebra system and Mathematica is a mathematical computer algebra software program.

The preprogrammed functions in a calculator are good examples of a function as a machine. For example, the square root key on your calculator represents a built-in, or preprogrammed, function. In general, you press the key labeled $\sqrt{\ }$ (or $\sqrt{x}$) and enter the input x. If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input, and the calculator will indicate an error. If $x \geq 0$, then a decimal *approximation* to $\sqrt{x}$ will appear in the display, or most CAS machines will produce an exact, symbolic simplified answer. Thus, the $\sqrt{x}$ key on your calculator is an example of a function, but it is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$.

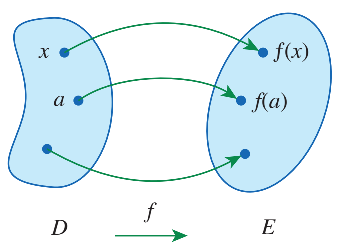


Figure 1.3
Arrow diagram illustrating a function f.

Another way to visualize a function is by an **arrow diagram** as in Figure 1.3. Each arrow connects, or indicates an assignment of, an element in the set D to an element in the set E. One arrow indicates that x is assigned to, or mapped to $f(x)$, another that a is assigned to $f(a)$, and so on.

The most common method for visualizing a function is its graph. If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

Notice that these are input-output pairs. In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

The graph of a function f provides a helpful visualization of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as simply the height of the graph above the value x (see Figure 1.4). The graph of f also allows us to visualize the domain of f on the x-axis and its range on the y-axis, as shown in Figure 1.5.

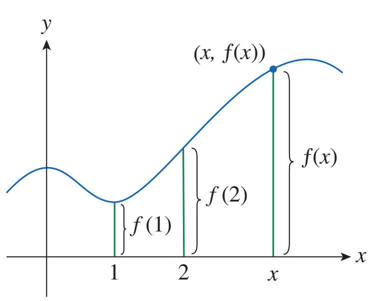


Figure 1.4
The value $f(x)$ is the height of the graph above x.

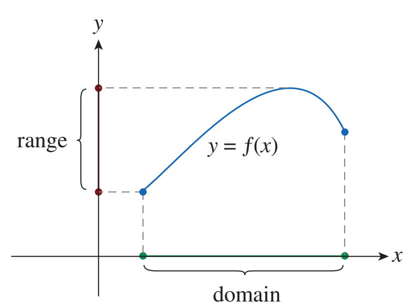


Figure 1.5
A visualization of the domain and range of a function f.

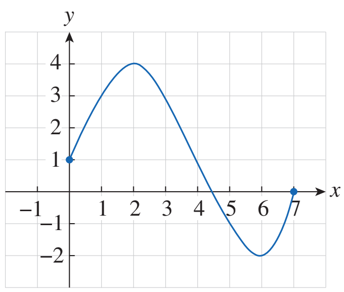


Figure 1.6
The graph of $y = f(x)$.

Example 1 Information from the Graph of *f*

The graph of a function f is shown in Figure 1.6.

(a) Find the values of $f(1)$ and $f(5)$.

(b) What are the domain and range of f?

Solution

(a) We can see from Figure 1.6 that the point $(1, 3)$ lies on the graph of f; the point on the graph that lies above $x = 1$ is 3 units above the x-axis. Therefore, the value of f at 1 is $f(1) = 3$. The function f maps the value 1 to 3.

When $x = 5$, the graph lies 1 unit below the x-axis. Therefore, $f(5) = -1$.

(b) The graph also shows that f is defined when $0 \le x \le 7$, so the domain is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4, so the range of f is $\{y \mid -2 \le y \le 4\} = [-2, 4]$. ■

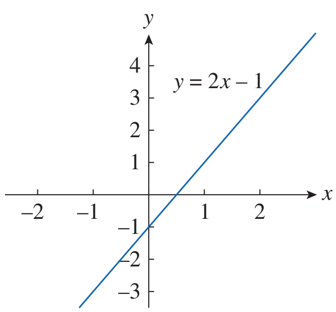


Figure 1.7
Graph of $y = 2x - 1$.

Example 2 Domain, Range, and a Sketch

Sketch the graph and find the domain and range of each function.

(a) $f(x) = 2x - 1$ (b) $g(x) = x^2$

Solution

(a) The equation of the graph is $y = 2x - 1$, and we recognize this as being the equation of a line with slope 2 and y-intercept -1. (Recall the slope-intercept form of the equation of a line: $y = mx + b$. See Appendix B.) This enables us to sketch a portion of the graph of f in Figure 1.7. The expression $2x - 1$ is defined for all real numbers, so the domain of f is the set of all real numbers, which we denote by $\mathbb{R}$. The graph shows that the range is also $\mathbb{R}$.

(b) We can use the rule for g to determine a few points on the graph and then join them to produce a rough sketch of the graph. For example, $g(2) = 2^2 = 4$ and $g(-1) = (-1)^2 = 1$. So, the points $(2, 4)$ and $(-1, 1)$ are on the graph of g. Figure 1.8 shows these two points and others and a sketch of the graph of g.

The graph of the equation $y = x^2$ is a parabola (see Appendix B). The domain is $\mathbb{R}$. The range consists of all values of $g(x)$, that is, all numbers of the form x^2. Since $x^2 \ge 0$ for all values x and any positive number y is a square, the range of g is $\{y \mid y \ge 0\} = [0, \infty)$. The domain and range of g are confirmed by using the graph (Figure 1.8). ■

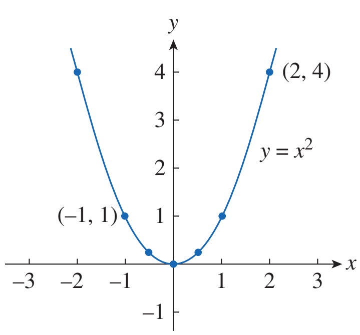


Figure 1.8
Graph of $y = x^2$.

The expression

$$\frac{f(a+h) - f(a)}{h}$$

in Example 3 is called a **difference quotient** and occurs frequently in calculus. We will see in Chapter 2 that this expression represents the average rate of change of $f(x)$ between $x = a$ and $x = a + h$.

Example 3 Difference Quotient

If $f(x) = 2x^2 - 5x + 1$ and $h \ne 0$, evaluate $\dfrac{f(a+h) - f(a)}{h}$.

Solution

Evaluate $f(a + h)$ by replacing x by $a + h$ in the expression for $f(x)$:

$$\begin{aligned} f(a+h) &= 2(a+h)^2 - 5(a+h) + 1 && \text{Evaluate } f \text{ at } a + h. \\ &= 2(a^2 + 2ah + h^2) - 5(a+h) + 1 && \text{Expand } (a+h)^2. \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 && \text{Distribute.} \end{aligned}$$

Substitute into the given expression and simplify.

$$\frac{f(a+h)-f(a)}{h} = \frac{(2a^2+4ah+2h^2-5a-5h+1)-(2a^2-5a+1)}{h}$$

Use expressions for $f(a+h)$ and $f(a)$.

$$= \frac{2a^2+4ah+2h^2-5a-5h+1-2a^2+5a-1}{h}$$

Distribute -1.

$$= \frac{4ah+2h^2-5h}{h} = 4a+2h-5$$

Simplify.

■

Representations of Functions

There are four possible ways to represent a function (the Rule of Four):

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

If a single function can be represented in all four ways, it's often useful to consider the multiple representations to gain additional insight into the characteristics of the function. In Example 2, for instance, we started with algebraic formulas and then obtained the graphs. But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

Year	Population (in millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870
2020	7755

A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula $A(r) = \pi r^2$, though it is possible to compile a table of values or to sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is $\{r \mid r > 0\} = (0, \infty)$, and the range is also $(0, \infty)$.

B. We are given a description of the function in words: $P(t)$ is the human population of the world (in millions) at time t. Let's measure t so that $t = 0$ corresponds to the year 1900. The table of values of world population provides a convenient representation of this function. If we plot these values, we obtain a *scatter plot* in Figure 1.9. It too is a useful representation; the graph allows us to visualize all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t. But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, using methods explained in Section 1.2, we obtain the approximation

$$P(t) \approx f(t) = 1358.03 \cdot (1.01478)^t$$

Figure 1.10 shows that it is a reasonably good *fit*. The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

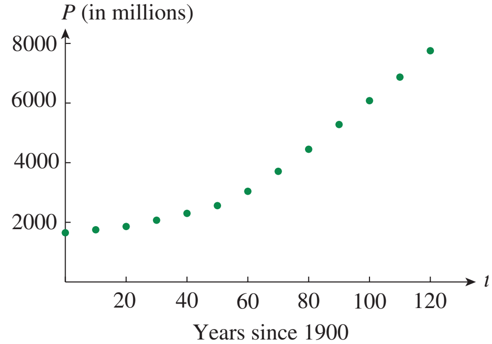


Figure 1.9
A scatter plot of the population data.

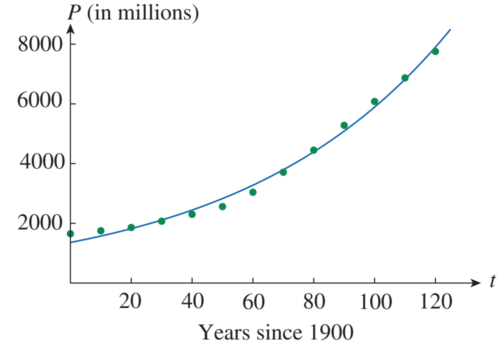


Figure 1.10
Graph of f added to the scatter plot.

The function P is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don't have complete knowledge of the values of the function, we will see throughout the book that it is still possible to perform the operations of calculus on such a function.

The function C may be given by, or represented by, a table of values.

d (miles)	$C(d)$ (dollars)
$0 < d \le 1$	5:00
$1 < d \le 2$	7:50
$2 < d \le 3$	10:00
$3 < d \le 4$	12:50
$4 < d \le 5$	15:00
$\vdots$	$\vdots$

C. The function representing the cost of a ride is described in words. Let $C(d)$ be the cost of riding an Uber from the Fort Myers airport to a location d miles away. The table of values in the margin represents possible values of this function at a specific time of day and is a convenient way to represent this function. It is also possible to sketch a graph of this function. See Example 10.

D. The graph shown in Figure 1.1 is a very natural representation of the rate of water usage $r(t)$. We could certainly compile a table of values, and it is even possible to construct an approximate formula for $r(t)$. But everything the water department needs to know, amplitudes and patterns, can be seen easily from the graph. The same is true for the patterns observed in electrocardiograms of heart patients, polygraphs for lie-detection, and even the vertical acceleration of the ground during an earthquake.

In the next example, we will use a verbal description to sketch the graph of a function.

Example 4 Function Defined Verbally

When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

Solution

The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. As hot water flows from the tank, T is constant at the temperature of the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t in Figure 1.11. ■

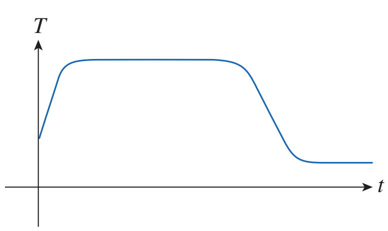


Figure 1.11
Graph of the temperature of the water versus time.

In Example 5 we start with a verbal description of a function in a physical situation and obtain an explicit algebraic formula. This skill, translating a verbal description into an explicit formula, is extremely useful in solving calculus problems that ask for the maximum or minimum values of quantities.

Example 5 Cost of Materials

A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

Solution

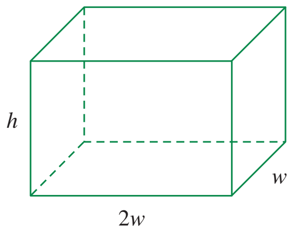


Figure 1.12
Rectangular storage container.

We begin with the diagram of a rectangular storage container in Figure 1.12 and introduce notation by letting w and $2w$ be the width and length of the base, respectively, and h be the height.

The area of the base is $(2w)w = 2w^2$, so the cost, in dollars, of the material for the base is $10(2w^2)$. Two of the sides have area wh and the other two have area $2wh$, so the cost of the material for the sides is $6[2(wh) + 2(2wh)]$. The total cost is therefore

$$C = 10(2w^2) + 6[2(wh) + 2(2wh)] = 20w^2 + 36wh.$$

To express C as a function of w alone, we need to eliminate h. We do so by using the fact that the volume is 10 m^3. Thus

$$w(2w)h = 10.$$

Solving for h, we have

$$h = \frac{10}{2w^2} = \frac{5}{w^2}.$$

In constructing applied functions as in Example 5, it may be useful to review the principles of problem solving as discussed on page 85, particularly *Step 1*: *Understand the Problem*.

Substituting this into the expression for C, we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}.$$

Therefore, the equation

$$C(w) = 20w^2 + \frac{180}{w} \quad w > 0$$

is an expression for C as a function of w. ■

The next example involves the domain of a function.

Example 6 Domain of a Function

Find the domain of each function.

(a) $f(x) = \sqrt{x + 2}$ (b) $g(x) = \dfrac{1}{x^2 - x}$

Domain Convention

If a function is given by a formula and the domain is not explicitly stated, the convention is that the domain is the set of all (real) numbers for which the function is defined.

Solution

(a) The square root of a negative number is not defined (as a real number). Therefore, the domain of f consists of all values of x such that $x + 2 \geq 0$. This is equivalent to $x \geq -2$. The domain is the interval $[-2, \infty)$.

(b) Factor the denominator in g.

$$\frac{1}{x^2 - x} = \frac{1}{x(x-1)}$$

Since we cannot divide by 0, $g(x)$ is not defined when $x = 0$ or $x = 1$.

Therefore, the domain of g is $\{x \,|\, x \neq 0, x \neq 1\}$.

The domain can also be written using interval notation:

$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$ ■

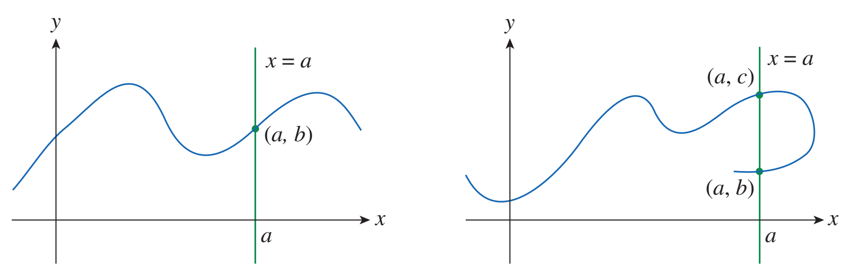
The graph of a function is a curve in the xy-plane. But suppose instead we start with a curve. How do we know if the curve is the graph of a function? We can answer this question by using the Vertical Line Test.

The Vertical Line Test

A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

The Vertical Line Test is illustrated in Figure 1.13. If every vertical line $x = a$ intersects a curve at most once, at (a, b), then exactly one function value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve cannot represent a function because a function cannot assign two different values to a.

Note: There is no violation of the Vertical Line Test if a line $x = a$ does not intersect the curve. The curve could still be the graph of a function such that $x = a$ is not in the domain.

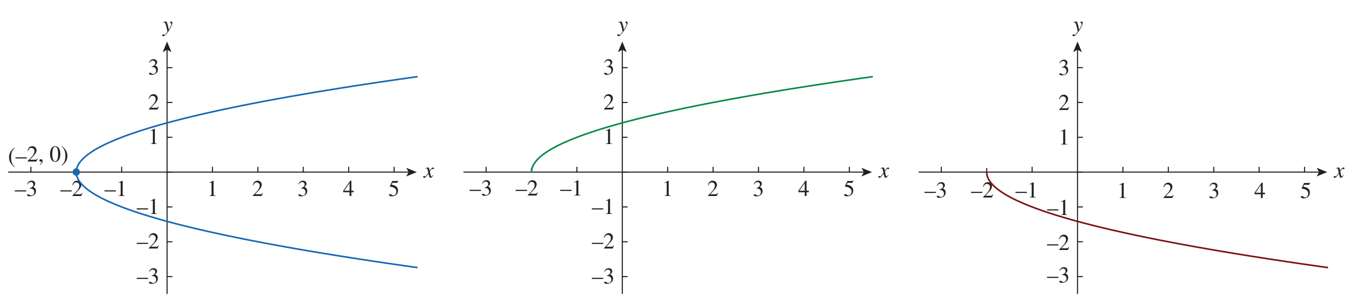


(a) This curve represents a function.

(b) This curve does not represent a function.

Figure 1.13
An illustration of the Vertical Line Test.

For example, the parabola $x = y^2 - 2$ shown in Figure 1.14(a) is not the graph of a function of x because there are vertical lines that intersect the parabola twice.



(a) $x = y^2 - 2$

(b) $y = \sqrt{x+2}$

(c) $y = -\sqrt{x+2}$

Figure 1.14
A curve may contain the graph of one or more functions.

The parabola, however, does contain the graphs of *two* functions of x. Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x + 2}$.

Therefore, the upper and lower halves of the parabola are the graphs of the functions $f(x) = \sqrt{x + 2}$ [from Example 6(a); see Figure 1.14(b)] and $g(x) = -\sqrt{x + 2}$. See Figure 1.14(c).

If we reverse the roles of x and y, then the equation $x = h(y) = y^2 - 2$ *does* define x as a function of y (with y as the independent variable and x the dependent variable) and the parabola in Figure 1.14(a) is the graph of the function h.

Piecewise Defined Functions

The functions in the following four examples are defined by different formulas in different parts of their domains. These functions are called **piecewise defined functions**.

Example 7 Evaluate and Graph a Piecewise Defined Function

The function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$, and sketch the graph of f.

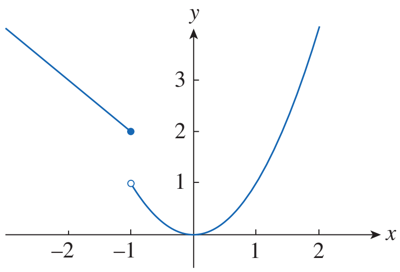


Figure 1.15
Graph of f.

Solution

Remember that a function is a rule. For this particular function, the rule is the following:

Consider the value of the input x.

If $x \leq -1$, then the value of $f(x)$ is $1 - x$.

However, if $x > -1$, then the value of $f(x)$ is x^2.

Since $-2 \leq -1$, then $f(-2) = 1 - (-2) = 3$.

Since $-1 \leq -1$, then $f(-1) = 1 - (-1) = 2$.

Since $0 > -1$, then $f(0) = 0^2 = 0$.

To sketch the graph of f, notice that if $x \leq -1$, then $f(x) = 1 - x$. Therefore, the part of the graph of f that lies to the left of the vertical line $x = -1$ must coincide with the line $y = 1 - x$. This line has slope -1 and y-intercept 1.

If $x > -1$, then $f(x) = x^2$. So, the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola.

The graph of f is shown in Figure 1.15. The solid dot indicates that the point $(-1, 2)$ is included on the graph. The open dot indicates that the point $(-1, 1)$ is excluded from the graph. ■

For a more extensive review of the absolute value function, see Appendix A.

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geq 0 \quad \text{for every number } a$$

For example,

$$|3| = 3, \quad |-3| = 3, \quad |0| = 0, \quad |\sqrt{2} - 1| = \sqrt{2} - 1, \quad |3 - \pi| = \pi - 3$$

In general, we have

$$\begin{aligned} |a| &= a && \text{if } a \geq 0 \\ |a| &= -a && \text{if } a < 0 \end{aligned}$$

Remember that if a is negative, then $-a$ is positive.

Example 8 Absolute Value Function

Sketch the graph of the absolute value function $f(x) = |x|$.

Solution

The preceding argument implies

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 7, the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the graph of $y = -x$ to the left of the y-axis. See Figure 1.16. ■

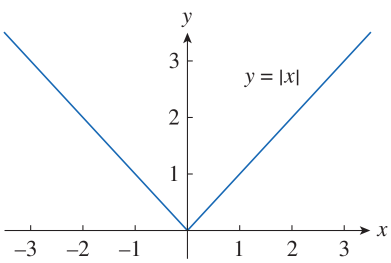


Figure 1.16
Graph of the absolute value function.

Example 9 Formula from a Graph

Find a formula for the function whose graph is given in Figure 1.17.

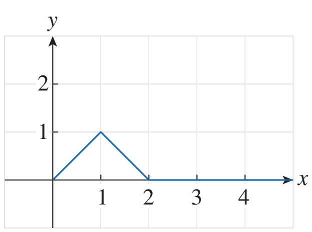


Figure 1.17
Graph of the function f.

Recall that the point-slope form of the equation of a line is $y - y_1 = m(x - x_1)$. See Appendix B.

Solution

The line through (0, 0) and (1, 1) has slope $m = 1$ and y-intercept $b = 0$. Its equation is $y = x$. Thus, for the part of the graph of f that joins (0, 0) to (1, 1), we have

$f(x) = x \quad \text{if } 0 \leq x \leq 1.$

The line through (1, 1) and (2, 0) has slope $m = -1$, so its point-slope form is

$y - 0 = (-1)(x - 2) \quad \text{or} \quad y = 2 - x.$

Therefore, $f(x) = 2 - x \quad \text{if } 1 < x \leq 2.$

We also see that the graph of f coincides with the x-axis for $x > 2$. Using this information, we have the following piecewise definition for f:

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \\ 0 & \text{if } x > 2 \end{cases}$$ ■

Example 10 An Uber Ride

In Example C at the beginning of this section, we considered the cost, $C(d)$, of an Uber ride from the Fort Myers airport to a destination d miles away. The cost is a piecewise defined function because, using the table of values given we have

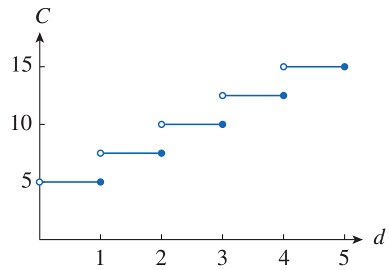


Figure 1.18
Graph of the cost function C.

$$C(d) = \begin{cases} 5.00 & \text{if } 0 < d \leq 1 \\ 7.50 & \text{if } 1 < d \leq 2 \\ 10.00 & \text{if } 2 < d \leq 3 \\ 12.50 & \text{if } 3 < d \leq 4 \\ \vdots & \end{cases}$$

The graph of C is shown in Figure 1.18.

You can see why functions similar to this one are called **step functions**–they jump, or step, from one value to the next. We will study these function in more detail in Chapter 2. ■

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For example, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

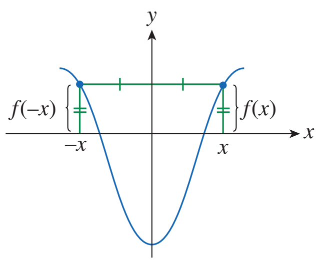


Figure 1.19
Graph of an even function.

The graph of an even function is symmetric with respect to the y-axis (see Figure 1.19). This means that if we plot the graph of f for $x \geq 0$, we can obtain the entire graph simply by reflecting this portion about the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

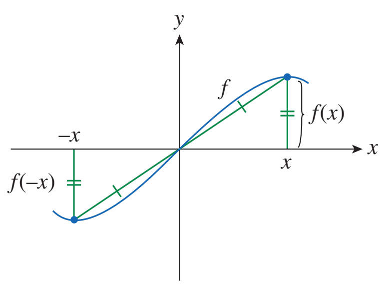


Figure 1.20
Graph of an odd function.

The graph of an odd function is symmetric about the origin (see Figure 1.20). If we already have the graph of f for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

Example 11 Even, Odd, or Neither

Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

Solution

(a) $f(-x) = (-x)^5 + (-x) = (-1)^5 x^5 + (-x)$ Evaluate f at $-x$.
$= -x^5 - x = -(x^5 + x)$ Simplify.
$= -f(x)$ Compare result with $f(x)$.

Therefore, f is an odd function.

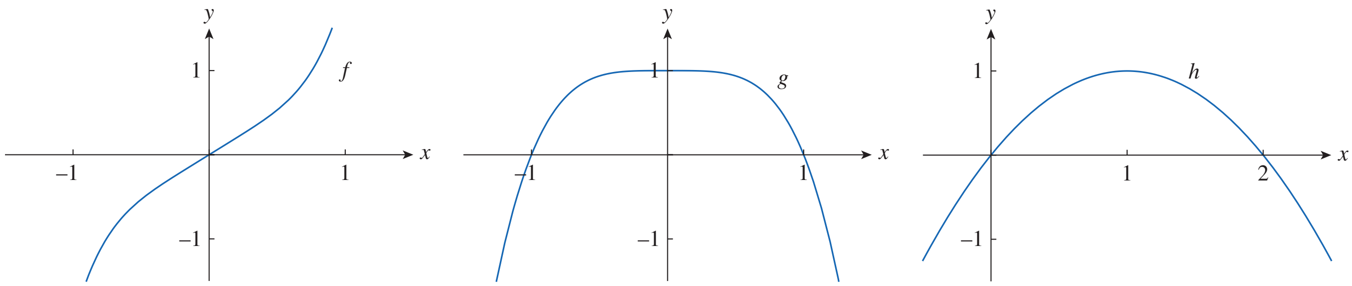
(b) $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

Therefore, g is an even function.

(c) $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

The graphs of the functions in Example 11 are shown in Figure 1.21. Notice that the graph of h is symmetric but not about the y-axis nor about the origin.



(a) Graph of f (b) Graph of g (c) Graph of h

Figure 1.21
Graphs of the functions in Example 11.

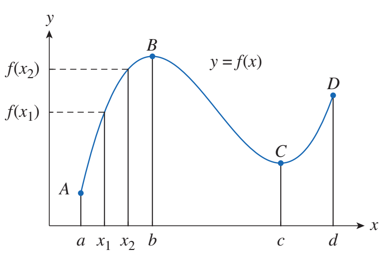


Figure 1.22
The graph is increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing on $[c, d]$.

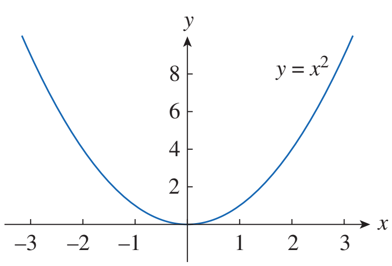


Figure 1.23
Graph of $y = x^2$.

Increasing and Decreasing Functions

The graph in Figure 1.22 rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$. Notice that if x_1 and x_2 are any two numbers in the interval $[a, b]$ with $x_1 < x_2$, then $f(x_1) < f(x_2)$. This is the defining property of an increasing function.

A function f is **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

The function is **decreasing** on I if

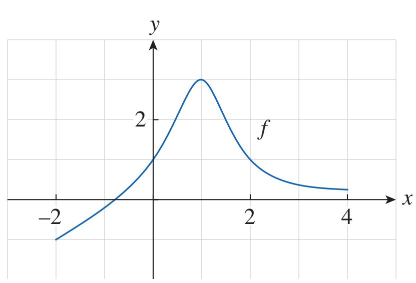
$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

In the definition of an increasing function, it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

The graph of the function $f(x) = x^2$ is given in Figure 1.23. Using the definition, the function f is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

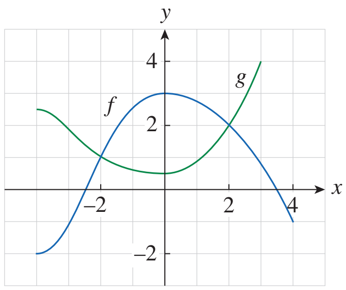
1.1 Exercises

1. The graph of a function f is given in the figure.



(a) Find the value of $f(1)$.
(b) Estimate the value of $f(-1)$.
(c) For what value(s) of x is $f(x) = 1$?
(d) Estimate the value of x such that $f(x) = 0$.
(e) State the domain and range of f.
(f) On what interval is f increasing?

2. The graphs of f and g are given in the figure below.



(a) Find the values of $f(-4)$ and $g(3)$.
(b) For what value(s) of x is $f(x) = g(x)$?
(c) Find all solutions to the equation $f(x) = -1$.
(d) On what interval is f decreasing?
(e) State the domain and range of f.
(f) State the domain and range of g.

3. If $f(x) = 3x^2 - x + 2$, find each of the following.
(a) $f(2)$ (b) $f(-2)$ (c) $f(a)$ (d) $f(-a)$
(e) $f(a+1)$ (f) $2f(a)$ (g) $f(2a)$ (h) $f(a^2)$
(i) $[f(a)]^2$ (j) $f(a+h)$

Evaluate the difference quotient for the given function. Simplify your answer.

4. $f(x) = 4 + 3x - x^2$, $\quad \dfrac{f(3+h) - f(3)}{h}$

5. $f(x) = x^3$, $\quad \dfrac{f(a+h) - f(a)}{h}$

6. $f(x) = \dfrac{1}{x}$, $\quad \dfrac{f(x) - f(a)}{x - a}$

7. $f(x) = \dfrac{x+3}{x+1}$, $\quad \dfrac{f(x) - f(1)}{x - 1}$

Find the domain of each function.

8. $f(x) = \dfrac{x+4}{x^2 - 9}$

9. $f(x) = \dfrac{2x^3 - 5}{x^2 + x - 6}$

10. $f(t) = \sqrt[3]{2t - 1}$

11. $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$

12. $h(x) = \dfrac{1}{\sqrt[4]{x^2 - 5x}}$

13. $F(p) = \sqrt{2 - \sqrt{p}}$

14. $f(u) = \dfrac{u+1}{1 + \dfrac{1}{u+1}}$

15. Sketch the graph of each of the following functions.
(a) $f(x) = |x + 1| - 2$ (b) $f(x) = |x| - 2$
(c) $f(x) = -|x + 2| + 3$ (d) $f(x) = 4 - |x|$
(e) $f(x) = 2 - |x + 4|$ (f) $f(x) = 5 + |x^2|$

16. Sketch the graph of each of the following functions.
(a) $g(x) = (x-1)^2 + 2$ (b) $g(x) = x^2 - 3$
(c) $g(x) = -(x-2)^2 + 3$ (d) $g(x) = 5 - (x+3)^2$
(e) $g(x) = 4 + (x+2)^2$

17. Find the domain and range, and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

Find the domain and sketch the graph of each function.

18. $f(x) = 1.6x - 2.4$

19. $g(t) = \dfrac{t^2 - 1}{t + 1}$

20. $f(x) = \dfrac{x - 1}{x^2 - 1}$

21. $f(x) = x^3 - 1$

22. $h(x) = \dfrac{x^2}{x + 1}$

23. $g(x) = \dfrac{\sqrt{x^2}}{x^3}$

For each piecewise defined function, evaluate $f(-3)$, $f(0)$, and $f(2)$, and sketch the graph of the function.

24. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 1 - x & \text{if } x \geq 0 \end{cases}$

25. $f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x < 2 \\ 2x - 5 & \text{if } x \geq 2 \end{cases}$

26. $f(x) = \begin{cases} x + 1 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

27. $f(x) = \begin{cases} -1 & \text{if } x \leq 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$

Find a algebraic expression for the function whose graph is described.

28. The line segment joining the points $(1, -3)$ and $(5, 7)$.

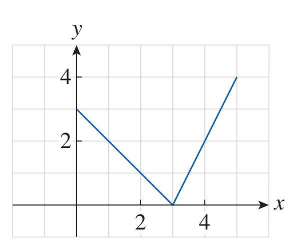
29. The line segment joining the points $(-5, 10)$ and $(7, -10)$.

30. The bottom half of the parabola $x + (y-1)^2 = 0$.

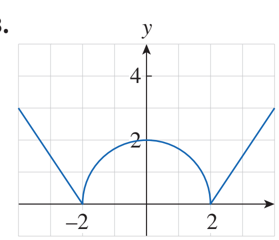
31. The top half of the circle $x^2 + (y-2)^2 = 4$.

Find a piecewise definition for the function whose graph is shown.

32.



33.

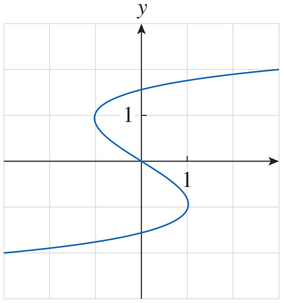


34. Figure 1.1 shows the graph of water usage in New York City around the time of the 2018 Super Bowl. Use this figure to estimate the range of the rate of water usage during this time period.

35. In this section we discussed examples of ordinary, everyday functions: population is a function of time, the cost of an Uber ride is a function of distance, and water temperature is a function of time. Give three examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a graph of each function.

Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

36.



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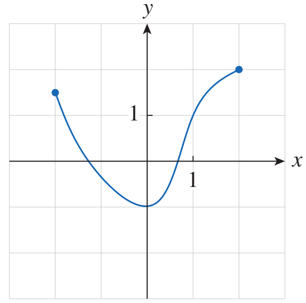

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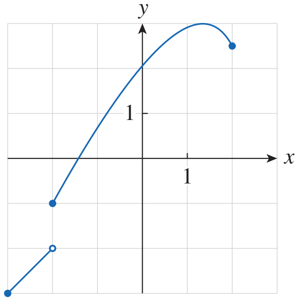

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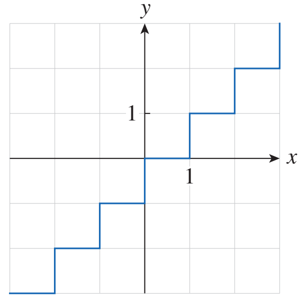
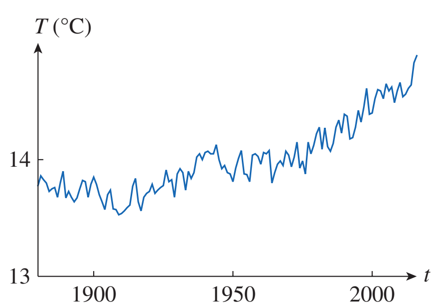
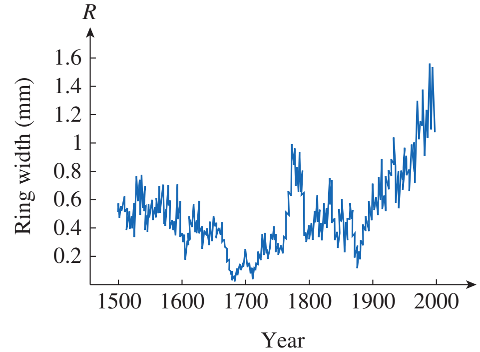


40. The graph below shows the global average temperature T from 1880 to 2016.



Estimate the following.
(a) The global average temperature in 1950.
(b) The year when the average temperature was 14.2˚C.
(c) The year when the temperature was highest. Lowest.
(d) The range of T.

41. Trees grow faster and form wider rings in warm years and grow more slowly and form narrower rings in cooler years. The figure shows ring widths of Siberian pine from 1500 to 2000.

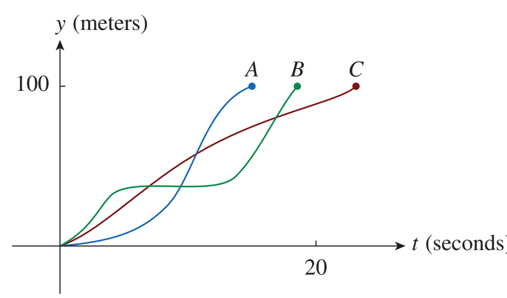


Source: Adapted from G. Jacoby et al., "Mongolian Tree Rings and 20th-Century Warming," *Science* 273 (1996): 771–73.

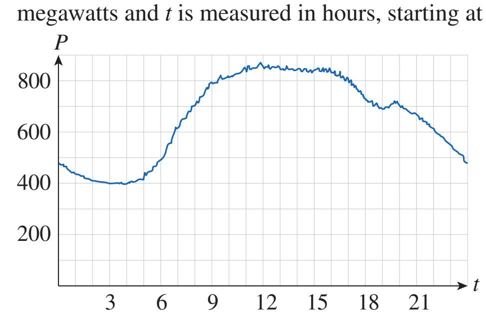
(a) What is the range of the ring width function?
(b) What does the graph tend to say about the temperature of Earth? Does the graph reflect the volcanic eruptions of the mid-19th century? Justify your answer.

42. Suppose you put some ice cubes in a glass, fill the glass with water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Sketch a graph of the temperature of the water as a function of the elapsed time.

43. Three runners compete in a 100 meter race. The graph depicts the distance run as a function of time for each runner. Describe in words what the graph tells you about this race. Who won the race? Did each runner finish the race?



44. The graph shows the power consumption for a day in September in San Francisco where P is measured in megawatts and t is measured in hours, starting at midnight.



(a) What was the power consumption at 6 AM? At 6 PM?
(b) When was the power consumption the lowest? When was it the highest? Do these times seem reasonable? Justify your answer.

45. Sketch a graph of the number of hours of daylight as a function of the time of year.

46. Sketch a graph of the outdoor temperature as a function of time during a typical spring day.

47. Sketch a graph of the market value of a new car as a function of time for a period of 20 years. Assume the car is well maintained.

48. Sketch a rough graph of the amount of a particular brand of coffee sold by a store as a function of the price of the coffee.

49. Suppose you place a frozen pie in an oven and bake it for an hour, then take it out and let it cool. Describe how the temperature of the pie changes as time passes. Sketch a graph of the temperature of the pie as a function of time.

50. A homeowner mows the lawn every Wednesday afternoon. Sketch a graph of the height of the grass as a function of time over the course of a 4-week period.

51. An airplane takes off from an airport and lands an hour later at another airport, 400 mi away. If t represents the time in minutes since the plane has left the terminal, let $x(t)$ be the horizontal distance traveled and $y(t)$ be the altitude of the plane.
(a) Sketch a possible graph of x as a function of t.
(b) Sketch a possible graph of y as a function of t.
(c) Sketch a possible graph of the ground speed of the airplane as a function of t.
(d) Sketch a possible graph of the vertical velocity of the airplane as a function of t.

52. Temperature readings T (in °F) were recorded every 2 hours from midnight to 2:00 PM in Charleston, SC, on January 7, 2020. The time t was measured in hours from midnight. The values are given in the table.

t	0	2	4	6	8	10	12	14
T	44	43	45	46	47	58	66	66

(a) Use the values in the table to sketch a graph of T as a function of t.
(b) Use your graph to estimate the temperature at 9:00 AM.

53. Researchers measured the blood alcohol concentration (BAC) of eight adult male subjects after rapid consumption of 30 mL of ethanol (corresponding to two standard alcoholic drinks). The table shows the data obtained by averaging the BAC (in mg/mL) of the eight males.

t (hours)	BAC	t (hours)	BAC
0.00	0.00	1.75	0.22
0.20	0.25	2.00	0.18
0.50	0.41	2.25	0.15
0.75	0.40	2.50	0.12
1.00	0.33	3.00	0.07
1.25	0.29	3.50	0.03
1.50	0.24	4.00	0.01

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

(a) Use the readings to sketch the graph of the BAC as a function of t.
(b) Use your graph to describe how the effect of alcohol varies with time.

54. A spherical balloon with radius r inches has volume $V(r) = \frac{4}{3}\pi r^3$. Find a function that represents the amount of air required to inflate the balloon from a radius of r inches to a radius of $r + 1$ inches.

Find a formula for the function described, and state its domain.

55. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.

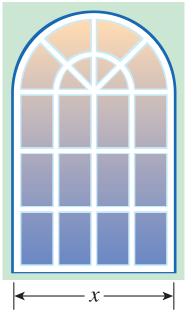
56. A rectangle has area 16 m^2. Express the perimeter of the rectangle as a function of the length of one of its sides.

57. Express the area of an equilateral triangle as a function of the length of a side.

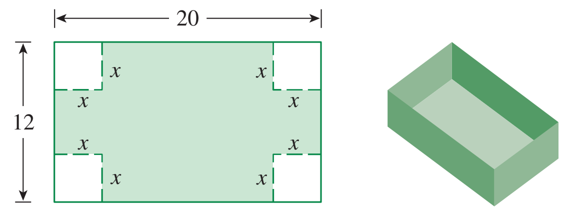
58. A closed rectangular box with volume 8 ft^3 has length twice its width. Express the height of the box as a function of the width.

59. An open rectangular box with volume 2 m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

60. A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.



61. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as shown in the figure. Express the volume V of the box as a function of x.



62. A cell phone plan has a basic charge of \$35 a month. The plan includes 2 gb of data and charges \$5 for each additional gb of data use. Write the monthly cost C as a function of the amount x of data used and graph C as a function of x for $0 \leq x \leq 10$.

63. In a certain state, the maximum speed permitted on freeways is 65 mi/h and the minimum speed is 40 mi/h. The fine for violating these limits is \$15 for every mile per hour above the maximum speed or below the minimum speed. Express the amount of the fine F as a function of the driving speed x and graph F for $0 \leq x \leq 100$.

64. An electricity company charges its customers a base rate of \$10 a month, plus 6 cents per kilowatt-hour (kWh) for the first 1200 kWh and 7 cents per kWh for all usage over 1200 kWh. Express the monthly cost E as a function of the amount x of electricity used. Then graph the function E for $0 \leq x \leq 2000$.

65. In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.

(a) Sketch the graph of the tax rate R as a function of the income I.

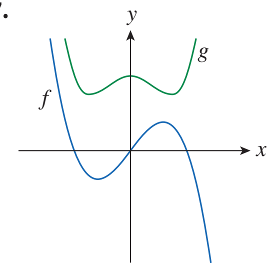
(b) How much tax is assessed on an income of \$14,000?

(c) Sketch the graph of the total assessed tax T as a function of the income I.

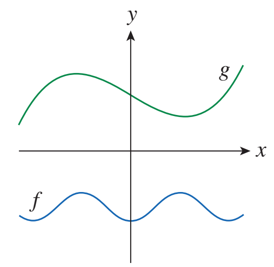
66. The function in Example 10 is called a *step function* because its graph looks like stairs. Give two other examples of step functions that arise in everyday life.

Graphs of f and g are shown. Decide whether each function is even, odd, or neither. Explain your reasoning.

67.



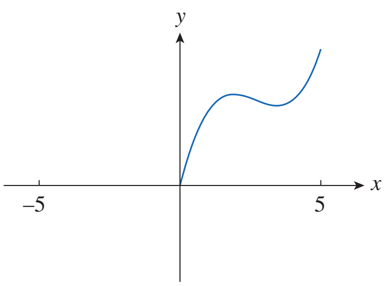
68.



69. (a) If the point (5, 3) is on the graph of an even function, what other point must also be on the graph?

(b) If the point (5, 3) is on the graph of an odd function, what other point must also be on the graph?

70. A function f has domain $[-5, 5]$ and a portion of its graph is shown in the figure.



(a) Complete the graph of f if it is known that f is even.

(b) Complete the graph of f if it is known that f is odd.

Determine whether f is even, odd, or neither. Use technology to check your answer graphically.

71. $f(x) = \dfrac{x}{x^2 + 1}$

72. $f(x) = \dfrac{x^2}{x^4 + 1}$

73. $f(x) = \dfrac{x}{x + 1}$

74. $f(x) = x|x|$

75. $f(x) = 1 + 3x^2 - x^4$

76. $f(x) = 1 + 3x^3 - x^5$

---

77. If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.

78. If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

1.2 Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Figure 1.24 illustrates the process of mathematical modeling. Given a real-world problem, the first task is to formulate a mathematical model by identifying and naming the independent and dependent variables and making assumptions that simplify the phenomenon enough to describe mathematically. We use our knowledge of the physical situation and our mathematical skills to obtain equations that relate the variables. In situations where there is no physical law to guide us, we may need to collect data (either from a reliable source or by conducting our own experiments) and examine the data in the form of a table in order to discern patterns. From this numerical representation of a function, we may wish to obtain a graphical representation by plotting the data. The graph might even suggest a suitable algebraic formula in some cases.

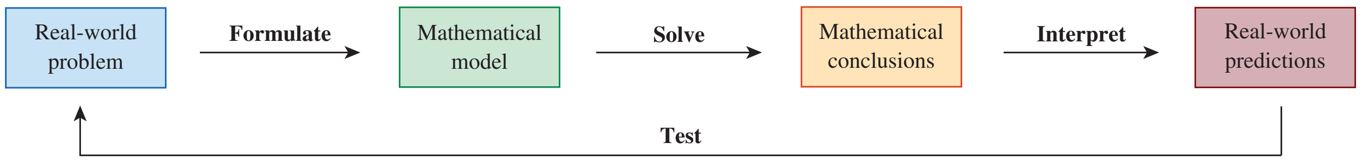


Figure 1.24
The modeling process.

The second step is to apply the mathematics that we know (such as the calculus that will be developed throughout this text) to the mathematical model that we have formulated in order to derive mathematical conclusions. Then, in the third stage, we take those mathematical conclusions and interpret them as information about the original real-world phenomenon by way of offering explanations or making predictions. The final step is to test our predictions by checking against new real-world data. If the predictions do not compare well with reality, we need to refine our model or formulate a new model and start the cycle again.

A mathematical model is never a completely accurate representation of a physical situation– it is an *idealization*. A good model simplifies reality enough to permit mathematical calculations and is accurate enough to provide valuable conclusions. It is important to realize the limitations of the model. In the end, Mother Nature has the final say.

There are many different types of functions that can be used to model relationships observed in the real world. In this section, we discuss the behavior and graphs of some of these functions and give examples of situations appropriately modeled by such functions.

Linear Models

The coordinate geometry of lines is reviewed in Appendix B.

If y is a **linear function** of x, then the graph of the function is a line. We can use the slope-intercept form of the equation of a line to write an algebraic expression for the function as

$$y = f(x) = mx + b$$

Another way to write the equation of a line is $y = a + bx$. This representation is helpful in the study of linear approximations and Taylor polynomials.

where m is the slope of the line and b is the y-intercept.

A characteristic feature of linear functions is that they change at a constant rate. For instance, Figure 1.25 shows a table of selected function values and a graph of the linear function $f(x) = 3x - 2$. Notice that whenever x increases by 1, the value of $f(x)$ increases by 3. So, $f(x)$ increases three times as fast as x. The slope of the graph $y = 3x - 2$, namely 3, can be interpreted as the rate of change of $f(x)$ with respect to x.

Note that we use the same scale on both axes to present a graph without any distortion. However, we often use different scales along the x- and y-axes in order to present a more *complete* graph.

x	$f(x) = 3x - 2$
1	1
2	4
3	7
4	10
5	13
6	16

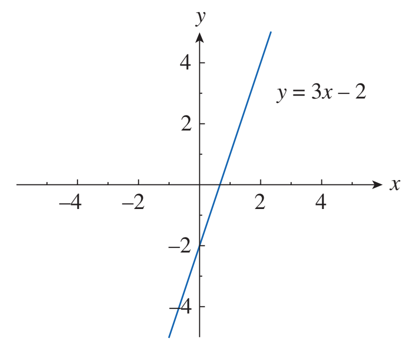


Figure 1.25
Table of values and a graph of $y = 3x - 2$.

Example 1 Linear Model Interpretation

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.

(b) Draw the graph of the function in part (a). What does the slope represent?

(c) What is the temperature at a height of 2.5 km?

Solution

(a) Because we are assuming that T is a linear function of h, we can write

$T = mh + b.$

We are given that $T = 20$ when $h = 0$. Therefore, $20 = m \cdot 0 + b = b$

So, the T-intercept is $b = 20$.

We are also given that $T = 10$ when $h = 1$. So $10 = m \cdot 1 + 20$.

The slope of the line is $m = 10 - 20 = -10$, and the linear function is

$T = -10h + 20.$

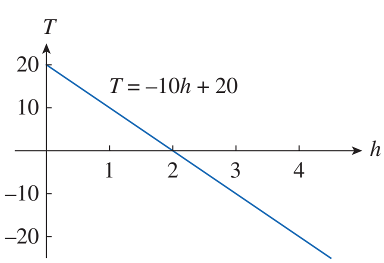


Figure 1.26
Graph of the function T.

(b) A graph of the function T is shown in Figure 1.26. The slope is $m = -10°\text{C/km}$, and this represents the rate of change of temperature with respect to height.

(c) At a height of $h = 2.5$ km, the temperature is

$T = -10(2.5) + 20 = -5°\text{C}.$ ■

If there is no physical law or principle to help us formulate a model, we construct an **empirical model**, which is based entirely on collected data. We would like to construct a curve that *fits* the data in the sense that it captures the basic trend of the data points.

Example 2 Linear Regression Model

Table 1.1 lists the average carbon dioxide level in the atmosphere, measured in parts per million, at Mauna Loa Observatory from 1980 to 2018. Use the data in Table 1.1 to find a model for the carbon dioxide level.

Solution

Use the data in Table 1.1 to construct a scatter plot as shown in Figure 1.27, where t represents the year and C represents the CO_2 level (in parts per million, ppm).

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	2000	369.4
1982	341.2	2002	373.2
1984	344.4	2004	377.5
1986	347.2	2006	381.9
1988	351.5	2008	385.6
1990	354.2	2010	389.9
1992	356.3	2012	393.8
1994	358.6	2014	398.6
1996	362.4	2016	404.2
1998	366.5	2018	408.5

Table 1.1
Table of average CO_2 levels. (Mauna Loa Observatory)

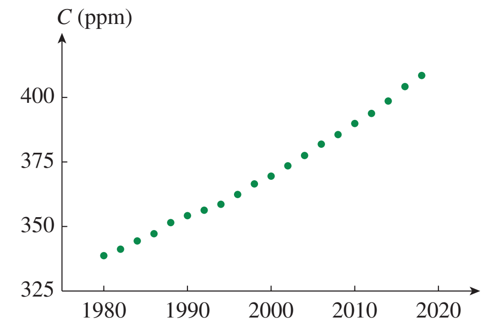


Figure 1.27
Scatter plot of the average CO_2 level versus year.

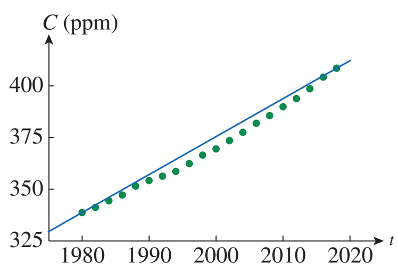


Figure 1.28
Scatter plot and a graph of the linear model constructed using the first and last data points.

A computer or graphing calculator finds the regression line by the method of **least squares**, which is to minimize the sum of the squares of the vertical distances between the data points and the line.

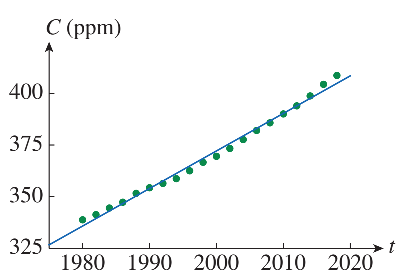


Figure 1.29
Scatter plot and a graph of the regression line.

Notice that the data points appear to lie close to a straight line, so it's intuitive and natural to choose a linear model in this case. But there are many possible lines that approximate these data, so which one should we use? One possibility is the line that passes through the first and last data points. The slope of this line is

$$\frac{408.5 - 338.7}{2018 - 1980} = \frac{69.8}{38} = 1.83684 \approx 1.837.$$

The equation of the line is

$$C - 338.7 = 1.837(t - 1980) \quad \text{or}$$

$$C = 1.837t - 3298.56 \tag{1}$$

Equation 1 is a possible linear model for the average carbon dioxide in the atmosphere. Figure 1.28 shows the scatter plot and a graph of this linear model.

Notice that this model produces higher values than most of the actual average CO_2 levels. A better model may be obtained by a statistical procedure called *linear regression*. Using Mathematica and the Fit command, the least squares model for the CO_2 level is

$$C = 1.81579t - 3259.58 \tag{2}$$

Figure 1.29 shows a scatter plot and a graph of the regression line. Comparing this graph with Figure 1.28, we see that the regression line appears to provide a better fit than the linear model obtained using only two data points. ■

Example 3 Using a Linear Model for Prediction

Use the linear model given by Equation 2 to estimate the average CO_2 level for 1987 and to predict the level for the year 2020. According to this model, when will the CO_2 level exceed 420 parts per million?

Solution

Use Equation 2 with $t = 1987$. The estimate for the average CO_2 level in 1987 is $C(1987) = (1.81579)(1987) - 3259.58 \approx 348.4$.

This is an example of *interpolation* because we have estimated a value *between* observed values. In fact, the Mauna Loa Observatory reported that the average CO_2 level in 1987 was 348.93 ppm, so our estimate is quite accurate.

With $t = 2020$, we get

$C(2020) = (1.81579)(2020) - 3259.58 \approx 408.31.$

Therefore, using this linear model, we predict that the average CO_2 level in the year 2020 will be 408.31 ppm. This is an example of *extrapolation* because we have predicted a value *outside* the given range of observations. Consequently, we are less certain about the accuracy of this prediction.

The actual average CO_2 level in the year 2020 was 414.24 greater than our predicted value.

Use Equation 2 to write an inequality that describes when the level of CO_2 is predicted to exceed 420 ppm.

$1.81579t - 3259.58 > 420$

Solve this inequality.

$t > \dfrac{3679.59}{1.81579} \approx 2026.43$

Therefore, using this model we predict that the CO_2 level will exceed 420 ppm by the year 2027. This prediction is uncertain because it involves a time (year) that is outside the range of years in which the level of CO_2 is known. In fact, Figure 1.29 suggests that CO_2 levels have been increasing more rapidly in recent years, so the level might exceed 420 ppm well before 2027. ■

Polynomials

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

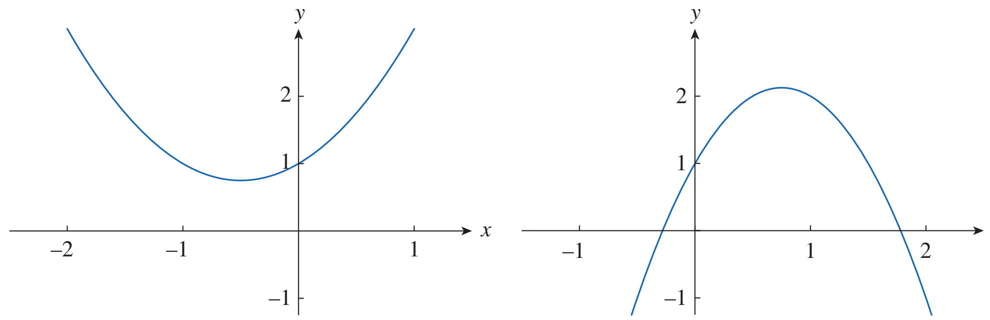
where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

A polynomial of degree 1 is of the form $P(x) = mx + b$ and is a **linear function**. A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$, where $a \neq 0$, and is called

a **quadratic function**. Its graph is always a parabola obtained by shifting, or transforming, the graph of the parabola $y = ax^2$, as we will see in Section 1.4. The parabola opens upward if $a > 0$ and downward if $a < 0$ (see Figure 1.30).



(a) Graph of $y = x^2 + x + 1$

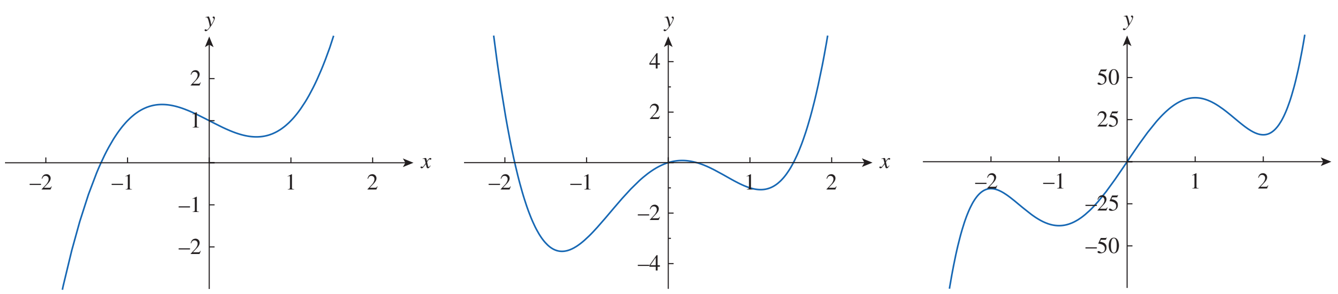
(b) Graph of $y = -2x^2 + 3x + 1$

Figure 1.30
The graph of a quadratic function is a parabola.

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \qquad a \neq 0$$

and is called a **cubic function**. Figure 1.31 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c). We will analyze the shapes of these graphs later.



(a) Graph of $y = x^3 - x + 1$

(b) Graph of $y = x^4 - 3x^2 + x$

(c) Graph of $y = 3x^5 - 25x^3 + 60x$

Figure 1.31
The graphs of polynomials of degree 3, 4, and 5.

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

Table 1.2
The height of the ball at one-second intervals.

Polynomials are commonly used to model various quantities that occur in the natural and social sciences. For instance, in Section 3.9 we will explain why economists often use a polynomial $P(x)$ to represent the cost of producing x units of a commodity. In the next example, we use a quadratic function to model the height of an object as it falls.

Example 4 The CN Tower and a Quadratic Model

A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground. The height, h, of the ball above the ground is recorded at one-second intervals and these values are given in Table 1.2. Find a model to fit the data and use this model to predict the time at which the ball hits the ground.

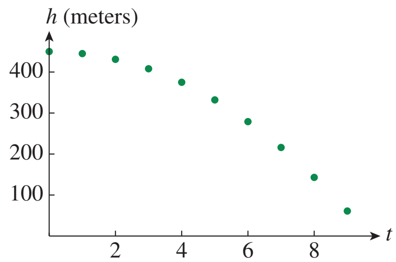


Figure 1.32
Scatter plot of the falling-ball data.

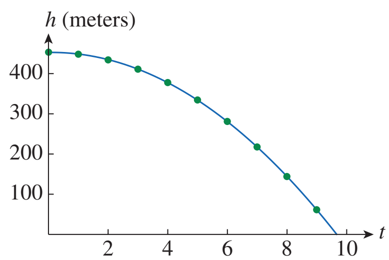


Figure 1.33
Graph of the quadratic model and the scatter plot.

Solution

A scatter plot of the data is shown in Figure 1.32. It appears that a linear model is inappropriate. However, it looks as if these points might lie on a parabola. This suggests that we try a quadratic model. Using a graphing calculator or computer algebra system (and the method of least squares), we obtain the quadratic model

$$h = 449.36 + 0.96t - 4.90t^2 \tag{3}$$

Figure 1.33 shows the graph of Equation 3 and the scatter plot of the data. This graph suggests that the quadratic model is a very good fit.

The ball hits the ground when $h = 0$. Solve the quadratic equation

$$-4.90t^2 + 0.96t + 449.36 = 0.$$

for t to find the time at which the ball hits the ground.

Using the quadratic formula,

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)} \quad \Rightarrow \quad t = -9.47887, \quad 9.67479.$$

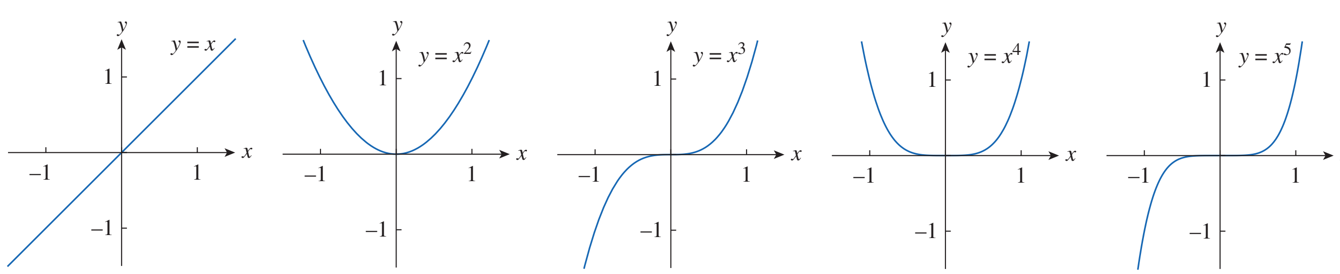
The positive root is $t \approx 9.67$, so this model predicts that the ball will hit the ground after about 9.7 seconds. ■

Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. Let's consider several cases based on the value of the constant a.

(i) $a = n$, where n is a positive integer

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 1.34. Note that these functions are all polynomials with only one term. We already know the shape of the graphs of $y = x^1 = x$ (line through the origin with slope 1) and $y = x^2$ (a parabola).



(a) Graph of $y = x$ (b) Graph of $y = x^2$ (c) Graph of $y = x^3$ (d) Graph of $y = x^4$ (e) Graph of $y = x^5$

Figure 1.34
The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$.

A **family of functions** is a collection of functions whose equations are related. Figure 1.35 shows two families of power functions, one with even powers and one with odd powers.

The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd. If n is even, then $f(x) = x^n$ is an even function and its graph opens upward, similar to the graph of the parabola $y = x^2$. If n is odd, then $f(x) = x^n$ is an odd function and its graph starts and ends in opposite directions, similar to the graph of $y = x^3$. Notice from Figure 1.35, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| > 1$. If x is small, that is, $|x| < 1$, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.

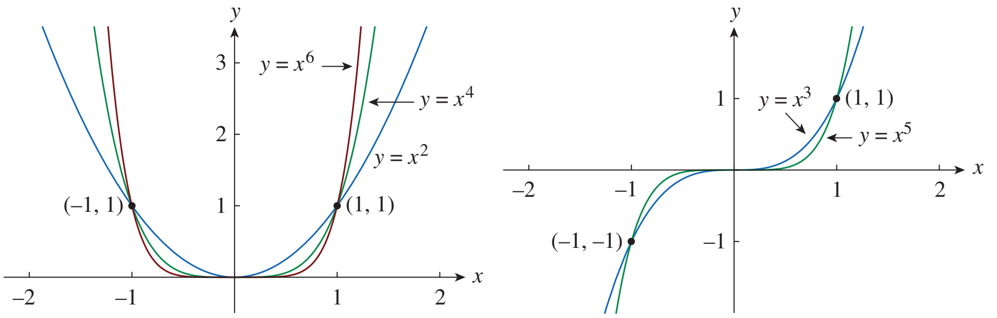


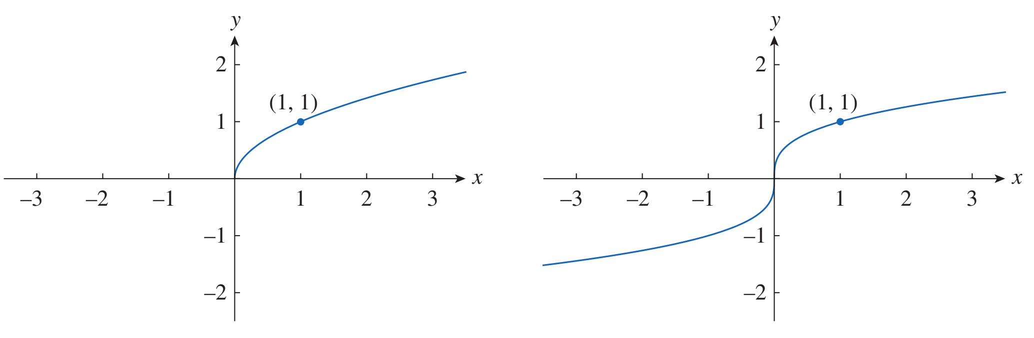
Figure 1.35
Graphs of two families of functions.

(ii) $a = \dfrac{1}{n}$, where n is a positive integer

Common Error
$\sqrt{x^2} = \pm x$. For example, $\sqrt{9} = \pm 3$

Correct Method
The square root function is a *function* and, therefore, returns only one value. The square root function always returns the positive square root.

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For $n = 2$, it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$. [See Figure 1.36(a).] For other even values of n, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$, the upper half of a parabola. For $n = 3$, we have the cube root function $f(x) = \sqrt[3]{x}$, whose domain is $\mathbb{R}$ (recall that every real number has a cube root) and whose graph is shown in Figure 1.36(b). The graph of $y = \sqrt[n]{x}$ for n odd ($n > 3$) is similar to the graph of $y = \sqrt[3]{x}$.



(a) Graph of $f(x) = \sqrt{x}$

(b) Graph of $f(x) = \sqrt[3]{x}$

Figure 1.36
Graphs of root functions.

(iii) $a = -1$

The graph of the reciprocal function $f(x) = x^{-1} = 1/x$ is shown in Figure 1.37. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes. The domain is $\{x \mid x \neq 0\}$ and the range is $\{y \mid y \neq 0\}$. This function arises in physics and chemistry in connection with Boyle's Law, which says that when the temperature is constant, the volume V of a gas is inversely proportional to the pressure P:

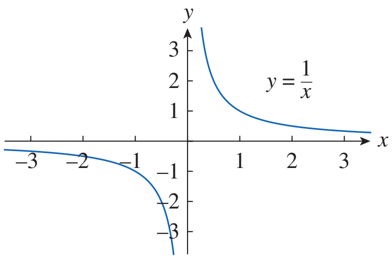


Figure 1.37
Graph of the reciprocal function.

$$V = \frac{C}{P}$$

where C is a constant. Thus, the graph of V as a function of P (see Figure 1.38) has the same general shape as the right half of Figure 1.37; however, the domain is $\{P \mid P > 0\}$.

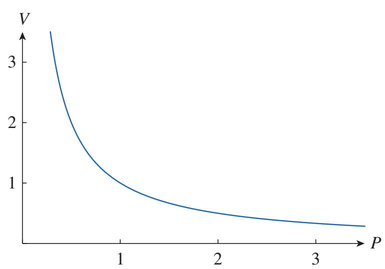


Figure 1.38
Volume as a function of pressure at a constant temperature.

Power functions are also used to model species-area relationships (Exercise 60), illumination as a function of distance from a light source (Exercise 59), and the period of revolution of a planet as a function of its distance from the sun (Exercise 61).

Rational Functions

A **rational function** f is the ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$. If P and Q have no factors in common, then the graph of f has a vertical asymptote at the roots of Q. Suppose P and Q have a common factor so that $(x - r)^n$ is a factor in P and $(x - r)^m$ is a factor in Q. If $m > n$ then the graph of f has a vertical asymptote at $x = r$. If $m \leq n$, then the graph of f has a hole at $x = r$.

A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 1.37. The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 1.39.

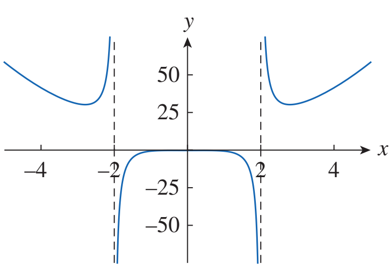


Figure 1.39
Graph of $y = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$.

Algebraic Functions

A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is, therefore, an algebraic function. Here are two more examples.

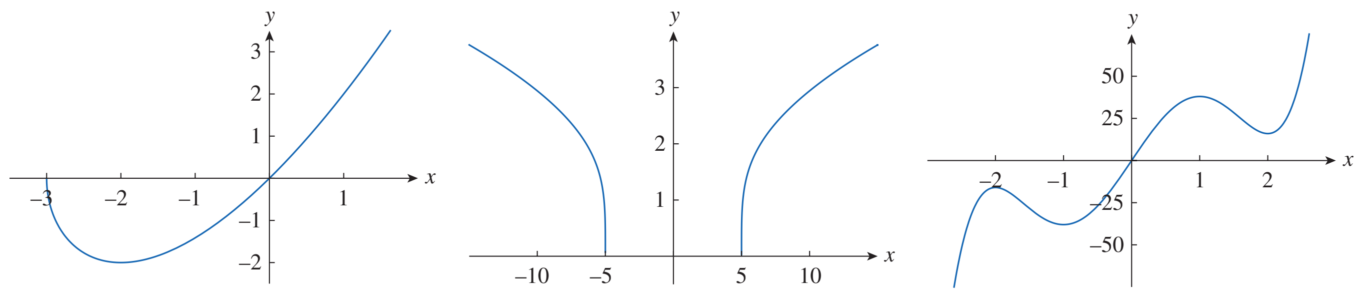
$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

When we sketch algebraic functions in Chapter 4, we will see that their graphs can assume a variety of shapes. Figure 1.40 illustrates some of the possibilities.

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and $c = 3.0 \times 10^5 \text{ km/s}$ is the speed of light in a vacuum.



(a) Graph of $f(x) = x\sqrt{x+3}$

(b) Graph of $g(x) = \sqrt[4]{x^2 - 25}$

(c) Graph of $h(x) = x^{2/3}(x-2)^2$

Figure 1.40
Graphs of algebraic functions.

Trigonometric Functions

Trigonometry and trigonometric functions are reviewed in Appendix C. In calculus, the convention is that radian measure is always used (except when otherwise indicated). For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x. The graphs of the sine and cosine functions are shown in Figure 1.41.

Notice that for both the sine and cosine functions, the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

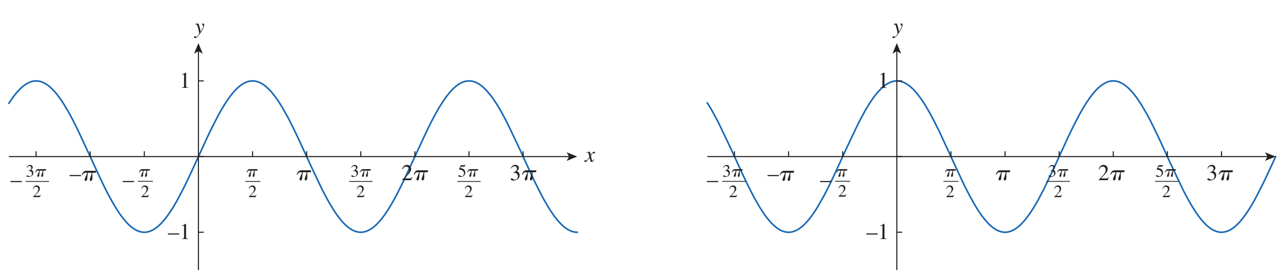
$$-1 \le \sin x \le 1 \qquad -1 \le \cos x \le 1$$

or, in terms of absolute values,

$$|\sin x| \le 1 \qquad |\cos x| \le 1$$

The zeros of the sine function occur at the integer multiples of π; that is,

$$\sin x = 0 \quad \text{when} \quad x = n\pi \quad n \text{ is an integer}$$



(a) Graph of $f(x) = \sin x$

(b) Graph of $g(x) = \cos x$

Figure 1.41
Graphs of trigonometric functions.

An important property of the sine and cosine functions is that they are periodic functions and have period 2π. This means that, for all values of x,

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$

The periodic nature of these functions makes them suitable for modeling repetitive phenomena such as tides, vibrating springs, and sound waves. For instance, in Example 4 in Section 1.3, we will see that a reasonable model for the number of hours of daylight in Philadelphia t days after January 1 is given by the function

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

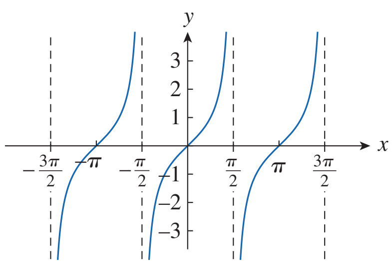


Figure 1.42
Graph of $y = \tan x$.

and its graph is shown in Figure 1.42. The tangent function is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm 3\pi/2, \ldots$. Its range is $(-\infty, \infty)$. Notice that the tangent function has period π:

$$\tan(x + \pi) = \tan x \qquad \text{for all } x$$

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions. Their graphs are shown in Appendix C.

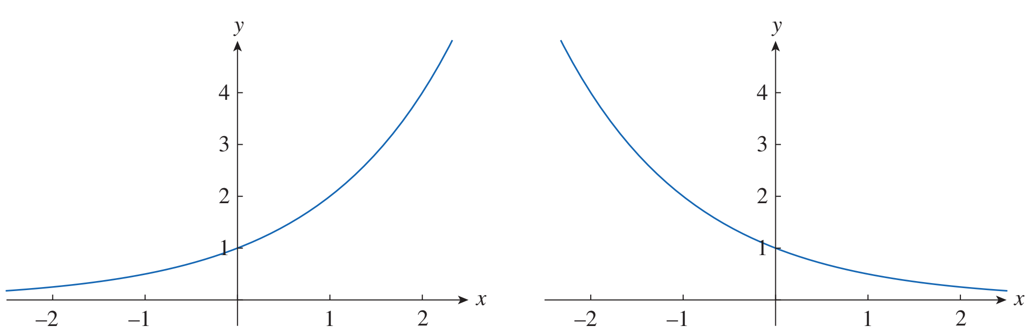
Exponential Functions

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant. The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 1.43. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Exponential functions will be discussed in detail in Section 1.4, and we will see that they are useful for modeling many natural phenomena, such as population growth (if $a > 1$) and radioactive decay (if $a < 1$).

Logarithmic Functions

The **logarithmic functions** $f(x) = \log_a x$, where the base $a > 0$ and $a \neq 1$, are the inverse functions of the exponential functions. They will be discussed in Section 1.5.



(a) Graph of $y = 2^x$

(b) Graph of $y = (0.5)^x$

Figure 1.43
Graphs of exponential functions.

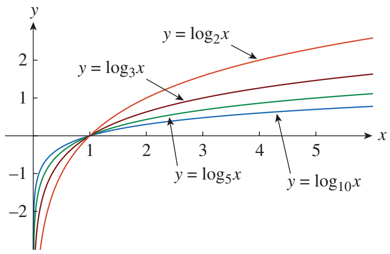


Figure 1.44
Graphs of logarithmic functions.

Figure 1.44 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

Example 5 Function Classification

Classify the following functions as one of the types of functions discussed in this section.

(a) $f(x) = 5^x$ (b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ (d) $u(t) = 1 - t + 5t^4$

Solution

(a) $f(x) = 5^x$ is an exponential function. The x is the exponent.

(b) $g(x) = x^5$ is a power function. The x is the base. We could also consider g to be a polynomial of degree 5.

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ is an algebraic function.

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4. ■

1.2 Exercises

Classify each function as a power function, root function, polynomial (state its degree), rational function, or algebraic function.

1. (a) $f(x) = \log_2 x$ (b) $g(x) = \dfrac{2x^3}{1 - x^2}$

(c) $u(t) = 1 - 2t + 3t^2$ (d) $u(t) = 1 - 1.1t + 2.54t^2$

(e) $h(x) = x^{-2/7}$ (f) $k(x) = (x - 3)^4(x + 2)^2$

(g) $v(t) = 5^t$ (h) $w(t) = \sin t \cos^2 t$

2. (a) $y = x^4$ (b) $y = x^2(2 - x^3)$

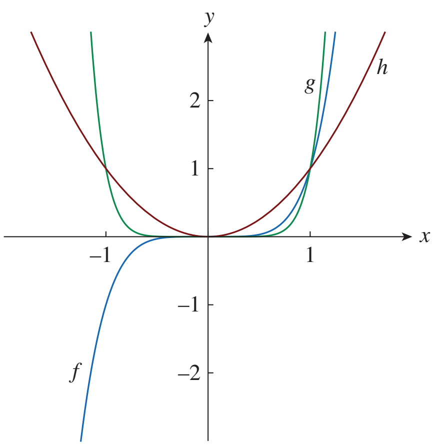
(c) $y = \dfrac{x}{1 + x}$ (d) $y = \dfrac{\sqrt{x^3 - 1}}{1 + \sqrt[3]{x}}$

(e) $y = x^{3/5} + 7$ (f) $y = \sqrt{\dfrac{x^2}{x^2 + 1}}$

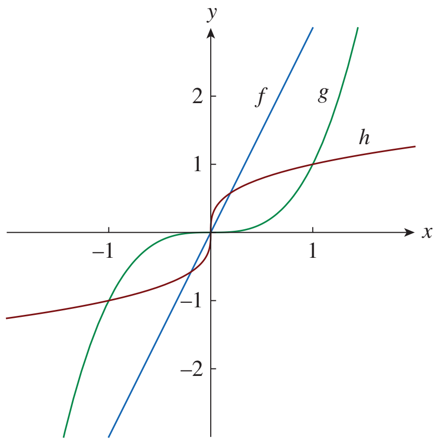
(g) $y = \dfrac{x^2 - 7x + 5}{x^3 + x - 1}$ (h) $y = \tan t - \cos t$

Match each equation with its graph. Explain your reasoning. (Do not use technology.)

3. (a) $y = x^2$ (b) $y = x^5$ (c) $y = x^8$



4. (a) $y = 3x$ (b) $y = x^3$ (c) $y = \sqrt[3]{x}$



Find the domain and range of the function.

5. $f(x) = 2x - 3$

6. $f(x) = x^2 + 4$

7. $g(x) = x^3 + 1$

8. $g(x) = \dfrac{x - 2}{x^2 - 4}$

9. $h(x) = x^3 + 2x + 2$

10. $h(x) = \sqrt{x} + 2$

11. $f(x) = \sqrt{x^2 + 4}$

12. $f(x) = \dfrac{1}{|x + 2|}$

Find the horizontal and vertical asymptotes on the graph of each function (or state that none exist).

13. $f(x) = \dfrac{x}{x + 4}$

14. $f(x) = \dfrac{x^2}{x + 3}$

15. $g(x) = \dfrac{x - 2}{(x + 2)(x - 2)}$

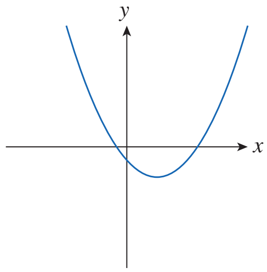
16. $g(x) = \dfrac{x}{x^3 - 1}$

17. $h(x) = \dfrac{|x + 2|}{x - 2}$

18. $h(x) = \dfrac{x + 1}{x^3 - 1}$

Use the Vertical Line Test to determine whether the given graph is the graph of y as a function of x.

19.



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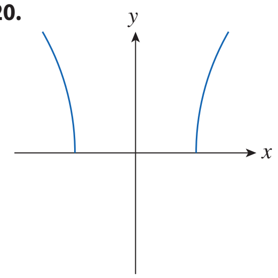

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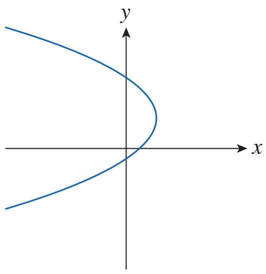
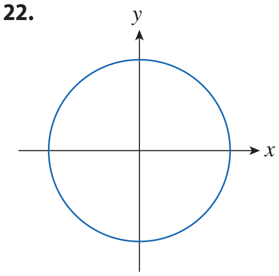


22.



23. Find a linear function f such that $f(3) = 1$ and $f(7) = 19$.

24. If $f(x) = 3x + 5$ and a and b are any distinct values in the domain of f, explain why the quotient

$$\frac{f(b) - f(a)}{b - a}$$

will always have the same value. What is this value?

25. If $f(x) = 5x + 2$, find $\dfrac{f(x + h) - f(x)}{h}$.

26. Find the domain and range of the quadratic function $f(x) = 2(x - 3)^2 + 5$.

27. Find a quadratic function f whose range is $\{y \mid y \leq 6\}$ and such that $f(4 + d) = f(4 - d)$ for all real values d.

28. Find a root function g for which $g(64) = 4$.

29. Find a rational function h with a linear denominator such that the graph of h has an x-intercept at 4 and a vertical asymptote at $x = 3$.

30. Find a rational function f that is undefined at $x = 2$ but does not have a vertical asymptote at $x = 2$.

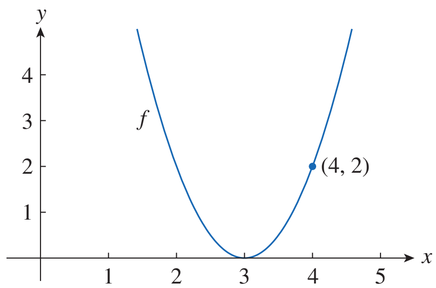
31. (a) Find an equation for the family of linear functions with slope 2 and sketch several members of this family.
(b) Find an equation for the family of linear functions such that $f(2) = 1$ and sketch several members of this family.
(c) Which function belongs to both families?

32. What do all members of the family of linear functions $f(x) = 1 + m(x + 3)$ have in common? Sketch several members of this family of functions.

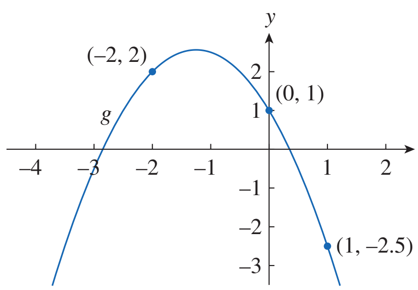
33. What do all members of the family of linear functions $f(x) = c - x$ have in common? Sketch several members of this family of functions.

34. Find an expression for each of the quadratic functions whose graphs are shown.

(a)



(b)



35. Find an expression for a cubic function f if $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.

Sketch the graph of a function that satisfies the given criteria.

36. $f(x)$ is a cubic function that is increasing on the interval $(-\infty, \infty)$.

37. $f(x)$ is a quadratic polynomial that intersects the x-axis at -2.

38. $f(x)$ is a fourth degree polynomial that has zeros $-2, 0, 1, 6$.

---

39. Recent studies indicate that the average global surface temperature of Earth has been rising steadily. Some scientists have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in °C and t represents years since 1900.

(a) What do the slope and T-intercept represent in the context of this problem?

(b) Use the equation to predict the average global surface temperature in 2100.

40. If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a (in years), pharmacists use the equation $c = 0.0417D(a + 1)$. Suppose the dosage for an adult is 200 mg.

(a) Find the slope of the graph of c. What does it represent?

(b) What is the dosage for a newborn?

41. The function $d = kv^2$ gives the approximate stopping distance d, in feet, for a certain car traveling at the rate of v miles per hour. It is known that a car traveling 20 mi/h requires 28 ft to stop. Use this information to determine how many feet would be required to stop the same car traveling at the rate of 40 mi/h.

42. The U.S. federal minimum wage for three years is given in the table below.

Year	Hourly wage
1981	\$3.35
1996	\$4.75
2015	\$7.25

Use the data for 1981 and 2015 to construct a linear model for the hourly wage for a given year. Use your model to estimate the minimum wage in 1996. How much does your estimate differ from the actual minimum wage in 1996?

43. The manager of a weekend market knows from past experience that if they charge x dollars for a rental space at the market, then the number y of spaces they can rent is given by the equation $y = 200 - 4x$.

(a) Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented can't be negative quantities. In addition, y must be a whole number.)

(b) What do the slope, the y-intercept, and the x-intercept of the graph represent in the context of this problem?

44. The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.

(a) Sketch a graph of this function.

(b) What is the slope of the graph, and what does it represent? What is the F-intercept, and what does it represent?

45. A driver leaves Naples, FL, at 2:00 PM and drives at a constant speed east along I-75. They pass through Big Cypress National Preserve, 40 mi from Naples, at 2:50 PM.

(a) Express the distance traveled in terms of the time elapsed.

(b) Sketch the graph of the equation in part (a).

(c) What is the slope of this line? What does it represent in the context of this problem?

46. Biologists have noticed that the chirping rate of crickets is related to temperature, and the relationship appears to be approximately linear. A cricket produces 113 chirps per minute at 70°F and 173 chirps per minute at 80°F.

(a) Find a linear equation that models the temperature T as a function of the number of chirps per minute N.

(b) What is the slope of the graph? What does it represent in the context of this problem?

(c) If the crickets are chirping at 150 chirps per minute, estimate the temperature.

47. The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in 1 day and \$4800 to produce 300 chairs in 1 day.

(a) Express the cost as a function of the number of chairs produced, assuming that the relationship is linear. Sketch the graph.

(b) What is the slope of the graph, and what does it represent in the context of this problem?

(c) What is the y-intercept of the graph, and what does it represent in the context of this problem?

48. At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 $\text{lb}/\text{in.}^2$. Below the surface, the water pressure increases by 4.34 $\text{lb}/\text{in.}^2$ for every 10 ft of descent.

(a) Express the water pressure as a function of the depth below the ocean surface.

(b) At what depth is the pressure 100 $\text{lb}/\text{in.}^2$?

49. The monthly cost of driving a car depends on the number of miles driven. A driver found that in May it cost them \$380 to drive 480 mi and in June it cost them \$460 to drive 800 mi.

(a) Express the monthly cost C as a function of the distance driven d, assuming that a linear relationship provides a suitable model.

(b) Use part (a) to predict the cost of driving 1500 mi per month.

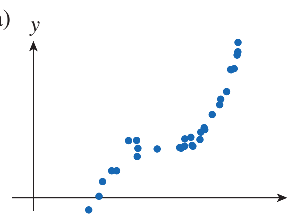
(c) Draw the graph of the linear function. What does the slope represent in the context of this problem?

(d) What does the C-intercept represent in the context of this problem?

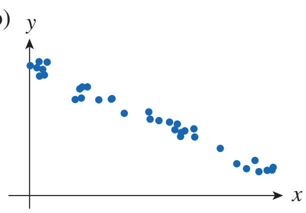
(e) Why does a linear function give a suitable model in this situation?

For each scatter plot, decide what type of function might provide a reasonable model for the data. Explain your reasoning.

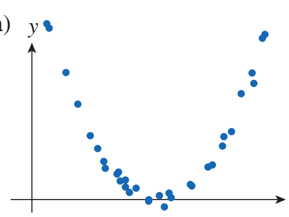
50. (a)



(b)



51. (a)



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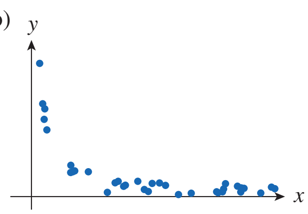


52. The table shows (lifetime) peptic ulcer rates (per 100 population) for various family incomes as reported by the National Health Interview Survey.

Income	Ulcer rate (per 100 population)
\$4,000	14.1
\$6,000	13.0
\$8,000	13.4
\$12,000	12.5
\$16,000	12.0
\$20,000	12.4
\$30,000	10.5
\$45,000	9.4
\$60,000	8.2

(a) Construct a scatter plot of these data and decide whether a linear model is appropriate. Explain your reasoning.

(b) Find and graph a linear model using the first and last data points.

(c) Use technology to find the least squares regression line. Graph this line.

(d) Use the linear model in part (c) to estimate the ulcer rate for an income of \$25,000.

(e) According to the model, how likely is someone with an income of \$80,000 to suffer from peptic ulcers?

(f) Do you think it would be reasonable to apply the model to someone with an income of \$200,000? Explain your reasoning.

53. Biologists have observed that the chirping rate of the snowy tree cricket appears to be related to temperature. The table shows the chirping rates for various temperatures.

Temperature (°F)	Chirping rate (chirps/min)	Temperature (°F)	Chirping rate (chirps/min)
50	20	75	140
55	46	80	173
60	79	85	198
65	91	90	211
70	113		

(a) Construct a scatter plot of the data.

(b) Find and graph the regression line.

(c) Use the linear model in part (b) to estimate the chirping rate at 100°F.

54. Anthropologists use a linear model that relates human femur (thighbone) length to height. The model allows an anthropologist to determine the height of an individual when only a partial skeleton (including the femur) is found. Consider a model by analyzing the data on femur length and height for the eight individuals given in the table.

Femur length (cm)	Height (cm)	Femur length (cm)	Height (cm)
50.1	178.5	44.5	168.3
48.3	173.6	42.7	165.0
45.2	164.8	39.5	155.4
44.7	163.7	38.0	155.8

(a) Construct a scatter plot of the data.
(b) Use technology to find the regression line that models the data, and sketch a graph of this line.
(c) An anthropologist finds a human femur of length 53 cm. Use your model to predict the height of this person.

55. The table gives the winning heights for the men's Olympic pole vault competitions up to 2016.

Year	Height (m)	Year	Height (m)
1896	3.30	1964	5.10
1900	3.30	1968	5.40
1904	3.50	1972	5.64
1908	3.71	1976	5.64
1912	3.95	1980	5.78
1920	4.09	1984	5.75
1924	3.95	1988	5.90
1928	4.20	1992	5.87
1932	4.31	1996	5.92
1936	4.35	2000	5.90
1948	4.30	2004	5.95
1952	4.55	2008	5.96
1956	4.56	2012	5.97
1960	4.70	2016	6.03

(a) Construct a scatter plot of the data and determine whether a linear model is appropriate. Justify your answer.
(b) Use technology to find the regression line that models this data, and sketch a graph of this line.
(c) Use your linear model to predict the height of the winning pole vault at the 2020 Olympics and compare with the actual winning height, 6.02 m.
(d) Is it reasonable to use the model to predict the winning height of the 2100 Olympics? Why or why not?

56. When laboratory rats are exposed to asbestos fibers, some of them develop lung tumors. The table lists the results of several experiments by different scientists.

Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors	Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors
50	2	1600	42
400	6	1800	37
500	5	2000	38
900	10	3000	50
1100	26		

(a) Use technology to find the regression line for the data.
(b) Construct a scatter plot and graph the regression line. Does the regression line appear to be a suitable model for the data? Explain your reasoning.
(c) What does the y-intercept of the regression line represent in the context of this problem?

57. The table shows the world average daily oil consumption from 1985 to 2019 measured in billions of barrels per day.

Years since 1985	Billions of barrels per day	Years since 1985	Billions of barrels per day
0	59.25	18	79.91
1	60.99	19	82.65
2	62.29	20	83.89
3	64.27	21	84.92
4	65.60	22	86.10
5	66.74	23	85.17
6	66.86	24	84.08
7	67.88	25	86.86
8	67.62	26	87.82
9	69.33	27	88.78
10	70.09	28	90.15
11	71.64	29	90.90
12	73.74	30	92.61
13	74.12	31	94.40
14	75.67	32	96.10
15	76.48	33	97.35
16	77.37	34	98.27
17	78.24		

(a) Construct a scatter plot and decide whether a linear model is appropriate. Explain your reasoning.
(b) Use technology to find the regression line, and sketch a graph of this line.
(c) Use the linear model to estimate the oil consumption in 2002.
(d) Use the linear model to estimate the oil consumption in 2020. Find the actual world average daily oil consumption in 2020 and compare with your estimate.

58. The table shows average U.S. retail residential prices of electricity from 2000 to 2019, measured in cents per kilowatt hour.

Years since 2000	Cents/kWh	Years since 2000	Cents/kWh
0	8.24	10	11.54
1	8.58	11	11.72
2	8.44	12	11.88
3	8.72	13	12.13
4	8.95	14	12.52
5	9.45	15	12.65
6	10.40	16	12.55
7	10.65	17	12.89
8	11.26	18	12.87
9	11.51	19	13.01

(a) Construct a scatter plot. Is a linear model appropriate? Explain your reasoning.
(b) Use technology to find the regression line and graph the regression line.
(c) Use your linear model from part (b) to estimate the average retail price of electricity in 2005 and 2017.

59. Many physical quantities are connected by *inverse square laws,* that is, by power functions of the form $f(x) = kx^{-2}$. In particular, the illumination of an object by a light source is inversely proportional to the square of the distance from the source. Suppose that after dark you are in a room with just one lamp and you are trying to read a book. The light is too dim and so you move halfway to the lamp. How much brighter is the light?

60. Many ecologists have modeled the species-area relation with a power function. In particular, the number of species S of bats living in caves in central Mexico has been related to the surface area A of the caves by the equation $S = 0.7\,A^{0.3}$.
(a) The cave called *Misión Imposible* near Puebla, Mexico, has a surface area of $A = 60\text{ m}^2$. How many species of bats would you expect to find in that cave?
(b) If you discover that four species of bats live in a cave, estimate the area of the cave.

61. The table shows the mean (average) distances d of the planets from the sun (taking the unit of measurement to be the distance from Earth to the sun) and their periods T (time of revolution in years).

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784

(a) Use technology to fit a power model to the data.
(b) Kepler's Third Law of Planetary Motion states that "The square of the period of revolution of a planet is proportional to the cube of its mean distance from the sun." Does your model agree with Kepler's Third Law? Explain your reasoning.

62. Rewrite the function $f(x) = |x| + |x + 3| + 2$ without any absolute value symbols.

1.3 New Functions from Old Functions

In this section we start with the basic functions presented in Section 1.2 and construct new functions by shifting, stretching, and reflecting their graphs. We also show how to combine pairs of functions by the standard arithmetic operations and by composition.

Transformations of Functions

By applying certain transformations to the graph of a given function, we can obtain the graphs of related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for these related graphs.

We will first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y-coordinate is increased by the same constant c). Similarly, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x). Therefore, the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1.45).

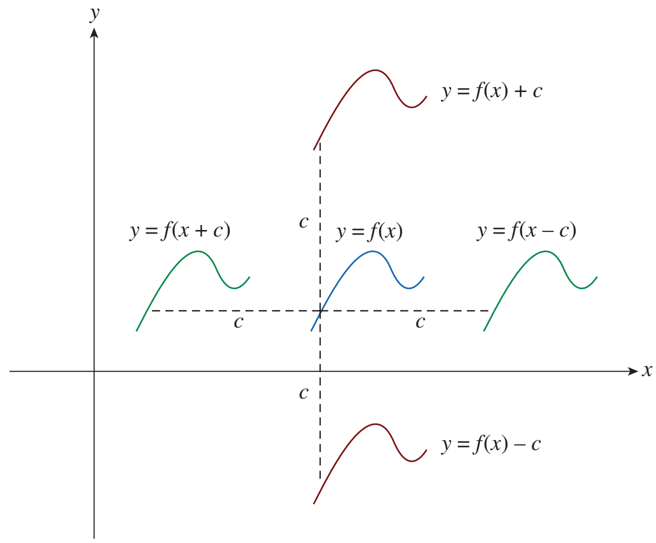


Figure 1.45
Graphs of functions obtained by translating the graph of *f*.

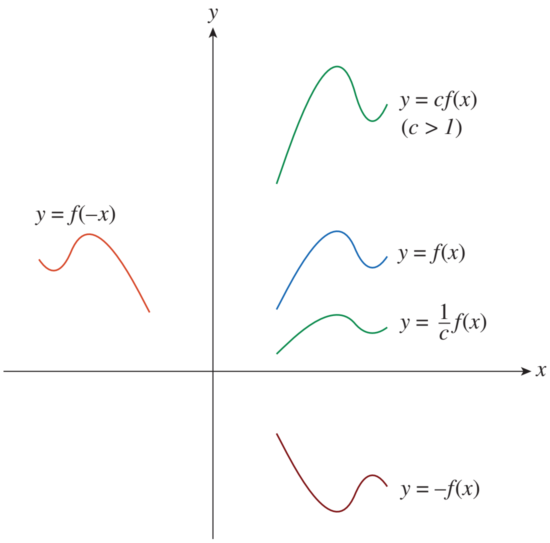


Figure 1.46
Graphs of functions obtained by stretching and reflecting the graph of *f*.

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y-coordinate is multiplied by the same constant c). The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis because the point (x, y) is replaced by the point $(x, -y)$. (See Figure 1.46 and the following summary, where the results of other stretching, shrinking, and reflecting transformations are presented.)

Vertical and Horizontal Stretching and Reflecting

Suppose $c > 1$. To obtain the graph of
$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c.
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c.
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c.
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c.
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.

Figure 1.47 illustrates these stretching transformations when applied to the cosine function with $c = 2$. For example, in order to sketch the graph of $y = 2\cos x$, we multiply the y-coordinate of each point on the graph of $y = \cos x$ by 2. This changes the *amplitude* of the cosine graph and means that the graph of $y = \cos x$ is stretched vertically by a factor of 2.

In order to sketch the graph of $y = \cos 2x$, we multiply the x-coordinate by 2. This changes the *period* of the cosine graph and shrinks the graph of $y = \cos x$ by a factor of 2.

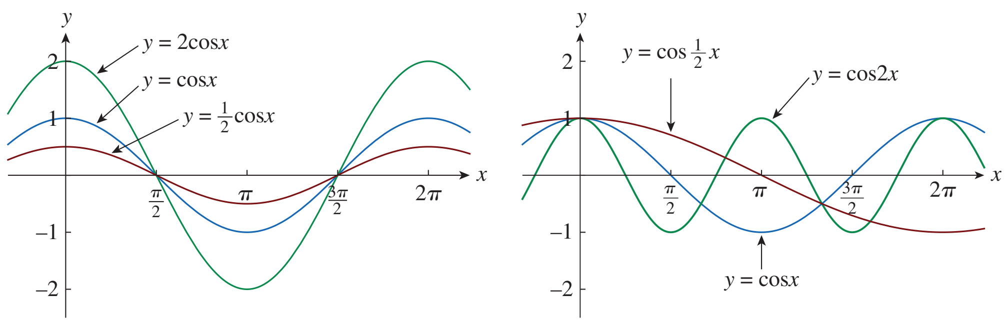


Figure 1.47
Stretching transformations applied to the cosine function.

Example 1 Transforming the Root Function

Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

Solution

The graph of the square root function $y = \sqrt{x}$ is shown in Figure 1.48 as Graph A.

The graph of $y = \sqrt{x} - 2$ is obtained by shifting this graph 2 units downward.

The graph of $y = \sqrt{x - 2}$ is obtained by sifting this graph 2 units to the right.

The graph of $y = -\sqrt{x}$ is obtained by reflecting this graph about the x-axis.

The graph of $y = 2\sqrt{x}$ is obtained by stretching this graph vertically by a factor of 2.

The graph of $y = \sqrt{-x}$ is obtained by reflecting this graph about the y-axis.

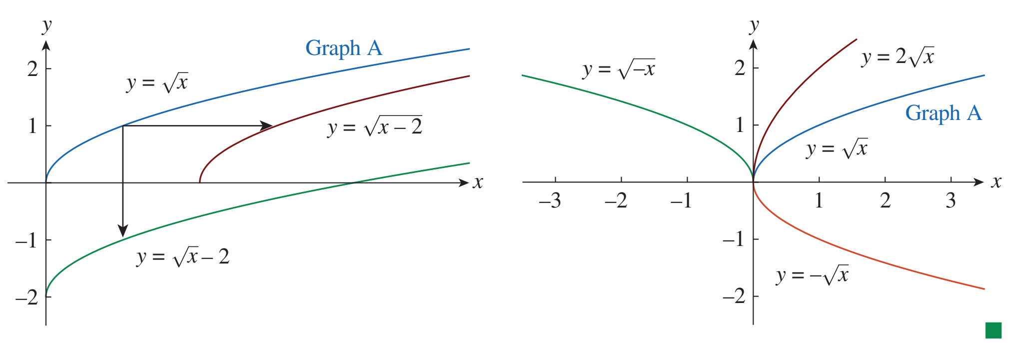


Figure 1.48
Graph of the square root and related functions.

Example 2 Use Transformations

Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

Solution

$y = x^2 + 6x + 10 = (x + 3)^2 + 1$ Complete the square.

The graph of f can be obtained from the graph of the parabola $y = x^2$ by shifting this graph 3 units to the left and then 1 unit upward. See Figure 1.49.

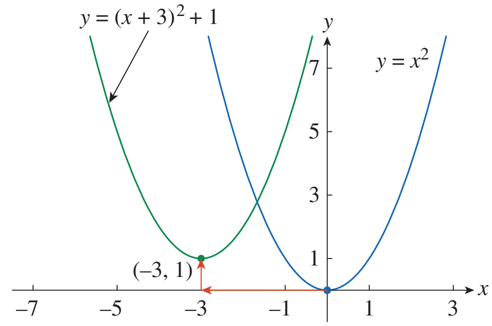


Figure 1.49
Graph of $y = (x + 3)^2 + 1$ obtained from the graph of $y = x^2$ by shifting left and upward.

Example 3 Transforming the Sine Function

Sketch the graph of each of the following functions.

(a) $y = \sin 2x$ (b) $y = 1 - \sin x$

Solution

(a) The graph of $y = \sin x$ is shown in Figure 1.50.

Shrink this graph horizontally by a factor of 2 to sketch the graph of $y = \sin 2x$, shown in Figure 1.51.

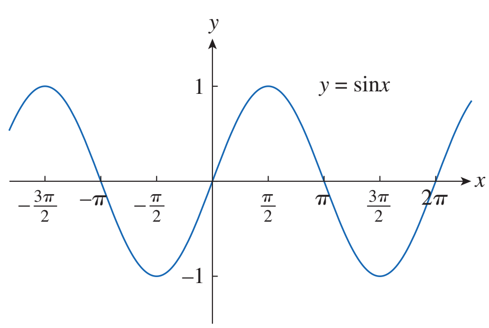


Figure 1.50
Graph of $y = \sin x$.

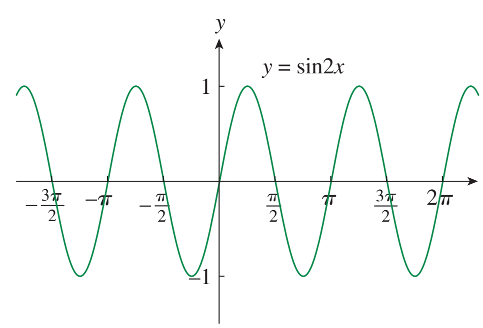


Figure 1.51
Graph of $y = \sin 2x$.

(b) Reflect the graph of $y = \sin x$ about the x-axis to obtain the graph of $y = -\sin x$. Shift this graph 1 unit upward to obtain the graph of $y = 1 - \sin x$, as shown in Figure 1.52.

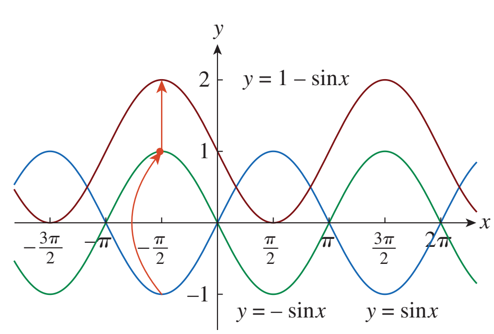


Figure 1.52
Graph of $y = 1 - \sin x$ obtained from $y = \sin x$ and $y = -\sin x$.

Example 4 A Model for the Amount of Daylight

Figure 1.53 shows the graph of the number of hours of daylight as a function of the time of the year at several latitudes. Given that Philadelphia is located at approximately 40°N latitude, find a function that models the length of daylight at Philadelphia.

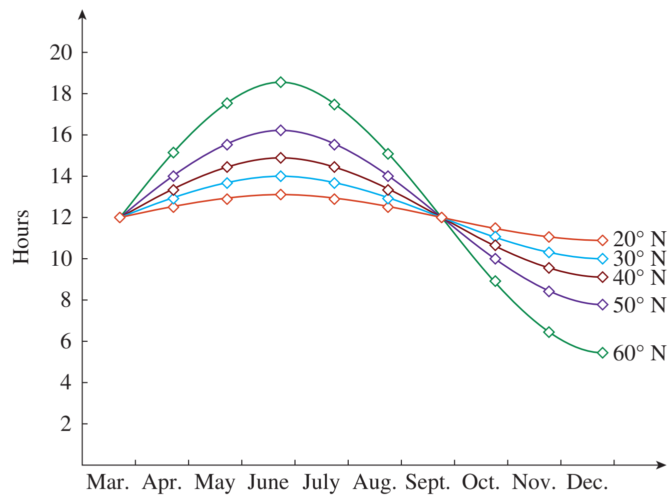


Figure 1.53
Graph of the length of daylight from March 21 through December 21 at various latitudes.
Source: Adapted from L. Harrison, *Daylight, Twilight, Darkness, and Time* (New York: Silver Burdett, 1935), page 40.

Solution

Notice that each curve resembles a shifted and stretched sine function. The curve representing the hours of daylight at the latitude of Philadelphia, 40°N, indicates daylight lasts about 14.8 hours on June 21 and 9.2 hours on December 21, so the amplitude of the curve (the factor by which we have to stretch the sine curve vertically) is $\frac{1}{2}(14.8 - 9.2) = 2.8$.

Because there are about 365 days in a year, the period of our model should be 365. But the period of $y = \sin t$ is 2π, so the horizontal stretching factor is $2\pi/365$.

We also notice that the curve begins its cycle on March 21, the 80th day of the year, so we have to shift the curve 80 units to the right.

Finally, we need to shift the curve 12 units upward.

Therefore, we model the length of daylight in Philadelphia on the tth day of the year by the function

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$ ■

Another transformation that is important in calculus involves taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geq 0$ and $y = -f(x)$ when $f(x) < 0$. This definition allows us to sketch the graph of $y = |f(x)|$ from the graph of $y = f(x)$: the part of the graph that lies *above* the x-axis remains the same, and the part that lies *below* the x-axis is reflected about the x-axis.

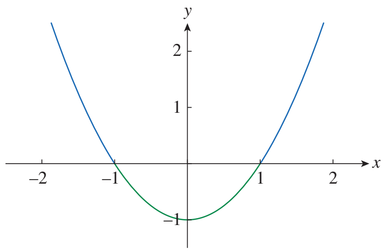


Figure 1.54
Graph of $y = x^2 - 1$.

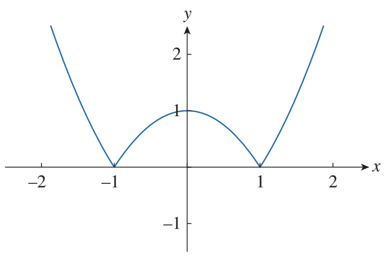


Figure 1.55
Graph of $y = |x^2 - 1|$.

Example 5 Absolute Value Effect

Sketch the graph of the function $y = |x^2 - 1|$.

Solution

Graph the parabola $y = x^2 - 1$ by shifting the graph of $y = x^2$ downward 1 unit (Figure 1.54). The graph lies below the x-axis for $-1 < x < 1$.

Reflect this part of the graph about the x-axis to obtain the graph of $y = |x^2 - 1|$ (Figure 1.55). ■

Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \qquad (f - g)(x) = f(x) - g(x)$$

If the domain of f is A and the domain of g is B, then the domain of $f + g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined. For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$. Since we cannot divide by 0, the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$. For example, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

There is one additional common way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We find y in terms of x by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

This method of combining functions is called *composition* because the new function is *composed* of the two given functions f and g.

In general, given any two functions f and g, we start with a number x in the domain of g and calculate $g(x)$. If this number $g(x)$ is in the domain of f, then we can calculate the value of $f(g(x))$. Notice that the output of one function is used as the input to the next function. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ ("f circle g").

Definition • Composite Function

Given two function f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

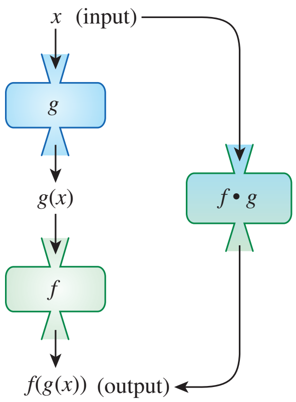


Figure 1.56
The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. Figure 1.56 shows how to picture $f \circ g$ in terms of machines.

Example 6 Composing Functions

If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

Solution

$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$

$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$ ■

Note: You can see from Example 6, that, in general $f \circ g \neq g \circ f$. Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 6, $f \circ g$ is the function that *first* subtracts 3 and *then* squares, and $g \circ f$ is the function that *first* squares and *then* subtracts 3.

Example 7 Practice with Composite Functions

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find each of the following functions and their domains.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

Solution

(a) $(f \circ g)(x) = f(g(x)) = f(\sqrt{2 - x}) = \sqrt{\sqrt{2 - x}} = \sqrt[4]{2 - x}$

The domain of $f \circ g$ is $\{x \mid 2 - x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2]$.

(b) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2 - \sqrt{x}}$

For $\sqrt{x}$ to be defined, we must have $x \geq 0$.

For $\sqrt{2 - \sqrt{x}}$ to be defined, we must have

$2 - \sqrt{x} \geq 0 \quad \Rightarrow \quad \sqrt{x} \leq 2 \quad \Rightarrow \quad x \leq 4.$

Therefore, the domain of $g \circ f$ is the closed interval $[0, 4]$.

(c) $(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$

The domain of $f \circ f$ is $[0, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g(\sqrt{2 - x}) = \sqrt{2 - \sqrt{2 - x}}$

This expression is defined when both $2 - x \geq 0$ and $2 - \sqrt{2 - x} \geq 0$.

$2 - x \geq 0 \quad \Rightarrow \quad x \leq 2$

$2 - \sqrt{2 - x} \geq 0 \quad \Rightarrow \quad \sqrt{2 - x} \leq 2 \quad \Rightarrow \quad 2 - x \leq 4 \quad \Rightarrow \quad x \geq -2$

Therefore, the domain of $g \circ g$ is the closed interval $[-2, 2]$. ■

It is possible to extend the idea of composition to three of more functions. For example, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Example 8 Extended Composition

Find $f \circ g \circ h$ if $f(x) = \dfrac{x}{x+1}$, $g(x) = x^{10}$, and $h(x) = x + 3$.

Solution

$$(f \circ g \circ h)(x) = (f(g(h))) = f(g(x+3))$$
$$= f((x+3)^{10}) = \frac{(x+3)^{10}}{(x+3)^{10}+1}$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as illustrated in Example 9.

Example 9 Decomposing a Function

Given $F(x) = \cos^2(x+9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

Solution

Translate the formula for F into words:

First add 9, then take the cosine of the result, and finally square.

This suggests we let $h(x) = x + 9 \quad g(x) = \cos x \quad f(x) = x^2$

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+9))$$
$$= f(\cos(x+9))$$
$$= [\cos(x+9)]^2$$
$$= \cos^2(x+9) = F(x)$$

Note that this *decomposition* is not unique.

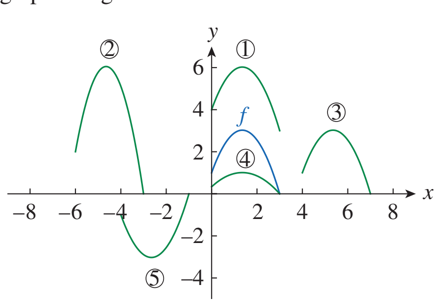
1.3 Exercises

1. Suppose the graph of f is known. Write an equation for each graph described below, obtained from the graph of f.
(a) Shift 3 units upward.
(b) Shift 3 units downward.
(c) Shift 3 units to the right.
(d) Shift 3 units to the left.
(e) Reflect about the x-axis.
(f) Reflect about the y-axis.
(g) Stretch vertically by a factor of 3.
(h) Shrink vertically by a factor of 3.

2. Explain how each graph is obtained from the graph of $y = f(x)$.

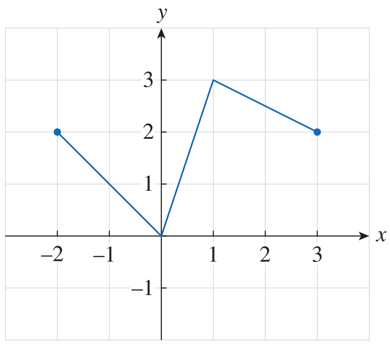
(a) $y = f(x) + 8$ (b) $y = f(x + 8)$
(c) $y = 8f(x)$ (d) $y = f(8x)$
(e) $y = -f(x) - 1$ (f) $y = 8f\left(\dfrac{1}{8}x\right)$

3. The graph of f is given in the figure. Match each equation with its graph and give a reason for each choice.



(a) $y = f(x - 4)$ (b) $y = f(x) + 3$
(c) $y = \dfrac{1}{3}f(x)$ (d) $y = -f(x + 4)$
(e) $y = 2f(x + 6)$

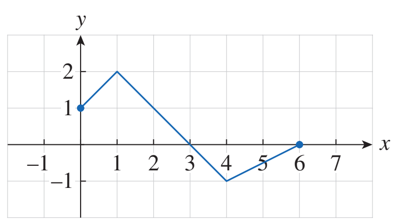
4. The graph of f is given in the figure. Draw the graph of each of the following functions.



(a) $y = f(x) - 3$ (b) $y = f(x + 1)$

(c) $y = \frac{1}{2}f(x)$ (d) $y = -f(x)$

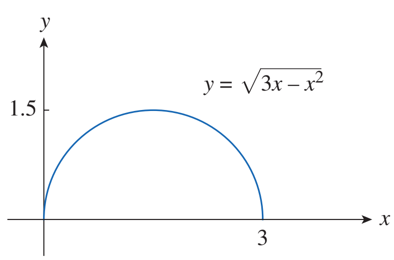
5. The graph of f is given in the figure. Use the appropriate transformations to sketch the graph of each related function.



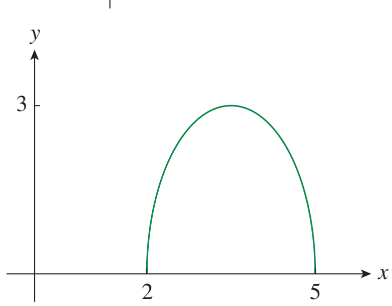
(a) $y = f(2x)$ (b) $y = f\left(\frac{1}{2}x\right)$

(c) $y = f(-x)$ (d) $y = -f(-x)$

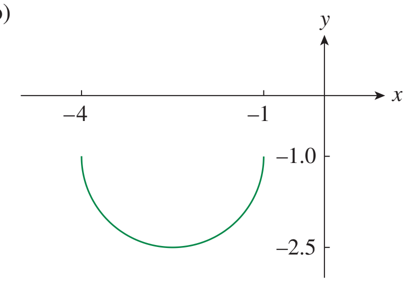
6. The graph of $y = \sqrt{3x - x^2}$ is shown in the figure. Use transformations to construct a function whose graph is as shown.



(a)

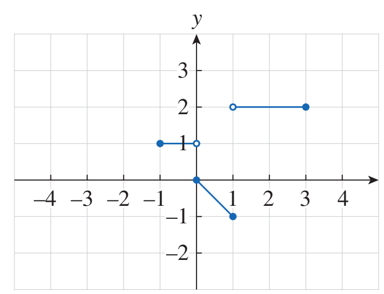


(b)



7. Let $f = \{(1, 4), (0, 2), (-4, 3), (2, 9)\}$ and $g(x) = \dfrac{1}{x}$. Find the ordered pairs in $g \circ f$.

8. The graph of $y = f(x)$ is shown in the figure.

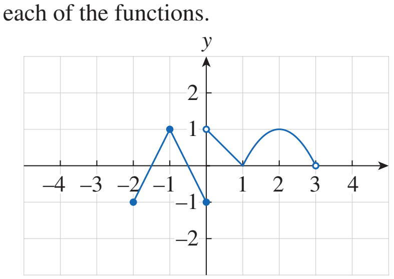


State the range of each function.

(a) $y = f(|x|)$ (b) $y = f(-x)$

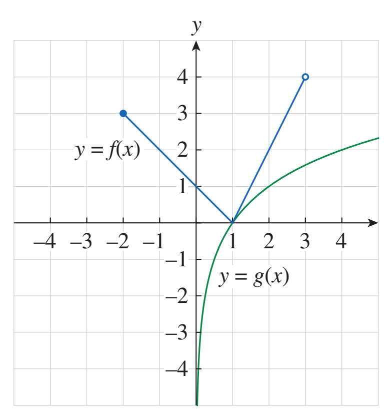
(c) $f(f(x))$

9. Use the graph of $y = f(x)$ shown in the figure to sketch the graph of each of the functions.



(a) $y = |2f(x)|$ (b) $y = -f\left(\frac{1}{2}x\right) + 1$

10. The graphs of $y = f(x)$ and $y = g(x)$ are shown in the figure.



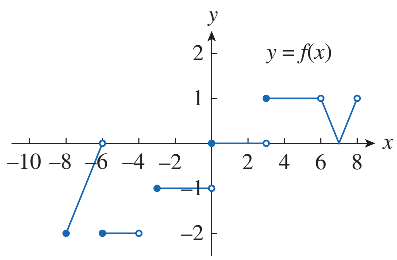
Find the domain of each of the following functions.

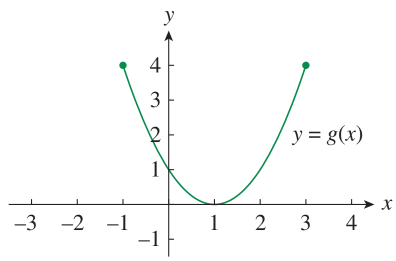
(a) $y = g(x)$ (b) $y = (f + g)(x)$

(c) $y = f(g(x))$ (d) $y = f(|x|)$

(e) $y = f(2x)$

11. The graphs of $y = f(x)$ and $y = g(x)$ are shown in the figures.





Find the domain of each of the following functions.

(a) $y = \left(\dfrac{f}{g}\right)(x)$ (b) $y = (g \circ f)(x)$

(c) $y = f(|x|)$ (d) $y = f(2x)$

12. Sketch the graphs of $f(x) = -2|x|$ and $g(x) = |2x|$ on the same coordinate axes. Explain the relationship between the two graphs.

13. Use methods of transformations to sketch the graph of $y = \sqrt{x+1} + 2$.

14. (a) How is the graph of $y = 2 \sin x$ related to the graph of $y = \sin x$? Use your answer and Figure 1.50 to sketch the graph of $y = 2 \sin x$.

(b) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 1.48 to sketch the graph of $y = 1 + \sqrt{x}$.

Graph the function by starting with the graph of one of the standard functions given in Section 1.2 and then applying the appropriate transformations.

15. $y = -x^3$ **16.** $y = (x - 3)^2$

17. $y = x^3 + 1$ **18.** $y = 1 - \dfrac{1}{x}$

19. $y = 2 \cos 3x$ **20.** $y = 2\sqrt{x+1}$

21. $y = x^2 - 4x + 5$ **22.** $y = 1 + \sin \pi x$

23. $y = 2 - \sqrt{x}$ **24.** $y = 3 - 2 \cos x$

25. $y = \sin\left(\dfrac{1}{2}x\right)$ **26.** $y = |x| - 2$

27. $y = |x - 2|$ **28.** $y = \dfrac{1}{4}\tan\left(x - \dfrac{\pi}{4}\right)$

29. $y = |\sqrt{x} - 1|$ **30.** $y = |\cos \pi x|$

31. The city of New Orleans is located at latitude 30°N. Use Figure 1.53 to find a function that models the number of hours of daylight at New Orleans as a function of the time of year. To check the accuracy of your model, use the fact that on March 31 the sun rises at 5:51 AM and sets at 6:18 PM in New Orleans.

32. A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between periods of maximum brightness is 5.4 days, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.

33. Some of the highest tides in the world occur in the Bay of Fundy on the Atlantic Coast of Canada. At Hopewell Cape the water depth at low tide is about 2.0 m and at high tide it is about 12.0 m. The natural period of oscillation is about 12 hours and on January 26, 2020, high tide occurred at 1:02 PM. Find a cosine function that models the water depth $D(t)$ (in meters) as a function of time t (in hours after midnight) on that day.

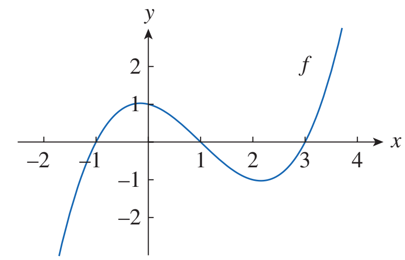
34. In a normal respiratory cycle, the volume of air that moves into and out of the lungs is about 500 mL. The reserve and residue volumes of air that remain in the lungs occupy about 2000 mL, and a single respiratory cycle for an average human takes about four seconds. Find a model for the total volume of air $V(t)$ in the lungs as a function of time.

35. (a) Explain how the graph of $y = f(|x|)$ is related to the graph of f?

(b) Sketch the graph of $y = \sin |x|$.

(c) Sketch the graph of $y = \sqrt{|x|}$.

36. Use the graph of f to sketch the graph of $y = \dfrac{1}{f(x)}$. Which features of f are most important in sketching $y = \dfrac{1}{f(x)}$? Explain your reasoning.



Find the functions (a) $f+g$, (b) $f-g$, (c) fg, and (d) f/g and state their domains.

37. $f(x) = x^3 + 2x^2, \quad g(x) = 3x^2 - 1$

38. $f(x) = \sqrt{3-x}, \quad g(x) = \sqrt{x^2-1}$

Find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and state their domains.

39. $f(x) = 3x + 5, \quad g(x) = x^2 + x$

40. $f(x) = x^3 - 2, \quad g(x) = 1 - 4x$

41. $f(x) = \sqrt{x+1}, \quad g(x) = 4x - 3$

42. $f(x) = \sin x, \quad g(x) = x^2 + 1$

43. $f(x) = x + \dfrac{1}{x}, \quad g(x) = \dfrac{x+1}{x+2}$

44. $f(x) = \dfrac{x}{1+x}, \quad g(x) = \sin 2x$

Find $f \circ g \circ h$.

45. $f(x) = 3x - 2, \quad g(x) = \sin x, \quad h(x) = x^2$

46. $f(x) = |x-4|, \quad g(x) = x^2, \quad h(x) = \sqrt{x}$

47. $f(x) = \sqrt{x-3}, \quad g(x) = x^2, \quad h(x) = x^3 + 2$

48. $f(x) = \tan x, \quad g(x) = \dfrac{x}{x-1}, \quad h(x) = \sqrt[3]{x}$

Find functions f and g such that $F = f \circ g$.

49. $F(x) = (2x + x^2)^4$

50. $F(x) = \cos^2 x$

51. $F(x) = \dfrac{\sqrt[3]{x}}{1+\sqrt[3]{x}}$

52. $F(x) = \sqrt[3]{\dfrac{x}{1+x}}$

53. $F(t) = \sec(t^2)\tan(t^2)$

54. $F(t) = \dfrac{\tan t}{1+\tan t}$

Find functions f, g, and h such that $F = f \circ g \circ h$.

55. $F(x) = \sqrt{\sqrt{x} - 1}$

56. $F(x) = \sqrt[8]{2 + |x|}$

57. $F(t) = \sin^2(\cos t)$

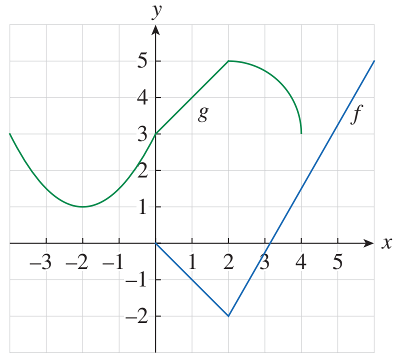
58. $F(t) = \tan\left|\dfrac{1}{t^3}\right|$

59. Use the table to evaluate each expression.

x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

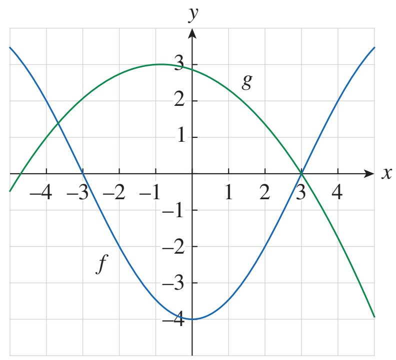
(a) $f(g(1))$ (b) $g(f(1))$ (c) $f(f(1))$
(d) $g(g(1))$ (e) $(g \circ f)(3)$ (f) $(f \circ g)(6)$

60. Use the graphs of f and g to evaluate each expression, or explain why it is undefined.



(a) $f(g(2))$ (b) $g(f(0))$ (c) $(f \circ g)(0)$
(d) $(g \circ f)(6)$ (e) $(g \circ g)(-2)$ (f) $(f \circ f)(4)$

61. Use the graphs of f and g to estimate the value of $f(g(x))$ for $x = -5, -4, -3, \ldots, 5$. Use these estimates to sketch a graph of $f \circ g$.



62. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
(a) Express the radius r of this circle as a function of time t (in seconds).
(b) If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret this expression in the context of the problem.

63. A spherical balloon is being inflated and the radius of the balloon is increasing at a rate of 2 cm/s.
(a) Express the radius r of the balloon as a function of the time t (in seconds).
(b) If V is the volume of the balloon as a function of the radius, find $V \circ r$ and interpret this expression in the context of the problem.

64. A ship is moving at a speed of 30 km/h parallel to a straight shoreline. The ship is 6 km from shore, and it passes a lighthouse at noon.
(a) Express the distance s between the lighthouse and the ship as a function of d, the distance the ship has traveled since noon; that is, find f so that $s = f(d)$.

(b) Express d as a function of t, the time elapsed since noon; that is, find g so that $d = g(t)$.
(c) Find $f \circ g$. What does this function represent in the context of this problem?

65. An airplane is flying at a speed of 350 mi/h at an altitude of 1 mi and passes directly over a radar station at time $t = 0$.
(a) Express the horizontal distance d (in miles) that the plane has flown as a function of t.
(b) Express the distance s between the plane and the radar station as a function of d.
(c) Use composition to express s as a function of t.

66. The **Heaviside function** H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.
(a) Sketch the graph of the Heaviside function.
(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.
(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

67. The Heaviside function defined in Exercise 66 can also be used to define the **ramp function** $y = ctH(t)$, where c is a positive constant, which represents a gradual increase in voltage or current in a circuit.
(a) Sketch the graph of the ramp function $y = tH(t)$.
(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 60$.
(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 32$.

68. Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. Is $f \circ g$ also a linear function? If so, what is the slope of its graph? If not, why?

69. If you invest x dollars at 4% interest compounded annually, then the amount $A(x)$ of the investment after 1 year is $A(x) = 1.04x$. Find $A \circ A$, $A \circ A \circ A$, and $A \circ A \circ A \circ A$. What do these compositions represent in the context of this problem? Find a formula for the composition of n copies of A.

70. (a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$.
(b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

71. If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.

72. Suppose g is an even function and let $h = f \circ g$. Is h always an even function? Explain your reasoning.

73. Suppose g is an odd function and let $h = f \circ g$. Is h always an odd function? What if f is odd? What if f is even?

74. (a) Suppose f is an even function, $g(x) = \sin x$, and $h = f \circ g$. Is h an even function, odd function, or neither? Justify your answer.
(b) Suppose f is an odd function, $g(x) = \cos x$, and $h = g \circ f$. Is h an even function, odd function, or neither? Justify your answer.
(c) Suppose f, g, and h are even functions, and $k = f \circ g \circ h$. Is k an even function, odd function, or neither? Justify your answer.

1.4 Exponential Functions

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x, is in the exponent. Note the difference between this and the power function, $g(x) = x^2$, in which the variable is in the base.

In general, an **exponential function** is a function of the form

$$f(x) = b^x$$

where b is a positive constant. Here is how we evaluate this function.

If $x = n$, a positive integer, then $b^n = \underbrace{b \cdot b \cdot \cdots \cdot b}_{n \text{ factors}}$.

If $x = 0$, then $b^0 = 1$.

If $x = -n$, where n is a positive integer, then $b^{-n} = \dfrac{1}{b^n}$.

If x is a rational number, $x = \dfrac{p}{q}$, where p and q are integers and $q > 0$, then

$$b^x = b^{p/q} = \sqrt[q]{b^p} = (\sqrt[q]{b})^p.$$

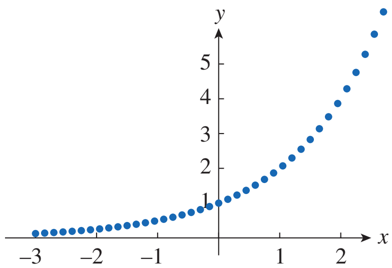


Figure 1.57
Representation of the graph of $y = 2^x$, x rational.

To complete the definition of an exponential function, we need to determine the meaning of b^x when x is an irrational number. For example, what do we mean by $2^{\sqrt{3}}$ or 5^{π}? And how do we find these values?

Consider the graph of $y = 2^x$ where x is rational. A representation of this graph is shown in Figure 1.57. We need to enlarge the domain of $f(x) = 2^x$ to include both rational and irrational numbers.

There are holes in the graph of Figure 1.57 corresponding to irrational values of x. We would certainly like to *fill in* these holes to produce an unbroken graph by defining $f(x) = 2^x$ when x is irrational such that the graph of f is *smooth* and increasing for all real numbers.

To do this, consider the irrational number $\sqrt{3}$.

Since $1.7 < \sqrt{3} < 1.8$, it seems reasonable that $2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$.

We can calculate $2^{1.7}$ and $2^{1.8}$ because 1.7 and 1.8 are rational numbers.

If we use a better approximation for $\sqrt{3}$, then it also seems reasonable that we can obtain a better approximation for $2^{\sqrt{3}}$.

$$\begin{array}{rcl}
1.73 < \sqrt{3} < 1.74 & \Rightarrow & 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\
1.732 < \sqrt{3} < 1.733 & \Rightarrow & 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\
1.7320 < \sqrt{3} < 1.7321 & \Rightarrow & 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\
1.73205 < \sqrt{3} < 1.73206 & \Rightarrow & 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206} \\
\vdots & & \vdots
\end{array}$$

A proof of this fact is given in J. Marsden and A. Weinstein, *Calculus Unlimited* (Menlo Park, CA: Benjamin Cummings, 1981).

It can be shown that there is exactly one number that is greater than all of the numbers

$$2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \quad 2^{1.7320}, \quad 2^{1.73205}, \quad \ldots$$

and less than all of the numbers

$$2^{1.8}, \quad 2^{1.74}, \quad 2^{1.733}, \quad 2^{1.7321}, \quad 2^{1.73206}, \quad \ldots$$

It seems reasonable to define $2^{\sqrt{3}}$ to be this unique number. Using this approximation process, we can find an estimate of this number.

$$2^{\sqrt{3}} \approx 3.321997$$

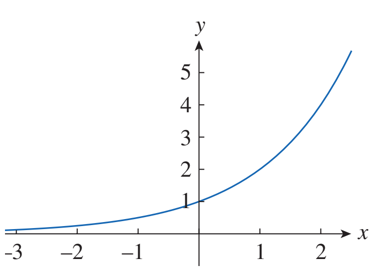


Figure 1.58
A smooth, complete graph of $f(x) = 2^x$, $x \in \mathbb{R}$.

Similarly, we can define 2^x (or b^x, if $b > 0$) where x is any irrational number. A complete graph of the function $f(x) = 2^x$, $x \in \mathbb{R}$, is shown in Figure 1.58.

The graphs of several functions in the family of functions of the form $y = b^x$ are shown in Figure 1.59 for various values of the base b. Notice that all of these graphs pass through the point (0, 1) because $b^0 = 1$ for $b \neq 0$. Notice also that as the base b increases, the exponential function grows more rapidly (for $x > 0$).

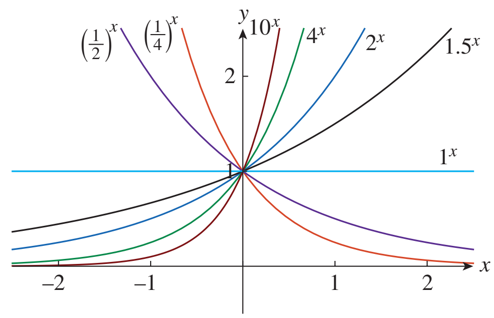
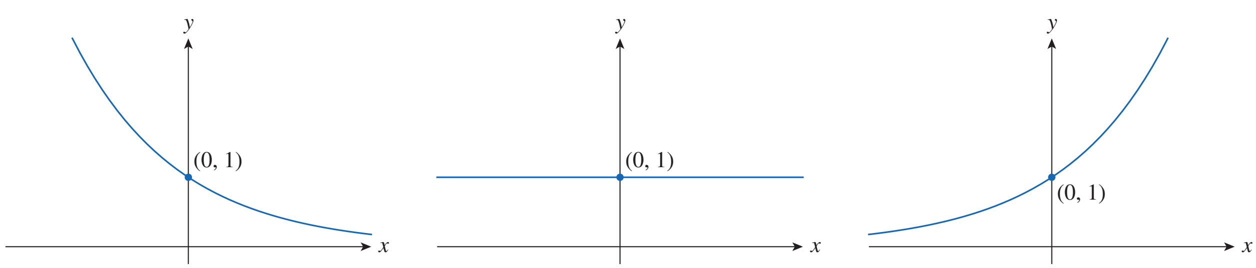


Figure 1.59
If $0 < b < 1$, then b^x approaches 0 as x becomes large. If $b > 1$, then b^x approaches 0 as x decreases through negative values (decreases without bound). In both cases the x-axis is a horizontal asymptote.

Figure 1.59 suggests that there are three kinds of exponential functions of the form $y = b^x$. If $0 < b < 1$, the exponential function decreases as x increases; if $b = 1$, the exponential function is constant; and if $b > 1$, the exponential function increases. These three cases are illustrated in Figure 1.60. Notice that if $b \neq 1$, then the exponential function $y = b^x$ has domain $\mathbb{R}$ and range $(0, \infty)$. In addition, since $(1/b)^x = 1/b^x = b^{-x}$, the graph of $y = (1/b)^x$ is a reflection of the graph of $y = b^x$ about the y-axis.



(a) Graph of $y = b^x, 0 < b < 1$ (b) Graph of $y = 1^x$ (c) Graph of $y = b^x, b > 1$

Figure 1.60
Graphs of the three kinds of exponential functions.

Exponential functions are important in the study of calculus. The following properties of exponential functions are true for all real numbers x and y.

Laws of Exponents

If a and b are positive numbers and x and y are any real numbers, then

1. $b^{x+y} = b^x b^y$ **2.** $b^{x-y} = \dfrac{b^x}{b^y}$ **3.** $(b^x)^y = b^{xy}$ **4.** $(ab)^x = a^x b^x$

Example 1 Graph, Domain, and Range

Sketch the graph of the function $y = 3 - 2^x$ and determine its domain and range.

Solution

We can sketch the graph using transformations: start with the graph of $y = 2^x$.

Reflect this graph about the x-axis to get the graph of $y = -2^x$.

Shift this graph upward 3 units to obtain the graph of $y = 3 - 2^x$. See Figure 1.61.

The domain is $\mathbb{R}$: we can evaluate the function for any real number.

The range is $(-\infty, 3)$: as x decreases without bound, y approaches 3, and as x increases without bound, y decreases without bound.

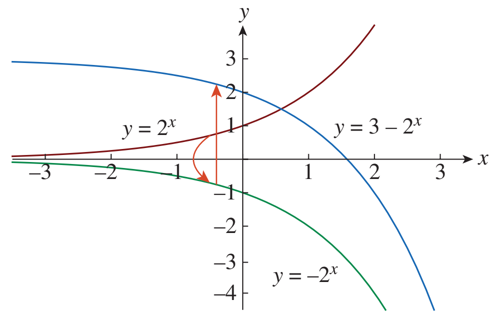


Figure 1.61
Graph of $y = 3 - 2^x$ obtained through transformations.

Example 2 Power Function and Exponential Function Comparison

Compare the graphs of the exponential function $y = 2^x$ and the power function $y = x^2$. Which function grows more quickly as x increases without bound?

Solution

Figure 1.62 shows the graphs of both functions. This figure suggest that the graphs intersect three times.

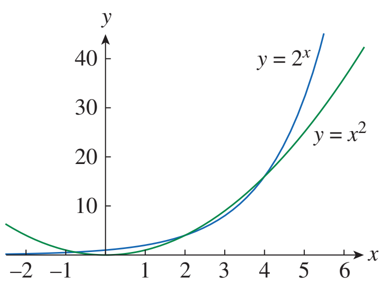


Figure 1.62
The graphs of $f(x) = 2^x$ and $g(x) = x^2$ intersect at three points.

For $x > 4$, the graph of $f(x) = 2^x$ stays above the graph of $g(x) = x^2$. Figure 1.63 supports this observation and shows that for large values of x, the exponential function $f(x) = 2^x$ grows much faster than the power function $g(x) = x^2$.

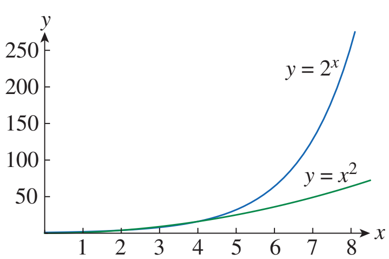


Figure 1.63
The exponential function $f(x) = 2^x$ grows much faster than the power function $g(x) = x^2$.

Applications of Exponential Functions

The exponential function is used often in mathematical models, for example, those involving nature or the lifetime of electronic components. In this section, we will briefly examine how the exponential function is used to describe population growth and radioactive decay. In later chapters, we will consider these and other applications in greater detail.

Here is an example of an exponential model. Consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the number of bacteria at time t is $p(t)$, where t is measured in hours, and the initial population is $p(0) = 1000$, then

$$p(1) = 2p(0) = 2 \times 1000$$
$$p(2) = 2p(1) = 2^2 \times 1000$$
$$p(3) = 2p(2) = 2^3 \times 1000$$

This pattern suggests that, in general,

$$p(t) = 2^t \times 1000$$

This (bacteria) population function is a constant multiple of the exponential function $y = 2^t$. Therefore, the population exhibits rapid growth as we observed earlier. Under ideal (growing) conditions (unlimited space and nutrition, and absence of disease), this type of exponential growth is typical of what actually occurs in nature.

Consider a model of the human population. Table 1.3 shows data for the population of the world at various years since 1900, and Figure 1.64 shows the corresponding scatter plot.

Year	Population (in millions)	Year	Population (in millions)
1900	1650	1970	3710
1910	1750	1980	4450
1920	1860	1990	5280
1930	2070	2000	6080
1940	2300	2010	6870
1950	2560	2020	7755
1960	3040		

Table 1.3
Table of the world population.

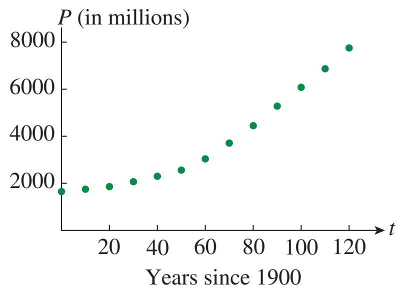


Figure 1.64
Scatter plot of the world population data.

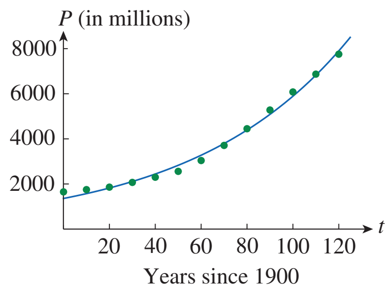


Figure 1.65
Graph of the exponential model for population growth and the scatter plot of the original data.

Day	Height (feet)	Day	Height (feet)
0	13.07	5	6.26
1	11.38	6	5.85
2	10.16	7	5.48
3	8.64	8	4.97
4	7.11	9	4.61

Table 1.4
Table of river height.

The pattern in the data suggests exponential growth. Using the method of least squares, we obtain an exponential model for this data:

$$P = (1358.03) \cdot (1.01478)^t$$

where $t = 0$ corresponds to 1900. Figure 1.65 shows the graph of this exponential function together with the scatter plot of the original data. The graph suggests that the exponential model describes, or models, the data reasonably well. The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

Here is an example of another type of exponential model. Table 1.4 shows the height $h(t)$ of the Susquehanna River at Bloomsburg, Pennsylvania, t days after a significant rainfall in November 2019 (U.S. Geological Survey). The corresponding scatter plot is shown in Figure 1.66.

The rapid decrease in river height and leveling off pattern in the scatter plot suggests an exponential model, $y = b^x$, where $b < 1$. An exponential model for this data, using the method of least squares, is

$$h = (12.8626) \cdot (0.880114)^t$$

Figure 1.67 shows the graph of this exponential function and the scatter plot of the original data. This model fits the data very well for this particular rain event.

Note that we could use the graph in Figure 1.67 (and the exponential model) to estimate the **half-life** of h, that is, the time required for the height of the river to decrease to half its initial value. In the next example, we are given the half-life of a radioactive element and will find the mass of a sample at any time.

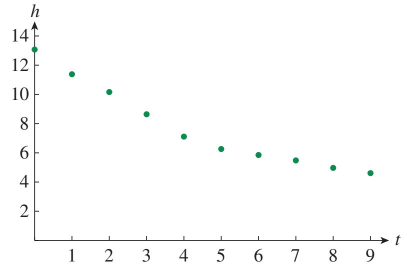


Figure 1.66
Scatter plot of the river height data.

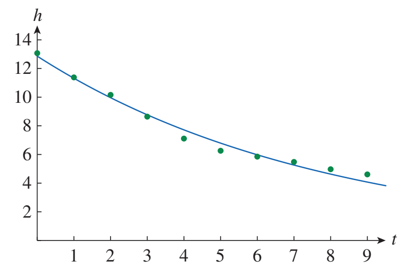


Figure 1.67
Graph of the exponential model for river height and the scatter plot of the original data.

Example 3 Half-Life and Exponential Decay

The half-life of strontium-90, ^{90}Sr, is 25 years. This means that half of any given quantity of ^{90}Sr will disintegrate in 25 years.

(a) If a sample of ^{90}Sr has a mass of 24 mg, find an expression for the mass $m(t)$ that remains after t years.

(b) Find the mass remaining after 40 years, correct to the nearest milligram.

(c) Sketch a graph of $m(t)$ and estimate the time required for the mass to be reduced to 5 mg.

Solution

(a) The mass is initially 24 mg and is reduced by half every 25 years. Therefore,

$$\begin{aligned} m(0) &= 24 \\ m(25) &= \frac{1}{2}(24) \\ m(50) &= \frac{1}{2}\cdot\frac{1}{2}(24) = \frac{1}{2^2}(24) \\ m(75) &= \frac{1}{2}\cdot\frac{1}{2^2}(24) = \frac{1}{2^3}(24) \\ m(100) &= \frac{1}{2}\cdot\frac{1}{2^3}(24) = \frac{1}{2^4}(24) \end{aligned}$$

This pattern suggests that the mass remaining after t years is

$$m(t) = \frac{1}{2^{t/25}}(24) = 24 \cdot 2^{-t/25} = 24 \cdot (2^{-1/25})^t.$$

This is an exponential function with base $b = 2^{-1/25} = \dfrac{1}{2^{1/25}}$.

(b) The mass remaining after 40 years is

$$m(40) = 24 \cdot 2^{-40/25} \approx 7.9 \text{ mg}.$$

(c) Figure 1.68 shows a graph of the function $m(t) = 24 \cdot 2^{-t/25}$ and the line $m = 5$.

To find the time when the mass is reduced to 5 mg, we need to solve the equation $m(t) = 5$ for t or, graphically, find the value of t where the graph of $m = m(t)$ and $m = 5$ intersect. The graphs in Figure 1.68 appear to intersect at $t \approx 57$. Using technology to solve the equation, $t \approx 57.576$. ■

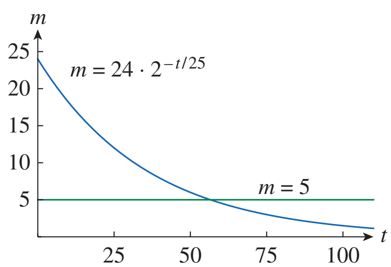


Figure 1.68
Graph of the exponential model and the line $m = 5$.

The Number e

Of all possible bases for an exponential function, there is one that is most convenient and useful in calculus. The choice of a base b determines the way the graph of $y = b^x$ crosses the y-axis. Figures 1.69 and 1.70 show the tangent lines to the graphs of $y = 2^x$ and $y = 3^x$ at the point (0, 1). Tangent lines will be defined precisely later. For now, you can think of the tangent line to a graph at a point as the line that touches the graph only at that point and has the same *direction* as the graph. If we consider the slopes of these tangent lines at (0, 1), we find that $m \approx 0.7$ for $y = 2^x$ and $m \approx 1.1$ for $y = 3^x$.

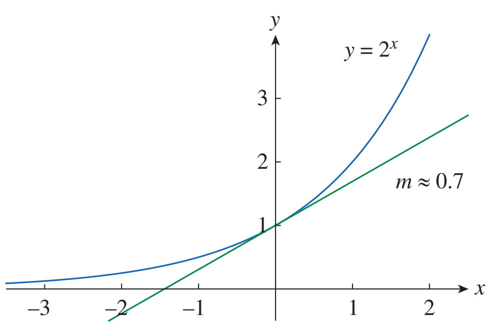


Figure 1.69
Graph of $y = 2^x$ and the tangent line to the graph at (0, 1).

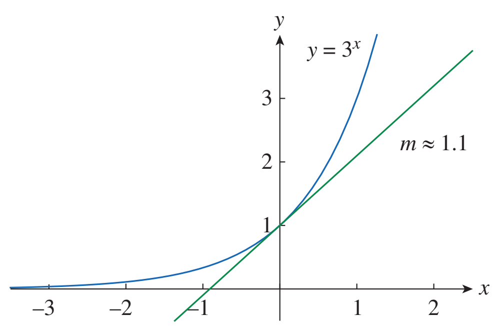


Figure 1.70
Graph of $y = 3^x$ and the tangent line to the graph at (0, 1).

These graphs suggest, and it seems reasonable to conclude, that there is some base b, $2 < b < 3$ such that the tangent line to the graph of $y = b^x$ at the point (0, 1) is exactly 1. This number does indeed exist and is denoted by the letter e. See Figure 1.71. The value of e, correct to five decimal places, is $e \approx 2.71828$, and the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. See Figure 1.72. The function $f(x) = e^x$ is called the **natural exponential function**.

This notation e was chosen by the Swiss mathematician Leonhard Euler in 1727.

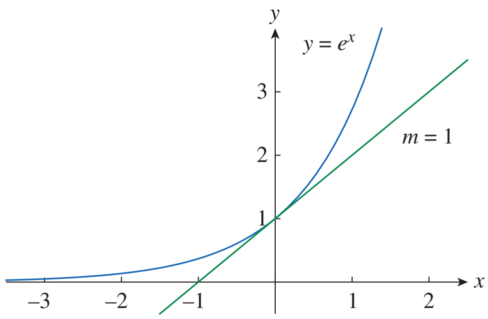


Figure 1.71
Graph of $y = e^x$ and the tangent line to the graph at (0, 1).

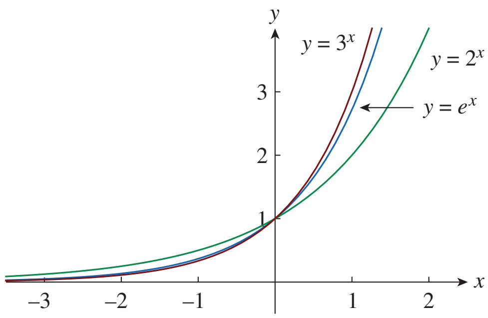


Figure 1.72
Graph of $y = 2^x$, $y = e^x$, and $y = 3^x$.

Example 4 Graph, Domain, Range, and *e*

Sketch the graph of the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

Solution

We can sketch the graph using transformations: start with the graph of $y = e^x$.

Reflect this graph about the y-axis to get the graph of $y = e^{-x}$.

Shrink the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$.

Finally, shift the graph 1 unit downward to obtain the graph of $y = \frac{1}{2}e^{-x} - 1$. Figure 1.73 illustrates these transformations.

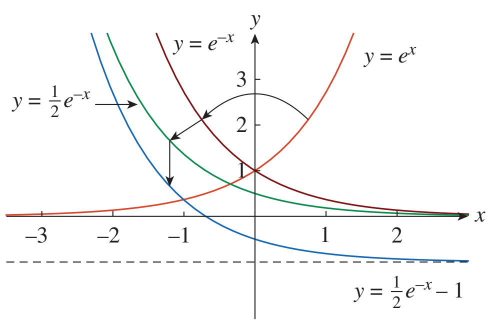


Figure 1.73
The graph of $y = \frac{1}{2}e^{-x} - 1$ obtained through transformations of the graph $y = e^x$.

The domain is $\mathbb{R}$: we can evaluate the function for any real number.

The range is $(-1, \infty)$: as x increases without bound, y approaches -1, and as x decreases without bound, y increases without bound.

1.4 Exercises

Use the Laws of Exponents to rewrite and simplify the expression.

1. (a) $\dfrac{4^{-3}}{2^{-8}}$ (b) $\dfrac{1}{\sqrt[3]{x^4}}$

2. (a) $8^{4/3}$ (b) $x(3x^2)^3$

3. (a) $b^8(2b)^4$ (b) $\dfrac{(6y^3)^4}{2y^5}$

4. (a) $5^{3-2\sqrt{7}} \cdot 25^{-1+\sqrt{7}}$ (b) $\dfrac{4^{n-2} \cdot 8^{2-n}}{16^{2-n}}$

5. (a) $(20)^{1/2} + 5^{3/2}$ (b) $8^{1/2} + 2^{3/2}$

6. (a) $(3^{x+1})(9^{2x})$ (b) $\dfrac{3^{1/3}}{3^{-2/3}}$

7. (a) $\dfrac{x^{2n} \cdot x^{3n-1}}{x^{n+2}}$ (b) $\dfrac{\sqrt{a\sqrt{b}}}{\sqrt[3]{ab}}$

8. For each of the following exponential expressions identify an interval $[a, b]$, where a and b are consecutive integers, that contains the expression.
(a) $2^{\sqrt{5}}$ (b) $3^{\sqrt{2}}$ (c) $4^{1.6}$ (d) $\sqrt{2}^{\sqrt{2}}$

9. (a) Write an equation that defines the exponential function with base $b > 0$.
(b) What is the domain of this function?
(c) If $b \neq 1$, what is the range of this function?
(d) Sketch the general shape of the graph of the exponential function for each of the following cases.
(i) $b > 1$ (ii) $b = 1$ (iii) $0 < b < 1$

10. (a) Explain how the number e is defined.
(b) What is an approximate value for e?
(c) What is the natural exponential function?

Use technology to graph the given functions on the same coordinate axes. Explain how these graphs are related.

11. $y = 2^x$, $y = e^x$, $y = 5^x$, $y = 20^x$

12. $y = e^x$, $y = e^{-x}$, $y = 8^x$, $y = 8^{-x}$

13. $y = 3^x$, $y = 10^x$, $y = \left(\dfrac{1}{3}\right)^x$, $y = \left(\dfrac{1}{10}\right)^x$

14. $y = 0.9^x$, $y = 0.6^x$, $y = 0.3^x$, $y = 0.1^x$

Use transformations to sketch the graph of the function.

15. $y = 4^x - 1$

16. $y = (0.5)^{x-1}$

17. $y = -2^{-x}$

18. $y = e^{|x|}$

19. $y = 1 - \dfrac{1}{2}e^{-x}$

20. $y = 2(1 - e^x)$

21. $y = -2^{-|x|} + 1$

22. $y = |-e^x + 2|$

23. Start with the graph of $y = e^x$ and write an equation of the graph that results from
(a) shifting 2 units downward.
(b) shifting 2 units to the right.
(c) reflecting about the x-axis.
(d) reflecting about the y-axis.
(e) reflecting about the x-axis and then about the y-axis.

24. Start with the graph of $y = e^{-x}$ and find the equation of the graph that results from
(a) reflecting about the line $y = 4$.
(b) reflecting about the line $x = 2$.

Find the domain of the function.

25. $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$

26. $f(x) = \dfrac{1 + x}{e^{\cos x}}$

27. $g(t) = \sqrt{10^t - 100}$

28. $g(t) = \sin(e^t - 1)$

Solve the equation for x.

29. $9^x = \left(\dfrac{1}{27}\right)^{x-2}$

30. $(\sqrt{2})^{2x-1} = \left(\dfrac{1}{4}\right)^{x+5}$

31. $8^x = (\sqrt{2})^{2x^2+4}$

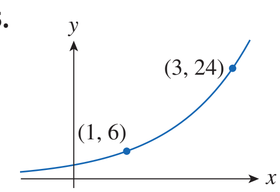
32. $(5^{3x})^2 = (5^x)^3 \cdot 5^{x+6}$

33. $3^{1/x} = 27^{x^2}$

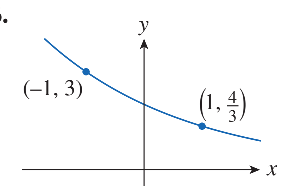
34. $4^{x+4} - 4^{x+3} = 96$

Find the exponential function $f(x) = Cb^x$ whose graph is given.

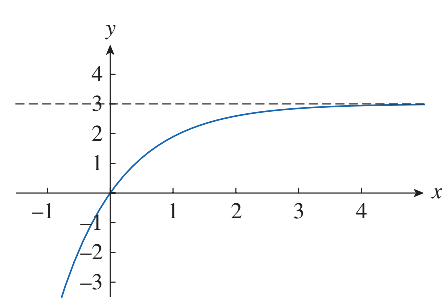
35.



36.



37. Find the exponential function $f(x) = Cb^x$ whose graph is given.



38. Sketch the graph of $y = 2^x$ and the graph of $y = \left(\frac{1}{2}\right)^x$ on the same coordinate axes. Explain the relationship between these two graphs.

39. Sketch the graph of $f(x) = \frac{x^2}{2^x}$. Use this graph to explain the behavior of the function f as x increases without bound.

40. If $f(x) = 5^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 5^x\left(\frac{5^h - 1}{h}\right)$$

41. Suppose you are offered a job that lasts 1 month. Which of the following methods of payment would you prefer? Explain your reasoning.

I. One million dollars at the end of the month.

II. One cent on the first day of the month, two cents on the second day, four cents on the third day, and in general, 2^{n-1} cents on the nth day.

42. Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch. Show that at a distance 2 ft to the right of the origin, the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.

43. Use technology to graph the functions $f(x) = x^5$ and $g(x) = 5^x$ on the same coordinate axes. Find all points of intersection of the graphs correct to three decimal places. Which function grows more rapidly as x increases?

44. Use technology to graph the functions $f(x) = x^{10}$ and $g(x) = e^x$ on the same coordinate axes. When does the graph of g finally surpass the graph of f?

45. Find the values of x such that $e^x > 1{,}000{,}000{,}000$.

46. One way to judge the value of an investment is by how quickly it doubles in value.

(a) If money is invested in a bank at a constant interest rate, one method to determine the time it will take to double is to divide 72 by the interest rate (called the Rule of 72). Use the Rule of 72 to determine how long it will take your money to double if the interest rate is 10%, 5%, or 2%.

(b) Another method of approximating the number of years required for the investment to double is to find the intersection point on the graphs of $y = 0.95e^{rt}$ and $y = 2$. Use technology to approximate the number of years required for your investment to double if the interest rate is 10%, 5%, or 2%.

47. A researcher is trying to determine the doubling time for a population of the bacterium *Giardia lamblia.* They start a culture in a nutrient solution and estimate the bacteria count every 4 hours.

(a) Construct a scatter plot of the data shown in the table.

Time (hours)	0	4	8	12	16	20	24
Bacteria count (CFU/mL)	37	47	63	78	105	130	173

(b) Use technology to find an exponential function $f(x) = a \cdot b^t$ that models the bacteria population after t hours.

(c) Use technology to determine how long it takes for the bacteria count to double.

G. lamblia

Sebastian Kaulitzki/Shutterstock.com

48. A bacteria culture starts with 500 bacteria and doubles in size every hour.

(a) How many bacteria are there after 3 hours?

(b) How many bacteria are there after t hours?

(c) How many bacteria are there after 40 minutes?

(d) Sketch the graph of the bacteria population function and estimate the time for the population to reach 100,000.

49. The half-life of bismuth-210, ^{210}Bi, is 5 days.

(a) If a sample has a mass of 200 mg, find the amount remaining after 15 days.

(b) Find the amount remaining after t days.

(c) Estimate the amount remaining after 3 weeks.

(d) Estimate the time required for the mass to be reduced to 1 mg.

50. An isotope of sodium, ^{24}Na, has a half-life of 15 hours. A sample of this isotope has mass 2 g.

(a) Find the amount remaining after 60 hours.

(b) Find the amount remaining after t hours.

(c) Estimate the amount remaining after 4 days.

(d) Estimate the time required for the mass to be reduced to 0.01 g.

51. Use the exponential model for river height, $h = (12.8626) \cdot (0.880114)^t$, to estimate when the river will be half of its original height.

52. After alcohol is fully absorbed into the body, it is metabolized with a half-life of about 1.5 hours. Suppose you have had three alcoholic drinks and an hour later, at midnight, your blood alcohol concentration (BAC) is 0.6 mg/mL.

(a) Find an exponential decay model for your BAC t hours after midnight.

(b) Graph your BAC and estimate when your BAC is 0.08 mg/mL.

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

53. The model $P = (1358.03) \times (1.01478)^t$ for the population of the world at various years since 1900 was found using the data from Table 1.3. Use this model to estimate the population in 1997 and to predict the population in 2022.

54. The table gives the population of the United States, in millions, for select years between 1900 and 2020.

Year	Population	Year	Population
1900	76	1970	203
1910	92	1980	227
1920	106	1990	250
1930	123	2000	281
1940	131	2010	310
1950	150	2020	331
1960	179		

Use technology with this data to find an exponential model for the U.S. population since 1900. Use this model to estimate the population in 1925 and to predict the population in the year 2025.

55. Sketch the graph of the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

Does the graph suggest that f is an even or odd function, or neither? Justify your answer.

56. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. Explain how the graph changes when b changes. Explain how the graph changes when a changes.

1.5 Inverse Functions and Logarithms

Inverse Functions

In this section we consider the general concept of inverse functions and then define a logarithmic function, the inverse of an exponential function.

Table 1.5 presents data from an experiment in which a bacteria culture started with 100 bacteria in a limited nutrient medium; the size of the bacteria population was recorded at hourly intervals. The number of bacteria N is a function of the time t: $N = f(t)$.

Suppose, however, that the biologist changes their point of view and becomes interested in the time required for the population to reach various levels. In other words, they are thinking of t as a function of N. This function is called the *inverse function* of f, denoted by f^{-1}, and is read as "f inverse." Thus, $t = f^{-1}(N)$ is the time required for the population level to reach N. The values of f^{-1} can be found by reading Table 1.5 from right to left or by considering Table 1.6. For example, $f^{-1}(550) = 6$ because $f(6) = 550$.

t (hours)	$N = f(t)$ = population at time t
0	100
1	168
2	259
3	358
4	445
5	509
6	550
7	573
8	586

Table 1.5
N as a function of t.

N	$t = f^{-1}(N)$ = time to reach N bacteria
100	0
168	1
259	2
358	3
445	4
509	5
550	6
573	7
586	8

Table 1.6
t as a function of N.

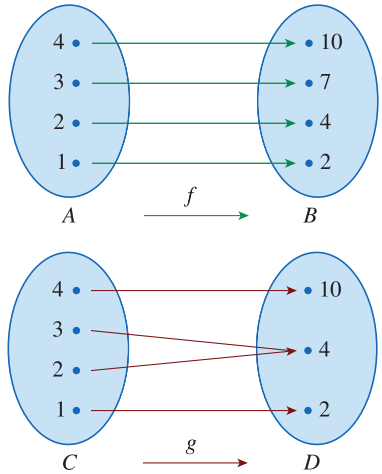


Figure 1.74
f is one-to-one; g is not.

In the language of inputs and outputs, this definition says that f is one-to-one if each output corresponds to only one input.

Not every function has an inverse. For example, compare the functions f and g represented by arrow diagrams shown in Figure 1.74. Note that f never takes on the same value twice (any two inputs in A have different outputs) but g does take on the same value twice (both 2 and 3 have the same output, 4). In symbols, $g(2) = g(3)$, but

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2$$

A function with this property (like f) is called a *one-to-one function*.

Definition • One-to-One Function

A function f is called a **one-to-one function**, or **injective**, if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2$$

We can examine the graph of a function to determine whether it is one-to-one. Suppose a horizontal line intersects the graph of a function f in more than one point (see Figure 1.75). There are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. Therefore, f is not one-to-one.

This suggests the following geometric method for determining whether a function is one-to-one.

Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

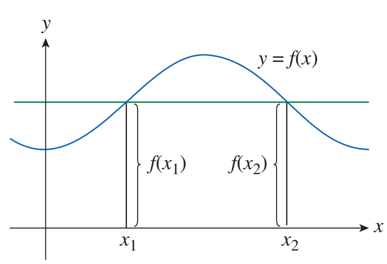


Figure 1.75
The function is not one-to-one because $f(x_1) = f(x_2)$.

It's straightforward to use the Horizontal Line Test to show that a function is *not* one-to-one. If you can draw a horizontal line that intersects the graph in more than one place, then the function is *not* one-to-one. However, in order to prove a function is one-to-one, we usually use an analytic approach. One typical method involves the contrapositive. That is, we show that if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Example 1 One-to-One Function

Is the function $f(x) = x^3$ one-to-one?

Solution

The graph of f is shown in Figure 1.76 along with several horizontal lines.

It certainly appears that f is a one-to-one function by the Horizontal Line Test.

To show this analytically:

Suppose $f(x_1) = x_1^3 = x_2^3 = f(x_2)$.

Then $x_1 = x_2$. (Take the cube root of both sides of $x_1^3 = x_2^3$.)

Therefore, f is one-to-one. ■

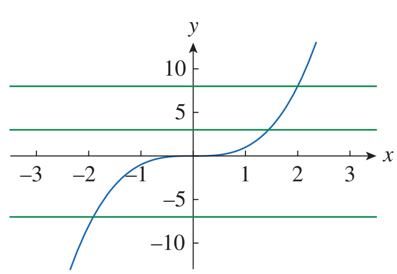


Figure 1.76
$f(x) = x^3$ is one-to-one.

Example 2 Not One-to-One

Is the function $g(x) = x^2$ one-to-one?

Solution

The graph of g is shown in Figure 1.77 along with two horizontal lines that intersect the graph of g more than once.

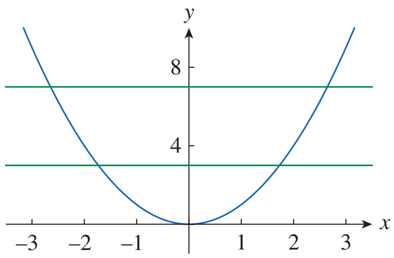


Figure 1.77
$g(x) = x^2$ is not one-to-one. Note that we could restrict the domain of $y = x^2$ to $x \geq 0$. This new function would be one-to-one.

Therefore, by the Horizontal Line Test, g is not one-to-one.

Analytically, we can find two different values in the domain of g that yield the same output.

For example, $g(-1) = 1 = g(1)$. ■

A one-to-one function is important in calculus because it has an inverse. The formal definition follows.

Definition • Inverse Function

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B.

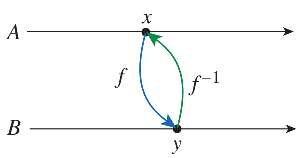


Figure 1.78
Arrow diagram showing the effect of f^{-1}.

This definition says that if f maps x into y, then f^{-1} maps y back to x. Whatever f does, f^{-1} undoes. Note that if f were not one-to-one, then f^{-1} would not be defined. The arrow diagram in Figure 1.78 is a visualization of the relationship between f and f^{-1}; f^{-1} reverses the effect of f. Note that

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$ then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

CAUTION: Do not mistake the -1 in f^{-1} for an exponent.

$$f^{-1}(x) \quad \text{does not mean} \quad \frac{1}{f(x)}$$

However, the reciprocal, $\dfrac{1}{f(x)}$ could be written as $[f(x)]^{-1}$.

Example 3 Evaluate an Inverse Function

Suppose $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$. Find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

Solution

Use the definition of the inverse function, f^{-1}.

$$f^{-1}(7) = 3 \text{ because } f(3) = 7$$

$$f^{-1}(5) = 1 \text{ because } f(1) = 5$$

$$f^{-1}(-10) = 8 \text{ because } f(8) = -10$$

The diagram in Figure 1.79 illustrates how f^{-1} undoes what f does in this case. ■

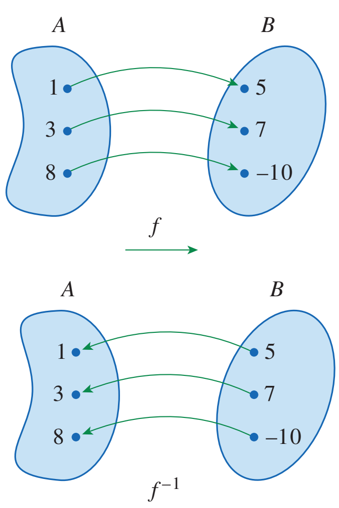


Figure 1.79
The inverse function reverses inputs and outputs.

In mathematics, and especially calculus, the letter x is traditionally used as the independent variable. Therefore, when we focus on the inverse function, f^{-1} rather than f, we usually reverse the roles of x and y in Definition 2. That is,

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x \qquad (1)$$

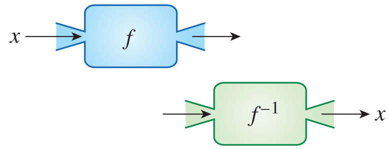


Figure 1.80
Whatever f does, f^{-1} undoes.

Using this conventional notation, we have the following **cancellation equations**:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A \tag{2}$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

The first cancellation equation says that if we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started (see the machine diagram in Figure 1.80). Thus, f^{-1} undoes what f does. The second equation says that f undoes what f^{-1} does.

For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$. The cancellation equations become

$$f^{-1}(f(x)) = (x^3)^{1/3} = x$$
$$f(f^{-1}(x)) = (x^{1/3})^3 = x$$

These equations simply show that the cube function and the cube root function cancel each other when applied in succession.

There is a very prescriptive method for finding an inverse function. Given a function $y = f(x)$, suppose we are able to solve this equation for x in terms of y. Then according to the definition of an inverse function, we must have $x = f^{-1}(y)$. If we want to call the independent variable x, we then interchange x and y and arrive at the equation $y = f^{-1}(x)$.

Finding the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 Interchange the variables x and y to express f^{-1} as a function of x. The resulting equation is $y = f^{-1}(x)$.

Example 4 Find an Inverse Function

Find the inverse function of $f(x) = x^3 + 2$.

Solution

Use the procedure for finding the inverse function.

$y = x^3 + 2$	Write $y = f(x)$.
$x^3 = y - 2$	Isolate term(s) involving x.
$x = \sqrt[3]{y - 2}$	Solve for x in terms of y.
$y = \sqrt[3]{x - 2}$	Interchange x and y.

Therefore, the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$. ■

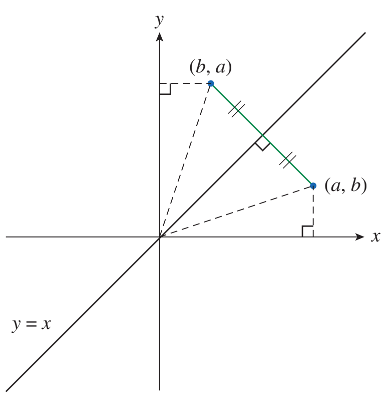


Figure 1.81
The point (b, a) is a reflection of the point (a, b) about the line $y = x$.

The principle of interchanging x and y to find the inverse function also provides a method for obtaining the graph of f^{-1} from the graph of f. Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. The point (b, a) is a reflection of the point (a, b) about the line $y = x$ (see Figure 1.81). Therefore, the graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$ (see Figure 1.82).

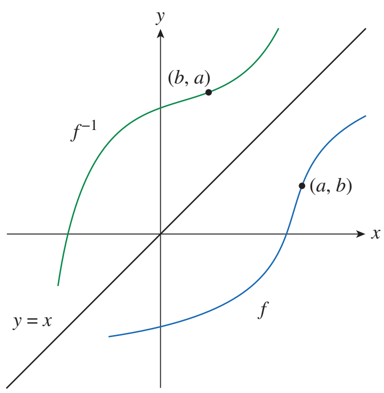


Figure 1.82
The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Example 5 f^{-1}, Domain, and Range

Let $f(x) = \sqrt{x-1} + 2$.

(a) Find the domain and range of f.

(b) Find f^{-1} and state the domain and range of f^{-1}.

(c) Sketch the graphs of f and f^{-1} on the same coordinate axes.

Solution

(a) $x - 1 \geq 0 \quad \Rightarrow \quad x \geq 1 \quad \Rightarrow \quad \text{domain of } f: [1, \infty)$

$$\sqrt{x-1} \in [0, \infty) \quad \Rightarrow \quad f(x) = \sqrt{x-1} + 2 \in [2, \infty) \quad \Rightarrow \quad \text{range of } f: [2, \infty)$$

(b)

$$y = \sqrt{x-1} + 2 \qquad \text{Let } y = f(x).$$

$$y - 2 = \sqrt{x-1} \qquad \text{Subtract 2 from both sides.}$$

$$(y-2)^2 = x - 1 \qquad \text{Square both sides.}$$

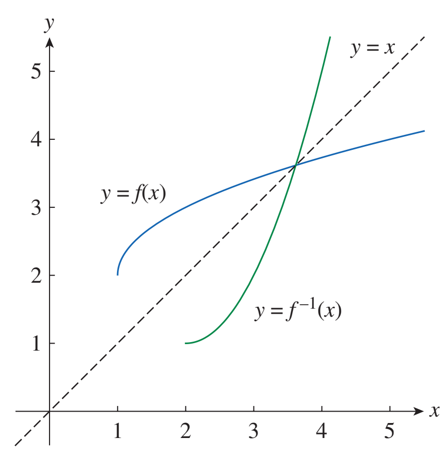
$$x = (y-2)^2 + 1 \qquad \text{Solve for } x.$$

$$y = (x-2)^2 + 1 \qquad \text{Interchange } x \text{ and } y.$$

The inverse function is $f^{-1}(x) = (x-2)^2 + 1$.

Domain of f^{-1}: $[2, \infty)$ (range of f) Range of f^{-1}: $[1, \infty)$ (domain of f)

(c) Graph of f and f^{-1}



Logarithmic Functions

If $b > 0$ and $b \neq 1$, the exponential function $f(x) = b^x$ is either always increasing or always decreasing. The Horizontal Line Test suggests, and we can prove, that f is one-to-one. Therefore, f has an inverse function f^{-1}, called the **logarithmic function with base *b***, denoted $\log_b$. Using the formula for an inverse function

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

then we have

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x \tag{3}$$

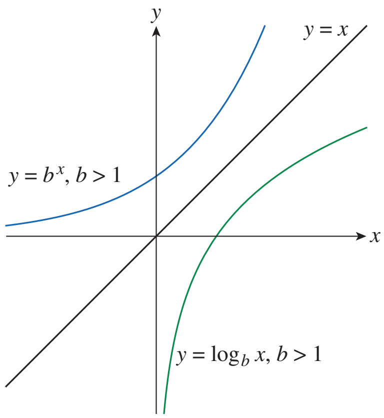


Figure 1.83
The graph of $y = \log_b x$ is a reflection of the graph of $y = b^x$ about the line $y = x$.

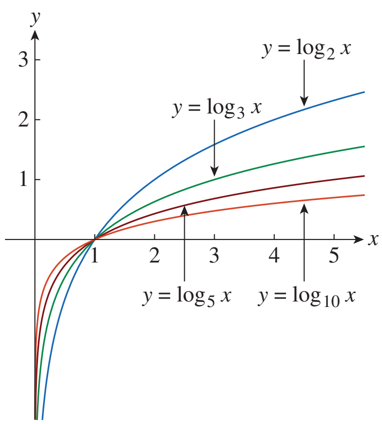


Figure 1.84
The graphs for several logarithmic functions.

Notation for Logarithms

Most textbooks in calculus and the sciences, as well as graphing calculators and some Computer Algebra Systems, use the notation $\ln x$ for the natural logarithm and $\log x$ for the "common logarithm," $\log_{10} x$. In the more advanced mathematical and scientific literature and in computer languages, however, the notation $\log x$ usually denotes the natural logarithm.

In words, if $x > 0$ then $\log_b x$ is the exponent to which the base b must be raised to produce x. For example, $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.

When applied to the functions $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, the cancellation equations become

$$\log_b(b^x) = x \quad \text{for every } x \in \mathbb{R} \tag{4}$$

$$b^{\log_b x} = x \quad \text{for every } x > 0$$

The logarithmic function $\log_b x$ has domain $(0, \infty)$ and range $\mathbb{R}$. Its graph is the reflection of the graph of $y = b^x$ about the line $y = x$. Figure 1.83 shows the case for $b > 1$. Note that since $y = b^x$ increases very rapidly for $x > 0$, the graph of $y = \log_b x$ increases slowly for $x > 1$.

Figure 1.84 shows the graphs of $y = \log_b x$ for values of the base $b > 1$. Since $\log_b 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

The following properties of logarithmic functions follow from the corresponding properties of exponential functions given in Section 1.4.

Laws of Logarithms

If x and y are positive numbers, then

1. $\log_b(xy) = \log_b x + \log_b y$

2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

3. $\log_b(x^r) = r \log_b x$ where r is any real number

Example 6 The Laws of Logarithms

Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

Solution

$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) \qquad \text{Laws of Logarithms 2.}$$

$$= \log_2 16 = 4 \qquad 2^4 = 16.$$

■

Natural Logarithms

Of all possible bases b for logarithms, the most convenient and useful in calculus is the number e, which was defined in Section 1.4. The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

If we let $b = e$ and use this special notation in Equations 5 and 6, then the defining properties of the natural logarithm function become

$$\ln x = y \quad \Leftrightarrow \quad e^y = x \tag{5}$$

and the cancellation equations are

$$\ln(e^x) = x \quad x \in \mathbb{R} \tag{6}$$
$$e^{\ln x} = x \quad x > 0$$

Note that if $x = 1$, then $\ln e = 1$.

Example 7 Equation Involving ln

Another way to solve this problem is to replace "ln" notation by $\log_e$. The equation can now be written as $\log_e x = 5$. By definition of logarithms, $e^5 = x$.

Find x if $\ln x = 5$.

Solution 1

$\ln x = 5$ means $e^5 = x$ — Use Equation 5.

Therefore, $x = e^5$.

Solution 2

$\ln x = 5$

$e^{\ln x} = e^5$ — Apply the exponential function to both sides of the equation.

$x = e^5$ — Use the second cancellation equation.

■

Example 8 Equation Involving *e*

Solve the equation $e^{5-3x} = 10$.

Solution

$e^{5-3x} = 10$

$5 - 3x = \ln 10$ — Take the natural logarithm of both sides of the equation.

$3x = 5 - \ln 10$ — Rearrange terms.

$x = \frac{1}{3}(5 - \ln 10)$ — Divide both sides by 3.

Note: Using technology, $x \approx 0.899$. ■

Example 9 Simplify a Logarithmic Expression

Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

Solution

$\ln a + \frac{1}{2}\ln b = \ln a + \ln b^{1/2}$ — Laws of Logarithms 3.

$= \ln a + \ln\sqrt{b}$ — Rewrite as square root.

$= \ln(a\sqrt{b})$ — Laws of Logarithms 1.

■

The following formula shows that a logarithm with any base can be expressed in terms of the natural logarithm.

Note that the Change of Base Formula could also be written using common logarithms: in calculus, it is more convenient and useful to use the natural logarithm.

Change of Base Formula

For any positive number b $(b \neq 1)$,

$$\log_b x = \frac{\ln x}{\ln b} \tag{7}$$

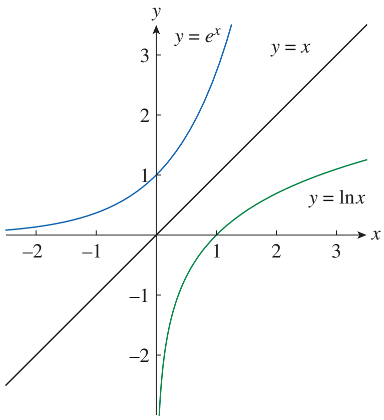


Figure 1.85
The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$.

Proof

Let $y = \log_b x$

$$b^y = x \qquad \text{Use Equation 5.}$$

$$y \ln b = \ln x \qquad \text{Take the natural logarithm of both sides.}$$

$$y = \frac{\ln x}{\ln b} \qquad \text{Solve for } y.$$

■

Most graphing calculators have a built-in function (or dedicated key) to compute a natural logarithm. Therefore, Equation 7 allows us to use a graphing calculator to compute a logarithm with any base. Most computer algebra systems have a built-in function for the logarithm to any base. Therefore, using technology we can graph and compute any logarithmic function, regardless of the base.

Graph and Growth of the Natural Logarithm

The graphs of the exponential function $y = e^x$ and its inverse function, the natural logarithm function, are shown in Figure 1.85. The graph of $y = \ln x$ is obtained by reflecting the graph of $y = e^x$ about the line $y = x$. Since (0, 1) is on the graph of $y = e^x$, the point (1, 0) is on the graph of $y = \ln x$.

As with all other logarithmic functions with base greater than 1, the natural logarithm is an increasing function defined on $(0, \infty)$, and the y-axis is a vertical asymptote. The values of $\ln x$ become very large negative, or decrease without bound, as x approaches 0.

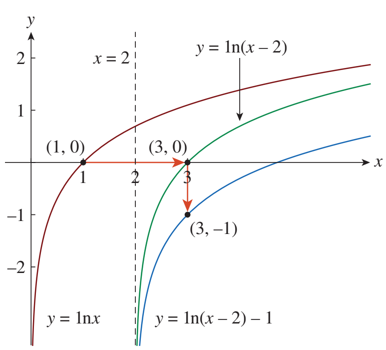


Figure 1.86
The graph of $y = \ln(x - 2) - 1$ obtained through transformations.

Example 10 Graph Involving the Natural Logarithm

Sketch the graph of the function $y = \ln(x - 2) - 1$.

Solution

We can sketch the graph using transformations: start with the graph of $y = \ln x$.
Shift this graph 2 units to the right to obtain the graph of $y = \ln(x - 2)$.
Finally, shift this graph 1 unit downward to obtain the graph of $y = \ln(x - 2) - 1$.
Figure 1.86 illustrates these transformations. ■

Although $\ln x$ is an increasing function, it grows *very* slowly when $x > 1$. In fact, $\ln x$ grows more slowly than any positive power of x. To illustrate this fact, consider the approximate values of the functions $y = \ln x$ and $y = x^{1/2} = \sqrt{x}$ in Table 1.7 and the graphs in Figures 1.87 and 1.88. Initially the graphs of $y = \sqrt{x}$ and $y = \ln x$ grow at similar rates, but eventually the growth of the root function far surpasses the logarithm.

x	1	2	5	10	50	100	500	1000	10,000	100,000
$\ln x$	0	0.69	1.61	2.30	3.91	4.6	6.2	6.9	9.2	11.5
$\sqrt{x}$	1	1.41	2.24	3.16	7.07	10.0	22.4	31.6	100	316
$\frac{\ln x}{\sqrt{x}}$	0	0.49	0.72	0.73	0.55	0.46	0.28	0.22	0.09	0.04

Table 1.7
Table of values for comparison of $\ln x$ and $\sqrt{x}$.

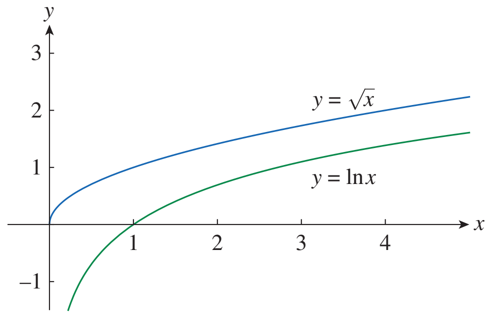


Figure 1.87
The graphs of $y = \sqrt{x}$ and $y = \ln x$ for small values of x.

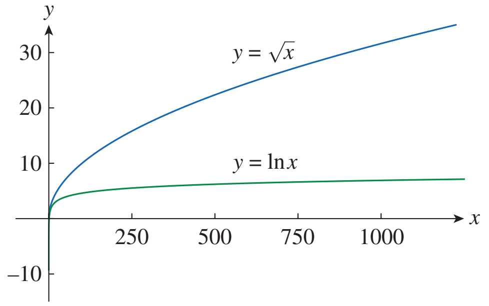


Figure 1.88
The graphs of $y = \sqrt{x}$ and $y = \ln x$ for large values of x.

1.5 Exercises

1. (a) Explain the concept of a one-to-one function.
(b) Explain how to use the graph of a function to determine whether it is one-to-one.

2. (a) Suppose f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What is the domain of f^{-1}? What is the range of f^{-1}?
(b) Given a formula for f, explain how to find a formula for f^{-1}.
(c) Given the graph of f, explain how to sketch the graph of f^{-1}.

A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether the function is one-to-one. Justify your answer.

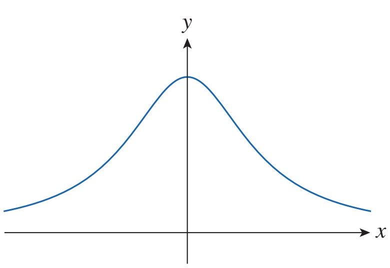
3.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

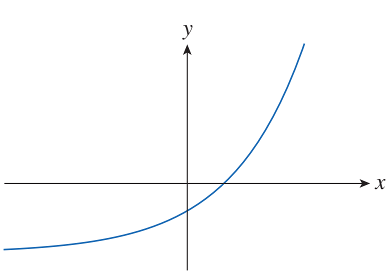
4.

x	1	2	3	4	5	6
$f(x)$	1.0	1.9	2.8	3.5	3.1	2.9

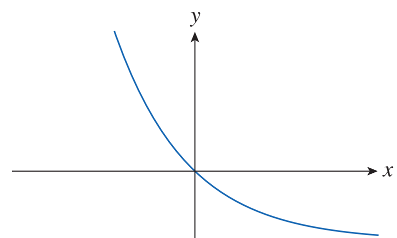
5.



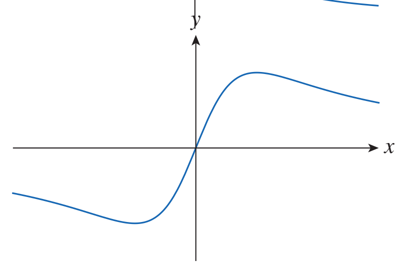
6.



7.



8.



9. $f(x) = 2x - 3$

10. $f(x) = x^4 - 16$

11. $g(x) = 1 - \sin x$

12. $g(x) = \sqrt[3]{x}$

13. $f(t)$ is the height of a football t seconds after kickoff.

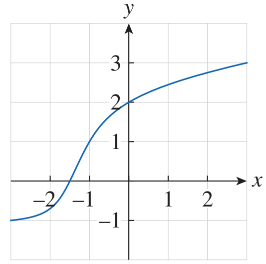
14. $f(t)$ is your height at age t.

15. Assume f is a one-to-one function.
(a) If $f(6) = 17$, find $f^{-1}(17)$.
(b) If $f^{-1}(3) = 2$, find $f(2)$.

16. If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f(f^{-1}(2))$.

17. If $g(x) = 3 + x + e^x$, find $g^{-1}(4)$.

18. Consider the graph of f.



(a) Explain why f is one-to-one.
(b) What are the domain and range of f^{-1}?
(c) Find $f^{-1}(2)$.
(d) Estimate the value of $f^{-1}(0)$.

19. The formula $C = \frac{5}{9}(F - 32)$, where $F \geq -459.67$, expresses the Celsius temperature C as a function of the Fahrenheit temperature F. Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

20. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

Find a formula for the inverse of the function.

21. $f(x) = 1 + \sqrt{2 + 3x}$

22. $f(x) = \dfrac{4x - 1}{2x + 3}$

23. $f(x) = e^{2x-1}$

24. $y = x^2 - x, \quad x \geq \dfrac{1}{2}$

25. $y = \ln(x + 3)$

26. $y = \dfrac{1 - e^{-x}}{1 + e^{-x}}$

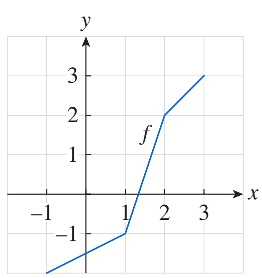
Sketch the graph of f. Use this graph to sketch the graph of f^{-1}. Check your work by finding an explicit formula for f^{-1}.

27. $f(x) = \sqrt{4x + 3}$

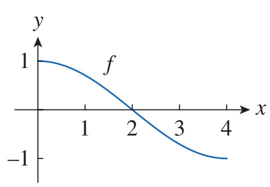
28. $f(x) = 1 + e^{-x}$

Use the graph of f to sketch the graph of f^{-1}.

29.



30.



31. Let $f(x) = \sqrt{1 - x^2}$, $0 \leq x \leq 1$.
(a) Find f^{-1}. Explain the curious relationship between f and f^{-1}.
(b) Describe the graph of f and explain how this supports the relationship between f and f^{-1}.

32. Let $g(x) = \sqrt[3]{1 - x^3}$.
(a) Find g^{-1}. Explain the relationship between g and g^{-1}.
(b) Sketch a graph of g and explain how this supports the relationship between g and g^{-1}.

33. (a) How is the logarithmic function $y = \log_b x$ defined?
(b) What is the domain of this function?
(c) What is the range of this function?
(d) Sketch the general shape of the graph of the function $y = \log_b x$ if $b > 1$.

34. (a) What is the natural logarithm?
(b) What is the common logarithm?
(c) Sketch the graphs of the natural logarithm function and the natural exponential function on the same coordinate axes.

Find the exact value of each expression.

35. (a) $\log_2 32$ (b) $\log_8 2$

36. (a) $\log_5 \dfrac{1}{125}$ (b) $\ln\left(\dfrac{1}{e^2}\right)$

37. (a) $\log_{10} 40 + \log_{10} 2.5$ (b) $\log_8 60 - \log_8 3 - \log_8 5$

38. (a) $e^{-\ln 2}$ (b) $e^{\ln(\ln e^3)}$

Rewrite each expression as a single logarithm.

39. $\ln 10 + 2 \ln 5$

40. $\ln b + 2 \ln c - 3 \ln d$

41. $\dfrac{1}{3}\ln(x + 2)^3 + \dfrac{1}{2}\left[\ln x - \ln(x^2 + 3x + 2)^2\right]$

Simplify each expression.

42. $\ln(1 + e^{2x}) - \ln(1 + e^{-2x})$

43. $(\ln p)(\log_p e)(\sqrt{e})^{\ln p}$

Use the Change of Base Formula to evaluate each logarithm correct to six decimal places.

44. $\log_5 10$

45. $\log_3 57$

Use the Change of Base Formula to graph each collection of functions on the same coordinate axes. Explain how these graphs are related.

46. $y = \log_{1.5} x, \quad y = \ln x, \quad y = \log_{10} x, \quad y = \log_{50} x$

47. $y = \ln x, \quad y = \log_{10} x, \quad y = e^x, \quad y = 10^x$

48. Use the laws of logarithms to show that

$$\frac{\ln(x+h)-\ln x}{h}=\ln\left(1+\frac{h}{x}\right)^{1/h}$$

49. Suppose the graph of $y=\log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

50. Sketch the graphs of $y=\ln|x|$ and $y=|\ln x|$ on the same coordinate axes. Explain why these graphs are different.

51. Use technology to compare the graphs of $f(x)=x^{0.1}$ and $g(x)=\ln x$. When does the graph of f finally surpass the graph of g?

Sketch the graph of the function without the use of technology. Use the basic graphs presented in this section and, if necessary, transformations.

52. $y=\log_{10}(x+5)$

53. $y=-\ln x$

54. $y=\ln(-x)$

55. $y=\ln|x|$

56. $y=\ln\dfrac{1}{|x+2|}$

For the given function, (a) find the domain and range, (b) find the x-intercept of the graph of f, and (c) sketch the graph of f.

57. $f(x)=\ln x+2$

58. $f(x)=\ln(x-1)-1$

Solve for x.

59. $e^{7-4x}=6$

60. $\ln(3x-10)=2$

61. $\ln(x^2-1)=3$

62. $e^{2x}-3e^x+2=0$

63. $2^{x-5}=3$

64. $\ln x+\ln(x-1)=1$

65. $\ln(\ln x)=1$

66. $e^{ax}=Ce^{bx}$, where $a\neq b$

67. $\log(x-5)-\ln 6>0$

68. $\log_4(x+6)+\log_4(x-3)\geq 1$

69. $2\ln x=(\ln x)^{-1}+1$

70. $\ln(\cos x)\leq 1$, for $0\leq x\leq 2\pi$

71. $|\ln x|<1$

Solve the inequality for x.

72. $\ln x<0$

73. $e^x>5$

74. $1<e^{3x-1}<2$

75. $1-2\ln x<3$

76. Let $f(x)=\ln(e^x-3)$.
(a) Find the domain of f.
(b) Find f^{-1} and its domain.

77. (a) What are the exact values of $e^{\ln 300}$ and $\ln(e^{300})$?
(b) Use technology to evaluate $e^{\ln 300}$ and $\ln(e^{300})$. Compare the results to part (a) and explain any differences.

78. Sketch a graph of the function $f(x)=\sqrt{x^3+x^2+x+1}$ and explain why it is one-to-one. Use a computer algebra system (CAS) to find an explicit expression for $f^{-1}(x)$.

79. Let $g(x)=x^6+x^4$, $x\geq 0$.
(a) Use a computer algebra system to find an expression for $g^{-1}(x)$.
(b) Graph $y=g(x)$, $y=x$, and $y=g^{-1}(x)$ on the same coordinate axes.

80. Suppose a population starts with 100 bacteria and doubles every 3 hours. The number of bacteria after t hours is $n=f(t)=100\cdot 2^{t/3}$.
(a) Find the inverse of the function, f^{-1}, and explain its meaning in the context of this problem.
(b) When will the bacteria population reach 50,000?

81. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, The amount of electricity stored by the capacitor is given by

$$Q(t)=Q_0(1-e^{-t/a})$$

where Q_0 is the maximum charge and t is measured in seconds since the flash went off.
(a) Find the inverse of this function, Q^{-1}, and explain its meaning in the context of this problem.
(b) How long does it take to recharge the capacitor to 90% of capacity if $a=2$?

82. Starting with the graph of $y=\ln x$, find the equation of the graph that results from
(a) shifting 3 units upward.
(b) shifting 3 units to the left.
(c) reflecting about the x-axis.
(d) reflecting about the y-axis.
(e) reflecting about the line $y=x$.
(f) reflecting about the x-axis and then about the line $y=x$.
(g) reflecting about the y-axis and then about the line $y=x$.
(h) shifting 3 units to the left and then reflecting about the line $y=x$.

83. (a) If we shift a graph to the left, what happens to its reflection about the line $y=x$? In view of this geometric principle, find an expression for the inverse of $g(x)=f(x+c)$, where f is a one-to-one function.
(b) Find an expression for the inverse of $h(x)=f(cx)$, where $c\neq 0$.

1.6 Parametric Curves

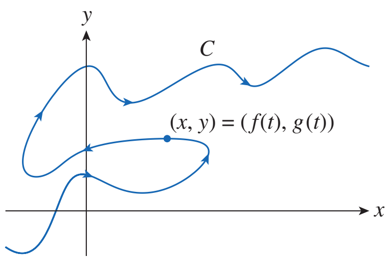


Figure 1.89
A curve C whose x- and y-coordinates are given by $x = f(t)$, and $y = g(t)$.

Suppose a particle moves along the curve C shown in Figure 1.89. We cannot describe the curve C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test. But we can think of the x- and y-coordinates of the particle as functions of a third variable t, often corresponding to time. Therefore, we can write $x = f(t)$ and $y = g(t)$. This pair of equations is often a convenient way to describe a curve, and it suggests the definition below.

Parametric Equations and Graphs

Suppose x and y are both given as functions of a third variable t, called a **parameter**, by the equations

$$x = f(t) \qquad y = g(t)$$

called **parametric equations**. Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which is called a **parametric curve**. The parameter t does not necessarily represent time. In fact, we could use another letter for the parameter. However, in many applications of parametric curves, t does indeed denote time, and in these cases we can interpret $(x, y) = (f(t), g(t))$ as the position of a particle at time t.

Example 1 Graph a Parametric Curve

Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

Solution

Each value of t produces a point on the curve.
Table 1.8 shows values for $x(t)$ and $y(t)$ for selected values of t.

For example, if $t = 0$, then $x = 0^2 - 2 \cdot 0 = 0$ and $y = 0 + 1 = 1$.
The corresponding point is $(0, 1)$.

The points determined in Table 1.8 are plotted in Figure 1.90. The points are joined by a smooth curve.

t	x	y
−2	8	−1
−1	3	0
0	0	1
1	−1	2
2	0	3
3	3	4
4	8	5

Table 1.8
Table of points (x, y) determined by values of the parameter t.

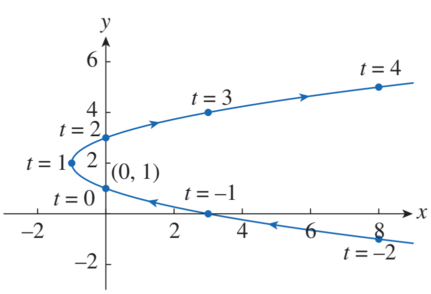


Figure 1.90
Graph of the curve defined by the parametric equations.

Consider a particle whose position is given by the parametric equations in this example. The particle moves along the curve in the direction of the arrows as t increases.

Notice that the consecutive points marked on the curve in Figure 1.90 appear at equal time intervals but not at equal distances. That is because the particle, in this case, slows down and then speeds up as t increases.

This equation in x and y describes *where* the particle has been, but it doesn't tell us *when* the particle was at a particular point. Parametric equations have an advantage for describing particle motion; they tell us *when* the particle was at a point and they indicate the *direction* of the motion.

It appears from Figure 1.90 that the curve traced out by the particle may be a parabola. This can be confirmed by eliminating the parameter t as follows.

$t = y - 1$ Solve for t in the second equation.

$x = t^2 - 2t = (y-1)^2 - 2(y-1) = y^2 - 4y + 3$ Substitute into the first equation.

Therefore, the curve represented by the given parametric equations is the graph of a parabola described by $x = y^2 - 4y + 3$. ■

In Example 1 no restriction was placed on the parameter t, so we assumed that t could be any real number. However, we often restrict t to lie in a specific finite interval. For example, the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1 \qquad 0 \le t \le 4$$

is shown in Figure 1.91.

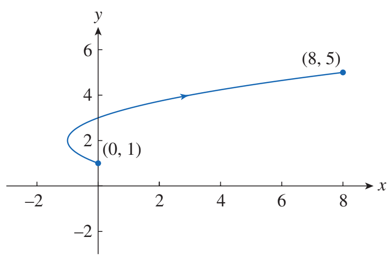


Figure 1.91
Graph of the curve described by the parametric equations $x = t^2 - 2t$, $y = t + 1, 0 \le t \le 4$.

The graph is the part of the parabola in Example 1 that starts at the point (0, 1) and ends at the point (8, 5). The arrowhead indicates the direction in which the curve is traced as t increases from 0 to 4.

In general, the curve described by the parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \le t \le b$$

has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

Example 2 Identifying a Parametric Curve

Describe the curve defined by the following parametric equations.

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$$

Solution

If we plot points, it appears that the curve is a circle, as shown in Figure 1.92.

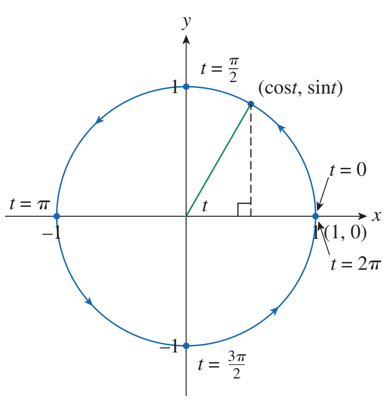


Figure 1.92
The curve described by these parametric equations appears to be a circle.

To confirm the curve is a circle, try to eliminate the variable t.

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ Trigonometric identity.

As t increases, the point (x, y) moves on the unit circle described by $x^2 + y^2 = 1$.

In this example the parameter t can be interpreted as the angle (in radians) as shown in Figure 1.92.

As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction, starting at the point (1, 0). ■

Example 3 Another Way to Describe a Circle

Describe the curve defined by the following parametric equations.

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \le t \le 2\pi$$

Solution

We can use the same method as in Example 2 to eliminate the variable t.

$$x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1$$

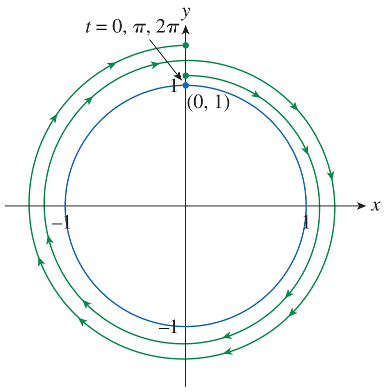


Figure 1.93
The unit circle, in blue, is described by the given parametric equations. The green curve illustrates the path of the point, moving around the unit circle twice in the clockwise direction.

The parametric equations again represent the unit circle described by $x^2 + y^2 = 1$.

However, as t increases from 0 to 2π, the point $(x, y) = (\sin 2t, \cos 2t)$ starts at $(0, 1)$ and moves *twice* around the circle in the clockwise direction, as indicated in Figure 1.93.

Examples 2 and 3 show that different sets of parametric equations can represent the same curve. Therefore, we need to distinguish between a *curve*, which is a set of points, and a *parametric curve*, in which the points are traced in a particular way.

Example 4 Determine the Parametric Equations for a Circle

Find the parametric equations for the graph of a circle centered at (h, k) with radius r.

Solution

Start with the equations that represent the unit circle: $x = \cos t$, $y = \sin t$.

Multiply the expression for x and y by r: $x = r\cos t$, $y = r\sin t$.

These equations represent a circle centered at the origin with radius r, traced counter-clockwise (as t increases). You should be able to verify this.

Shift h units in the x-direction and k units in the y-direction.

This gives us the parametric equations of the circle centered at (h, k) with radius r.

$$x = h + r\cos t \qquad y = k + r\sin t \qquad 0 \le t \le 2\pi$$

A graph of this circle is shown in Figure 1.94.

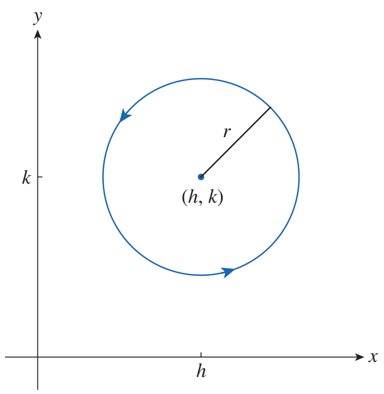


Figure 1.94
Graph of a circle centered at (h, k) with radius r described by the parametric equations $x = h + r\cos t$, $y = k + r\sin t$, $0 \le t \le 2\pi$.

Example 5 Part of a Parabola

Sketch the curve described by the parametric equations $x = \sin t$, $y = \sin^2 t$.

Solution

Notice that $y = (\sin t)^2 = x^2$.

This means that the point (x, y) moves on the parabola described by $y = x^2$.

Since $-1 \le \sin t \le 1$, then $-1 \le x \le 1$.

Therefore, the parametric equations represent only the part of the parabola for which $-1 \le x \le 1$.

Since $\sin t$ is periodic, the point $(x, y) = (\sin t, \sin^2 t)$ moves back and forth infinitely often along the parabola from $(-1, 1)$ to $(1, 1)$, as indicated in Figure 1.95.

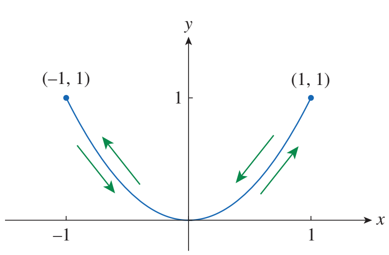


Figure 1.95
The part of the parabola described by $y = x^2$ is traced out infinitely often by the parametric equations.

Technology

Most graphing calculators and mathematics software can be used to sketch graphs described by parametric equations. In fact, it is instructive to watch a parametric curve being drawn by a graphing calculator because the points are plotted in order as the corresponding parameter increases from its minimum value to its maximum value.

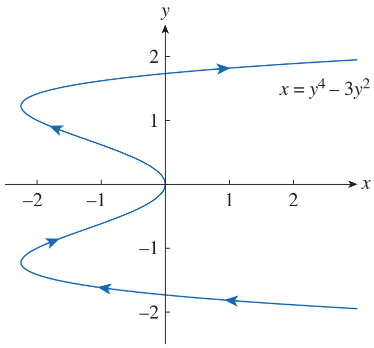


Figure 1.96
A sketch of the curve, using technology, defined by the parametric equations. The arrows indicate the direction the point (x, y) moves along the curve as t increases.

Example 6 Graph *x* as a Function of *y*

Use technology to sketch the graph described by $x = y^4 - 3y^2$.

Solution

Although it is possible to solve the given equation for y as four functions of x and graph them individually, parametric equations provide a much easier method.

Let the parameter be $t = y$ and then transform to parametric equations

$$x = t^4 - 3t^2 \qquad y = t$$

Use these parametric equations to sketch the graph.

If we use technology to sketch the graph described by parametric equations, as shown in Figure 1.96, in addition to specifying the viewing rectangle, we may also need to select a *range* of values for the parameter t. It is important to select an interval of values for t that will produce a graph showing all the important features. ■

In general, we can graph an equation of the form $x = g(y)$ by using the parametric equations

$$x = g(t) \qquad y = t$$

The graph of an equation of the form we are most familiar with, $y = f(x)$, can also be obtained by using the parametric equations

$$x = t \qquad y = f(t)$$

This transformation to parametric equations might be useful in problems involving particle motion.

Technology is extremely helpful for sketching the graph described by parametric equations. For example, the graphs shown in Figures 1.97, 1.98, and 1.99 are almost impossible to produce without using technology.

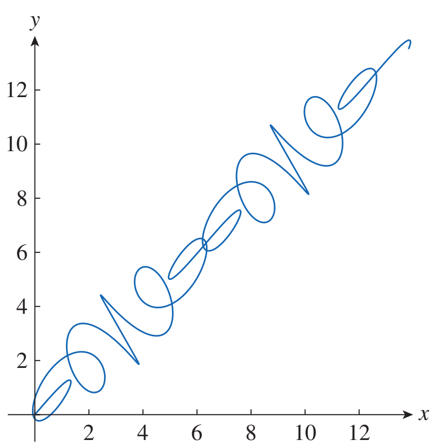


Figure 1.97
$x = t + \sin 5t$
$y = t + \sin 6t.$

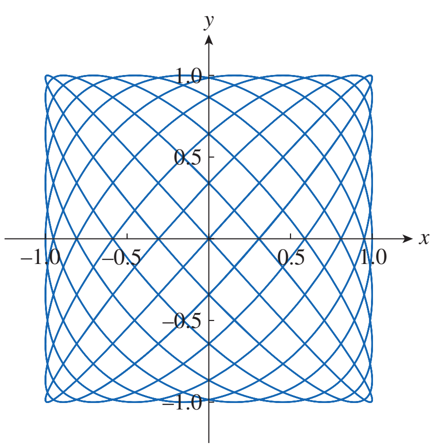


Figure 1.98
$x = \sin 9t$
$y = \sin 10t.$

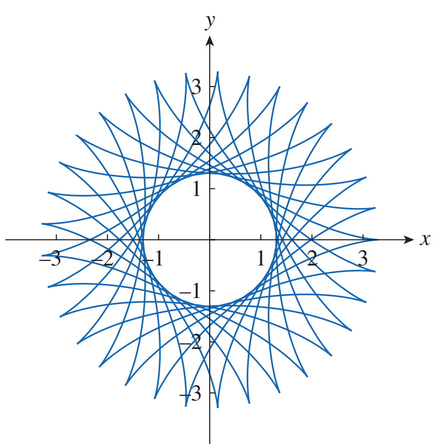


Figure 1.99
$x = 2.3 \cos 10t + \cos 23t$
$y = 2.3 \sin 10t - \sin 23t.$

One of the most important uses of parametric curves is in computer-aided design. The parametric equations that represent **Bézier curves** are used extensively in manufacturing, especially in the automotive industry. These curves are also used in specifying the shapes of letters and other symbols in laser printers and in documents viewed electronically.

The Cycloid

Example 7 Cycloid Equations

Suppose the point P lies on the circumference of a circle. The curve traced out by P as the circle rolls along a straight line is called a **cycloid** and is shown in Figure 1.100.

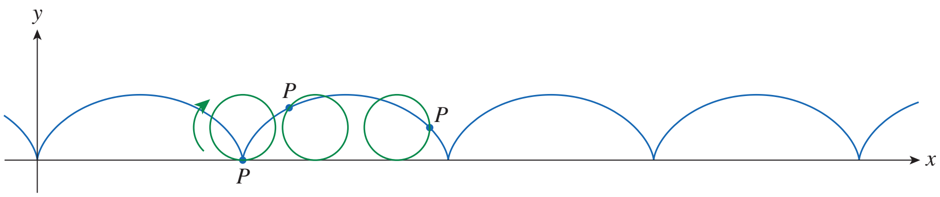


Figure 1.100
As the circle rolls along the x-axis, the curve traced out by the point P is a cycloid.

If the circle has radius r and rolls along the x-axis, and if one position of P is the origin, find the parametric equations for the cycloid.

Solution

Let θ be the parameter, equal to the angle of rotation of the circle, with $\theta = 0$ when P is at the origin.

Suppose the circle has rotated through θ radians. See Figure 1.101.

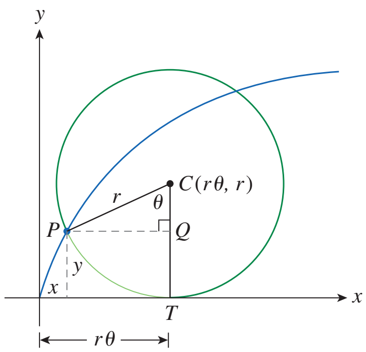


Figure 1.101
A circle rolls along the x-axis. The parameter is θ, the angle of rotation of the circle.

Since the circle is in contact with the line, the distance it has rolled is

$|OT| = \text{arc } PT = r\theta.$

The center of the circle is, $C(r\theta, r)$.
Let the coordinates of P be (x, y).

Use Figure 1.101 to find expressions for x and y.

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Therefore, the parametric equations of the cycloid are

$$x = r(\theta - \sin\theta) \quad y = r(1 - \cos\theta) \quad \theta \in \mathbb{R} \tag{1}$$

One arch of the cycloid is produced from one rotation of the circle for $0 \leq \theta \leq 2\pi$. Although the equations for the cycloid were derived using Figure 1.101, which illustrates the case in which $0 < \theta < \dfrac{\pi}{2}$, it can be shown that these equations are valid for other values of θ.

It is possible to eliminate the parameter θ from the equations in 1, but the resulting equation in x and y is complicated and not as convenient to work with as the parametric equations. ■

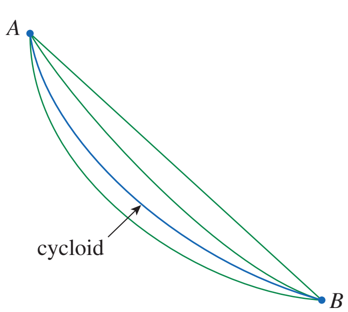


Figure 1.102
The path that produces the shortest time from A to B is an inverted cycloid.

Galileo was one of the first people to study the cycloid. He proposed that bridges be built in the shape of cycloids, and he tried to find the area under one arch of a cycloid. The cycloid also arose in connection with the **brachistochrone problem**: Find the curve along which a particle will slide in the shortest time, under the influence of gravity, from a point A to a lower point B not directly beneath A. The Swiss mathematician John Bernoulli, who posed the problem in 1696, showed that among all possible curves that join A to B, as shown in Figure 1.102, the particle will take the least time sliding from A to B if the curve is part of an inverted arch of a cycloid.

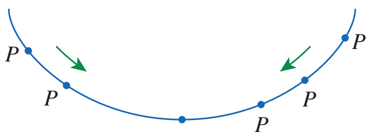


Figure 1.103
No matter where a particle P starts, it takes the same time to slide to the bottom.

The Dutch physicist Huygens showed that the cycloid is the solution to the **tautochrone problem**; that is, no matter where a particle P is placed on an inverted cycloid, it takes the same time to slide to the bottom, as illustrated in Figure 1.103. Huygens proposed that pendulum clocks, which he invented, should swing in cycloidal arcs because then the pendulum would take the same time to make a complete oscillation whether it swings through a wide or a small arc.

1.6 Exercises

Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

1. $x = t^2 + t,\ y = t^2 - t,\ -2 \le t \le 2$

2. $x = t^2,\ y = t^3 - 4t,\ -3 \le t \le 3$

3. $x = \cos^2 t,\ y = 1 - \sin t,\ 0 \le t \le \pi/2$

4. $x = e^{-t} + t,\ y = e^t - t,\ -2 \le t \le 2$

For the set of parametric equations:
(a) Sketch the curve represented by the parametric equations.
(b) Eliminate the parameter to find a Cartesian equation that represents the curve.

5. $x = 2t - 1, \quad y = \dfrac{1}{2}t + 1$

6. $x = 3t + 2, \quad y = 2t + 3$

7. $x = t^2 - 3, \quad y = t + 2, \quad -3 \le t \le 3$

8. $x = \sin t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi$

9. $x = \sqrt{t}, \quad y = 1 - t$

10. $x = t^2, \quad y = t^3$

For the set of parametric equations:
(a) Eliminate the parameter to find a Cartesian equation that represents the curve.
(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

11. $x = \sin\dfrac{1}{2}\theta, \quad y = \cos\dfrac{1}{2}\theta, \quad -\pi \le \theta \le \pi$

12. $x = \dfrac{1}{2}\cos\theta, \quad y = 2\sin\theta, \quad 0 \le \theta \le \pi$

13. $x = \sin t, \quad y = \csc t, \quad 0 < t < \dfrac{\pi}{2}$

14. $x = e^t, \quad y = e^{-2t}$

15. $x = t^2, \quad y = \ln t$

16. $x = \sqrt{t + 1}, \quad y = \sqrt{t - 1}$

17. $x = \tan^2\theta, \quad y = \sec\theta, \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

Describe the motion of a particle with position (x, y) as t varies in the given interval.

18. $x = 3 + 2\cos t, \quad y = 1 + 2\sin t, \quad -\pi/2 \le t \le 3\pi/2$

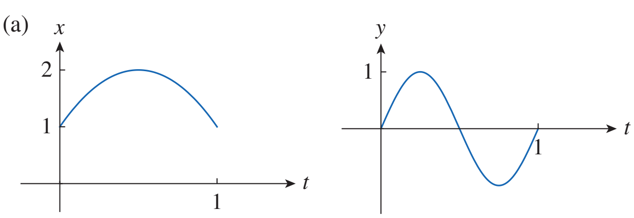
19. $x = 2\sin t, \quad y = 4 + \cos t, \quad 0 \le t \le 3\pi/2$

20. $x = 5\sin t, \quad y = 2\cos t, \quad -\pi \le t \le 5\pi$

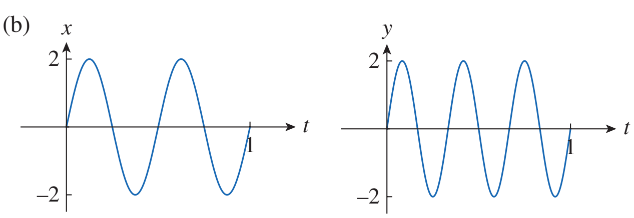
21. $x = \sin t, \quad y = \cos^2 t, \quad -2\pi \le t \le 2\pi$

22. Match the graph of the parametric equations $x = f(t)$ and $y = g(t)$ in (a)–(d) with the corresponding parametric curve labeled I–IV. Give a reason for each choice.

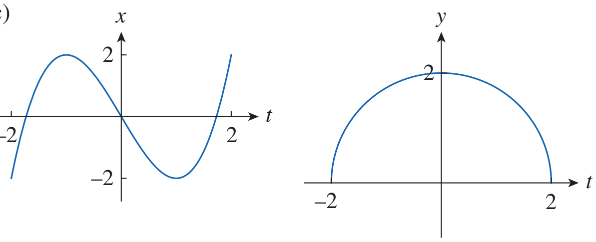
(a)



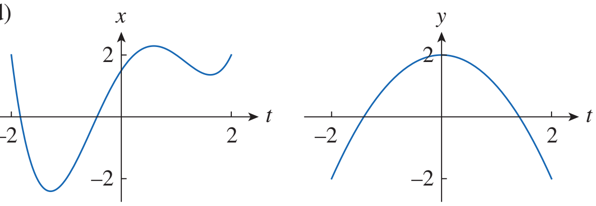
(b)



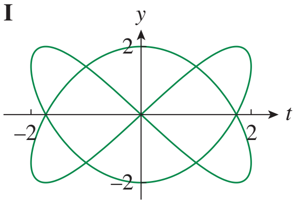
(c)



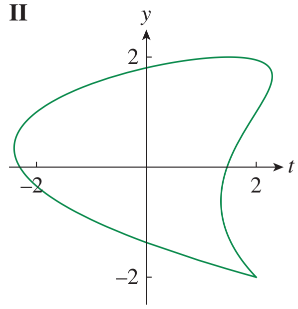
(d)



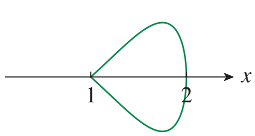
I



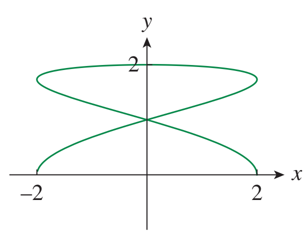
II



III

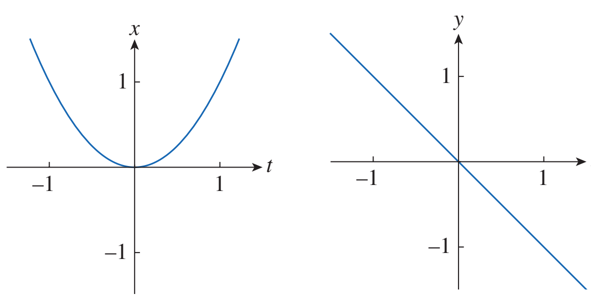


IV



Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the graph described by the parametric equations $x = f(t)$, $y = g(t)$. Indicate with arrow the direction in which the curve is traced as t increases.

23.



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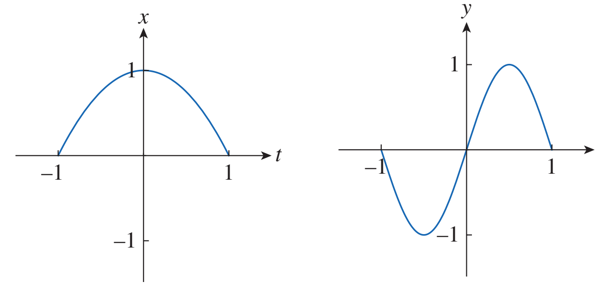

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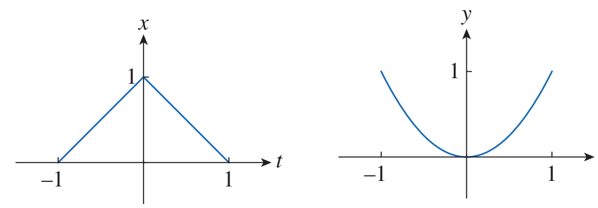


26. Without using technology to sketch the curve, match the parametric equations with the graphs labeled I–VI. Give a reason for each choice.

(a) $x = t^4 - t + 1, \quad y = t^2$

(b) $x = t^2 - 2t, \quad y = \sqrt{t}$

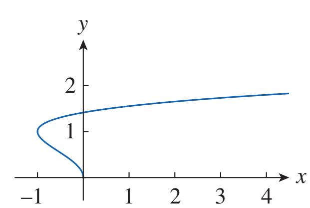
(c) $x = \sin 2t, \quad y = \sin(t + \sin 2t)$

(d) $x = \cos 5t, \quad y = \sin 2t$

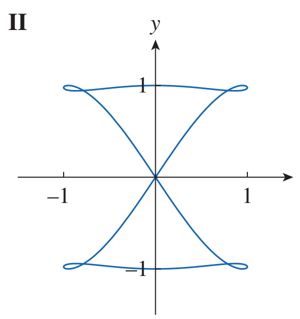
(e) $x = t + \sin 4t, \quad y = t^2 + \cos 3t$

(f) $x = \dfrac{\sin 2t}{4 + t^2}, \quad y = \dfrac{\cos 2t}{4 + t^2}$

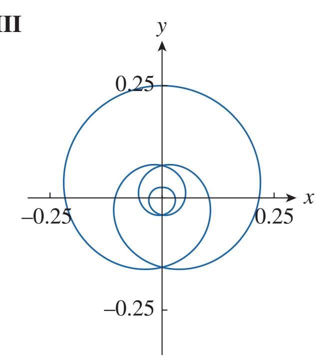
I



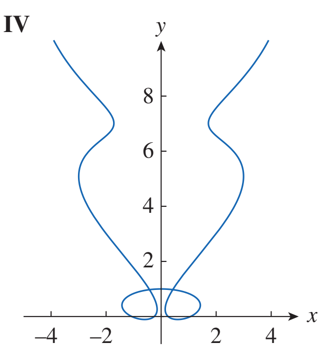
II



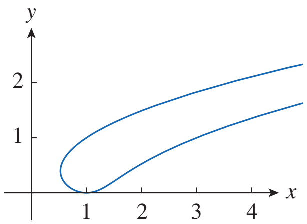
III



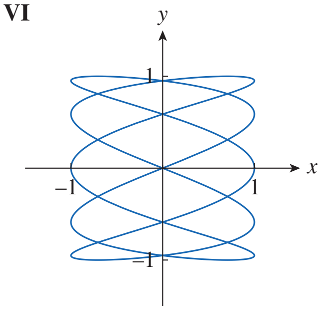
IV



V



VI



27. Sketch the graph described by $x = y - 2 \sin \pi y$.

28. Sketch the graphs described by $y = x^3 - 4x$ and $x = y^3 - 4y$ and find the points of intersection.

29. (a) Show that the parametric equations

$$x = x_1 + (x_2 - x_1)t \quad y = y_1 + (y_2 - y_1)t$$

where $0 \le t \le 1$ describe the line segment that joins the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

(b) Find the parametric equations that represent the line segment from $(-2, 7)$ to $(3, -1)$.

30. Use technology and the result in Exercise 29 to draw the triangle with vertices $A(1, 1)$, $B(4, 2)$, and $C(1, 5)$.

31. Find parametric equations for the path of a particle that moves along the circle described by $x^2 + (y - 1)^2 = 4$ in the manner described.

(a) Once around clockwise, starting at the point $(2, 1)$.

(b) Three times around counterclockwise, starting at the point $(2, 1)$.

(c) Halfway around counterclockwise, starting at the point $(0, 3)$.

32. (a) Find parametric equations to represent the graph of the ellipse described by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

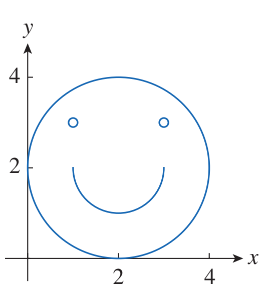
Hint: Modify the equations that describe a circle.

(b) Use these parametric equations to graph the ellipse when $a = 3$ and $b = 1, 2, 4$, and 8.

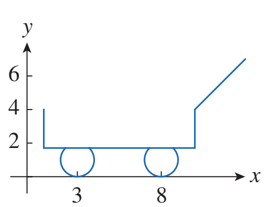
(c) How does the shape of the ellipse change as b varies?

Use technology to sketch the image, and explain your method.

33.



34.



Compare the graphs represented by the parametric equations. Explain how they differ.

35. (a) $x = t^3, \quad y = t^2$ (b) $x = t^6, \quad y = t^4$

(c) $x = e^{-3t}, \quad y = e^{-2t}$

36. (a) $x = t, \quad y = t^{-2}$ (b) $x = \cos t, \quad y = \sec^2 t$

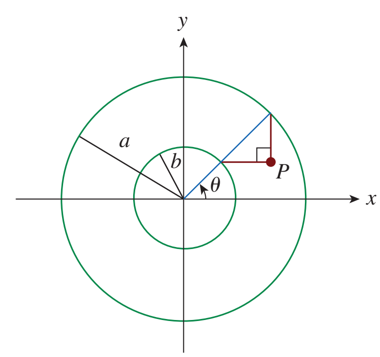
(c) $x = e^t, \quad y = e^{-2t}$

37. Let P be a point at a distance d from the center of a circle of radius r. The curve traced out by P as the circle rolls along a straight line is called a **trochoid**. Think of the motion of a point on a spoke of a bicycle wheel. The cycloid is the special case of a trochoid with $d = r$. Using the same parameter θ as for the cycloid, and assuming that the line is the x-axis and $\theta = 0$ when P is at one of its lowest points, show that parametric equations of the trochoid are

$$x = r\theta - d \sin \theta \quad y = r - d \cos \theta$$

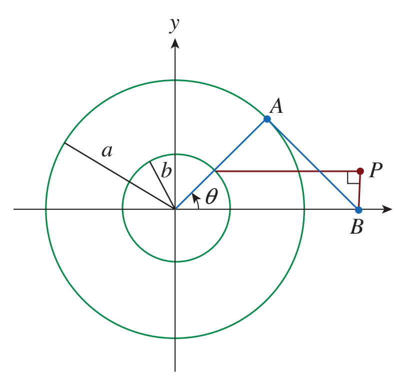
Sketch the trochoid for the cases $d < r$ and $d > r$.

38. If a and b are fixed numbers, find parametric equations that represent the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter.



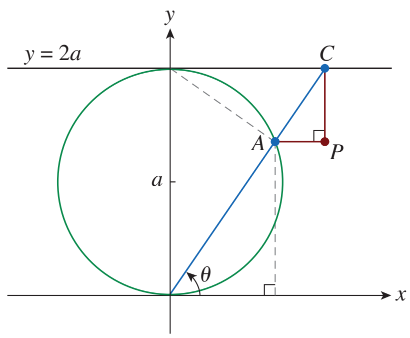
Then eliminate the parameter and identify the curve.

39. If a and b are fixed numbers, find parametric equations that represent the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter.



The line segment AB is tangent to the larger circle.

40. A curve, called a **witch of Maria Agnesi**, consists of all possible positions of the point P as shown in the figure.



Show that the parametric equations that represent this curve can be written as

$$x = 2a \cot\theta \quad y = 2a \sin^2\theta$$

Sketch the curve.

41. Suppose that the position of one particle at time t is given by

$$x_1 = 3 \sin t \quad y_1 = 2 \cos t \quad 0 \le t \le 2\pi$$

and the position of a second particle is given by

$$x_2 = -3 + \cos t \quad y_2 = 1 + \sin t \quad 0 \le t \le 2\pi$$

(a) Graph the paths of both particles. How many points of intersection are there?

(b) Are any of these points of intersection *collision points*? That is, are the particles ever at the same place at the same time? If so, find the collision points.

(c) Describe what happens if the path of the second particle is given by

$$x_2 = 3 + \cos t \quad y_2 = 1 + \sin t \quad 0 \le t \le 2\pi$$

42. If an object is projected with an initial velocity v_0 meters per second at an angle α above the horizontal and air resistance is assumed to be negligible, then its position after t seconds is given by the parametric equations

$$x = (v_0 \cos\alpha)t \quad y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

where g is the acceleration due to gravity, 9.8 m/s^2.

(a) If a baseball is thrown from the outfield with $\alpha = 30°$ and $v_0 = 40$ m/s, when will the ball hit the ground? How far from the outfielder will it hit the ground? What is the maximum height reached by the ball?

(b) Use technology to check your answers to part (a). Graph the path of the baseball for several other values of the angle α to see where it hits the ground. Summarize your findings.

(c) Show that the path of the baseball is parabolic by eliminating the parameter.

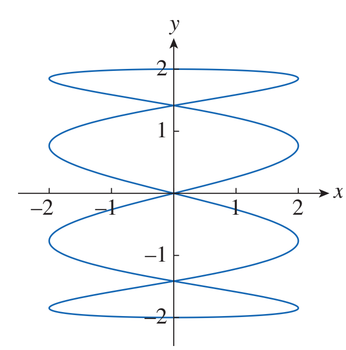
43. Investigate the family of curves defined by the parametric equations $x = t^2$, $y = t^3 - ct$. How does the shape change as c increases? Illustrate by graphing several members of the family.

44. The **swallowtail catastrophe curves** are defined by the parametric equations $x = 2ct - 4t^3$, $y = -ct^2 + 3t^4$. Graph several of these curves. What features do the curves have in common? How do they change when c increases?

45. Graph several members of the family of curves with parametric equations $x = t + a \cos t$, $y = t + a \sin t$, where $a > 0$. How does the shape change as a increases? For what values of a does the curve have a loop?

46. Graph several members of the family of curves $x = \sin t + \sin nt$, $y = \cos t + \cos nt$, where n is a positive integer. What features do the curves have in common? What happens as n increases?

47. The curves described by the equations $x = a \sin nt$, $y = b \cos t$ are called **Lissajous figures**. An example is shown in the figure.



Investigate how these curves vary when a, b, and n vary, where n is a positive integer.

48. Investigate the family of curves described by the parametric equations $x = \cos t$, $y = \sin t - \sin ct$, where $c > 0$. Start by letting c be a positive integer and see what happens to the shape as c increases. Then explore some of the possibilities that occur when c is a fraction.

Laboratory Project | Motion of a Point on a Circle

In this project we investigate families of curves, called *hypocycloids* and *epicycloids*, that are generated by the motion of a point on a circle that rolls inside or outside another circle.

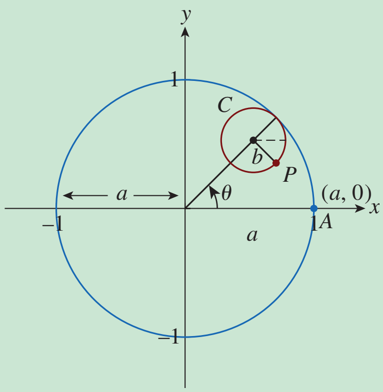


Figure 1.104
The geometry for constructing a hypocycloid.

1. A **hypocycloid** is a curve traced out by a fixed point P on a circle C of radius b as C rolls on the inside of a circle with center O and radius a, as shown in Figure 1.104.

   Show that if the initial position of P is $(a, 0)$ and the parameter θ is chosen as in the figure, then parametric equations of the hypocycloid are

   $$x = (a - b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right) \qquad y = (a - b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$$

2. Use technology to draw the graphs of hypocycloids such that a is a positive integer and $b = 1$. How does the value of a affect the graph? Show that if $a = 4$, then the parametric equations of the hypocycloid reduce to

   $$x = 4\cos^3\theta \qquad y = 4\sin^3\theta$$

   This curve is called a **hypocycloid of four cusps**, or an **astroid**.

3. Draw the graphs of hypocycloids such that $b = 1$ and $a = n/d$, a fraction where n and d have no common factor. First let $n = 1$ and try to determine graphically the effect of the denominator d on the shape of the graph. Then let n vary while keeping d constant. What happens when $n = d + 1$?

4. What happens if $b = 1$ and a is an irrational number? Experiment with an irrational number like $\sqrt{2}$ or $e - 2$. Take larger and larger values for θ and speculate on what would happen if we were to graph the hypocycloid for all real values of θ.

5. If the circle C rolls on the *outside* of the fixed circle, the curve traced out by P is called an **epicycloid**. Find parametric equations for the epicycloid.

6. Investigate the possible shapes for epicycloids. Use methods similar to Problems 2–4.

1 Review

Concepts and Vocabulary

1. (a) In your own words, define a function. What are the domain and range of a function?
   (b) Explain how to obtain the graph of a function.
   (c) How can you determine whether a curve is the graph of a function?
2. Explain the four ways to represent a function. Illustrate each with an example.
3. (a) What is an even function? How can you tell if a function is even by looking at its graph? Give three examples of an even function.
   (b) What is an odd function? How can you tell if a function is odd by looking at its graph? Give three examples of an odd function.
4. Explain what is meant by an increasing function.
5. Explain the notion and purpose of a mathematical model.
6. Give an example of each type of function.
   (a) Linear function (b) Power function
   (c) Exponential function (d) Quadratic function
   (e) Polynomial function (f) Rational function
7. Sketch the graph of the following functions on the same coordinate axes.
   $f(x) = x, \quad g(x) = x^2, \quad h(x) = x^3, \quad j(x) = x^4$
8. Sketch the graph of each function.
   (a) $y = \sin x$ (b) $y = \tan x$
   (c) $y = e^x$ (d) $y = \ln x$
   (e) $y = 1/x$ (f) $y = |x|$
   (g) $y = \sqrt{x}$ (h) $y = \tan^{-1} x$
9. Suppose f has domain A and g has domain B.
   (a) What is the domain of $f + g$?
   (b) What is the domain of fg?
   (c) What is the domain of f/g?
10. How is the composite function $f \circ g$ defined? What is its domain?
11. Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.
   (a) Shift 2 units upward.
   (b) Shift 2 units downward.
   (c) Shift 2 units to the right.
   (d) Shift 2 units to the left.
   (e) Reflect about the x-axis. Reflect about the y-axis.
   (f) Stretch vertically by a factor of 2.
   (g) Shrink vertically by a factor of 2.
   (h) Stretch horizontally by a factor of 2.
   (i) Shrink horizontally by a factor of 2.
12. Suppose $f(x) = x^2$. Write the function that results from the following transformations.
   (a) Shift 1 unit upward.
   (b) Shift 1 unit to the left.
   (c) Shift 2 units downward and 1 unit to the right.
   (d) Shift 2 units upward, then reflect about the x-axis.
   (e) Shift 1 unit downward, 2 units to the right, then reflect about the y-axis.
13. (a) What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?
   (b) If f is a one-to-one function, how is its inverse function f^{-1} defined? How is the graph of f^{-1} obtained from the graph of f?
14. (a) How is the inverse sine function, $f(x) = \sin^{-1} x$, defined? What are its domain and range?
   (b) How is the inverse cosine function, $f(x) = \cos^{-1} x$, defined? What are its domain and range?
   (c) How is the inverse tangent function, $f(x) = \tan^{-1} x$, defined? What are its domain and range?

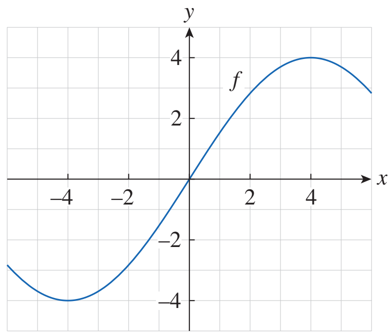
True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. If f is a function, then $f(s + t) = f(s) + f(t)$.
2. If $f(s) = f(t)$, then $s = t$.
3. If f is a function, then $f(3x) = 3f(x)$.
4. If $x_1 < x_2$ and f is a decreasing function, then $f(x_1) > f(x_2)$.
5. If f and g are functions, then $f \circ g = g \circ f$.
6. If f is one-to-one, then $f^{-1}(x) = \dfrac{1}{f(x)}$.
7. When solving an equation, it is always mathematically permissible to divide both sides by e^x.
8. If $0 < a < b$, then $\ln a < \ln b$.
9. If $x > 0$, then $(\ln x)^6 = 6 \ln x$.
10. If $x > 0$ and $a > 1$, then $\dfrac{\ln x}{\ln a} = \ln\left(\dfrac{x}{a}\right)$.
11. If x is any real number, then $\sqrt{x^2} = x$.
12. The graph of $y = f(x)$ and $y = |f(x)|$ are the same.

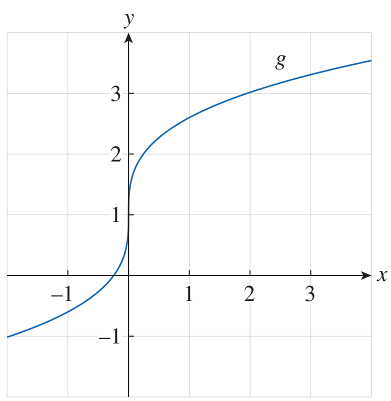
Exercises

1. Let f be a function whose graph is shown.



(a) Estimate the value of $f(2)$.
(b) Estimate the values of x such that $f(x) = 3$.
(c) State the domain of f.
(d) State the range of f.
(e) On what interval is f increasing?
(f) Is f one-to-one? Explain your reasoning.
(g) Is f even, odd, or neither even nor odd? Justify your answer.

2. Let g be the function whose graph is shown.



(a) Estimate the value of $g(2)$.
(b) Explain why this graph suggests that g is one-to-one.
(c) Estimate the value of $g^{-1}(2)$.
(d) Estimate the domain of g^{-1}.
(e) Sketch the graph of g^{-1}.

3. If $f(x) = x^2 - 2x + 3$, evaluate and simplify the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

Find the domain and range of the function. Write your answers in interval notation.

4. $f(x) = \dfrac{2}{3x - 1}$

5. $g(x) = \sqrt{16 - x^4}$

6. $h(x) = \ln(x + 6)$

7. $F(t) = 3 + \cos 2t$

8. $f(x) = \dfrac{3}{x + 2}$

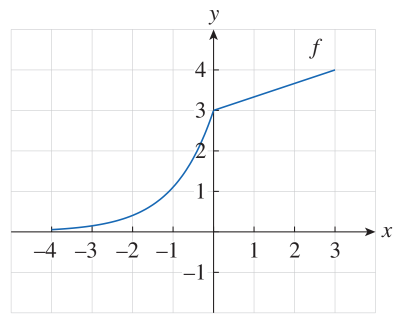
9. $f(x) = 3^x + 2$

10. $f(x) = \dfrac{\sin x}{x}$

11. $f(x) = \tan(x + 1)$

12. Suppose the graph of f is given. Describe how the graphs of the following functions can be obtained from the graph of f.
(a) $y = f(x) - 8$ (b) $y = f(x - 8)$
(c) $y = 1 + 2f(x)$ (d) $y = f(x - 2) - 2$
(e) $y = -f(x)$ (f) $y = f^{-1}(x)$

13. Use the graph of f to sketch the graph of each of the following functions.
(a) $y = f(x - 8)$ (b) $y = -f(x)$
(c) $y = 2 - f(x)$ (d) $y = \frac{1}{2}f(x) - 1$
(e) $y = f^{-1}(x)$ (f) $y = f^{-1}(x + 3)$



Use transformations to sketch the graph of the function.

14. $y = (x - 2)^3$

15. $y = 2\sqrt{x}$

16. $y = x^2 - 2x + 2$

17. $y = \ln(x + 1)$

18. $f(x) = -\cos 2x$

19. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geq 0 \end{cases}$

20. Determine whether f is even, odd, or neither even nor odd.
(a) $f(x) = 2x^5 - 3x^2 + 2$ (b) $f(x) = x^3 - x^7$
(c) $f(x) = e^{-x^2}$ (d) $f(x) = 1 + \sin x$

21. Simplify each expression.
(a) $\dfrac{2^b 2^a 3^a}{2^{-a}}$ (b) $e^{\ln x - \ln y}$
(c) $2 \ln a + 3 \ln b - 4 \ln c$

22. Sketch the curve corresponding to each equation. Use the Vertical Line Test to determine whether the curve is the graph of a function.
(a) $y = x^2 - 1$ (b) $x + y^2 = 4$
(c) $\sqrt{x^2 + y^2} = 1$ (d) $y = \dfrac{\sin x}{x}$
(e) $y = \dfrac{\ln x}{x}$

23. If $f(x) = x^2 + 2x + 1$, evaluate and simplify each of the following expressions.

(a) $f(a+1) - f(a)$ (b) $\dfrac{f(a+1) - f(a-1)}{2}$

(c) $\dfrac{f(a+2) - f(a-2)}{4}$ (d) $\dfrac{f(a+h) - f(a-h)}{2h}$

24. Find an expression for the function whose graph consists of the line segment from the point $(-2, 2)$ to the point $(-1, 0)$ together with the top half of the circle with center the origin and radius 1.

25. If $f(x) = \ln x$ and $g(x) = x^2 - 9$, find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, (d) $g \circ g$ and their domains.

26. Express the function $F(x) = \dfrac{1}{\sqrt{1 + \sqrt{x}}}$ as a composition of three functions.

27. Life expectancy improved dramatically in the 20th century. The table below gives the life expectancy at birth (in years) of males born in the United States.

Birth year	Life expectancy	Birth year	Life expectancy
1900	48.3	1970	67.1
1910	51.1	1980	70.0
1920	55.2	1990	71.8
1930	57.4	2000	74.3
1940	62.5	2010	76.2
1950	65.0	2020	76.1
1960	66.6		

Use a scatter plot to choose an appropriate type of mathematical model. Use your model to predict the life expectancy of a male born in the United States in the year 2030.

28. A small-appliance manufacturer finds that it costs \$9000 to produce 1000 toaster ovens a week and \$12,000 to produce 1500 toaster ovens a week.

(a) Express the cost as a function of the number of toaster ovens produced, assuming that the function is linear. Sketch the graph.

(b) What is the slope of the graph? Explain its meaning in the context of this problem.

(c) What is the y-intercept of the graph? Explain its meaning in the context of this problem.

29. If $f(x) = 2x + \ln x$, find $f^{-1}(2)$.

30. Find the inverse function of $f(x) = \dfrac{x+1}{2x+1}$.

31. Find the exact value of each expression.

(a) $e^{2 \ln 3}$ (b) $\log_{10} 25 + \log_{10} 4$

32. Solve each equation for x.

(a) $e^x = 5$ (b) $\ln x = 2$

(c) $e^{e^x} = 2$ (d) $e^{3x} = 2e^x$

(e) $\dfrac{\ln x}{x} = 3$ (f) $\ln(\ln x) = 5$

33. The half-life of palladium-100, ^{100}Pd, is 4 days. Therefore, half of any given quantity of ^{100}Pd will disintegrate in 4 days. The initial mass of a sample is 1 gram.

(a) Find the mass that remains after 16 days.

(b) Find the mass $m(t)$ that remains after t days.

(c) Find the inverse of this function and explain its meaning in the context of this problem.

34. The population of a certain species in a limited environment with initial population 100 and carrying capacity 1000 is

$$P(t) = \frac{100{,}000}{100 + 900e^{-t}}$$

where t is measured in years.

(a) Graph this function and use the graph to estimate how long it takes for the population to reach 900.

(b) Find the inverse of this function and explain its meaning in the context of this problem.

(c) Use the inverse function to find the time required for the population to reach 900. Compare this answer with your estimate in part (a).

35. Let $f(x) = 6e^{-x^2}$.

(a) Is f even, odd, or neither even nor odd?

(b) Find the average rate of change of f over the interval $[1, 3]$.

(c) Use technology to sketch the graph of f.

36. Graph the function $f(x) = \ln(x^2 - c)$ for several values of c. Explain how the graph changes as c varies.

37. Graph the three functions $y = x^a$, $y = a^x$, and $y = \log_a x$ on the same coordinate axes for several values of $a > 1$. As x increases without bound, which of these functions has the largest values and which has the smallest values?

38. (a) Sketch the curve represented by the parametric equations $x = e^t$, $y = \sqrt{t}$, $0 \le t \le 1$, and indicate with an arrow the direction in which the curve is traced as t increases.

(b) Eliminate the parameter to find a Cartesian equation of the curve.

39. (a) Find parametric equations for the path of a particle that moves counterclockwise halfway around the circle $(x-2)^2 + y^2 = 4$, from the top to the bottom.

(b) Use the equations from part (a) to graph the semicircular path.

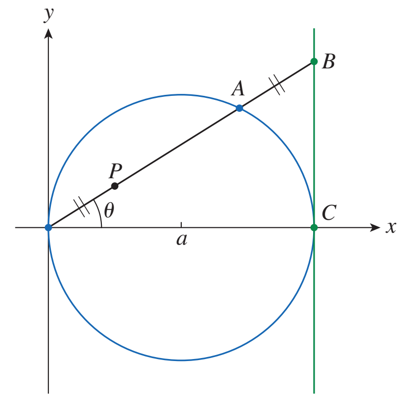
40. Use parametric equations to graph the function

$$f(x) = 2x + \ln x$$

and its inverse on the same coordinate axes.

41. (a) Find parametric equations for the set of all points P determined as shown in the figure such that $|OP| + |AB|$. (This curve is called the **cissoid of Diocles** after the Greek scholar Diocles, who introduced the cissoid as a graphical method for constructing the edge of a cube whose volume is twice that of a given cube.)

(b) Use the geometric description of the curve to draw a rough sketch of the curve. Check your work by using the parametric equations to graph the curve.



Principles of Problem Solving

There are no set rules that will ensure success in solving problems and, therefore, problem solving is probably one of the most difficult topics to teach. However, it is possible to outline some general steps in the problem-solving process and to give some principles that may be useful in the solution of certain problems. These principles are just common sense and intuitive steps made explicit. They have been adapted from George Polya's book *How To Solve It*.

1. Understand the Problem

The first step is to read the problem and make sure that you understand it clearly. Ask yourself the following questions:

What is the unknown?

What are the given quantities?

What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram.

Usually it is necessary to

introduce suitable notation.

In choosing symbols for the unknown quantities, we often use letters such as a, b, c, m, n, x, and y, but in some cases it helps to use initials as suggestive symbols; for instance, V for volume or t for time.

2. Think of a Plan

Find a connection between the given information and the unknown that will enable you to calculate the unknown; create a vision for solving the problem. It often helps to ask yourself explicitly: "How can I relate the given information to the unknown?" If you don't see a connection immediately, the following ideas may be helpful in devising a plan.

Try to Recognize Something Familiar Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown. Look for key words and phrases.

Try to Recognize Patterns Some problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, or numerical, or algebraic. If you can see regularity or repetition in a problem, you might be able to guess what the continuing pattern is and then prove it.

Use Analogy Try to think of an analogous problem, that is, a similar problem, a related problem, but one that is easier than the original problem. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult problem. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem involves three-dimensional geometry, you could look for a similar problem in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

Introduce Something Extra It may sometimes be necessary to introduce something new, an auxiliary aid, to help make the connection between the given and the unknown. For instance, in a problem where a diagram is useful, the auxiliary aid could be a new line drawn in a diagram. In a more algebraic problem, it could be a new unknown that is related to the original unknown.

Take Cases We may sometimes have to split a problem into several cases and present a different argument for each of the cases. For instance, we often have to use this strategy in dealing with absolute value.

Work Backward Sometimes it is useful to imagine that your problem is solved and work backward, step by step, until you arrive at the given data. Then you may be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation $3x - 5 = 7$, we suppose that x is a number that satisfies $3x - 5 = 7$ and work backward. We add 5 to each side of the equation and then divide each side by 3 to get $x = 4$. Since each of these steps can be reversed, we have solved the problem.

Establish Subgoals In a complex problem, it is often useful to set subgoals (in which the desired situation is only partially fulfilled). If we can first reach these subgoals, then we may be able to build on them to reach our final goal.

Indirect Reasoning Sometimes it is appropriate to solve a problem indirectly. In using proof by contradiction to prove that P implies Q, we assume that P is true and Q is false and try to see why this can't happen. Somehow we have to use this information and arrive at a contradiction to what we absolutely know is true.

Mathematical Induction In proving statements that involve a positive integer n, it is frequently helpful to use the following principle.

Principle of Mathematical Induction

Let S_n be a statement involving the positive integer n. Suppose that

(1) S_1 is true.

(2) S_{k+1} is true whenever S_k is true.

Then S_n is true for all positive integers n.

This principle is reasonable because, since S_1 is true, it follows from condition 2 (with $k = 1$) that S_2 is true. Then, using condition 2 with $k = 2$, we see that S_3 is true. Again using condition 2, this time with $k = 3$, we have that S_4 is true. This procedure can be followed indefinitely.

3. Carry Out the Plan

A solution plan was developed in Step 2. Now we carry out that plan, check each stage of the plan, and write the details that confirm each stage is correct.

4. Look Back

Having completed our solution, it's always a good idea to look back over the results, partly to see if we have made errors in the solution and partly to see if we can think of an easier way to solve the problem. Another reason for looking back is that it will familiarize us with the method of solution and this may be useful for solving a future problem. Descartes said, "Every problem that I solved became a rule which served afterwards to solve other problems."

These principles of problem solving are illustrated in the following examples. Before you look at the solutions, try to solve these problems yourself, referring to these Principles of Problem Solving if you get stuck. You may find it useful to refer to this section from time to time as you solve the exercises in the remaining chapters of this book.

Example 1 Hypotenuse Function

Express the hypotenuse h of a right triangle with area 25 m^2 as a function of its perimeter P.

Solution

PS Understand the problem

First, identify the given information and the unknown quantity.

Given quantities: perimeter P, area 25 m^2

Unknown: hypotenuse h

PS Draw a diagram

It may be helpful to draw a diagram. Figure 1.105 illustrates the hypotenuse in a right triangle.

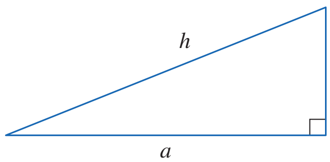


Figure 1.105
Right triangle with hypotenuse h.

In order to connect the given quantities to the unknown, we introduce two extra variables a and b, which represent the lengths of the other two sides of the right triangle. Since this is a right triangle, we can use the Pythagorean Theorem to write an expression that involves these three variables.

$$h^2 = a^2 + b^2$$

The other necessary connections among the variables are found by writing expressions for the area and perimeter:

PS Connect the given with the unknown

PS Introduce something extra

$$25 = \frac{1}{2}ab \qquad \text{Area of the triangle.}$$

$$P = a + b + h \qquad \text{Perimeter of the triangle.}$$

Since we assume that P is given, we now have three equations in three unknowns a, b, and h:

$$h^2 = a^2 + b^2 \qquad (1)$$

$$25 = \frac{1}{2}ab \qquad (2)$$

$$P = a + b + h \qquad (3)$$

PS Relate to the familiar

In theory, we can solve this system of three equations in three unknowns. But the solution technique isn't very obvious or straight forward. We can use the problem-solving strategy of pattern recognition, and then solve these equations using a simpler method.

Look at the right side of the three equations. These expressions should remind you of something familiar. They contain the parts of the binomial expansion

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Using this connection, write two expressions for $(a + b)^2$.
Using Equations 1 and 2:

$$\begin{aligned} (a + b)^2 &= (a^2 + b^2) + 2ab && \text{Group terms.} \\ &= h^2 + 4(25) && \text{Use Equations 1 and 2.} \end{aligned}$$

Using Equation 3:

$$\begin{aligned} (a + b)^2 &= (P - h)^2 && \text{Solve for } a + b \text{ in Equation 3.} \\ &= P^2 - 2Ph + h^2 && \text{Expand the expression.} \end{aligned}$$

Now, equate these two expressions for $(a + b)^2$.

$$h^2 + 100 = P^2 - 2Ph + h^2 \qquad \text{Equate expressions for } (a+b)^2.$$

$$2Ph = P^2 - 100 \qquad h^2 \text{ cancels isolate } h.$$

$$h = \frac{P^2 - 100}{2P} \qquad \text{Solve for } h.$$

This is the required expression for h as a function of P. ■

The next example illustrates that it is often necessary to use the problem-solving principle of *taking cases* when dealing with absolute values.

Example 2 Absolute Values and an Inequality

Solve the inequality $|x - 3| + |x + 2| < 11$.

Solution

Recall the definition of the absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Apply this definition to the two expressions in the inequality.

$$|x - 3| = \begin{cases} x - 3 & \text{if } x - 3 \geq 0 \\ -(x - 3) & \text{if } x - 3 < 0 \end{cases} = \begin{cases} x - 3 & \text{if } x \geq 3 \\ -x + 3 & \text{if } x < 3 \end{cases}$$

$$|x + 2| = \begin{cases} x + 2 & \text{if } x + 2 \geq 0 \\ -(x + 2) & \text{if } x + 2 < 0 \end{cases} = \begin{cases} x + 2 & \text{if } x \geq -2 \\ -x - 2 & \text{if } x < -2 \end{cases}$$

PS Take cases

These expressions suggest that we need to consider three cases:

$x < -2; \quad -2 \leq x < 3; \quad x \geq 3$

Case I: If $x < -2$, use the absolute value definitions to simplify the inequality.

$$\begin{aligned} |x - 3| + |x + 2| &< 11 \\ -x + 3 - x - 2 &< 11 \\ -2x &< 10 \\ x &> -5 \end{aligned}$$

Case II: If $-2 \leq x < 3$, the inequality becomes

$$\begin{aligned} -x + 3 + x + 2 &< 11 \\ 5 &< 11 \end{aligned} \qquad \text{Always true.}$$

Case III: If $x \geq 3$ the inequality becomes

$$\begin{aligned} x - 3 + x + 2 &< 11 \\ 2x &< 12 \\ x &< 6 \end{aligned}$$

Combine the results from cases I, II, and III. The inequality is satisfied when $-5 < x < 6$ or in interval notation $(-5, 6)$. ■

In the next example, we will first guess the answer by looking at a few special cases and recognizing a pattern. Then we will prove the result using mathematical induction.

There are three steps to the Principle of Mathematical Induction.

STEP 1 Prove that S_n is true when $n = 1$.

STEP 2 Assume that S_n is true when $n = k$ and deduce that S_n is true when $n = k + 1$.

STEP 3 Conclude that S_n is true for all n by the Principle of Mathematical Induction.

Example 3 Pattern Recognition

If $f_0(x) = x/(x + 1)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

Solution

Start by finding formulas for $f_n(x)$ for the special cases $n = 1, 2$, and 3.

$$f_1(x) = (f_0 \circ f_0)(x) = f_0(f_0(x)) = f_0\left(\frac{x}{x+1}\right)$$

$$= \frac{\dfrac{x}{x+1}}{\dfrac{x}{x+1} + 1} = \frac{\dfrac{x}{x+1}}{\dfrac{2x+1}{x+1}} = \frac{x}{2x+1}$$

$$f_2(x) = (f_0 \circ f_1)(x) = f_0(f_1(x)) = f_0\left(\frac{x}{2x+1}\right)$$

$$= \frac{\dfrac{x}{2x+1}}{\dfrac{x}{2x+1} + 1} = \frac{\dfrac{x}{2x+1}}{\dfrac{3x+1}{2x+1}} = \frac{x}{3x+1}$$

$$f_3(x) = (f_0 \circ f_2)(x) = f_0(f_2(x)) = f_0\left(\frac{x}{3x+1}\right)$$

$$= \frac{\dfrac{x}{3x+1}}{\dfrac{x}{3x+1} + 1} = \frac{\dfrac{x}{3x+1}}{\dfrac{4x+1}{3x+1}} = \frac{x}{4x+1}$$

PS Look for a pattern

Notice the pattern: the coefficient of x in the denominator of $f_n(x)$ is $n + 1$ in the three cases considered. So, a reasonable guess is, in general,

$$f_n(x) = \frac{x}{(n+1)x+1} \tag{4}$$

To prove that our guess is correct, we use the Principle of Mathematical Induction. We have already verified that Equation 4 is true for $n = 1$. Assume that it is true for $n = k$, that is,

$$f_k(x) = \frac{x}{(k+1)x+1}$$

Then

$$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{x}{(k+1)x+1}\right)$$

$$= \frac{\dfrac{x}{(k+1)x+1}}{\dfrac{x}{(k+1)x+1}+1} = \frac{\dfrac{x}{(k+1)x+1}}{\dfrac{(k+2)x+1}{(k+1)x+1}} = \frac{x}{(k+2)x+1}$$

This expression shows that Equation 4 is true for $n = k + 1$. Therefore, by mathematical induction, this equation is true for all positive integers n. ■

Problems

1. One of the legs of a right triangle has length 4 cm. Express the length of the altitude perpendicular to the hypotenuse as a function of the length of the hypotenuse.

2. The altitude perpendicular to the hypotenuse of a right triangle is 12 cm. Express the length of the hypotenuse as a function of the perimeter.

3. Solve the equation $|2x - 1| - |x + 5| = 3$.

4. Solve the inequality $|x - 1| - |x - 3| \geq 5$.

5. Sketch the graph of the function $f(x) = |x^2 - 4|x| + 3|$.

6. Sketch the graph of the function $g(x) = |x^2 - 1| - |x^2 - 4|$.

7. Draw the graph of the equation $x + |x| = y + |y|$.

8. Draw the graph of the equation $x^4 - 4x^2 - x^2y^2 + 4y^2 = 0$.

9. Sketch the region in the plane consisting of all points (x, y) such that $|x| + |y| \leq 1$.

10. Sketch the region in the plane consisting of all points (x, y) such that $|x - y| + |x| - |y| \leq 2$.

11. Evaluate $(\log_2 3)(\log_3 4)(\log_4 5) \cdots (\log_{31} 32)$.

12. (a) Show that the function $f(x) = \ln(x + \sqrt{x^2 + 1})$ is an odd function.
(b) Find the inverse function of f.

13. Solve the inequality $\ln(x^2 - 2x - 2) \leq 0$.

14. Use indirect reasoning to prove that $\log_2 5$ is an irrational number.

15. A driver begins a road trip. For the first half of the distance, they drive at the leisurely pace of 30 mi/h; they then drive the second half at 60 mi/h. What is their average speed on this trip?

16. Is it always true that $f \circ (g + h) = f \circ g + f \circ h$?

17. Prove that if n is a positive integer, then $7^n - 1$ is divisible by 6.

18. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

19. If $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

20. (a) If $f_0(x) = \dfrac{1}{2 - x}$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find an expression for $f_n(x)$ and use mathematical induction to prove it.

(b) Graph f_0, f_1, f_2, f_3 on the same coordinate axes and describe the effects of repeated composition on the graph of f_n.

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The world land speed record for wheel-driven class cars has steadily increased from 40 mph in 1898. In 1964, speeds obtained in jet propelled vehicles were recognized, and the land speed record continued to increase. The current land speed record, set in 1997, is 763.035 mph obtained in the ThrustSSC, a twin turbofan jet-powered car. This ride actually broke the sound barrier. Engineers continue to design faster vehicles. One such vehicle, the Bloodhound LSR, also uses a jet engine, and its team believes it can one day reach 1000 mph. But it certainly seems reasonable that there is a limit to the fastest land speed record.

2 Limits

Contents

The idea of a limit is central to many important concepts in calculus. Therefore, we begin the study of calculus by investigating limits and their properties. There are two basic, very important, problems in calculus involving the tangent line and the velocity of a moving object. The limits involved in solving these problems lead to the first central concept in calculus, the derivative.

2.1 The Tangent and Velocity Problems

Two very important problems in calculus involve the tangent line to a curve and the velocity (and acceleration) of a moving object. Each of these problems involves the concept of a limit. Let's consider the tangent line problem.

The Tangent Line Problem

Given a curve C and a point P on C, how can we describe the tangent line to C at P? What does it mean to say that a line is tangent to a curve? For a circle, we intuitively know that a tangent line should just glance off, or touch, the circle, as in Figure 2.1.

This argument suggests that a tangent line to a curve C at a point P intersects the curve in just one point, P. More precisely, Figure 2.2 suggests a tangent line intersects a curve at P and has the same *direction* as the curve near P.

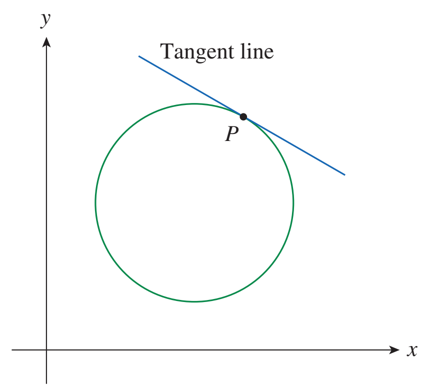


Figure 2.1
Tangent line to a circle at a point P.

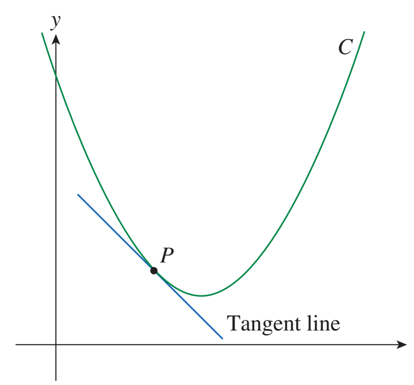


Figure 2.2
Tangent line to a curve C at a point P.

The tangentt line in Figure 2.3 seems to satisfy this working definition: the tangent line intersects the curve at P and has the same direction as the curve near P. However, in Figure 2.4, the tangent line has the same direction as the curve near P but certainly intersects the curve in at least one more point.

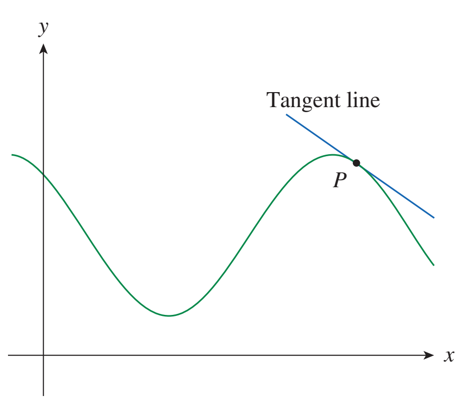


Figure 2.3
Tangent line to a curve at a point P seems to satisfy our working definition.

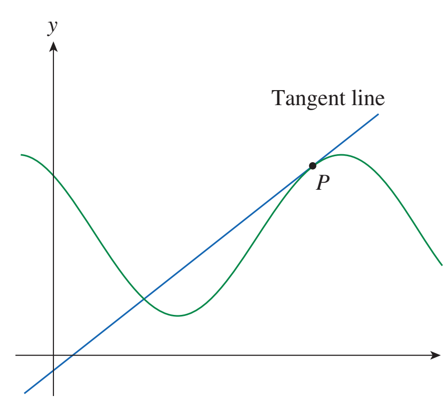


Figure 2.4
Tangent line to a curve at a point P intersects the curve in at least one more point.

In order to refine our definition and to find an equation of the tangent line to a curve at a point, we need the coordinates of the point and the slope, or direction, of the line at that point. Since we know only one point, P, on the tangent line, we will need to start by estimating the slope.

Example 1 Tangent Line Equation

Find an equation of the tangent line to the graph of $f(x) = x^2$ at the point $P(1, 1)$.

Solution

We know only one point, P, on the tangent line, l, but we need two points to compute the slope, $m_{\tan}$, of the tangent line.

A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts (intersects) the curve more than once.

It seems reasonable to use a secant line to estimate the slope of the tangent line. Select another point on the graph of $y = f(x)$, in this case, just a little to the right, with coordinates $Q = (1 + h, f(1 + h))$, where h is a small positive number.

The line connecting the two points P and Q is a secant line (Figure 2.5), and its slope is

$$m_{\sec} = \frac{\Delta y}{\Delta x} = \frac{f(1 + h) - f(1)}{(1 + h) - 1} = \frac{f(1 + h) - f(1)}{h} \qquad h > 0$$

To obtain a better estimate of the slope of the tangent line, let the point Q move along the graph of $y = f(x)$, closer to the point P (Figure 2.6). The associated secant line approaches the tangent line, and the slope of the secant line, $m_{\sec}$, approaches the slope of the tangent line, $m_{\tan}$.

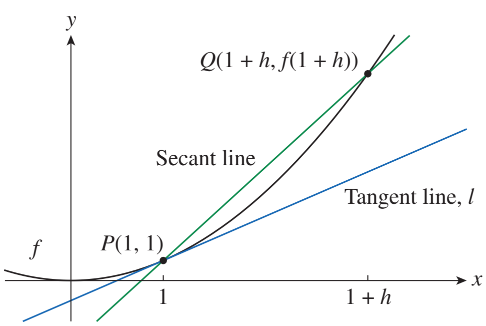


Figure 2.5
The slope of the secant line is $m_{\sec} = \dfrac{f(1 + h) - f(1)}{h}$.

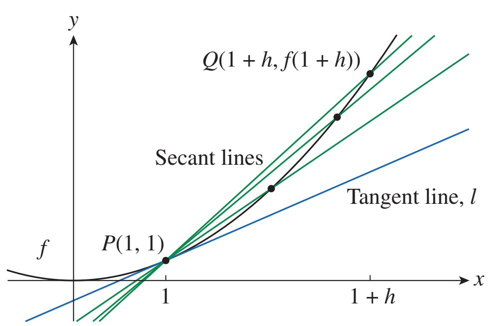


Figure 2.6
The secant line approaches the tangent line.

Common Error

$f(x + h) - f(x) = f(h)$

Correct Method

This simplification is true only under certain conditions. In general, we need to evaluate f at $x + h$ and at x and then simplify the difference if possible.

The table shows the values of $m_{\sec}$ for several values of h.

h	0.5	0.1	0.01	0.001	0.0001
$m_{\sec}$	2.5	2.1	2.01	2.001	2.0001

This table and Figure 2.6 suggest that as Q gets closer to P, or equivalently, as h gets closer to 0, the secant line approaches the tangent line, and the slope of the tangent line should be 2.

Note that we could also consider secant lines starting with a point Q to the left of P. In this case the values of h are (small) negative numbers.

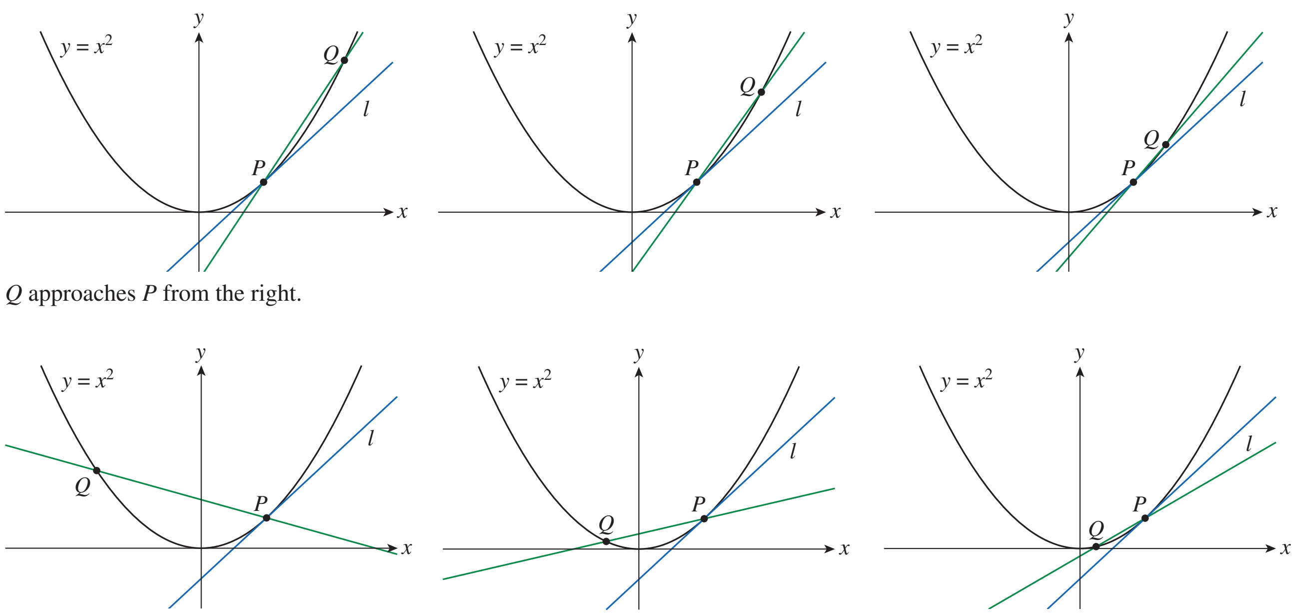
We say that the slope of the tangent line is the *limit* of the slopes of the secant lines and express this symbolically by writing

$$\lim_{Q \to P} m_{\text{sec}} = m_{\text{tan}} \quad \text{and} \quad \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = 2$$

Assuming that the slope of the tangent line is 2, use the point-slope form of the equation of a line to write an equation of the tangent line through (1, 1):

$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1$

Figure 2.7 illustrates the limiting process that occurs in this example. As Q approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line l.



Q approaches P from the right.

Q approaches P from the left.

Figure 2.7
Graphical illustration of the limiting process.

t	C
0	579
1	844
2	1312
3	2015
4	2801
5	4579
6	6061

We can never be certain of the characteristics of a function between two discrete points. In this case, however, historical evidence suggests the graph of this type of function is smooth and increasing.

Many functions that occur in science are not described by explicit equations; they are instead defined by experimental data. Example 2 shows how to estimate the slope of the tangent line to the graph of such a function.

Example 2 Estimate the Slope of a Tangent Line

In 2019–2020 the Centers for Disease Control and Prevention (CDC) began to monitor the outbreak of a unique respiratory illness caused by a coronavirus. The data in the table presents the number of reported cases C per day starting on January 22 ($t = 0$). Use the data to sketch a reasonable graph of the function C and estimate the slope of the tangent line at the point where $t = 4$. Note that the slope of the tangent line represents the rate of change in the number of reported cases per day.

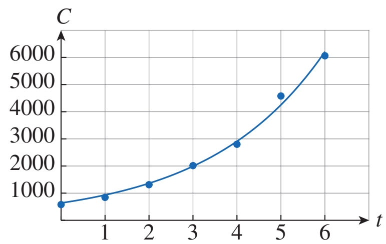


Figure 2.8
Scatter plot and a curve that approximates the function C.

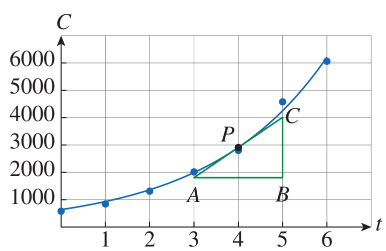


Figure 2.9
A graphical estimate of the slope.

Solution

Figure 2.8 shows a scatter plot of the data and a reasonable sketch of the curve that approximates the graph of the function C.

There are several ways in which we can estimate the slope of the tangent line.

If we consider the secant line through the points $P(4, 2801)$ and $Q(6, 6061)$,

$$m_{\text{sec}} = \frac{6061 - 2801}{6 - 4} = 1630.$$

We could also use other points for Q from the table. For example, for $P(4, 2801)$ and $Q(2, 1312)$

$$m_{\text{sec}} = \frac{2801 - 1312}{4 - 2} = 744.5.$$

Another way to estimate the slope is to first find an equation of best fit. Using the equation for the curve C, $C = 637.9 \cdot 1.46^t$, an estimate for the slope of the tangent line is

$$m_{\text{sec}} = \frac{C(5) - C(4)}{5 - 4} = \frac{4231.72 - 2898.44}{1} = 1333.28.$$

We can even estimate the slope of the tangent line graphically.

Draw an approximate tangent line at the point on the curve P where $t = 4$ and measure the sides of the triangle ABC, as illustrated in Figure 2.9.

An estimate of the slope of the tangent line is

$$m_{\text{tan}} \approx \frac{|BC|}{|AB|} = \frac{4015 - 1800}{5 - 3} = 1107.5.$$

Note that each of these estimates seems reasonable but we certainly need a more precise method to compute the slope of a tangent line. ■

The Velocity Problem

Suppose it takes 2 hours to travel 120 mi to an amusement park. Your average velocity for this trip is distance traveled divided by elapsed time, or $120/2 = 60$ mi/h. But the velocity of your car is not constant on a trip – you certainly didn't drive exactly 60 mi/h for the entire ride. The speedometer in your car provides a definite velocity at each moment. Finding this *instantaneous velocity* (or instantaneous rate of change) is an important calculus problem.

Example 3 Instantaneous Velocity of an Object

Suppose a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Solution

Through experiments conducted four centuries ago, Galileo discovered that the distance traveled by a freely falling object is proportional to the square of the time it has been falling. This model for free fall is pretty accurate even though it does not account for air resistance.

If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

We can estimate the instantaneous velocity at $t = 5$ by computing the average velocity over a short time interval, for example, from $t = 5$ to $t = 5.1$:

$$\begin{aligned} \text{average velocity} &= \frac{\text{change in position}}{\text{elapsed time}} \\ &= \frac{s(5.1) - s(5)}{0.1} \\ &= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s} \end{aligned}$$

To obtain a better estimate of the instantaneous velocity, we can compute the average velocity over smaller time intervals. The table shows average velocity calculations over successively smaller time intervals.

Time Interval	Average Velocity (m/s)
[5, 6]	53.9
[5, 5.1]	49.49
[5, 5.05]	49.245
[5, 5.01]	49.049
[5, 5.001]	49.0049

This table suggests that the average velocity is approaching 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time intervals that start at $t = 5$.

So it appears that the instantaneous velocity at $t = 5$ is $v = 49$ m/s. ■

The method and calculations used to solve this instantaneous velocity problem are very similar to those used earlier in this section to find the equation of a tangent line to a curve at a point. Indeed, there is a close connection between these two problems.

The graph of a distance function of the ball is given in Figure 2.10. Consider the points $P(a, 4.9a^2)$ and $Q(a + h, 4.9(a + h)^2)$ on the graph. The slope of the secant line is

$$m_{\text{sec}} = \frac{4.9(a + h)^2 - 4.9a^2}{(a + h) - a}$$

This is the same value as the average velocity over the time interval $[a, a + h]$.

Therefore, the velocity at time $t = a$ (the limit of the average velocities as h approaches 0) is the slope of the tangent line at P (the limit of the slopes of the secant lines). (See Figure 2.11.)

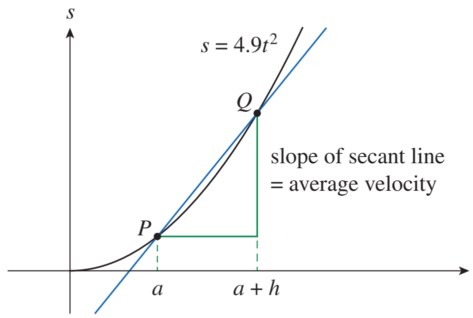


Figure 2.10
Graph of the distance function and a secant line. The slope of the secant line is the average velocity over the time interval $[a, a+h]$.

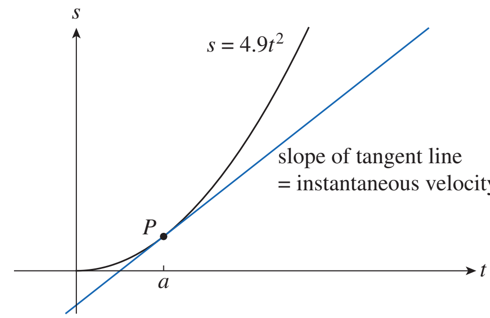


Figure 2.11
The slope of the tangent line is the instantaneous velocity.

The examples in this section show that in order to solve the tangent line and velocity problems, we must be able to find limits. In the next sections, we will learn how to compute limits and apply several limit properties. We will then return to the problems of finding the equation of a tangent line and velocities in later sections of this chapter.

2.1 Exercises

1. Write an expression for the slope of the secant line through each pair of points on the graph of the function $y = f(x)$.
(a) $(a, f(a))$ and $(b, f(b))$
(b) $(a, f(a))$ and $(a+1, f(a+1))$
(c) $(a, f(a))$ and $(a+h, f(a+h))$
(d) $(a+h, f(a+h))$ and $(a-h, f(a-h))$

2. Suppose $f(x) = kx^2 + bx$, where k and b are constants. Find the slope of the secant line through the following pairs of points. Simplify your answer.
(a) $(a+h, f(a+h))$ and $(a-h, f(a-h))$
(b) $(a+2h, f(a+2h))$ and $(a-2h, f(a-2h))$
(c) Compare your answers in parts (a) and (b). Can you explain this result?

3. A point P on the graph of the function $y = f(x)$ has coordinates $(1, f(1))$. Consider the points $Q_1(1.1, f(1.1))$, $Q_2(1.01, f(1.01))$, and $Q_3(1.001, f(1.001))$. For each function, compute the slope of the secant lines PQ_1, PQ_2, and PQ_3. Use these values to estimate the slope of the tangent line to the graph of f at the point P.
(a) $f(x) = 3x^2$
(b) $f(x) = 4x^2 + 3x - 1$
(c) $f(x) = x^3 - x$

4. Water drains from the bottom of a tank that initially holds 1000 gallons. The table that follows gives values of the volume, V, of water in the tank at various times t in minutes.

t (min)	5	10	15	20	25	30
V (gal)	694	444	250	111	28	0

(a) If P is the point (15, 250) on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5, 10, 20, 25$, and 30.
(b) Use your answers in part (a) to estimate the slope of the tangent line at P.
(c) Use a graph of the function to estimate the slope of the tangent line at P.
(d) Explain the meaning of the slope of the tangent line in the context of this problem.

5. A cardiac monitor is used to measure the heart rate of a patient after surgery. It computes the total number of heartbeats after t minutes. When the data are graphed, the slope of the tangent line represents the heart rate, in beats per minute.

t (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data in the table to find the slope of the secant line between the points with the given values of t.

(a) $t = 36$ and $t = 42$ (b) $t = 38$ and $t = 42$
(c) $t = 40$ and $t = 42$ (d) $t = 42$ and $t = 44$

Use your results to estimate the patient's heart rate at 42 minutes.

6. The point $P(2, -1)$ lies on the cure $y = \dfrac{1}{1-x}$.

(a) If Q is the point $\left(x, \dfrac{1}{1-x}\right)$, use technology to find the slope of the secant line PQ for the following values of x.
(i) 1.5 (ii) 1.9 (iii) 1.99 (iv) 1.999
(v) 2.5 (vi) 2.1 (vii) 2.01 (viii) 2.001

(b) Using your results from part (a), estimate the value of the slope of the tangent line to the curve at $P(2, -1)$.

(c) Using your slope from part (b), write an equation for the tangent line to the curve at $P(2, -1)$.

7. The point $P(1, 0)$ lies on the curve $y = \ln x$.

(a) If Q is the point $(x, \ln x)$, use technology to find the slope of the secant line PQ for the following values of x.
(i) 0.5 (ii) 0.9 (iii) 0.99 (iv) 0.999
(v) 1.5 (vi) 1.1 (vii) 1.01 (viii) 1.001

(b) Using your results from part (a), estimate the value of the slope of the tangent line to the curve at $P(1, 0)$.

(c) Using your slope from part (b), write an equation for the tangent line to the curve at $P(1, 0)$.

8. The point $P(0.5, 0)$ lies on the graph of $y = \cos \pi x$.

(a) If Q is the point $(x, \cos \pi x)$, use technology to find the slope of the secant line PQ for the following values of x.
(i) 0 (ii) 0.4 (iii) 0.49 (iv) 0.499
(v) 1 (vi) 0.6 (vii) 0.51 (viii) 0.501

(b) Using your results from part (a), estimate the value of the slope of the tangent line to the curve at $P(0.5, 0)$.

(c) Using your slope from part (b), write an equation for the tangent line to the curve at $P(0.5, 0)$.

(d) Sketch the graph of $y = \cos \pi x$, two secant lines, and the tangent line on the same coordinate axes.

9. If a ball is thrown into the air with initial velocity of 40 ft/s, its height in feet after t seconds is given by $y = 40t - 16t^2$.

(a) Find the average velocity of the ball for the time interval beginning at $t = 2$ and lasting
(i) 0.5 second (ii) 0.1 second
(iii) 0.05 second (iv) 0.01 second

(b) Use your answers from part (a) to estimate the instantaneous velocity of the ball at time $t = 2$.

10. If a rock is thrown upward on the planet Mars with initial velocity 10 m/s, its height in meters after t seconds is given by $y = 10t - 1.86t^2$.

(a) Find the average velocity of the rock over the given time intervals.
(i) [1, 2] (ii) [1, 1.5] (iii) [1, 1.1]
(iv) [1, 1.01] (v) [1, 1.001]

(b) Use your answers from part (a) to estimate the instantaneous velocity of the rock at time $t = 1$.

11. A motorcyclist accelerates from rest along a straight path. Selected values of their position s from the starting point are given in the table.

t (seconds)	0	1	2	3	4	5	6
s (feet)	0	4.9	20.6	46.5	79.5	124.8	176.7

(a) Find the average velocity for each time interval.
(i) [2, 4] (ii) [3, 4] (iii) [4, 5] (iv) [4, 6]

(b) Use the data in the table to sketch a reasonable graph of s as a function of t. Use this graph to estimate the instantaneous velocity at time $t = 3$.

12. The position of a particle, s, measured in meters, moving back and forth along a straight path is given by the equation $s = 2 \sin \pi t + 3 \cos \pi t$, where t is measured in seconds.

(a) Find the average velocity of the particle over each time interval.
(i) [1, 2] (ii) [1, 1.1]
(iii) [1, 1.01] (iv) [1, 1.001]

(b) Use your answers from part (a) to estimate the instantaneous velocity of the particle at time $t = 1$.

13. The height of a roller coaster car above the ground is modeled by the function

$$H(t) = \frac{200t \cos(0.3t)}{t+1} + 260$$

where H is the height in feet above the ground after t seconds.

(a) Graph the function H for $t \in [0, 10]$.

(b) Compute the average velocity of the roller coaster car on the interval $[2, 2+h]$ for $h = 0.1, 0.01, 0.001, 0.0001$, and 0.00001.

(c) Use your answers in part (a) to estimate the instantaneous velocity of the roller coaster car at time $t = 2$ seconds. Indicate the units of measure.

14. The point $P(1, 0)$ lies on the graph of $y = \sin\left(\dfrac{10\pi}{x}\right)$.

(a) If Q is the point $\left(x, \sin\left(\dfrac{10\pi}{x}\right)\right)$, use technology to find the slope of the secant line PQ for $x = 2, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8$, and 0.9. Do these slopes appear to be approaching a single value, or limit?

(b) Use a graph of the function to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P.

(c) Choose appropriate secant lines and use them to estimate the slope of the tangent line at P.

2.2 The Limit of a Function

In order to find either an equation of a tangent line to a curve at a point or the instantaneous velocity of an object, we need to evaluate a limit. In this section, we will examine limits in general, numerical and graphical methods of estimating limits, and analytical procedures for finding the exact value of a limit.

Let f be the function defined by $f(x) = x^2 - x + 2$ and consider the behavior of this function for values of x near 2. Table 2.1 gives values of $f(x)$ for values of x close to 2, but not equal to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

Table 2.1
Table of values for x close to 2.

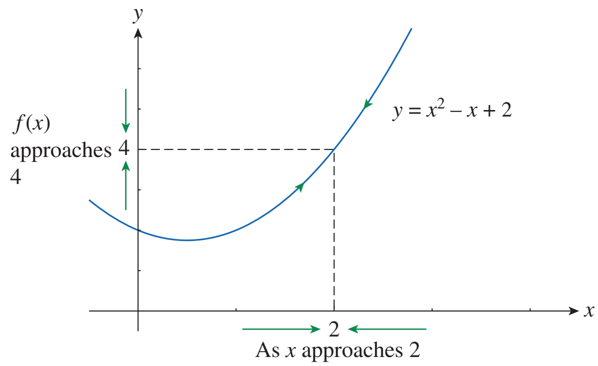


Figure 2.12
Graph of f close to $x = 2$.

The table and the graph of f (a parabola) in Figure 2.12 suggest that as x gets close to 2 (from either side), $f(x)$ gets closer to 4. It appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2. We express this function behavior by saying

"the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4"

The notation for this concept is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

The next definition introduces more general notation.

Definition • Limit of a Function

Suppose $f(x)$ is defined for values of x near a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) The expression

$$\lim_{x \to a} f(x) = L$$

is read as "the limit of $f(x)$, as x approaches a, equals L"

and means we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

This limit expression implies that the values of $f(x)$ approach L as x approaches a. In other words, the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side) but $x \neq a$.

Another way to express the limit

$$\lim_{x \to a} f(x) = L$$

is

$$f(x) \to L \quad \text{as} \quad x \to a$$

which is usually read as "$f(x)$ approaches L as x approaches a."

It is important to interpret the phrase "but $x \neq a$" correctly in this definition. In finding the limit of $f(x)$ as x approaches a, we are interested in the values of $f(x)$ *near a*, *but not at* $x = a$; that is, how $f(x)$ is behaving *around* a. In fact, $f(x)$ may not even be defined at $x = a$, but we can still examine values of $f(x)$ close to $x = a$. Remember, we are examining the behavior of the function around and close to a, but not at a.

Figure 2.13 shows the graphs of three functions. In each case, $\lim_{x \to a} f(x) = L$. However, there is an open circle at the point (a, L) in both Figures 2.13(b) and 2.13(c). The difference is that the function is defined at $x = a$ in (b), but not in (c).

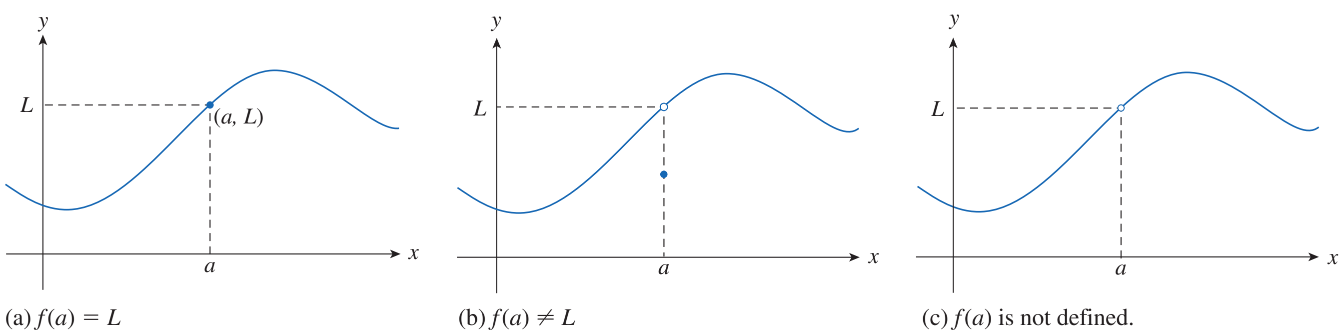


Figure 2.13
$\lim_{x \to a} f(x) = L$ in all three cases.

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

Table 2.2
Values of $f(x)$ for x close to 1.

Example 1 Guess a Limit from Numerical Values

Use a table of values and a graph to estimate the value of $\lim_{x \to 1} \dfrac{x-1}{x^2-1}$.

Solution

Notice that the function $f(x) = \dfrac{x-1}{x^2-1}$ is not defined at $x = 1$ or $x = -1$.

However, the limit expression is all about what's happening near 1, not at 1.

Table 2.2 shows values of $f(x)$ for values of x close to 1, and Figure 2.14 is a graph of $y = f(x)$ near 1. Note the hole in the graph of f.

The table of values and graph both suggest that $\lim\limits_{x \to 1} \dfrac{x-1}{x^2-1} = 0.5$.

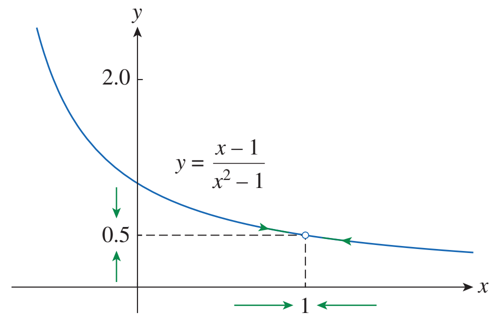


Figure 2.14
Graph of f close to $x = 1$.

Note that we can factor the denominator of f and simplify the expression.

Note: Factoring and canceling, or algebraic manipulation, is an important strategy for finding limits. This intuitive method for finding limits is presented in more detail in the next section.

$$\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}$$

The functions $\dfrac{x-1}{x^2-1}$ and $\dfrac{1}{x+1}$ are different functions because they have different domains.

They are the same everywhere, except at $x = 1$.

Therefore, if we want to know what $\dfrac{x-1}{x^2-1}$ is doing near $x = 1$, it seems reasonable to look at what $\dfrac{1}{x+1}$ is doing near $x = 1$.

Notice that $x = 1$ is in the domain of this new rational function. If we substitute $x = 1$, $\dfrac{1}{1+1} = \dfrac{1}{2} = 0.5$, which is our estimate from above.

Now let's change f slightly by giving it the value 2 when $x = 1$ and calling the resulting function g:

$$g(x) = \begin{cases} \dfrac{x-1}{x^2-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

The new function g still has the same limit as x approaches 1, as illustrated in Figure 2.15. ■

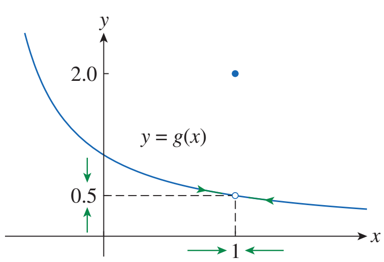


Figure 2.15
Graph of g close to $x = 1$.

Example 2 Estimate a Limit

Use a table of values to estimate the value of $\lim\limits_{x \to 0} \dfrac{\sqrt{x^2+9}-3}{x^2}$.

Solution

Table 2.3 lists the values of the function for several values of x near 0.

x	$\dfrac{\sqrt{x^2+9}-3}{x^2}$
± 1.0	0.162277 . . .
± 0.5	0.165525 . . .
± 0.1	0.166620 . . .
± 0.05	0.166655 . . .
± 0.01	0.166666 . . .

Table 2.3
Table of values for x near 0.

This table suggests that as x approaches 0 (from either side), the function approaches 0.1666666. . . .

Using the limit notation, $\lim\limits_{x \to 0} \dfrac{\sqrt{x^2+9}-3}{x^2} = \dfrac{1}{6}$. ■

Suppose in Example 2 we consider values of x even closer to 0. Table 2.4 shows a table of values obtained using technology for very small values of x. This table appears to contradict our previous estimate.

x	$\dfrac{\sqrt{x^2+9}-3}{x^2}$
± 0.0005	0.16800
± 0.0001	0.20000
± 0.00005	0.00000
± 0.00001	0.00000

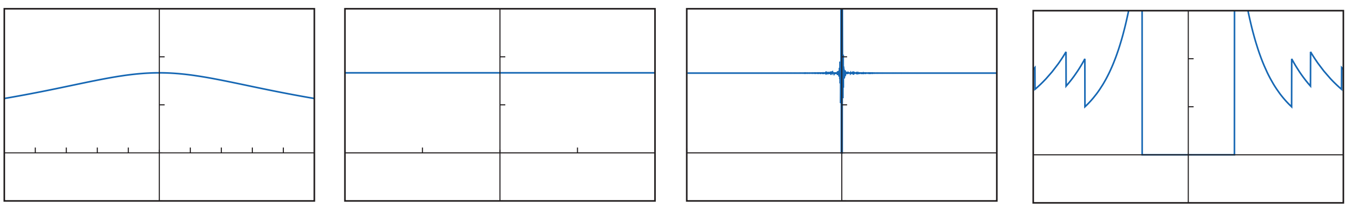
Table 2.4
Table of values for x very close to 0.

Every graphing calculator and even computer software may produce slightly different values; however, if x is sufficiently close to 0, it is likely technology will return a function value of 0. This seems to suggest that the limit is 0, not $\frac{1}{6}$. But this simply isn't true, and we will find this limit analytically in Section 2.3.

Some similar misleading results occur when we use technology to graph the function

$$f(x) = \frac{\sqrt{x^2+9}-3}{x^2}$$

Parts (a) and (b) of Figure 2.16 show accurate graphs of f. We could use a trace feature on many calculators to estimate the limit near 0; however, as we zoom in near $x = 0$, the technology rounding errors are transferred to the graph, as shown in Figure 2.16(c) and (d).



(a) $[-5, 5] \times [-0.1, 0.3]$ (b) $[-0.1, 0.1] \times [-0.1, 0.3]$ (c) $[-10^{-5}, 10^{-5}] \times [-0.1, 0.3]$ (d) $[-10^{-7}, 10^{-7}] \times [-0.1, 0.3]$

Figure 2.16
Graph of the function f near 0 in four viewing windows.

Therefore, we can certainly use technology to help explore and even confirm our results, but we need some definitive analytical procedures for finding limits.

Example 3 Estimate Another Limit

Use a table of values and a graph to estimate the value of $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$.

Solution

Note that the function $f(x) = \dfrac{\sin x}{x}$ is not defined at $x = 0$.

The limit in Example 3 has important consequences throughout calculus. It is essential to use your calculator in radian mode to find this limit and to use this technology setting throughout the text and course. A common error involves using a graphing calculator set to degree mode.

Table 2.5 shows values of $f(x)$ for x close to 0, and Figure 2.17 shows a graph of f near 0.

x	$\dfrac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

Table 2.5
Table of values for x near 0.

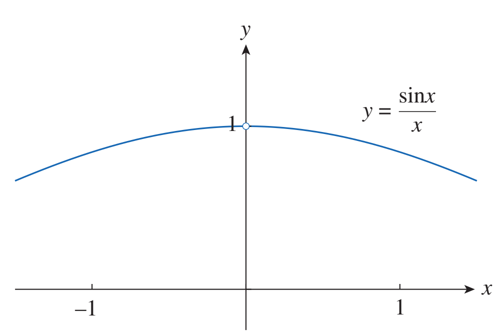


Figure 2.17
Graph of f near 0.

The table and the graph both suggest that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

This estimate is correct and can be proved either by using a geometric argument or analytically. ■

Most computer algebra systems have built-in commands to evaluate limits. The algorithms used to compute limits are more sophisticated than numerical experimentation. Try to find the limits in this section, both in the examples and the exercises, using a Computer Algebra System (CAS).

Example 4 Infinite Oscillation

Use a table of values and a graph to estimate the value of $\lim_{x\to 0} \sin\left(\dfrac{\pi}{x}\right)$.

Solution

Note that the function $f(x) = \sin\left(\dfrac{\pi}{x}\right)$ is not defined for $x = 0$.

Table 2.6 shows values of $f(x)$ for x close to 0, and Figure 2.18 shows a graph of f near 0.

x	$\sin\left(\dfrac{\pi}{x}\right)$
−1.000	0.0000
−0.500	0.0000
−0.100	0.0000
−0.010	0.0000
−0.001	0.0000
1.000	0.0000
0.500	0.0000
0.100	0.0000
0.010	0.0000
0.001	0.0000

Table 2.6
Table of values for x near 0.

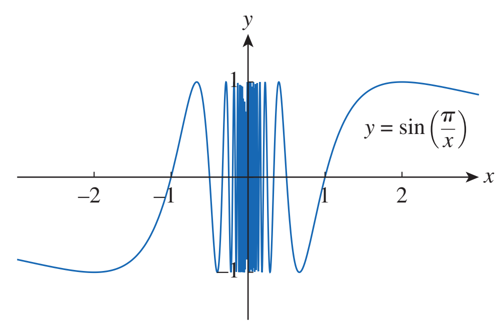


Figure 2.18
The graph of f oscillates between −1 and +1 near 0.

The table certainly suggests that the value of the limit is 0, but the graph tells a different story: the values of $f(x)$ do not approach any single value as x gets close to 0.

For n a positive integer, $f\left(\dfrac{2}{n}\right) = \sin\left(\dfrac{n\pi}{2}\right)$ is either −1, 0, or +1.

Since $\dfrac{2}{n}$ can be made arbitrarily small, this suggests the graph of the function f oscillates back and forth between −1 and +1, infinitely many times near 0.

Since the values of $f(x)$ do not approach a fixed number as x approaches 0, $\lim_{x\to 0} \sin\left(\dfrac{\pi}{x}\right)$ does not exist (DNE). ■

Example 5 Estimate a Limit Numerically

Use a table of values to find $\lim_{x\to 0}\left(x^3 + \dfrac{\cos 5x}{10{,}000}\right)$.

Solution

Consider the values shown in Table 2.7. It appears that the limit is 0.

x	$x^3 + \dfrac{\cos 5x}{10{,}000}$
1.0	1.000028
0.5	0.124920
0.1	0.001088
0.05	0.000222
0.01	0.000101

Table 2.7
Initial table of values.

However, Table 2.8, with smaller values of x, suggests that

x	$x^3 + \dfrac{\cos 5x}{10{,}000}$
0.005	0.00010009
0.004	0.00010004
0.003	0.00010002
0.002	0.00010000
0.001	0.00010000

Table 2.8
Table for smaller values of x.

$$\lim_{x\to 0}\left(x^3 + \frac{\cos 5x}{10{,}000}\right) = 0.000100 = \frac{1}{10{,}000}.$$

Later we will learn that $\lim_{x \to 0} \cos 5x = 1$ and so the overall limit is indeed 0.0001. ■

Examples 1–5 show that numerical and graphical tools can help us explore and estimate a limit; however, a table of values or a graph can be misleading. We need more precise methods for computing limits. In Section 2.3 we will learn a series of limit properties that will allow us to compute limits precisely.

One-Sided Limits

In the previous examples, we considered limits as x approaches a value a from *both sides*, that is, as x approaches a through values just a little greater than a and just a little less than a. But it is often necessary to examine a *one-sided* limit.

The function H is named after the electrical engineer Oliver Heaviside (1850–1925) and can be used to describe an electric current that is switched on at time $t = 0$.

Example 6 The Heaviside Function and One-Sided Limits

The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

The graph of H is shown in Figure 2.19.

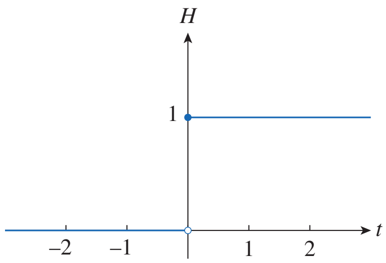


Figure 2.19
Graph of the Heaviside function.

As t approaches 0 from the left, $H(t) = 0$ and therefore approaches 0.

As t approaches 0 from the right, $H(t) = 1$ and therefore approaches 1.

There is no single number that $H(t)$ approaches as t gets close to 0 (from both sides).

Therefore, $\lim_{t \to 0} H(t)$ does not exist (DNE). ■

In Example 6, $H(t)$ approaches 0 as t approaches 0 from the left, and $H(t)$ approaches 1 as t approaches 0 from the right. These results are written mathematically as

$$\lim_{t \to 0^-} H(t) = 0 \quad \text{and} \quad \lim_{t \to 0^+} H(t) = 1$$

The notation $t \to 0^-$ indicates that we consider only values of t that are less than 0. Similarly, $t \to 0^+$ means that we consider only values of t that are greater than 0.

Definition • One-Sided Limits

The expression

$$\lim_{x \to a^-} f(x) = L$$

is read as the **left-hand limit of $f(x)$ as x approaches a**, or the **limit of $f(x)$ as x approaches a from the left**, is equal to L. This means we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a, with $x < a$. The expression

$$\lim_{x \to a^+} f(x) = L$$

is read as the **right-hand limit of $f(x)$ as x approaches a**, or the **limit of $f(x)$ as x approaches a from the right**, is equal to L. This means we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a, with $x > a$.

A Closer Look

1. The notation $x \to a^-$ means that we consider only values less than a. Similarly, the notation $x \to a^+$ means we consider only values greater than a.
2. Figures 2.20 and 2.21 illustrate the definition of one-sided limits.

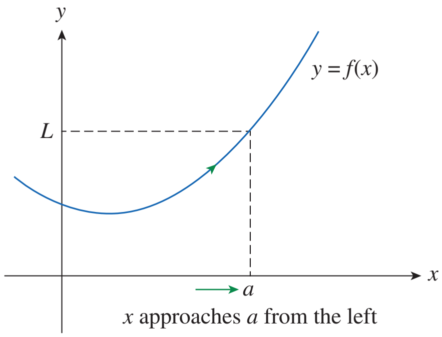


Figure 2.20
Illustration of the definition of $\lim_{x \to a^-} f(x) = L$.

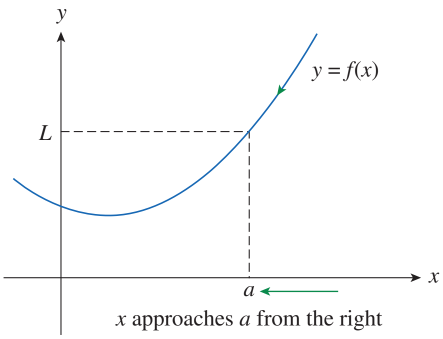


Figure 2.21
Illustration of the definition of $\lim_{x \to a^+} f(x) = L$.

3. The left-hand limit and the right-hand limit could be different values. Example 6 illustrates this concept. ■

It seems reasonable that if the two one-sided limits are the same value, then the overall (two-sided) limit is that same number. This concept is expressed mathematically as

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L \tag{1}$$

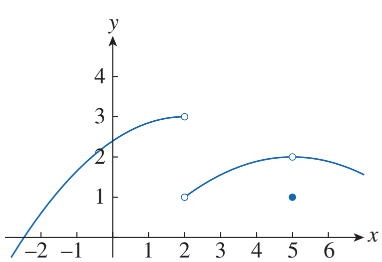


Figure 2.22
Graph of $y = g(x)$.

Example 7 Use a Graph to Find One-Sided Limits

The graph of the function g is shown in Figure 2.22.

Use this graph to find the value of each of the following limits (if it exists).

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

Solution

(a) $\lim_{x \to 2^-} g(x) = 3$

From the graph, the values of $g(x)$ approach 3 as x approaches 2 from the left.

(b) $\lim_{x \to 2^+} g(x) = 1$

From the graph, the values of $g(x)$ approach 1 as x approaches 2 from the right.

(c) $\lim_{x \to 2} g(x)$ DNE

Since the left-hand and the right-hand limits are different, the overall limit does not exist.

(d) $\lim_{x \to 5^-} g(x) = 2$

From the graph, the values of $g(x)$ approach 2 as x approaches 5 from the left.

(e) $\lim_{x\to 5^+} g(x) = 2$

From the graph, the values of $g(x)$ approach 2 as x approaches 5 from the right.

(f) $\lim_{x\to 5} g(x) = 2$

The left-hand and the right-hand limits are the same.

Note that $g(5) \neq 2$. This again illustrates the fundamental concept of a limit. We consider only values near 5, but not equal to 5. ■

We have already seen several examples in which a limit does not exist. Example 8 illustrates a special case.

Example 8 Determine if the Limit Exists

Use a table of values and a graph to find $\lim_{x\to 0} \frac{1}{x^2}$, if it exists.

Solution

The function $f(x) = \frac{1}{x^2}$ is not defined for $x = 0$.

Table 2.9 shows values of $f(x)$ for x close to 0 and Figure 2.23 shows a graph of f near 0.

x	$\frac{1}{x^2}$
± 1.0	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

Table 2.9
Table of values for x close to 0.

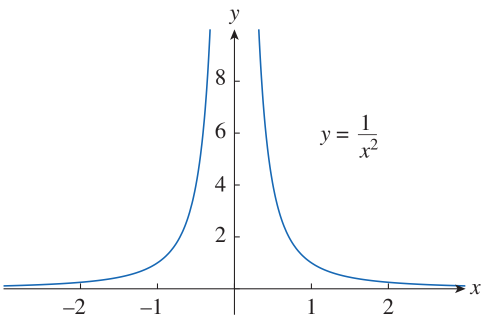


Figure 2.23
The graph of f close to $x = 0$.

The table of values and the graph suggest:

as x approaches 0, x^2 also approaches 0, and $\frac{1}{x^2}$ becomes large.

It appears from the graph that values of $f(x)$ can be made arbitrarily large by taking x close enough to 0.

The values of $f(x)$ do not approach any specific number.

Therefore, $\lim_{x\to 0} \frac{1}{x^2}$ does not exist (DNE). ■

At the beginning of this section, we considered the function $f(x) = x^2 - x + 2$ and, based on numerical and graphical evidence, we concluded that

$$\lim_{x \to 2}(x^2 - x + 2) = 4$$

According to the definition of a limit, this means that the values of $f(x)$ can be made as close to 4 as we like, provided that we take x sufficiently close to 2. In the following example, we use graphical methods to determine just how close is sufficiently close.

Example 9 Graphical and Numerical Interpretation of Sufficiently Close

If $f(x) = x^2 - x + 2$, how close to 2 does x have to be to ensure that $f(x)$ is within a distance 0.1 of the number 4?

Solution

If the distance from $f(x)$ to 4 is less than 0.1, then $f(x)$ lies between 3.9 and 4.1.

So the requirement is $3.9 < x^2 - x + 2 < 4.1$.

We need to determine the values of x such that the curve $y = x^2 - x + 2$ lies between the horizontal lines $y = 3.9$ and $y = 4.1$.

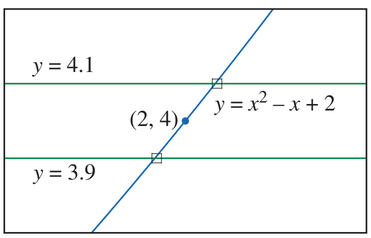


Figure 2.24
$[1.8, 2.2] \times [3.7, 4.3]$. Graphs of the curve and lines.

Figure 2.24 shows a typical graphing calculator screen illustration (with annotations) of the curve and the lines near the point (2, 4).

We can either trace along the curve or use a built-in function to estimate the x-coordinate of the point of intersection of the line $y = 3.9$ and the curve $y = x^2 - x + 2$: $x \approx 1.966$.

Similarly, the curve intersects the line $y = 4.1$ when $x \approx 2.033$.

Therefore, if we round to be conservative, we can conclude

$3.9 < x^2 - x + 2 < 4.1$ when $1.97 < x < 2.03$.

So $f(x)$ is within a distance of 0.1 of 4 when x is within a distance of 0.03 of 2. ■

The idea behind Example 9 can be used to formulate the precise definition of a limit that is discussed in Appendix D.

2.2 Exercises

1. Explain in your own words the meaning of the expression

$$\lim_{x \to 2} f(x) = 5$$

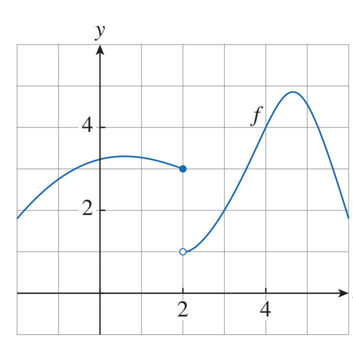
Is it possible for this statement to be true, yet $f(2) = 3$? Explain your reasoning.

2. Explain the meaning of the two expressions

$$\lim_{x \to 1^-} f(x) = 3 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 7$$

In this situation, is it possible that $\lim_{x \to 1} f(x)$ exists? Explain your reasoning.

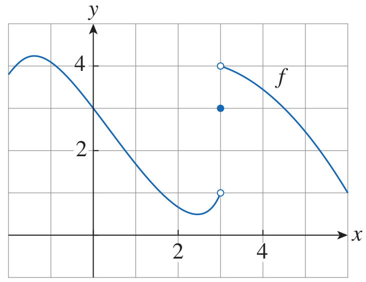
3. Use the graph of f to find the value of each expression. If a limit does not exist, explain why.



(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to 2^+} f(x)$ (c) $\lim_{x \to 2} f(x)$

(d) $f(2)$ (e) $\lim_{x \to 4} f(x)$ (f) $f(4)$

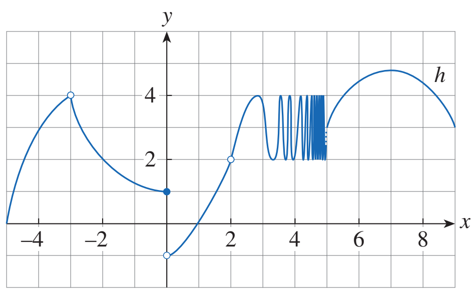
4. Use the graph of f to find the value of each expression. If a limit does not exist, explain why.



(a) $\lim_{x \to 0} f(x)$ (b) $\lim_{x \to 3^-} f(x)$ (c) $\lim_{x \to 3^+} f(x)$

(d) $\lim_{x \to 3} f(x)$ (e) $f(3)$

5. Use the graph of h to find the value of each expression. If a limit does not exist, explain why.



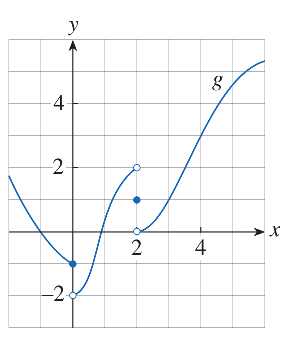
(a) $\lim_{x \to -3^-} h(x)$ (b) $\lim_{x \to -3^+} h(x)$ (c) $\lim_{x \to -3} h(x)$

(d) $h(-3)$ (e) $\lim_{x \to 0^-} h(x)$ (f) $\lim_{x \to 0^+} h(x)$

(g) $\lim_{x \to 0} h(x)$ (h) $h(0)$ (i) $\lim_{x \to 2} h(x)$

(j) $h(2)$ (k) $\lim_{x \to 5^+} h(x)$ (l) $\lim_{x \to 5^-} h(x)$

6. Use the graph of g to find the value of each expression. If a limit does not exist, explain why.



(a) $\lim_{x \to 0^-} g(x)$ (b) $\lim_{x \to 0^+} g(x)$ (c) $\lim_{x \to 0} g(x)$

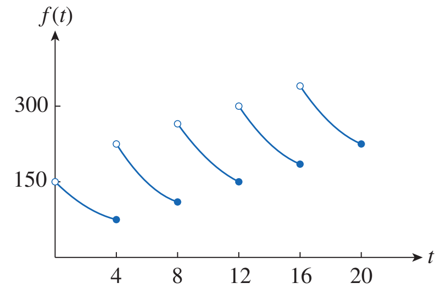
(d) $\lim_{x \to 2^-} g(x)$ (e) $\lim_{x \to 2^+} g(x)$ (f) $\lim_{x \to 2} g(x)$

(g) $g(2)$ (h) $\lim_{x \to 4} g(x)$

7. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the blood stream after t hours. Find

$$\lim_{t \to 12^-} f(t) \quad \text{and} \quad \lim_{t \to 12^+} f(t)$$

and explain the significance of these one-sided limits in the context of this problem.



8. Let $f(x) = \begin{cases} 1 + x & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x < 1 \\ 2 - x & \text{if } x \ge 1 \end{cases}$

Sketch the graph of f and use it to evaluate each limit.

(a) $\lim_{x \to -1^-} f(x)$ (b) $\lim_{x \to -1^+} f(x)$ (c) $\lim_{x \to -1} f(x)$

(d) $\lim_{x \to 1^-} f(x)$ (e) $\lim_{x \to 1^+} f(x)$ (f) $\lim_{x \to 1^+} f(x)$

9. Let $g(x) = \begin{cases} 1 + \sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \le x \le \pi \\ \sin x & \text{if } x > \pi \end{cases}$

Sketch the graph of g and use it to evaluate each limit.

(a) $\lim_{x \to 0^-} g(x)$ (b) $\lim_{x \to 0^+} g(x)$ (c) $\lim_{x \to 0} g(x)$

(d) $\lim_{x \to \pi^-} g(x)$ (e) $\lim_{x \to \pi^+} g(x)$ (f) $\lim_{x \to \pi} g(x)$

Use technology to graph each function and find the value of each limit, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 0^-} f(x)$ (b) $\lim_{x \to 0^+} f(x)$ (c) $\lim_{x \to 0} f(x)$

10. $f(x) = \dfrac{1}{1 + e^{1/x}}$

11. $f(x) = \dfrac{x^2 + x}{\sqrt{x^3 + x^2}}$

12. $f(x) = \dfrac{\sqrt{2 - 2\cos 2x}}{x}$

Sketch the graph of a function f that satisfies all of the given conditions.

13. $\lim_{x \to 3^+} f(x) = 2, \quad \lim_{x \to 3^-} f(x) = 1$

14. $(0, 3)$ lies on the graph of f, $\lim_{x \to 0} f(x) = 2$

15. $\lim_{x\to 0^-} f(x) = -2$, $\lim_{x\to 0^+} f(x)$ does not exist $f(0) = -3$

16. $\lim_{x\to 0^-} f(x) = -1$, $\lim_{x\to 0^+} f(x) = 2$, $f(0) = 1$

17. $\lim_{x\to 0} f(x) = 1$, $\lim_{x\to 3^-} f(x) = -2$, $\lim_{x\to 3^+} f(x) = 2$, $f(0) = -1$, $f(3) = 1$

18. $\lim_{x\to 3^+} f(x) = 4$, $\lim_{x\to 3^-} f(x) = 2$, $\lim_{x\to -2} f(x) = 2$, $f(3) = 3$, $f(-2) = 1$

19. $\lim_{x\to 0^-} f(x) = 2$, $\lim_{x\to 0^+} f(x) = 0$, $\lim_{x\to 4^-} f(x) = 3$, $\lim_{x\to 4^+} f(x) = 0$, $f(0) = 2$, $f(4) = 1$

Use a table of values to estimate the value of the limit, if it exists.

20. $\lim_{x\to 3} \dfrac{x^2 - 3x}{x^2 - 9}$

21. $\lim_{x\to -3} \dfrac{x^2 - 3x}{x^2 - 9}$

22. $\lim_{x\to 0} \dfrac{x}{\sin x}$

23. $\lim_{x\to 1} \ln x$

24. $\lim_{t\to 0} \dfrac{e^{5t} - 1}{t}$

25. $\lim_{h\to 0} \dfrac{(2 + h)^5 - 32}{h}$

Use a table of values and a (technology) graph to estimate the value of the limit.

26. $\lim_{x\to 4} \dfrac{\ln x - \ln 4}{x - 4}$

27. $\lim_{p\to -1} \dfrac{1 + p^9}{1 + p^{15}}$

28. $\lim_{\theta\to 0} \dfrac{\sin 3\theta}{\tan 2\theta}$

29. $\lim_{t\to 0} \dfrac{5^t - 1}{t}$

30. $\lim_{x\to 0} \dfrac{\sqrt{x + 4} - 2}{x}$

31. $\lim_{x\to 0} \dfrac{9^x - 5^x}{x}$

32. $\lim_{x\to 0^+} x^x$

33. $\lim_{x\to 0^+} x^2 \ln x$

34. Let $f(x) = \dfrac{\cos 2x - \cos x}{x^2}$.

(a) Graph the function f and zoom in near the point where the graph crosses the y-axis. Estimate the value of $\lim_{x\to 0} f(x)$.

(b) Use a table of values to confirm your answer in part (a).

35. Let $f(x) = \dfrac{\sin x}{\sin \pi x}$.

(a) Sketch the graph of the function f and use the graph to estimate the value of $\lim_{x\to 0} f(x)$.

(b) Use a table of values to confirm your answer in part (a).

36. Let $f(x) = (1 + x)^{1/x}$.

(a) Use a table of values to estimate the value of $\lim_{x\to 0} f(x)$. Does this number look familiar?

(b) Sketch the graph of f to confirm your answer in part (a).

37. The slope of the tangent line to the graph of the exponential function $y = 2^x$ at the point $(0, 1)$ is $\lim_{x\to 0} \dfrac{2^x - 1}{x}$. Use a table of values to estimate this limit.

38. Let $f(x) = x^2 - \dfrac{2^x}{1000}$.

(a) Evaluate the function f for $x = 1, 0.8, 0.6, 0.4, 0.2, 0.1, 0.05$. Use these results to estimate the value of $\lim_{x\to 0} f(x)$.

(b) Evaluate the function f for $x = 0.04, 0.02, 0.01, 0.005, 0.003, 0.001$. Use these results to estimate the value of $\lim_{x\to 0} f(x)$.

39. Let $h(x) = \dfrac{\tan x - x}{x^3}$.

(a) Evaluate the function h for $x = 1, 0.5, 0.1, 0.05, 0.01, 0.005$. Use these results to estimate the value of $\lim_{x\to 0} h(x)$.

(b) Evaluate $h(x)$ for values of x closer and closer to 0. Explain why you eventually obtained values of 0 for $h(x)$. Do these function values change your estimate of the limit in part (a)?

(c) Graph the function h in the viewing window $[-1, 1] \times [0, 1]$. Zoom in near the point where the graph crosses the y-axis and estimate the limit in part (a). What happens to the graph of h as you continue to zoom in? How does this graph support your results in part (b)?

40. Graph the function $f(x) = \sin\left(\dfrac{\pi}{x}\right)$ in the viewing window $[-1, 1] \times [-1, 1]$. Zoom in near the origin and explain the behavior of the graph of f near $x = 0$.

41. Use a graph to determine how close to 2 we have to take x to ensure that $x^3 - 3x + 4$ is within a distance of 0.2 of the number 6. What if we insist that $x^3 - 3x + 4$ be within 0.1 of 6?

42. Let $f(x) = \dfrac{x^3 - 1}{\sqrt{x} - 1}$.

(a) Use a table of values and a graph of f to estimate the value of $\lim_{x\to 1} f(x)$.

(b) How close to 1 does x have to be to ensure that $f(x)$ is within 0.5 of the limit in part (a)?

43. Let $f(x) = \tan \dfrac{1}{x}$.

(a) Show that $f(x) = 0$ for $x = \dfrac{1}{\pi}, \dfrac{1}{2\pi}, \dfrac{1}{3\pi}, \ldots$.

(b) Show that $f(x) = 1$ for $x = \dfrac{4}{\pi}, \dfrac{4}{5\pi}, \dfrac{4}{9\pi}, \ldots$.

(c) Using the results from parts (a) and (b), what can you conclude about $\lim_{x\to 0^+} f(x)$?

44. Let $f(x) = \tan(2 \sin x)$ for $-\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$.

Use a table of values and a graph of the function f to find each limit, if it exists.

(a) $\lim_{x\to -(\pi/4)^+} f(x)$ (b) $\lim_{x\to (\pi/4)^+} f(x)$

2.3 Calculating Limits Using the Limit Laws

In Section 2.2, we considered a numerical approach (using a table of values) and a graphical approach to investigate limits. We also discovered that these intuitive methods are not always accurate or easy to use without technology. In this section we will learn how to find the limits of various functions algebraically. We will use many properties of limits, called the *Limit Laws* to calculate limits of functions.

Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$ Limit of a sum

Sum Law: The limit of a sum is the sum of the limits.

Note that these first two limit laws can be extended to more than two functions.

2. $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$ Limit of a difference

Difference Law: The limit of a difference is the difference of the limits.

3. $\lim\limits_{x \to a} [cf(x)] = c \lim\limits_{x \to a} f(x)$ Limit of a constant times a function

Constant Multiple Law: The limit of a constant times a function is the constant times the limit of the function. Another way to think about this: constants pass freely through limit symbols.

4. $\lim\limits_{x \to a} [f(x) \cdot g(x)] = [\lim\limits_{x \to a} f(x)] \cdot [\lim\limits_{x \to a} g(x)]$ Limit of a product

Product Law: The limit of a product is the product of the limits.

5. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} \quad \text{if } \lim\limits_{x \to a} g(x) \neq 0$ Limit of a quotient

Quotient Law: The limit of a quotient is the quotient of the limits.

A Closer Look

1. The limits of these new functions involving f and g can be computed only if the limit of f and the limit of g exist.
2. All of these laws can be proved using the precise definition of a limit. The proof of Law 1 (the Sum Law) is in Appendix E.
3. These limit laws are intuitive. For example, if $f(x)$ is close to L and $g(x)$ is close to M, it seems reasonable to conclude that $f(x) + g(x)$ is close to $L + M$.
4. The limit laws are also valid for one-sided limits. ■

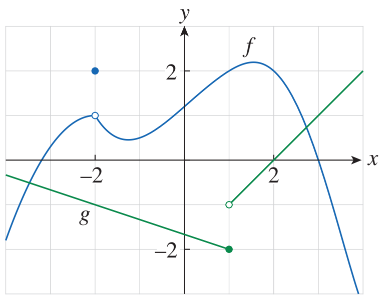


Figure 2.25
Graphs of f and g.

Example 1 Graphs and the Limit Laws

The graphs of f and g are given in Figure 2.25.

Use the Limit Laws to evaluate each limit if it exists.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 3} \dfrac{g(x)}{f(x)}$

Solution

(a) Use the graphs of f and g.

$$\lim_{x \to -2} f(x) = 1 \quad \text{and} \quad \lim_{x \to -2} g(x) = -1$$

$$\lim_{x \to -2} [f(x) + 5g(x)] = \lim_{x \to -2} f(x) + \lim_{x \to -2} [5g(x)] \qquad \text{Limit Law 1.}$$

$$= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) \qquad \text{Limit Law 3.}$$

$$= 1 + 5(-1) = -4 \qquad \text{Use known limits.}$$

(b) Use the graphs of f and g.

$$\lim_{x \to 1} f(x) = 2 \quad \lim_{x \to 1^-} g(x) = -2 \quad \lim_{x \to 1^+} g(x) = -1$$

Therefore, $\lim_{x \to 1} g(x)$ does not exist and we cannot use Limit Law 4.

However, we can consider one-sided limits.

$$\lim_{x \to 1^-} [f(x)g(x)] = \left[\lim_{x \to 1^-} f(x)\right] \cdot \left[\lim_{x \to 1^-} g(x)\right] = 2 \cdot (-2) = -4$$

$$\lim_{x \to 1^+} [f(x)g(x)] = \left[\lim_{x \to 1^+} f(x)\right] \cdot \left[\lim_{x \to 1^+} g(x)\right] = 2 \cdot (-1) = -2$$

Therefore, $\lim_{x \to 1} [f(x)g(x)]$ does not exist since the limits from the left and right are not equal.

(c) Use the graphs of f and g.

$$\lim_{x \to 3} f(x) = 0 \quad \text{and} \quad \lim_{x \to 3} g(x) = 1$$

Because the limit of the denominator is 0, we cannot use Law 5.

The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number. ■

There are several additional limit properties that are used to evaluate the limits of more complex expressions. Many are derived from the five Limit Laws. For example, let $g(x) = f(x)$ in the Product Law to obtain the following property.

Power Law: Suppose $\lim_{x \to a} f(x)$ exists and let n be a positive integer.

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ Limit of a power

There are two special limits that seem reasonable and intuitive.

Let c be a constant.

7. $\lim_{x \to a} c = c$ — Limit of a constant

8. $\lim_{x \to a} x = a$ — Limit of the Identity Function

The proofs of these properties are based on the precise definition of a limit. For now, it is important to be able to apply these special limits in a variety of ways.

Example 2 Basic Limit Law Applications

Here are some basic applications of Limit Laws 7 and 8.

The limit of a constant (function):

$$\lim_{x \to 3} 7 = 7 \qquad \lim_{x \to -2} \sqrt{3} = \sqrt{3} \qquad \lim_{x \to 1^+} 73 = 73$$

The limit of the identity function:

$$\lim_{x \to 4} x = 4 \qquad \lim_{x \to \pi} x = \pi \qquad \lim_{x \to -1^-} x = -1$$

If we let $f(x) = x$ in Limit Law 6, and apply Law 8, we obtain the following property.

Let n be a positive integer.

9. $\lim_{x \to a} x^n = a^n$

And finally, there are some similar properties for the limit of a root

Let n be a positive integer.

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$

If n is even, we assume that $a > 0$.

11. Root Law:

$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ — Limit of a root

If n is even, we assume that $\lim_{x \to a} f(x) > 0$.

Example 3 Apply the Limit Laws

Evaluate each limit.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$

(b) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

Solution

(a)
$$\begin{aligned}
\lim_{x \to 5}(2x^2 - 3x + 4) &= \lim_{x \to 5}(2x^2) - \lim_{x \to 5}(3x) + \lim_{x \to 5} 4 && \text{Limit Laws 2 and 1.} \\
&= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4 && \text{Limit Law 3.} \\
&= 2(5^2) - 3(5) + 4 && \text{Limit Laws 9, 8, and 7.} \\
&= 39
\end{aligned}$$

(b) In the first step of this solution, we will use Limit Law 5; however, we are not justified in using this law until we know for certain that the limits of the numerator and the denominator exist and the limit of the denominator is not 0.

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \to -2} (x^3 + 2x^2 - 1)}{\lim_{x \to -2} (5 - 3x)} \qquad \text{Limit Law 5.}$$

$$= \frac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} 1}{\lim_{x \to -2} 5 - 3 \lim_{x \to -2} x} \qquad \text{Limit Laws 1, 2, and 3.}$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} \qquad \text{Limit Laws 9, 8, and 7.}$$

$$= -\frac{1}{11}$$

■

Examples 2 and 3 suggest a very important concept that in some cases $\lim_{x \to a} f(x) = f(a)$; that is, we can determine the limit by direct substitution. We would get the correct answer in Example 3(a) by substituting 5 for x. Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 and 3 are polynomial and rational functions, and similar use of the Limit Laws proves that direct substitution always works for such functions. We state this fact as follows.

Direct Substitution Property

If f is a polynomial function or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

In general, direct substitution works if the function is *continuous* at a, and we will study continuity in Section 2.4. Examples 4 and 5 demonstrate the method of direct substitution and a technique for finding a limit if direct substitution fails.

Example 4 Direct Substitution

(a) $\lim_{x \to 3} (x^2 - 7x + 4) = 3^2 - 7 \cdot 3 + 4 = -8$

(b) $\lim_{x \to -2} (x^4 - 3x^2 + 7) = (-2)^4 - 3(-2)^2 + 7 = 11$

(c) $\lim_{x \to 5} \dfrac{3x + 1}{x^2 + 1} = \dfrac{3(5) + 1}{5^2 + 1} = \dfrac{16}{26} = \dfrac{8}{13}$

(d) $\lim_{x \to -1} \dfrac{3x^3 - 4}{x^2 + 2x + 5} = \dfrac{3(-1)^3 - 4}{(-1)^2 + 2(-1) + 5} = \dfrac{-7}{4} = -\dfrac{7}{4}$ ■

Example 5 Indeterminate Form of a Limit

Find $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$.

Solution

Let $f(x) = \dfrac{x^2 - 1}{x - 1}$.

We cannot use direct substitution because $x = 1$ is not in the domain of f: $f(1)$ is not defined. Consider the limit in the numerator and the limit in the denominator separately.

$$\lim_{x\to 1}(x^2 - 1) = 1^2 - 1 = 0 \quad \text{and} \quad \lim_{x\to 1}(x - 1) = 1 - 1 = 0$$

Therefore, the limit $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$ is in the indeterminate form $\dfrac{0}{0}$.

This is an indication that we need to do something else to find the limit.

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2-1}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1} && \text{Factor the numerator.}\\
&= \lim_{x\to 1}(x+1) && \text{Cancel common factor.}\\
&= 1 + 2 = 2 && \text{Use direct substitution.}
\end{aligned}$$

Note: Factoring and canceling like factors, or in general, algebraic manipulation, is another important strategy for finding limits.

$\dfrac{x^2-1}{x-1}$ and $x + 1$ are different functions because they have different domains.

However, the domains are the same everywhere except at $x = 1$.

Therefore, if we want to know what $\dfrac{x^2-1}{x-1}$ is doing near $x = 1$, we can simply look at what $x + 1$ is doing near $x = 1$.

And $x = 1$ is in the domain of this polynomial function, so we can use direct substitution. ■

In Example 5 we were able to find the limit by associating two functions that are the same everywhere except at $x = 1$. Remember, this is a valid technique because we consider only what is happening near 1, not at 1. Here is a summary of this solution technique.

If $f(x) = g(x)$ for $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limit exists.

Example 6 Indeterminate Form of a Limit

Find $\lim_{x\to -3}\dfrac{x^2+x-6}{x^2-2x-15}$.

Solution

Consider the limit in the numerator and the limit in the denominator.

$$\lim_{x\to -3}(x^2 + x - 6) = (-3)^2 + (-3) - 6 = 0 \quad \text{and}$$

$$\lim_{x\to -3}(x^2 - 2x - 15) = (-3)^2 - 2(-3) - 15 = 0$$

Therefore, the limit $\lim_{x\to -3}\dfrac{x^2+x-6}{x^2-2x-15}$ is in the indeterminate form $\dfrac{0}{0}$.

Note that $x = -3$ is not in the domain of the function $\dfrac{x^2+x-6}{x^2-2x-15}$.

The strategy here is algebraic manipulation: factor and cancel.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 - 2x - 15} = \lim_{x \to -3} \frac{(x+3)(x-2)}{(x+3)(x-5)}$$ Factor the numerator and the denominator.

$$= \lim_{x \to -3} \frac{x-2}{x-5}$$ Cancel common factor.

$$= \frac{-3-2}{-3-5} = \frac{5}{8}$$ Direct substitution; simplify.

■

Example 7 Indeterminate Form of a Limit

Find $\lim_{x \to 1} \dfrac{x^3 - 1}{x^3 - 7x + 6}$.

Solution

$\lim_{x \to 1}(x^3 - 1) = 0$ and $\lim_{x \to 1}(x^3 - 7x + 6) = 0$

Therefore, the limit $\lim_{x \to 1} \dfrac{x^3 - 1}{x^3 - 7x + 6}$ is in the indeterminate form $\dfrac{0}{0}$.

$$\lim_{x \to 1} \frac{x^3 - 1}{x^3 - 7x + 6} = \lim_{x \to 1} \frac{(x-1)(x^2 + x + 1)}{(x-1)(x-2)(x+3)}$$ Factor the numerator and the denominator.

$$= \lim_{x \to 1} \frac{x^2 + x + 1}{(x-2)(x+3)}$$ Cancel common factor.

$$= \frac{1^2 + 1 + 1}{(1-2)(1+3)} = -\frac{3}{4}$$ Direct substitution; simplify.

■

Example 8 involves a type of limit that we will see frequently in Chapter 3.

Example 8 Limit of a Difference Quotient

Find $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$.

Solution

The limit $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$ is in the indeterminate form $\dfrac{0}{0}$.

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} \frac{(9 + 6h + h^2) - 9}{h}$$ Expand the expression in the numerator.

$$= \lim_{h \to 0} \frac{6h + h^2}{h}$$ Simplify.

$$= \lim_{h \to 0} \frac{h(6+h)}{h}$$ Factor the numerator.

$$= \lim_{h \to 0} (6 + h) = 6 + 0 = 6$$ Cancel common factor; direct substitution.

■

Example 9 Calculate a Limit by Rationalizing

Find $\lim_{x \to 0} \dfrac{\sqrt{x^2 + 9} - 3}{x^2}$.

Solution

The limit $\lim_{x \to 0} \dfrac{\sqrt{x^2 + 9} - 3}{x^2}$ is in the indeterminate form $\dfrac{0}{0}$.

The strategy here is algebraic manipulation, but this time, rationalize the numerator.

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2} = \lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2} \cdot \frac{\sqrt{x^2 + 9} + 3}{\sqrt{x^2 + 9} + 3}$$ Multiply by 1 in a convenient form.

$$= \lim_{x \to 0} \frac{(x^2 + 9) - 9}{x^2(\sqrt{x^2 + 9} + 3)}$$ Difference of two squares in numerator.

$$= \lim_{x \to 0} \frac{x^2}{x^2(\sqrt{x^2 + 9} + 3)} = \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 9} + 3}$$ Simplify and cancel common factor.

$$= \frac{1}{\sqrt{0^2 + 9} + 3} = \frac{1}{3 + 3} = \frac{1}{6}$$ Direct substitution; simplify.

Note: This analytical argument confirms our numerical estimate in Example 2 in Section 2.2. ■

Remember that the Limit Laws also apply to one-sided limits.

Often we need to find the left- and right-hand limits in order to evaluate an overall limit. Theorem 1 says that a two-sided limit (or overall limit) exists if and only if both of the one-sided limits exist and are equal.

Theorem • Overall Limit and One-Sided Limits

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Example 10 Consider One-Sided Limits

Let $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ -x + 2 & \text{if } x \geq 1 \end{cases}$

A graph of f is shown in Figure 2.26.

Find $\lim_{x \to 1} f(x)$ if it exists.

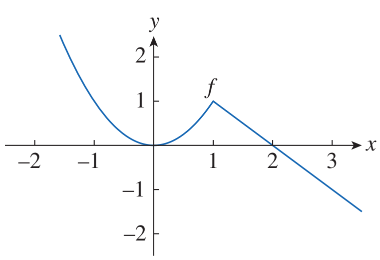


Figure 2.26
Graph of f.

Solution

The function f is defined differently on the left and right sides of $x = 1$.

Therefore, we must find the overall limit by considering one-sided limits and then using the theorem above.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1$$ Use the definition of f for $x < 1$.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-x + 2) = -1 + 2 = 1$$ Use the definition of f for $x > 1$.

Since both one-sided limits exist and are equal, by the theorem above, $\lim_{x \to 1} f(x) = 1$. ■

Example 11 A Limit That Does Not Exist

Let $f(x) = \dfrac{|x|}{x}$. Show that $\lim\limits_{x \to 0} f(x)$ does not exist.

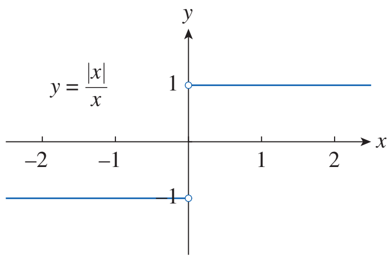


Figure 2.27
Graph of $y = \dfrac{|x|}{x}$.

Solution

$x = 0$ is not in the domain of f.

Use the definition of the absolute value function to rewrite f as a piecewise defined function. Figure 2.27 shows a graph of f.

$$f(x) = \begin{cases} \dfrac{|x|}{x} = \dfrac{-x}{x} = -1 & \text{if } x < 0 \\[2ex] \dfrac{|x|}{x} = \dfrac{x}{x} = 1 & \text{if } x > 0 \end{cases}$$

Consider the one-sided limits.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1 \qquad \text{Use the definition of } f \text{ for } x < 0.$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1 \qquad \text{Use the definition of } f \text{ for } x > 0.$$

Since $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$, it follows that $\lim\limits_{x \to 0} f(x)$ does not exist (DNE). ■

Example 12 Piecewise Defined Function and One-Sided Limits

Let $f(x) = \begin{cases} 8 - 2x & \text{if } x < 4 \\ \sqrt{x - 4} & \text{if } x > 4 \end{cases}$

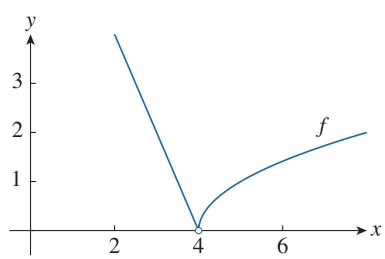


Figure 2.28
Graph of the piecewise defined function f.

A graph of f is shown in Figure 2.28.

Find $\lim\limits_{x \to 4} f(x)$, if it exists.

Note that $x = 4$ is not in the domain of f and this may (or may not) impact the limit of f at $x = 4$.

Solution

Consider the one-sided limits.

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8 - 2x) = 8 - 2(4) = 0 \qquad \text{Use the definition of } f \text{ for } x < 4.$$

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x - 4} = \sqrt{4 - 4} = 0 \qquad \text{Use the definition of } f \text{ for } x > 4.$$

The left- and right-hand limits exist and are equal.

Therefore, the limit exists and $\lim\limits_{x \to 4} f(x) = 0$. ■

Example 13 Limits and the Greatest Integer Function

The **greatest integer function** is defined by $[\![x]\!]$ = the largest integer less than or equal to x.

For example, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, and $[\![-\tfrac{1}{2}]\!] = -1$.

Show that $\lim\limits_{x \to 3} [\![x]\!]$ does not exist.

Solution

A graph of the greatest integer function is shown in Figure 2.29.

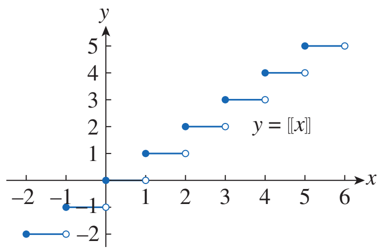


Figure 2.29
Graph of the greatest integer function.

Consider the one-sided limits.

Use the graph and definition of the greatest integer function.

$$\lim_{x\to 3^-} [\![x]\!] = \lim_{x\to 3^-} 2 = 2$$

$$\lim_{x\to 3^+} [\![x]\!] = \lim_{x\to 3^+} 3 = 3$$

Since the one-sided limits are not equal, $\lim_{x\to 3} [\![x]\!]$ does not exist.

The following two theorems provide useful additional properties of limits. Theorem 2 says that if f is always less than or equal to g near a, then the limit of f must be less than or equal to the limit of g.

Theorem • Limit Inequality

If $f(x) \leq g(x)$ for all x near a, except possibly at a, and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

The Squeeze Theorem is useful in a variety of contexts.

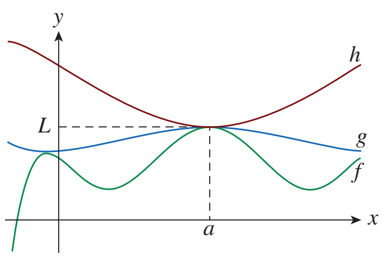


Figure 2.30
Illustration of the Squeeze Theorem.

The Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ for x near a, except possibly at a, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

The Squeeze Theorem, or the Sandwich Theorem, or the Pinching Theorem, is illustrated in Figure 2.30. It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is *squeezed* to the same limit L at a.

Example 14 Squeeze Theorem Application

Show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Solution

We cannot use direct substitution: $\sin\left(\frac{1}{x}\right)$ is not defined at $x = 0$.

We cannot use the Product Law because $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Let's try the Squeeze Theorem. We need to find a function f less than $g(x) = x^2 \sin\left(\frac{1}{x}\right)$ and a function h greater than g as x approaches 0.

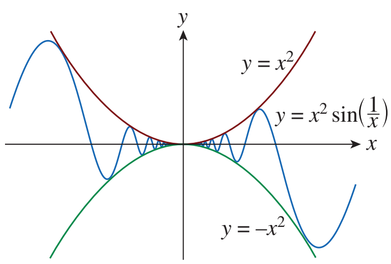


Figure 2.31

The function $g(x) = x^2 \sin\left(\dfrac{1}{x}\right)$ is squeezed between $-x^2$ and x^2 near 0.

Use the properties of the sine function.

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \qquad \text{Sine of any number lies between } -1 \text{ and } 1.$$

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2 \qquad \text{Multiply all parts of the inequality by a positive number. The direction of the inequalities remain the same.}$$

Since $\lim\limits_{x\to 0}(-x^2) = 0$ and $\lim\limits_{x\to 0} x^2 = 0$, by the Squeeze Theorem, $\lim\limits_{x\to 0} x^2 \sin\left(\dfrac{1}{x}\right) = 0$.

This limit and the Squeeze Theorem are illustrated in Figure 2.31. ■

2.3 Exercises

1. Given that

$$\lim_{x\to 2} f(x) = 4 \qquad \lim_{x\to 2} g(x) = -2 \qquad \lim_{x\to 2} h(x) = 0$$

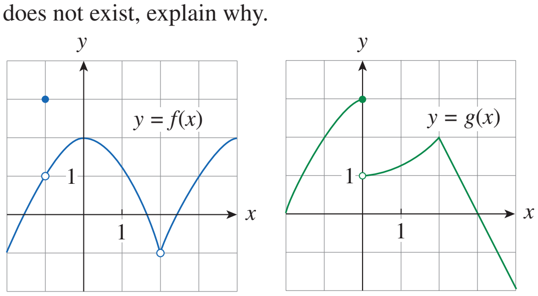
find each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim\limits_{x\to 2} [f(x) + 5g(x)]$ (b) $\lim\limits_{x\to 2} [g(x)]^3$

(c) $\lim\limits_{x\to 2} \sqrt{f(x)}$ (d) $\lim\limits_{x\to 2} \dfrac{3f(x)}{g(x)}$

(e) $\lim\limits_{x\to 2} \dfrac{g(x)}{h(x)}$ (f) $\lim\limits_{x\to 2} \dfrac{g(x)h(x)}{f(x)}$

2. Use the graphs of f and g to evaluate each limit. If the limit does not exist, explain why.



(a) $\lim\limits_{x\to 2} [f(x) + g(x)]$ (b) $\lim\limits_{x\to 0} [f(x) - g(x)]$

(c) $\lim\limits_{x\to -1} [f(x)g(x)]$ (d) $\lim\limits_{x\to 3} \dfrac{f(x)}{g(x)}$

(e) $\lim\limits_{x\to 2} [x^2 f(x)]$ (f) $f(-1) + \lim\limits_{x\to -1} g(x)$

Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim\limits_{x\to 3} (5x^3 - 3x^2 + x - 6)$

4. $\lim\limits_{x\to -1} (x^4 - 3x)(x^2 + 5x + 3)$

5. $\lim\limits_{t\to -2} \dfrac{t^4 - 2}{2t^2 - 3t + 2}$

6. $\lim\limits_{u\to -2} \sqrt{u^4 + 3u + 6}$

7. $\lim\limits_{x\to 8} (1 + \sqrt[3]{x})(2 - 6x^2 + x^3)$

8. $\lim\limits_{t\to 2} \left(\dfrac{t^2 - 2}{t^3 - 3t + 5}\right)^2$

9. $\lim\limits_{x\to 2} \sqrt{\dfrac{2x^2 + 1}{3x - 2}}$

10. (a) Explain the error in the following equation.

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) Explain why the following equation is correct.

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2}(x + 3)$$

Find the limit, if it exists.

11. $\lim\limits_{x\to 0} (\sqrt{x} + 2)(\sqrt{x} - 2)$

12. $\lim\limits_{x\to 2} \sqrt[3]{x + 6}$

13. $\lim\limits_{x\to -1^-} \dfrac{x + 1}{x^2 - 1}$

14. $\lim\limits_{x\to (\pi/2)^+} x \sin x$

15. $\lim\limits_{x\to \pi/2} x^2 \cos x$

16. $\lim\limits_{x\to 5} \dfrac{x^2 - 6x + 5}{x - 5}$

17. $\lim\limits_{x\to -3} \dfrac{x^2 + 3x}{x^2 - x - 12}$

18. $\lim\limits_{t\to -3} \dfrac{t^2 - 9}{2t^2 + 7t + 3}$

19. $\lim\limits_{x\to -1} \dfrac{2x^2 + 3x + 1}{x^2 - 2x - 3}$

20. $\lim\limits_{h\to 0} \dfrac{(-5 + h)^2 - 25}{h}$

21. $\lim\limits_{h\to 0} \dfrac{(2 + h)^3 - 8}{h}$

22. $\lim\limits_{x\to -2} \dfrac{x + 2}{x^3 + 8}$

23. $\lim_{t \to 1} \frac{t^4 - 1}{t^3 - 1}$

24. $\lim_{h \to 0} \frac{\sqrt{9 + h} - 3}{h}$

25. $\lim_{u \to 2} \frac{\sqrt{4u + 1} - 3}{u - 2}$

26. $\lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$

27. $\lim_{h \to 0} \frac{(3 + h)^{-1} - 3^{-1}}{h}$

28. $\lim_{t \to 0} \frac{\sqrt{1 + t} - \sqrt{1 - t}}{t}$

29. $\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$

30. $\lim_{x \to 16} \frac{4 - \sqrt{x}}{16x - x^2}$

31. $\lim_{x \to 2} \frac{x^2 - 4x + 4}{x^4 - 3x^2 - 4}$

32. $\lim_{t \to 0} \left(\frac{1}{t\sqrt{1 + t}} - \frac{1}{t} \right)$

33. $\lim_{x \to -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4}$

34. $\lim_{h \to 0} \frac{4(x + h) - 4x}{h}$

35. $\lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}$

36. $\lim_{h \to 0} \frac{\frac{1}{(x + h)^2} - \frac{1}{x^2}}{h}$

37. Let $f(x) = \frac{x}{\sqrt{1 + 3x} - 1}$.

(a) Graph the function $y = f(x)$ and use the graph to estimate $\lim_{x \to 0} f(x)$.

(b) Use a table of values to estimate the value of $\lim_{x \to 0} f(x)$.

(c) Use the Limit Laws to find the exact value of this limit.

38. Let $g(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$.

(a) Use a graph of $y = g(x)$ to estimate the value of $\lim_{x \to 0} g(x)$.

(b) Use a table of values to estimate the value of $\lim_{x \to 0} g(x)$.

(c) Use the Limit Laws to find the exact value of this limit.

39. Let $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$. Use the Squeeze Theorem to show that $\lim_{x \to 0} g(x) = 0$. Illustrate this result by graphing f, g, and h on the same coordinate axes.

40. Let $g(x) = \sqrt{x^3 + x^2} \sin\left(\frac{\pi}{x}\right)$. Find appropriate functions f and h, and use the Squeeze Theorem to show that $\lim_{x \to 0} g(x) = 0$. Illustrate this result by graphing f, g, and h on the same coordinate axes.

41. If $4x - 9 \le f(x) \le x^2 - 4x + 7$ for $x \ge 0$, find $\lim_{x \to 4} f(x)$.

42. If $2x \le g(x) \le x^4 - x^2 + 2$ for all x, find $\lim_{x \to 1} g(x)$.

43. Prove that $\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$.

44. Prove that $\lim_{x \to 0^+} \sqrt{x}\, e^{\sin(\pi/x)} = 0$.

Find the limit, if it exists. If the limit does not exist, explain why.

45. $\lim_{x \to 3} (2x + |x - 3|)$

46. $\lim_{x \to -6} \frac{2x + 12}{|x + 6|}$

47. $\lim_{x \to (1/2)^-} \frac{2x - 1}{|2x^3 - x^2|}$

48. $\lim_{x \to -2} \frac{2 - |x|}{2 + x}$

49. $\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

50. $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

51. The *signum*, or *sign*, *function*, denoted sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.

(b) Find each limit, or explain why it does not exist.

(i) $\lim_{x \to 0^+} \operatorname{sgn} x$ (ii) $\lim_{x \to 0^-} \operatorname{sgn} x$

(iii) $\lim_{x \to 0} \operatorname{sgn} x$ (iv) $\lim_{x \to 0} |\operatorname{sgn} x|$

52. Let $g(x) = \operatorname{sgn}(\sin x)$.

(a) Find each limit, or explain why it does not exist.

(i) $\lim_{x \to 0^+} g(x)$ (ii) $\lim_{x \to 0^-} g(x)$ (iii) $\lim_{x \to 0} g(x)$

(iv) $\lim_{x \to \pi^+} g(x)$ (v) $\lim_{x \to \pi^-} g(x)$ (vi) $\lim_{x \to \pi} g(x)$

(b) Find all values a such that $\lim_{x \to a} g(x)$ does not exist.

(c) Sketch the graph of g.

53. Let $g(x) = \frac{x^2 + x - 6}{|x - 2|}$.

(a) Find $\lim_{x \to 2^+} g(x)$ and $\lim_{x \to 2^-} g(x)$.

(b) Does $\lim_{x \to 2} g(x)$ exist? Justify your answer.

(c) Sketch the graph of g.

54. Let $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x - 2)^2 & \text{if } x \ge 1 \end{cases}$

(a) Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

(b) Does $\lim_{x \to 1} f(x)$ exist? Justify your answer.

(c) Sketch the graph of f.

55. Let $g(x) = \begin{cases} 4 - \frac{x}{2} & \text{if } x < 2 \\ \sqrt{x + c} & \text{if } x \ge 2 \end{cases}$

Find the value of c such that $\lim_{x \to 2} g(x)$ exists.

56. Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \le 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Find each of the following, if it exists.

(i) $\lim_{x \to 1^-} g(x)$ (ii) $\lim_{x \to 1} g(x)$ (iii) $g(1)$

(iv) $\lim_{x \to 2^-} g(x)$ (v) $\lim_{x \to 2^+} g(x)$ (vi) $\lim_{x \to 2} g(x)$

(b) Sketch the graph of g.

57. Consider the greatest integer function as defined in this section.

(a) Find each limit.

(i) $\lim_{x \to -2^+} [\![x]\!]$ (ii) $\lim_{x \to -2} [\![x]\!]$ (iii) $\lim_{x \to -2.4} [\![x]\!]$

(b) If n is an integer, evaluate each limit.

(i) $\lim_{x \to n^-} [\![x]\!]$ (ii) $\lim_{x \to n^+} [\![x]\!]$

58. Let $f(x) = [\![\cos x]\!]$ for $-\pi \le x \le \pi$.

(a) Sketch the graph of f.

(b) Find each limit, if it exists.

(i) $\lim_{x \to 0} f(x)$ (ii) $\lim_{x \to (\pi/2)^-} f(x)$

(iii) $\lim_{x \to (\pi/2)^+} f(x)$ (iv) $\lim_{x \to \pi/2} f(x)$

(c) Find all values a such that $\lim_{x \to a} f(x)$ exists.

59. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

60. Let

$$f(x) = \begin{cases} ax^2 & \text{if } x < -2 \\ 2 + bx & \text{if } -2 \le x \le 2 \\ 2bx + 3a + 2 & \text{if } x > 2 \end{cases}$$

Find the values of a and b such that both $\lim_{x \to -2} f(x)$ and $\lim_{x \to 2} f(x)$ exist.

61. Evaluate the following limit in terms of a and b, where $a, b \neq 0$ and $b > 0$.

$$\lim_{x \to 0} \frac{\sqrt{b}}{\sqrt{a^2x^2 + b^2} - ax}$$

---

62. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

is an expression for the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light.

(a) Find $\lim_{v \to c^-} L(v)$. Interpret the result in the context of this problem.

(b) Explain why a left-hand limit is necessary in the context of this problem.

63. If p is a polynomial, show that $\lim_{x \to a} p(x) = p(a)$.

64. If r is a rational function, show that $\lim_{x \to a} r(x) = r(a)$ for every value a in the domain of r.

65. If $\lim_{x \to 1} \dfrac{f(x) - 8}{x - 1} = 10$, find $\lim_{x \to 1} f(x)$.

66. If $\lim_{x \to 0} \dfrac{f(x)}{x^2} = 5$, find each of the following limits.

(a) $\lim_{x \to 0} f(x)$ (b) $\lim_{x \to 0} \dfrac{f(x)}{x}$

67. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that $\lim_{x \to 0} f(x) = 0$.

68. Find functions f and g such that neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exist but $\lim_{x \to a} [f(x) + g(x)]$ exists.

69. Find functions f and g such that neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exist but $\lim_{x \to a} [f(x) \cdot g(x)]$ exists.

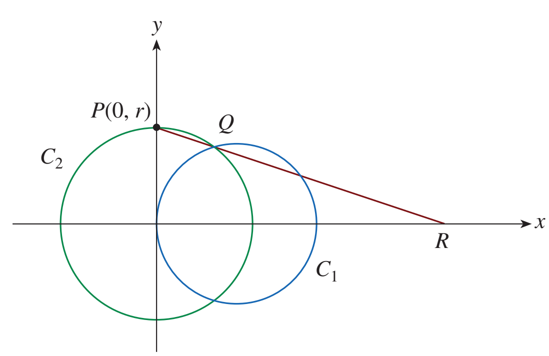
70. Evaluate $\lim_{x \to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

71. Is there a value a such that

$$\lim_{x \to 2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value a and evaluate the limit.

72. The figure shows the graphs of a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 centered at the origin with radius r. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as the circle C_2 shrinks, that is, as $r \to 0^+$?



2.4 Continuity

The continuity of a function is an important concept and is defined in terms of limits. Intuitively, a function f is continuous if you can draw the graph of f without lifting your pencil. There is no break in the graph. The mathematical definition of continuity corresponds closely with this idea and with the meaning of the word continuity in everyday language. A continuous process is one that takes place gradually, without interruption or abrupt changes. We need to translate this idea into precise mathematics.

Definition • Continuity of a Function at a Value

A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

A Closer Look

1. This definition implicitly requires three things if f is continuous at a:
   (1) $f(a)$ is defined (that is, a is in the domain of f).
   (2) $\lim_{x \to a} f(x)$ exists.
   (3) $\lim_{x \to a} f(x) = f(a)$.
2. Here is a translation of this definition. The function f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. This means that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small. Figure 2.32 illustrates this concept.
3. To determine whether a function is continuous at a, check that it meets each condition stated in A Closer Look 1. If any one of these conditions is not satisfied, then the function is **discontinuous at *a*** (or, f has a **discontinuity** at a). ■

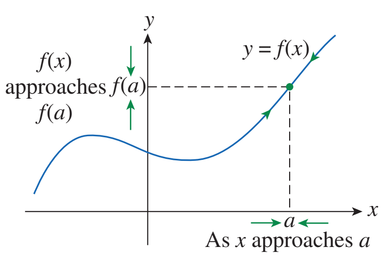


Figure 2.32
If f is continuous at a, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph of f. There is no *gap*, or *hole*, in the graph.

Physical phenomena are usually continuous. For example, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. However, discontinuities do occur in certain situations, for example, in electrical currents. Remember the Heaviside function? This function is discontinuous at 0 because $\lim_{t \to 0} H(t)$ does not exist.

Example 1 Check Continuity Using a Graph

The graph of a function f is shown in Figure 2.33. Use the graph to determine whether the function is continuous at $x = 1$, at $x = 3$, and at $x = 5$.

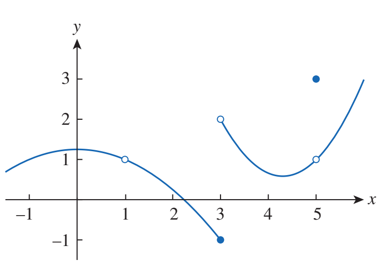


Figure 2.33
Graph of f.

Solution

Check the three conditions for continuity at each value of x.

(a) $x = 1$

(1) $f(1)$ is not defined; there is a hole in the graph of f at $x = 1$.

Therefore, f is discontinuous at $x = 1$.

(b) $x = 3$

(1) $f(3)$ is defined. From the graph, $f(3) = -1$.

(2) $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$. Therefore, $\lim_{x \to 3} f(x)$ does not exist.

The function f is discontinuous at $x = 3$.

(c) $x = 5$

(1) $f(5)$ is defined. From the graph, $f(5) = 3$.

(2) $\lim_{x \to 5} f(x)$ exists. The left and right limits are the same, 1.

(3) $\lim_{x \to 5} f(x) = 1 \neq 3 = f(5)$. Therefore, f is discontinuous at $x = 5$. ■

Now let's see how to detect discontinuities when a function is defined by a formula.

Example 2 Discontinuities from a Formula

Determine whether each function is continuous at the given value.

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ at $x = 2$

(b) $g(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ at $x = 0$ at $x = 2$

(c) $h(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $j(x) = [\![x]\!]$ at $x = 3$

Solution

(a) $x = 2$ is not in the domain of f, so $f(2)$ is not defined.

Therefore, f is discontinuous at $x = 2$.

(b) $g(0) = 1$. The function g is defined at $x = 0$.

$\lim_{x \to 0} g(x) = \lim_{x \to 0} \dfrac{1}{x^2}$ does not exist because the denominator approaches 0 while the numerator approaches a nonzero number (see Example 8 of Section 2.2).
Therefore, g is discontinuous at $x = 0$.

(c) $h(2) = 1$. The function h is defined at $x = 2$.

$$\lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2} (x + 1) = 3$$

$$\lim_{x \to 2} h(x) = 3 \neq 1 = h(2)$$

Therefore, h is discontinuous at $x = 2$.

(d) $j(3) = [\![3]\!] = 3$. The function j is defined at $x = 3$.

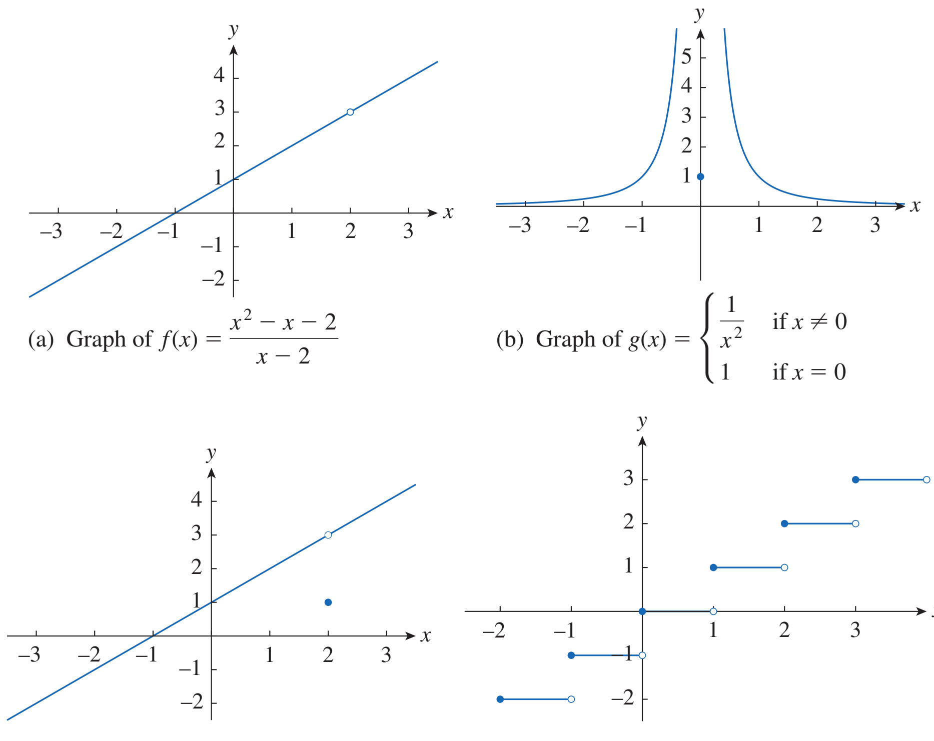
$\lim_{x \to 3^-} [\![x]\!] = 2$ and $\lim_{x \to 3^+} [\![x]\!] = 3$; $\lim_{x \to 3} [\![x]\!]$ does not exist.

Therefore, j is discontinuous at $x = 3$.

Recall that $\lim_{x \to n} [\![x]\!]$ does not exist for any integer n. ■

The graphs of the four functions in Example 2 are shown in Figure 2.34. In each case, we cannot draw the complete graph without lifting the pencil because there is a hole, a break, or a jump in the graph. These different kinds of discontinuities have special names.

The discontinuities illustrated in Figures 2.34(a) and 2.34(c) are called **removable**. We can *remove* the discontinuity because we can *fill in the hole* by redefining f at $x = 2$. The discontinuity illustrated in Figure 2.34(b) is called an **infinite discontinuity**. It is **nonremovable** since there is no value for $f(0)$ that would then make the function



(a) Graph of $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) Graph of $g(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) Graph of
$h(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) Graph of $j(x) = [\![x]\!]$

Figure 2.34
Graphs of the functions in Example 2.

continuous at $x = 0$. The discontinuities in Figure 2.34(d) are called **jump discontinuities**: the graph of the function *jumps* from one value to another, without taking on any value in between.

The graphs in Figure 2.34 and the different types of discontinuities suggest additional definitions involving continuity from one side.

Definition • Continuity From the Right or Left

A function f is **continuous from the right at *a*** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at *a*** if

$$\lim_{x \to a^-} f(x) = f(a)$$

Example 3 The Greatest Integer Function and Continuity

Let $f(x) = [\![x]\!]$, the greatest integer function.

For every integer n:

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [\![x]\!] = n = f(n)$$

and

$$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [\![x]\!] = n - 1 \neq f(n)$$

Therefore, f is continuous from the right but discontinuous from the left.

Definition 3 combines several ideas to extend the concept of continuity to an interval.

Definition • Continuity of a Function on an Interval

A function f is **continuous on an interval *I*** if it is continuous at every number in the interval I.

A Closer Look

There are four cases for the interval I: (a, b), $[a, b)$, $(a, b]$, and $[a, b]$.

A function f is continuous on the interval $(a, b]$, for example, if the function is continuous at every number in the interval (a, b) and continuous from the left at b.

Figure 2.35 shows the graphs of four functions that are continuous on the four given intervals.

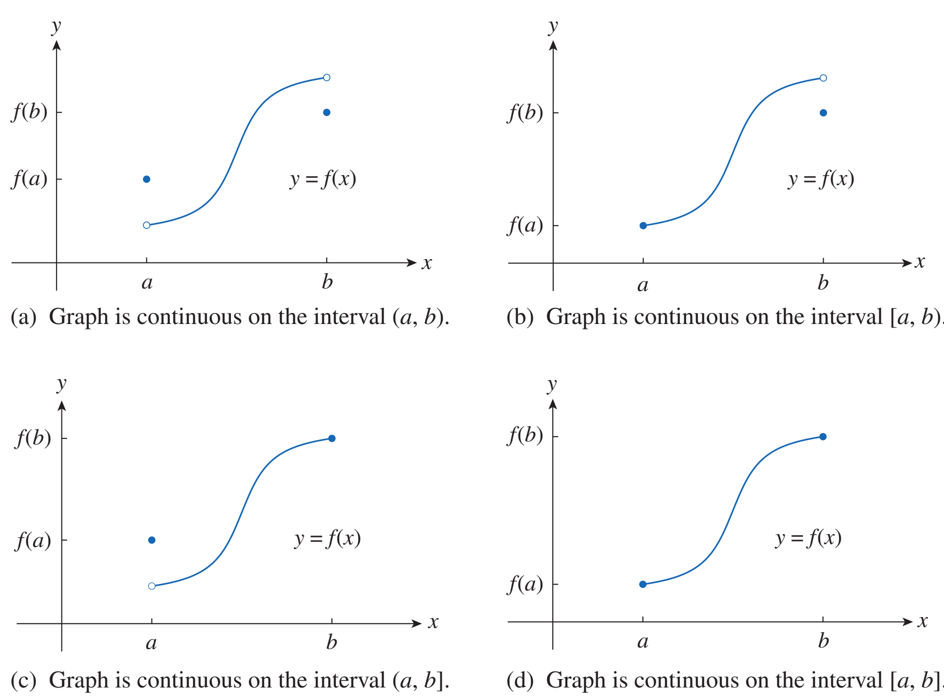


Figure 2.35
Graphs of functions that are continuous on the four cases of intervals.

Example 4 Continuous on an Interval

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

Solution

If $-1 < a < 1$: $f(a) = 1 - \sqrt{1 - a^2}$ is defined (exists).

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} (1 - \sqrt{1 - x^2}) \\
&= 1 - \lim_{x \to a} \sqrt{(1 - x^2)} && \text{Limit Laws 2 and 7.} \\
&= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} && \text{Limit Law 11.} \\
&= 1 - \sqrt{1 - a^2} \quad \Rightarrow \quad \text{The limit exists.} && \text{Limit Laws 2, 7, and 9.}
\end{aligned}$$

So, $\lim_{x \to a} f(x) = f(a)$

Therefore, f is continuous at a if $-1 < a < 1$.

Similar calculations show that

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \quad \text{and} \quad \lim_{x \to 1^-} f(x) = 1 = f(1).$$

So f is continuous from the right at -1 and from the left at 1.

Therefore, f is continuous on the interval $[-1, 1]$.

The graph of f is shown in Figure 2.36. Note that this is the graph of the lower half of a circle defined by $x^2 + (y - 1)^2 = 1$. ■

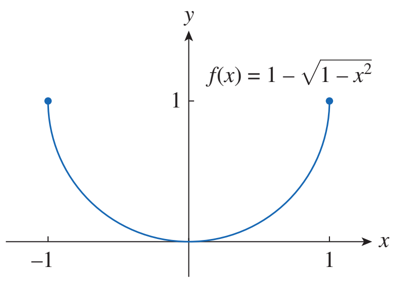


Figure 2.36
The function f is continuous on the interval $[-1, 1]$.

Theorem 4 is used to determine the continuity of combinations of functions.

Throughout the text, we will present several theorems in which properties of simple functions are *passed on* to more complex functions. Theorem 4 is the first of these theorems.

Theorem • Continuity of Combinations of Functions

If f and g are continuous at a, and c is a constant, then the following functions are also continuous at a.

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

Proof

Each part of this theorem follows from an associated Limit Law.

For example, in part 1, since f and g are continuous at a:

$$\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a} g(x) = g(a)$$

Therefore,

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) && \text{Limit Law 1.} \\ &= f(a)+g(a) && f \text{ and } g \text{ are continuous at } a. \\ &= (f+g)(a)\end{aligned}$$

This shows that the sum function, $f+g$, is continuous at a. ■

Continuity of Combinations of Functions

Using Theorem 4 and Definition 3, we can extend this result to show that combinations of functions are continuous on an interval: if f and g are continuous on an interval I, then so are the functions $f+g$, $f-g$, cf, fg, and f/g (where g is nonzero). Theorem 5 is a more detailed statement of the Direct Substitution Property, which was given in Section 2.3.

Theorem • Continuity of Polynomial and Rational Functions

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Proof

(a) Let p be a polynomial function of the form

$p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$, where $c_0, c_1, \ldots, c_n$ are constants.

Using the properties of limits.

$$\lim_{x\to a} c_0 = c_0 \qquad \text{Limit Law 7.}$$

$$\lim_{x\to a} x^m = a^m \qquad \text{Limit Law 9.}$$

Therefore, the function $f(x) = x^m$ is a continuous function, and the function $g(x) = cx^m$ is continuous.

Since p is the sum of functions of this form and a constant function, p is also continuous.

(b) A rational function is of the form $f(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomials.

The domain of f is $D = \{x \in \mathbb{R} \mid q(x) \neq 0\}$.

From part (a), p and q are continuous everywhere. Using the theorem above.

the quotient, $\dfrac{p}{q}$, is continuous at every number in D. ■

We can illustrate the theorem above with practical applications: the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ is a polynomial function of r. Similarly, if a ball is thrown vertically into the air with a velocity of 50 ft/s, then the height of the ball, in feet, t seconds later is given by the formula $h(t) = 50t - 16t^2$. This is also a polynomial function, so the height is a continuous function of the elapsed time.

Knowing which functions are continuous allows us to evaluate some limits very quickly. Consider Example 5.

Example 5 Limit Involving a Continuous Function

Let $f(x) = \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$. Find $\lim\limits_{x \to -2} f(x)$.

Solution

Since f is the quotient of two polynomials, it is a rational function.

Therefore, it is continuous on its domain, $D = \{x \mid x \neq \frac{5}{3}\}$.

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \lim_{x \to -2} f(x) = f(-2) \qquad f \text{ continuous on its domain.}$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11} \qquad \text{Evaluate limit by direct substitution.}$$

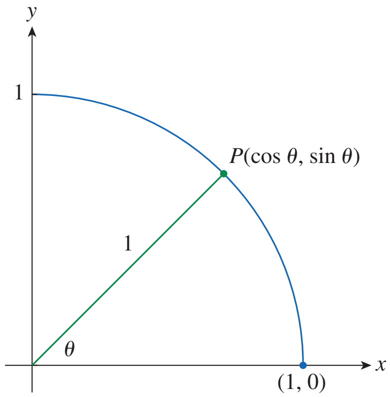


Figure 2.37
Illustration of the definition of the sine and cosine functions.

Another way to establish the limits in Equation 6 is to use the Squeeze Theorem with the inequality $\sin\theta < \theta$ for $\theta > 0$.

It seems reasonable, and the previous two theorems can be used to show, that most familiar algebraic functions are continuous at every number in their domains. For example, Limit Law 11 shows that every root function is continuous (on its domain).

We also need to consider continuity of trigonometric functions. The graphs certainly suggest the sine and cosine functions are continuous everywhere. Using the definitions of the sine and cosine, the coordinates of the point P in Figure 2.37 are $(\cos\theta, \sin\theta)$. As $\theta \to 0$, the point P approaches $(1, 0)$. Therefore, $\cos\theta \to 1$ and $\sin\theta \to 0$. Written more formally,

$$\lim_{\theta \to 0} \cos\theta = 1 \qquad \lim_{\theta \to 0} \sin\theta = 0 \tag{1}$$

Since $\cos 0 = 1$ and $\sin 0 = 0$, the limits in Equation 1 show that the cosine and sine function are continuous at 0. The addition formulas for cosine and sine can be used to show that these functions are indeed continuous everywhere.

The tangent function

$$\tan x = \frac{\sin x}{\cos x}$$

is a quotient function and is continuous except where $\cos x = 0$. This occurs when x is an odd integer multiple of $\dfrac{\pi}{2}$. Therefore, $f(x) = \tan x$ has infinitely many discontinuities at $x = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$. The graph of the tangent function is shown in Figure 2.38.

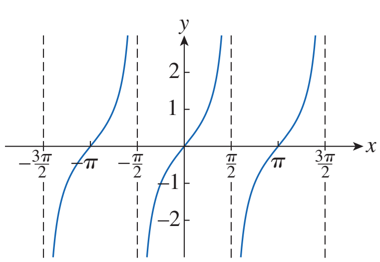


Figure 2.38
Graph of $f(x) = \tan x$.

The inverse function of any continuous, one-to-one function is also continuous. This fact can be formally proved, but the geometric intuition seems very reasonable. Recall that the graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$. If the graph of f has no break, then neither does the graph of f^{-1}.

Recall that the exponential function, $f(x) = b^x$, was defined to fill in the holes of the graph of $y = b^x$, where x is rational. This definition of f makes it a continuous function on $\mathbb{R}$. Therefore, its inverse function, $f(x) = \log_b x$, is also continuous on $(0, \infty)$.

Theorem 7 summarizes this discussion of familiar continuous functions.

Theorem • Continuity of Types of Functions

The following types of functions are continuous at every number in their domains.

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions
- Exponential functions
- Logarithmic functions

Example 6 Continuity on Intervals

Find all values where the function is continuous.

(a) $f(x) = \dfrac{x^3 + 1}{x^2 + 5x + 6}$ (b) $g(x) = \dfrac{\ln x + e^x}{x^2 - 1}$

Solution

(a) f is a rational function, the quotient of two polynomials.

In the denominator, $x^2 + 5x + 6 = (x + 2)(x + 3) = 0 \quad \Rightarrow \quad x = -2, -3$.

The function f is continuous on its domain, $\{x \mid x \neq -2, x \neq -3\}$.

(b) The function $y = \ln x$ is continuous for $x > 0$ and $y = e^x$ is continuous on $\mathbb{R}$.

So the numerator $\ln x + e^x$ is continuous on $(0, \infty)$.

The denominator, $x^2 - 1$, is a polynomial, continuous everywhere.

The function g is continuous at all positive numbers except where

$x^2 - 1 = 0 \quad \Rightarrow \quad x = -1, 1$.

Therefore, g is continuous on the intervals $(0, 1)$ and $(1, \infty)$. ■

Example 7 Limit of a Continuous Function

Find $\displaystyle\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$.

Solution

The numerator, $\sin x$, is continuous everywhere.

The denominator, $2 + \cos x$, is the sum of continuous functions.

Therefore, it is continuous everywhere.

And the denominator is never 0: $\cos x \geq -1 \quad \Rightarrow \quad 2 + \cos x \geq 1$.

The quotient function $f(x) = \dfrac{\sin x}{2 + \cos x}$ is continuous everywhere.

Since this is a continuous function, find the limit by direct substitution.

$$\lim_{x \to \pi} \frac{\sin x}{2 + \cos x} = \frac{\sin \pi}{2 + \cos \pi} = \frac{0}{2 - 1} = 0$$ ■

Another common way to combine two functions f and g to obtain a new function is composition, $f \circ g$. Theorem 8 shows that the composition of continuous functions is also continuous.

Theorem • Limit of a Composition Function

If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other notation,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

A Closer Look

1. This theorem seems reasonable because if x is close to a, then $g(x)$ is close to b. Since f is continuous at b, if $g(x)$ is close to b, then $f(g(x))$ is close to $f(b)$.
2. Here is a way to think about this theorem: if f is continuous and the limit involving g exists, then the limit symbol passes freely through the function.
3. This theorem is often expressed informally by saying "a continuous function of a continuous function is a continuous function." ■

This theorem can be used to determine the continuity of a composite function.

Theorem • Continuity of a Composite Function

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Proof

$\lim_{x\to a} g(x) = g(a)$ — g is continuous at a.

$\lim_{x\to a} f(g(x)) = f(g(a))$ — f is continuous at $b = g(a)$ by Theorem 8.

This second statement shows that the function $h(x) = f(g(x))$ is continuous at a; that is, $f \circ g$ is continuous at a. ■

Example 8 Continuity of a Composite Function

Find all values where the function is continuous.

(a) $F(x) = \sin(x^2)$ (b) $G(x) = \ln(1 + \cos x)$

Solution

(a) If $g(x) = x^2$ and $f(x) = \sin x$, then $f(g(x)) = f(x^2) = \sin(x^2) = F(x)$.

g is continuous on $\mathbb{R}$ because it is a polynomial, and f is also continuous on $\mathbb{R}$.

Therefore, by $F = f \circ g$ is continuous on $\mathbb{R}$.

(b) $f(x) = \ln x$ is continuous on its domain.

$g(x) = 1 + \cos x$ is continuous everywhere. — Sum of continuous functions.

$G(x) = f(g(x)) = \ln(1 + \cos x)$ is continuous wherever it is defined.

$1 + \cos x \geq 0$ and $1 + \cos x = 0 \Rightarrow \cos x = -1 \Rightarrow x = \pm\pi, \pm 3\pi, \ldots$.

G has discontinuities at odd multiples of π and is continuous everywhere else.

Figure 2.39 shows a graph of G.

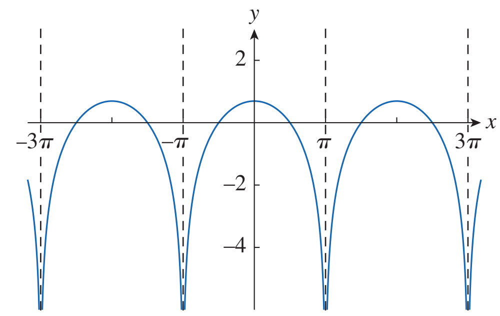


Figure 2.39
Graph of $G(x) = \ln(1 + \cos x)$.

The Intermediate Value Theorem

The next theorem is an important property of continuous functions. The proof is found in more advanced books on calculus.

The Intermediate Value Theorem (IVT)

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

A Closer Look

1. Another way to think about the IVT: f takes on every value between $f(a)$ and $f(b)$.
2. The IVT states there is *at least* one c. There may be more than one c.
3. This is an *existence theorem*. The IVT states there is a c, but it doesn't say where it is or how to find it. We will study other existence theorems throughout the text and will use these theorems as justification in solving problems.
4. Figure 2.40 is a visualization of the IVT and Figure 2.41 shows that the conclusion of the IVT is not necessarily true if the function is discontinuous anywhere in the closed interval.

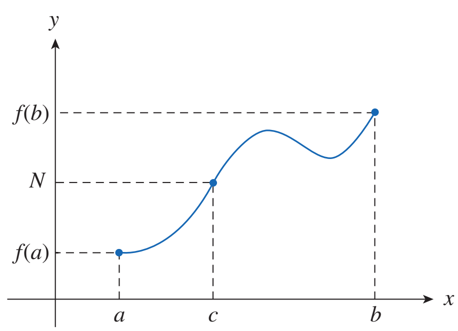


Figure 2.40
f is continuous on the closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$.

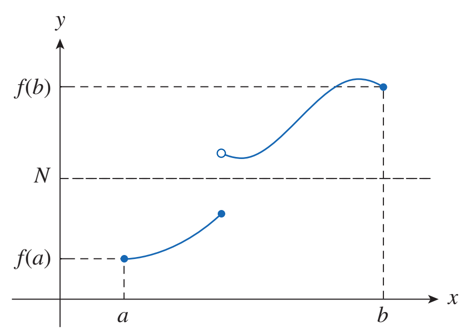


Figure 2.41
The function f is discontinuous in the interval $[a, b]$. There is no number c in (a, b) such that $f(c) = N$.

Example 9 Use the IVT to Show the Existence of a Zero

Use the Intermediate Value Theorem to show that $f(x) = 4x^3 - 6x^2 + 3x - 2$ has a zero between 1 and 2.

Solution

Since f is a polynomial, it is continuous on the closed interval [1, 2].

$f(1) = 4 - 6 + 3 - 2 = -1 < 0$

$f(2) = 32 - 24 + 6 - 2 = 12 > 0$

So the function values at the endpoints have opposite signs.

The IVT states that there is at least one c in (1, 2) such that $f(c) = 0$.

That is, f has at least one zero c in the interval (1, 2).

We can locate the zero more precisely by repeated use of the IVT.

$f(1.2) = -0.128 < 0$ and $f(1.3) = 0.548 > 0$

Therefore, a zero lies between 1.2 and 1.3.

Using the IVT once again:

$f(1.22) = -0.007008 < 0$ and $f(1.23) = 0.056068 > 0$

A zero lies between 1.22 and 1.23.

Figures 2.42 and 2.43 illustrate this repeated use of the Intermediate Value Theorem.

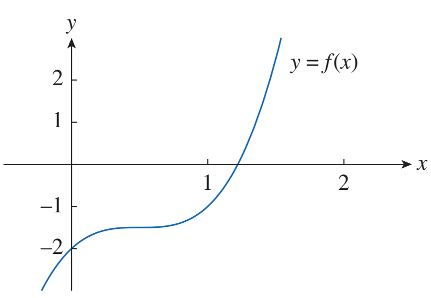


Figure 2.42
Graph of $y = f(x)$ shows a zero between 1 and 2.

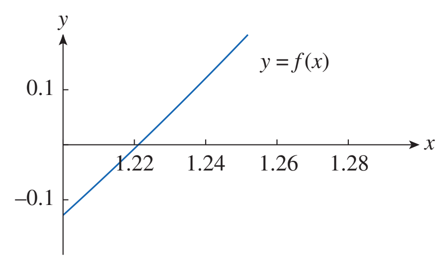


Figure 2.43
Graph of $y = f(x)$ shows a zero between 1.22 and 1.23. ■

Example 9 illustrates how the Intermediate Value Theorem can be used to approximate a zero by repeated application on increasingly smaller intervals. One specific technique is called the *bisection method*. Here is a real-world application: a car accelerating from 0 to 50 mi/h over a time interval I must travel at every speed in between during the interval I.

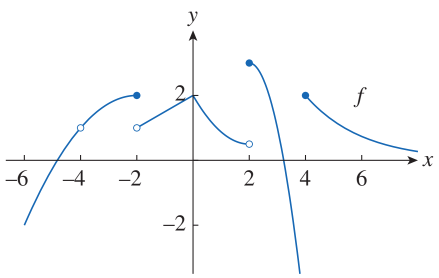
The Intermediate Value Theorem is also used by graphing calculators (in connected mode) and computer software to sketch the graph of a function. A graphing calculator first plots a finite number of points on the graph. It then assumes the function is continuous and therefore takes on all the intermediate values between two consecutive points. The graphing calculator then connects the dots to produce the final graph.

2.4 Exercises

1. Write an expression involving a limit that indicates the function f is continuous at 4.

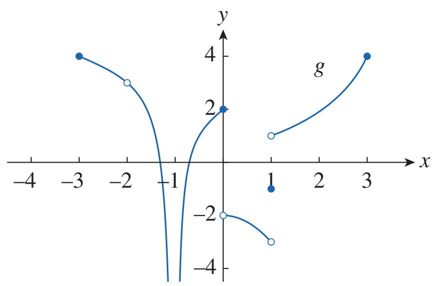
2. If f is continuous on $(-\infty, \infty)$, what can you say about the graph of f?

3. The graph of the function f is shown in the figure.



(a) Use the graph to determine the values at which f is discontinuous and explain why.
(b) For each value in part (a), determine whether f is continuous from the right, or from the left, or neither.

4. The graph of the function g is shown in the figure.



Find the intervals on which g is continuous.

Sketch the graph of a function f that is continuous except at the stated value(s).

5. Discontinuous, but continuous from the right, at 2.

6. Discontinuities at -1 and 4, but continuous from the left at -1 and from the right at 4.

7. Removable discontinuity at 3, jump discontinuity at 5.

8. Neither left nor right continuous at -2, continuous only from the left at 2.

9. Find a function f such that $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) \neq f(3)$, or explain why no such function exists.

10. The toll T charged for driving on a certain stretch of a road is \$5 except during rush hours (between 7 AM and 10 AM and between 4 PM and 7 PM) when the toll is \$7.

(a) Sketch a graph of T as a function of the time t, measured in hours past midnight.
(b) Discuss the discontinuities of this function and their significance to someone who uses the road.

11. Explain why each function is continuous or discontinuous.
(a) The temperature at a specific location as a function of time
(b) The temperature at a specific time as a function of the distance due west from New York City
(c) The altitude above sea level as a function of the distance due west from New York City
(d) The cost of a taxi ride as a function of the distance traveled
(e) The current in the circuit for the lights in a room as a function of time
(f) The speed of a city bus as a function of elapsed time since it left the first station

Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

12. $f(x) = (x + 2x^3)^4, \quad a = -1$

13. $g(t) = \dfrac{t^2 + 5t}{2t + 1}, \quad a = 2$

14. $p(v) = 2\sqrt{3v^2 + 1}, \quad a = 1$

15. $f(x) = 3x^4 - 5x + \sqrt[3]{x^2 + 4}, \quad a = 2$

Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

16. $f(x) = x + \sqrt{x - 4}, \quad [4, \infty)$

17. $g(x) = \dfrac{x - 1}{3x + 6}, \quad (-\infty, -2)$

Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

18. $f(x) = \begin{cases} \dfrac{1}{x + 2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases} \quad a = -2$

19. $f(x) = \begin{cases} x + 3 & \text{if } x \leq -1 \\ 2^x & \text{if } x > -1 \end{cases} \quad a = -1$

20. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \quad a = 1$

21. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases} \quad a = 0$

22. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ $\quad a = 3$

For the function f, how would you *remove the discontinuity*? In other words, how would you define $f(2)$ in order for f to be continuous at 2?

23. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

24. $f(x) = \dfrac{x^3 - 8}{x^2 - 4}$

25. $f(x) = \dfrac{4 - \sqrt{x^2 + x + 10}}{x - 2}$

Using the appropriate theorems, explain why the function is continuous at every number in its domain. Find the domain.

26. $F(x) = x^2 + \sqrt{2x - 1}$

27. $G(x) = \sqrt[3]{x}(1 + x^3)$

28. $H(x) = \dfrac{x^2 + 1}{2x^2 - x - 1}$

29. $Q(x) = \dfrac{e^{\sin x}}{2 + \cos \pi x}$

30. $L(t) = e^{-5t} \cos 2\pi t$

31. $h(x) = \dfrac{\tan x}{\sqrt{4 - x^2}}$

32. $g(x) = \ln(x^4 - 1)$

33. $f(x) = \sin(\cos(\sin x))$

Locate the discontinuities of the function and illustrate these values by sketching a graph of the function.

34. $f(x) = \dfrac{1}{1 + e^{1/x}}$

35. $f(x) = \ln(\tan^2 x)$

Use continuity to evaluate the limit.

36. $\lim\limits_{x \to 2} x\sqrt{20 - x^2}$

37. $\lim\limits_{x \to \pi} \sin(x + \sin x)$

38. $\lim\limits_{x \to 1} \ln\left(\dfrac{5 - x^2}{1 + x}\right)$

39. $\lim\limits_{x \to 1} e^{x^2 - x}$

Show that f is continuous on $(-\infty, \infty)$.

40. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 1 \\ \ln x & \text{if } x > 1 \end{cases}$

41. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

Find the values at which f is discontinuous. For each of these values, determine whether f is continuous from the right, from the left, or neither. Sketch the graph of f.

42. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \leq x < 1 \\ \dfrac{1}{x} & \text{if } x \geq 1 \end{cases}$

43. $f(x) = \begin{cases} 2^x & \text{if } x \leq 1 \\ 3 - x & \text{if } 1 < x \leq 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$

44. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

45. The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \dfrac{GM}{r^2} & \text{if } r \geq R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r? Justify your answer.

46. Let $f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$

Find the value of the constant c such that f is continuous on $(-\infty, \infty)$.

47. Let $f(x) = \begin{cases} \dfrac{x^3 - a^3}{x - a} & \text{if } x \neq a \\ c & \text{if } x = a \end{cases}$

Find the value of the constant c such that f is continuous at $x = a$.

48. Find the values of a and b such that f is continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$$

49. Can there exist a function f and a value a such that f is continuous at $x = a$ even though $\lim\limits_{x \to a^+} f(x) \neq \lim\limits_{x \to a^-} f(x)$? Explain your reasoning.

50. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim\limits_{x \to 2} [3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

51. Which of the following functions has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a.

(a) $f(x) = \dfrac{x^4 - 1}{x - 1}, \quad a = 1$

(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2}, \quad a = 2$

(c) $f(x) = [\![\sin x]\!], \quad a = \pi$

52. Suppose that a function f is continuous on [0, 1] except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypotheses).

53. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

54. Suppose that f is continuous on [1, 5] and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

55. $x^4 + x - 3 = 0, \quad (1, 2)$

56. $\ln x = x - \sqrt{x}, \quad (2, 3)$

57. $e^x = 3 - 2x, \quad (0, 1)$

58. $\sin x = x^2 - x, \quad (1, 2)$

For the given equation

(a) Show that there is at least one real root.
(b) Use technology to find an interval of length 0.01 that contains the root.

59. $\cos x = x^3$

60. $\ln x = 3 - 2x$

(a) Show that there is at least one real root.
(b) Use technology to find the root.

61. $100e^{-x/100} = 0.01x^2$

62. $\sqrt{x - 5} = \dfrac{1}{x + 3}$

63. $\ln x = e^{-2x}$

64. $\ln x + \sin x + 2 = \dfrac{1}{1 + x^2}$

Without using technology, show that the graph of the function has at least two x-intercepts in the specified interval.

65. $f(x) = \sin(x^2), \quad (1, 3)$

66. $f(x) = x^2 - 3 + \dfrac{1}{x}, \quad (0, 2)$

---

67. Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a + h) = f(a)$$

68. To prove that the sine function is continuous, we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. Using Exercise 67, we can instead show that the sine function is continuous if

$$\lim_{h \to 0} \sin(a + h) = \sin a$$

Use the limits in Equation 6 to show that this is true.

69. Prove that the cosine function is continuous everywhere.

70. Is there a number that is exactly 1 more that its cube?

71. Let $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

For what values is f continuous?

72. Let $g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$

For what values is g continuous?

73. If a and b are positive numbers, show that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

74. Show that the function

$$f(x) = \begin{cases} x^4 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, \infty)$.

75. Let $F(x) = |x|$, the absolute value function.
(a) Show that F is continuous everywhere.
(b) Show that if f is a continuous function on an interval I, then so is $|f|$.
(c) Is the converse of the statement in part (b) true? That is, if $|f|$ is continuous does it follow that f is continuous? If so, explain your reasoning. If not, find a counterexample.

76. A hiker starts walking at 7:00 AM and takes their usual path to the top of the mountain, arriving at 7:00 PM. The following morning, they starts at 7:00 AM at the top and take the same path back, arriving at the beginning of the trail at 7:00 PM. Use the Intermediate Value Theorem to show that there is a point on the path that the hiker will cross at exactly the same time of day on both days.

2.5 Limits Involving Infinity

We have so far developed an intuitive definition of $\lim_{x \to a} f(x) = L$ and used various Limit Laws to evaluate limits. Remember, in words, this limit expression means that if a function f is defined near a, except possibly at a, then the value of $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a.

In this section we will investigate the global behavior of functions and, in particular, whether their graphs have vertical or horizontal asymptotes.

Infinite Limits

In Example 8 in Section 2.2, we concluded that

$$\lim_{x \to 0} \frac{1}{x^2} \quad \text{does not exist}$$

by considering a table of values (Table 2.10) and the graph of $y = 1/x^2$ (Figure 2.44). In fact, the values of $1/x^2$ can be made arbitrarily large by taking x close enough to 0. The values of $f(x)$ do not approach a specific number, so $\lim_{x \to 0} \frac{1}{x^2}$ does not exist.

x	$\frac{1}{x^2}$
± 1.0	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

Table 2.10
Table of values for x close to 0.

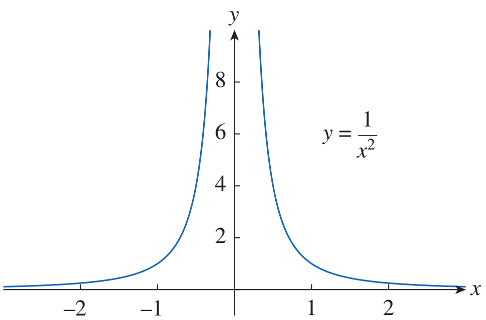


Figure 2.44
The graph of f close to $x = 0$.

A Closer Look

1. The *unbounded* nature of the function in Example 8 in Section 2.2 is written using the special mathematical notation
$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$
2. We use the symbol for infinity, ∞, to describe the behavior of this function near 0. Remember, ∞ is not a number, but a symbol used, in this instance, to describe this special function behavior.
3. Even though we use an equal sign, this notation does not mean that the limit exists. The limit does not exist but can be described using this special notation.

In general, we use the expression

$$\lim_{x \to a} f(x) = \infty$$

to indicate that the values of $f(x)$ tend to become larger and larger, or increase without bound, as x gets closer and closer to a. Since infinity (∞) is not a number, we can also say that the limit does not exist (DNE).

Definition • Intuitive Idea of an Infinite Limit

Let f be a function defined on both sides of a, except possibly at a itself. The expression

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

A Closer Look

1. Here is some other notation for $\lim_{x \to a} f(x) = \infty$: $f(x) \to \infty$ as $x \to a$.
2. The symbol ∞ is not a number.

   However, the expression $\lim_{x \to} f(x) = \infty$ is often read as

   "the limit of $f(x)$, as x approaches a, is infinity" or

   "$f(x)$ becomes infinite as x approaches a" or

   "$f(x)$ increases without bound as x approaches a."
3. Figure 2.45 shows a graph of a function f that illustrates this intuitive definition.

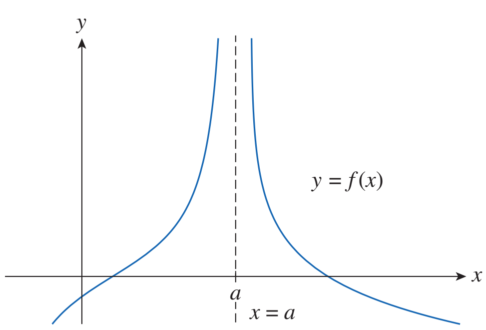


Figure 2.45
The graph of a function f with the property $\lim_{x \to a} f(x) = \infty$.

The next definition involves similar functions that become large *negative* as x gets close to a.

Definition • Intuitive Idea of a Negative Infinite Limit

Let f be a function defined on both sides of a, except possibly at a itself. The expression

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

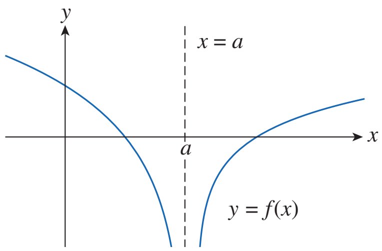


Figure 2.46
The graph of a function f with the property $\lim_{x \to a} f(x) = -\infty$.

The expression $\lim_{x \to a} f(x) = -\infty$ is read as
"the limit of $f(x)$, as x approaches a, is negative infinity" or
"$f(x)$ decreases without bound as x approaches a."
Figure 2.46 shows a graph of a function f that illustrates this definition.

Here is an analytical example of this infinite limit:

$$\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$$

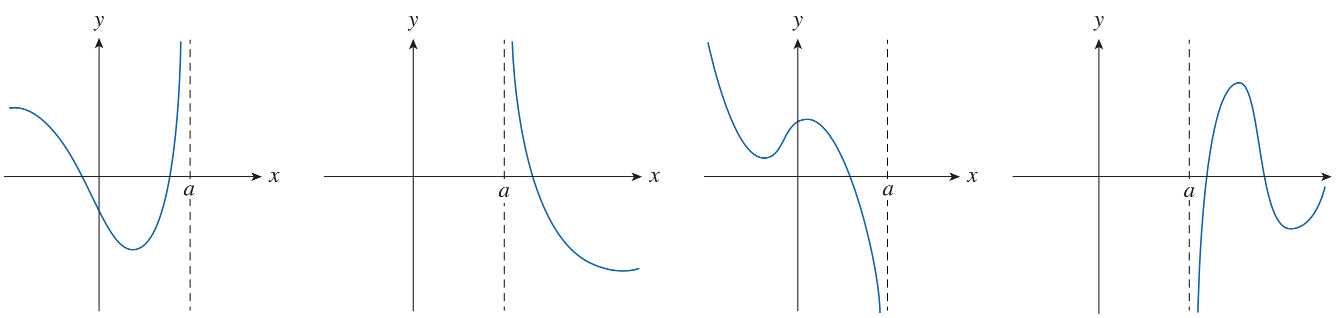
Try to sketch the graph of the function $f(x) = -\frac{1}{x^2}$ near 0 to illustrate this limit.

Remember, this is just a symbolic representation of special function behavior. The limit does not exist, but we all agree to describe the nature of the function in this way.

There are similar definitions involving one-sided limits.

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$
$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Remember that "$x \to a^-$" means that we consider only values of x that are less than a, and similarly "$x \to a^+$" means that we consider only $x > a$. The graphs in Figure 2.47 illustrate these four expressions.



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

Figure 2.47
Illustrations of one-sided limits involving infinity.

The dashed line in each of the graphs in Figure 2.47 is associated with an infinite limit and has a special name, which is given in the following definition.

Definition • Vertical Asymptote

The vertical line $x = a$ is called a **vertical asymptote** on the graph of $y = f(x)$ if at least one of the following limit statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{X \to a^+} f(x) = \infty$$
$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{X \to a^+} f(x) = -\infty$$

The graph of a function can never cross any of its vertical asymptotes.

For example, the y-axis is a vertical asymptote on the graph of $y = \dfrac{1}{x^2}$ since $\lim\limits_{x \to 0} \dfrac{1}{x^2} = \infty$. In Figure 2.47, the line $x = a$ is a vertical asymptote in each of the graphs. Vertical asymptotes are usually drawn as dashed lines, and knowing the location of any vertical asymptotes is helpful in sketching the graph of a function.

Example 1 One-Sided Infinite Limits

Find $\lim\limits_{x \to 3^+} \dfrac{2x}{x - 3}$ and $\lim\limits_{x \to 3^-} \dfrac{2x}{x - 3}$.

Solution

The graph of $f(x) = \dfrac{2x}{x - 3}$ is shown in Figure 2.48. The vertical asymptote on the graph suggests values of the one-sided limits.

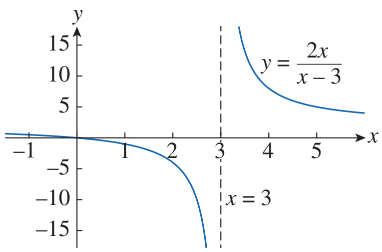


Figure 2.48

Graph of the function $f(x) = \dfrac{2x}{x - 3}$.

We can use an analytical method to evaluate a limit that results in a nonzero constant divided by 0 after direct substitution:

$x = 3$ is not in the domain of f. However, notice that if $x = 3$, the numerator is $2 \cdot 3 = 6 \neq 0$ (nonzero) and the denominator is $(3 - 3) = 0$.

As x approaches 3 from the right, $x - 3$ approaches 0 through small positive values.

$$\lim_{x \to 3^+} \frac{2x}{x - 3} = \infty$$

$x - 3 \to 0$ through small positive values.
Denominator gets small positive; fraction increases without bound.

Similarly

$$\lim_{x \to 3^-} \frac{2x}{x - 3} = -\infty$$

$x - 3 \to 0$ through small negative values.
Denominator gets small negative; fraction decreases without bound.

■

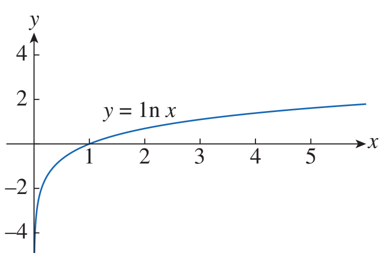


Figure 2.49
Graph of $y = \ln x$.

Two familiar functions whose graphs have vertical asymptotes are $y = \ln x$ and $y = \tan x$. From Figure 2.49 we see that

$$\lim_{x \to 0^+} \ln x = -\infty \tag{1}$$

and so the line $x = 0$ (the y-axis) is a vertical asymptote. In fact, the same is true for $y = \log_a x$ provided that $a > 1$. (See Figures 1.83 and 1.84 in Section 1.5.)

Figure 2.50 shows that

$$\lim_{x \to (\pi/2)^-} \tan x = \infty$$

and so the line $x = \pi/2$ is a vertical asymptote. In fact, the lines $x = (2n + 1)\pi/2$, where n is an integer, are all vertical asymptotes of $y = \tan x$.

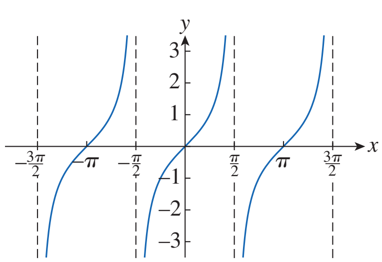


Figure 2.50
Graph of $y = \tan x$.

Example 2 Limit of a Composite Function

Find $\lim\limits_{x \to 0} \ln(\tan^2 x)$.

Solution

One way to solve this problem is to introduce a new variable, $t = \tan^2 x$.

Then $t \geq 0$ and $t = \tan^2 x \to \tan^2 0 = 0$ as $x \to 0$ because the tangent function is a continuous function.

Using Equation 1: $\lim\limits_{x \to 0} \ln(\tan^2 x) = \lim\limits_{t \to 0^+} \ln t = -\infty$ ■

Limits at Infinity

In computing infinite limits, we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). Here we examine functions in which x increases or decreases without bound, or limits at infinity; that is, we let $x \to \infty$ or $x \to -\infty$ and examine the behavior of $f(x)$. These limits can be used to determine the horizontal asymptotes on the graph of some functions.

Consider the function $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ as x increases without bound. A table of values and a graph of f are shown in Table 2.11 and Figure 2.51.

x	$f(x)$
0	-1.0
± 1	0.0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 10	0.999200
± 100	0.999800
± 1000	0.999998

Table 2.11
Table of values as x increases.

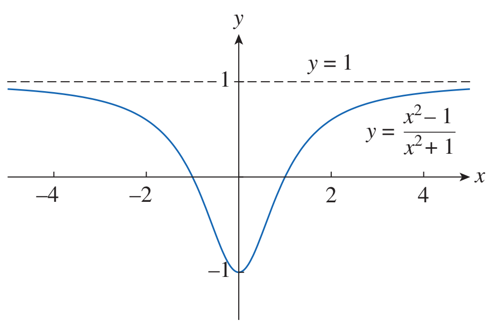


Figure 2.51
Graph of $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$.

The table and the graph suggest that as x increases without bound, the values of $f(x)$ approach 1. The graph of f approaches the horizontal line $y = 1$, called a *horizontal asymptote* on the graph of f. It appears that we can make the values of $f(x)$ as close to 1 as we want by taking x sufficiently large. This function behavior is expressed symbolically as

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x increases without bound.

This example leads to the following definition.

Definition • Limit at Infinity

Let f be a function defined on an interval $[a, \infty)$. Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

A Closer Look

1. We generally use the standard notation when evaluating a limit: $\lim_{x\to\infty} f(x) = L$. However, the following notation is sometimes used to indicate this kind of a limit: $f(x) \to L$ as $x \to \infty$.
2. Even though ∞ is not a number, we often read the expression $\lim_{x\to\infty} f(x) = L$ as

   "the limit of $f(x)$, as x approaches infinity, is L" or

   "the limit of $f(x)$, as x becomes infinite, is L" or

   "the limit of $f(x)$, as x increases without bound, is L."
3. Figure 2.52 shows several graphs of functions in which $\lim_{x\to\infty} f(x) = L$.

A horizontal asymptote is usually drawn as a dashed line.

Note that the graph of a function may cross a horizontal asymptote.

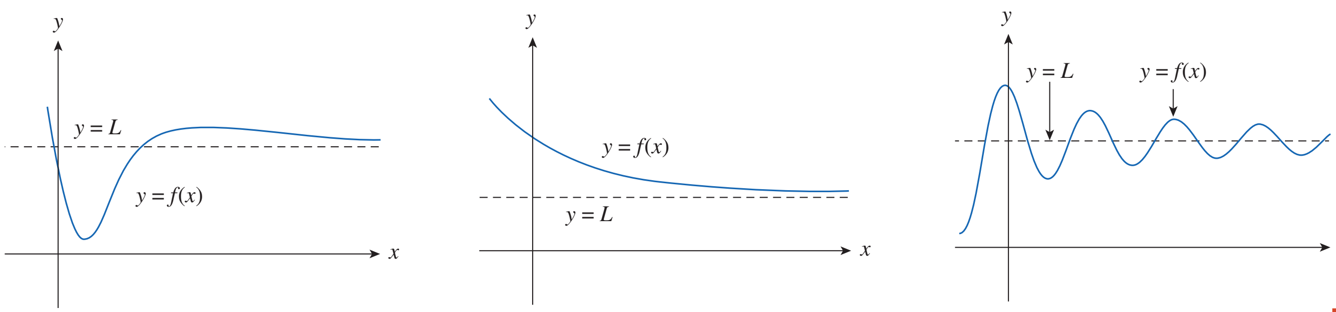


Figure 2.52
Examples illustrating $\lim_{x\to\infty} f(x) = L$.

In Table 2.11 and Figure 2.51, we can also see that as x becomes larger and larger negative, the values of $f(x)$ are close to 1. In fact, as x decreases without bound, we can make $f(x)$ as close to 1 as we like. This function behavior is expressed symbolically as

$$\lim_{x\to-\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, as illustrated in Figures 2.53 and 2.54, the notation

$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

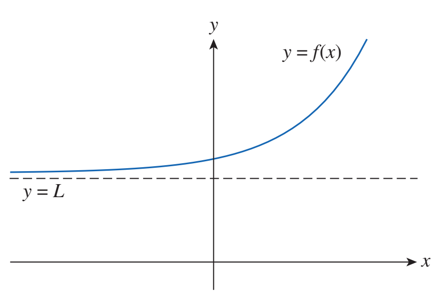


Figure 2.53
Example of the graph of a function f such that $\lim_{x\to-\infty} f(x) = L$.

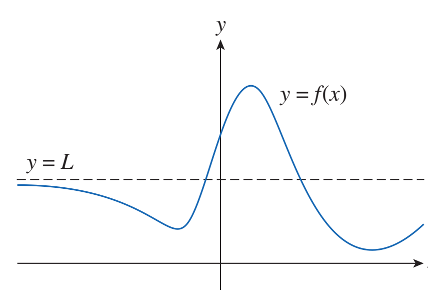


Figure 2.54
Example of the graph of a function f such that $\lim_{x\to-\infty} f(x) = L$. Note that the graph of f may cross a horizontal asymptote.

Again, the symbol $-\infty$ does not represent a number, but the expression $\lim_{x\to -\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches negative infinity, is L"

Notice that the graph of f in Figures 2.53 and 2.54 approaches, the line $y = L$ as we look to the left on the graph.

Definition • Horizontal Asymptote

The line $y = L$ is called a **horizontal asymptote** on the graph of $y = f(x)$ if either

$$\lim_{x\to \infty} f(x) = L \quad \text{or} \quad \lim_{x\to -\infty} f(x) = L$$

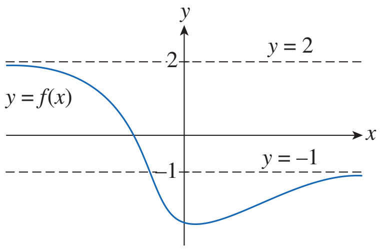


Figure 2.55
Graph of $y = f(x)$ has two horizontal asymptotes.

For example, if $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$, the line $y = 1$ is a horizontal asymptote on the graph of f because $\lim_{x\to \infty} f(x) = 1$, as illustrated in Figure 2.51. Note that $\lim_{x\to -\infty} f(x) = 1$ also, but only one of these limits is necessary for the line $y = 1$ to be a horizontal asymptote.

It is possible for the graph of a function to have two horizontal asymptotes. The curve $y = f(x)$ sketched in Figure 2.55 has both $y = -1$ and $y = 2$ as horizontal asymptotes because

$$\lim_{x\to \infty} f(x) = -1 \qquad \text{and} \qquad \lim_{x\to -\infty} f(x) = 2$$

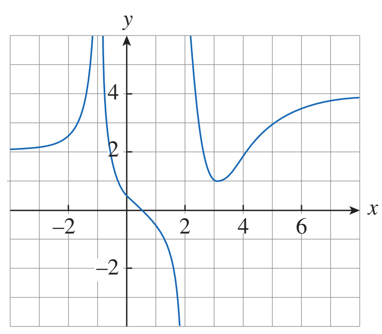


Figure 2.56
Graph of $y = f(x)$.

Example 3 Infinite Limits and Asymptotes from a Graph

Use the graph of the function f shown in Figure 2.56 to draw reasonable conclusions for the infinite limits, limits at infinity, and the equations for any horizontal or vertical asymptotes.

Solution

For values of x close to -1 and close to 2, the graph suggests:

$$\lim_{x\to -1^-} f(x) = \infty \qquad \text{and} \qquad \lim_{x\to -1^+} f(x) = \infty \qquad \text{Therefore, } \lim_{x\to -1} f(x) = \infty$$

$$\lim_{x\to 2^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 2^+} f(x) = \infty$$

Since these two "limits" are different, $\lim_{x\to 2} f(x)$ does not exist and we cannot write a single limit statement to describe the behavior of f near 2.

However, the lines $x = -1$ and $x = 2$ are vertical asymptotes.

The graph also suggests:

$$\lim_{x\to \infty} f(x) = 4 \qquad \text{and} \qquad \lim_{x\to -\infty} f(x) = 2$$

Therefore, the lines $y = 4$ and $y = 2$ are horizontal asymptotes. ■

Example 4 Limits at Infinity

Find $\lim\limits_{x\to\infty} \dfrac{1}{x}$ and $\lim\limits_{x\to-\infty} \dfrac{1}{x}$.

Solution

Construct a table of values for some numerical evidence.

A graph of $y = \dfrac{1}{x}$ is shown in Figure 2.57.

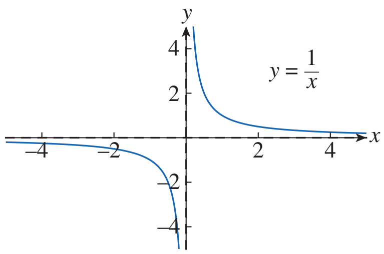


Figure 2.57

A graph of $y = \dfrac{1}{x}$. There is a vertical asymptote on this graph because $\lim\limits_{x\to0^-} \dfrac{1}{x} = -\infty$ $\left(\text{and } \lim\limits_{x\to0^+} \dfrac{1}{x} = \infty\right)$.

The graph suggests that as $x \to \infty$, $\dfrac{1}{x} \to 0$.

We can reason analytically to evaluate this limit, using arguments and symbols similar to those for infinite limits.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

As x increases without bound, the numerator is constant, 1. Denominator is increasing without bound; fraction is approaching 0.

Similarly,

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

As x decreases without bound, the numerator is constant, 1. Denominator is decreasing without bound; fraction is approaching 0.

These two limits indicate that the line $y = 0$ is a horizontal asymptote on the graph of $y = \dfrac{1}{x}$. ■

The Limit Laws presented in Section 2.3, with the exception of Laws 9 and 10, are also valid for limits at infinity. In practice this means that we can replace $x \to a$ with $x \to \infty$ or $x \to -\infty$. If we combine Law 6 with the results from Example 4, we obtain the following useful rule for calculating limits.

If n is a positive integer, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0 \tag{2}$$

Example 5 Limit at Infinity and Rational Functions

Find each limit.

(a) $\lim\limits_{x\to\infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$ (b) $\lim\limits_{x\to-\infty} \dfrac{3x^2 + 5x}{x^3 + 1}$

Solution

(a) We can try to reason analytically: as $x \to \infty$, the dominating term in the numerator is $3x^2$, and $3x^2 \to \infty$. In the denominator, $5x^2$ is the dominating term, and $5x^2 \to \infty$ as well. Therefore, this limit looks like $\frac{\infty}{\infty}$ but is not equal to 1. This is another indeterminate form, also an indication that you must do something.

This type of limit problem can often be solved by dividing each term in the numerator and the denominator by x raised to the highest power that appears in the denominator. In this problem, that's x^2.

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

$$= \lim_{x\to\infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}} = \lim_{x\to\infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

Divide numerator and denominator by x^2; simplify.

$$= \frac{\lim_{x\to\infty}\left[3 - \frac{1}{x} - \frac{2}{x^2}\right]}{\lim_{x\to\infty}\left[5 + \frac{4}{x} + \frac{1}{x^2}\right]}$$

Limit of a quotient.

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{2}{x^2}}{\lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{4}{x} + \lim_{x\to\infty}\frac{1}{x^2}} = \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{1}{x} - 2\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 5 + 4\lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}\frac{1}{x^2}}$$

$$= \frac{3-0-0}{5+0+0} = \frac{3}{5}$$

Limit Laws.

(b) Notice that the numerator is increasing without bound and the denominator is decreasing without bound. This limit is in the indeterminate form $\frac{\infty}{-\infty}$.

Once again, we divide the numerator and denominator by x raised to the highest power in the denominator, x^3.

$$\lim_{x\to-\infty} \frac{3x^2+5x}{x^3+1}$$

$$= \lim_{x\to-\infty} \frac{\frac{3x^2+5x}{x^3}}{\frac{x^3+1}{x^3}} = \lim_{x\to-\infty} \frac{\frac{3}{x}+\frac{5}{x^2}}{1+\frac{1}{x^3}}$$

Divide numerator and denominatorby x^3; simplify.

$$= \frac{\lim_{x\to-\infty}\left[\frac{3}{x}+\frac{5}{x^2}\right]}{\lim_{x\to-\infty}\left[1+\frac{1}{x^3}\right]} = \frac{0}{1} = 0$$

Limit Laws.

The graphs of $f(x) = \frac{3x^2-x-2}{5x^2+4x+1}$ and $g(x) = \frac{3x^2+5x}{x^3+1}$ are shown in Figures 2.58 and 2.59.

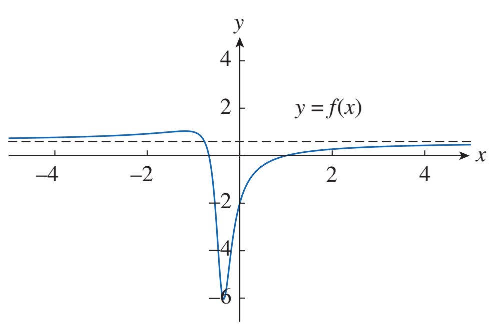


Figure 2.58

Graph of $y = f(x) = \dfrac{3x^2-x-2}{5x^2+4x+1}$.

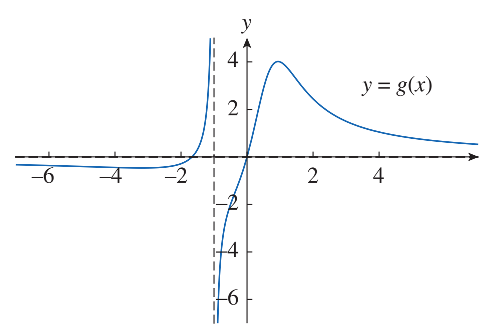


Figure 2.59

Graph of $y = g(x) = \dfrac{3x^2+5x}{x^3+1}$.

The graphs confirm the limit results we found in parts (a) and (b).

Notice that the graph of $y = g(x)$ also has a vertical asymptote at $x = -1$. ■

Example 6 Find Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes on the graph of the function

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$$

Solution

As $x \to \infty$, both the numerator and the denominator are increasing without bound. This limit is in the indeterminate form $\frac{\infty}{\infty}$. Even though the expression is not a rational function, we still use a similar strategy. Divide numerator and denominator by x raised to the highest power in the denominator.

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty}\frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{3x-5}{x}}$$ Divide numerator and denominator by x.

$$= \lim_{x\to\infty}\frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{3-\frac{5}{x}}$$ Since $x \to \infty$, $x \geq 0$, and $x = |x| = \sqrt{x^2}$.

$$= \frac{\lim\limits_{x\to\infty}\sqrt{2+\frac{1}{x^2}}}{\lim\limits_{x\to\infty}\left[3-\frac{5}{x}\right]} = \frac{\sqrt{2}}{3}$$ Limit Laws; evaluate; simplify.

Similarly,

$$\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty}\frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{3x-5}{x}}$$ Divide numerator and denominator by x.

$$= \lim_{x\to\infty}\frac{\frac{\sqrt{2x^2+1}}{-\sqrt{x^2}}}{3-\frac{5}{x}}$$ Since $x \to -\infty$, $x < 0$, and $-x = |x| = \sqrt{x^2}$.

$$= \frac{\lim\limits_{x\to-\infty}-\sqrt{2+\frac{1}{x^2}}}{\lim\limits_{x\to-\infty}\left[3-\frac{5}{x}\right]} = -\frac{\sqrt{2}}{3}$$ Limit Laws; evaluate; simplify.

The lines $y = \frac{\sqrt{2}}{3}$ and $y = -\frac{\sqrt{2}}{3}$ are horizontal asymptotes on the graph of f.

A vertical asymptote may occur where the denominator, $3x - 5$, is 0.

$3x - 5 = 0 \Rightarrow x = \frac{5}{3}$. Consider the one-sided limits at $x = \frac{5}{3}$.

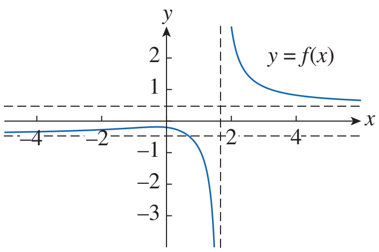


Figure 2.60
There are two horizontal asymptotes and one vertical asymptote on the graph of $y = f(x)$.

$$\lim_{x\to(5/3)^-} \frac{\sqrt{2x^2+1}}{3x-5} = -\infty$$ As $x \to (5/3)^-$, the numerator is approaching a nonzero, positive constant; denominator $\to$ 0 through small negative values.

$$\lim_{x\to(5/3)^+} \frac{\sqrt{2x^2+1}}{3x-5} = \infty$$ As $x \to (5/3)^+$, the numerator is approaching the same nonzero, positive constant; denominator $\to$ 0 through small positive values.

The line $x = \frac{5}{3}$ is a vertical asymptote on the graph of f.

All three asymptotes are shown in Figure 2.60. ■

Example 7 A Difference of Functions

Find $\lim\limits_{x\to\infty} (\sqrt{x^2+1} - x)$.

Solution

As $x \to \infty$, the expression $\sqrt{x^2+1}$ is increasing without bound and the other term in the difference, x, also increases without bound, $x \to \infty$. So, this expression is in another indeterminate form, $\infty - \infty$. The strategy here is to rationalize the numerator.

$$\lim_{x\to\infty} (\sqrt{x^2+1} - x)$$

$$= \lim_{x\to\infty} (\sqrt{x^2+1} - x) \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$ Multiply by 1 in a convenient form. Use the conjugate.

$$= \lim_{x\to\infty} \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x}$$ Simplify.

$$= 0$$ The numerator is constant. The denominator is increasing without bound.

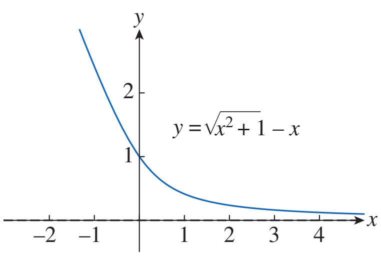


Figure 2.61
Graph of $y = \sqrt{x^2+1} - x$.
The horizontal asymptote is $y = 0$.

The line $y = 0$ is a horizontal asymptote on the graph of $y = \sqrt{x^2+1} - x$, shown in Figure 2.61. ■

Example 8 Change of Variable

Find $\lim\limits_{x\to 2^+} \arctan\left(\dfrac{1}{x-2}\right)$.

Solution

As $x \to 2^+$, the difference $x - 2 \to 0^+$, and the fraction $\dfrac{1}{x-2} \to \infty$.

In this case, it is easier to evaluate the limit by changing variables.

Transform the problem to an equivalent expression using a different variable.

Let $t = \dfrac{1}{x-2}$. Then, as $x \to 2^+$, $t \to \infty$.

$$\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right) = \lim_{t\to\infty} \arctan t = \frac{\pi}{2}$$ ■

Let $f(x) = e^x$, the natural exponential function. The table of values in Table 2.12 and the graph shown in Figure 2.62 suggest that the line $y = 0$ (the x-axis) is a horizontal asymptote on the graph of $y = e^x$.

x	e^x
0	1.00000
-1	0.36788
-2	0.13534
-3	0.04979
-5	0.00674
-8	0.00034
-10	0.00005

Table 2.12
Table of values as x decreases.

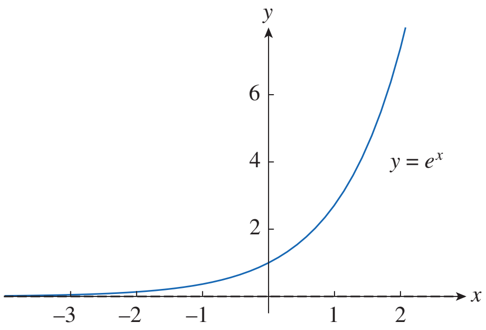


Figure 2.62
Graph of $y = e^x$.

Notice that the values of e^x approach 0 very quickly as x decreases. We can express this limiting behavior by

$$\lim_{x \to -\infty} e^x = 0 \tag{3}$$

Example 9 Change of Variable

Find $\lim_{x \to 0^-} e^{1/x}$.

Solution

As $x \to 0^-$, the expression $\frac{1}{x} \to -\infty$.

Therefore, a strategy here is to change the variable.

Let $t = \frac{1}{x}$, then as $x \to 0^-$, $t \to -\infty$.

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$ Use Equation 3.

Example 10 Oscillating Function

Find $\lim_{x \to \infty} \sin x$.

Solution

As x increases, the values of $\sin x$ oscillate between -1 and 1 infinitely often.

Therefore, the values of $\sin x$ do not approach a definite number, and $\lim_{x \to \infty} \sin x$ does not exist (DNE).

Note: This limit calculation is equivalent to $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$.

Infinite Limits at Infinity

In many of the preceding examples, the values of $f(x)$ approached a definite real number as x increased or decreased without bound, as $x \to \infty$ or as $x \to -\infty$. However, there are four cases in which $f(x)$ also increases or decreases without bound as $x \to \infty$ or $x \to -\infty$.

$$\lim_{x \to -\infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to \infty} f(x) = \infty$$

For example, for the natural exponential function, $\lim_{x \to \infty} e^x = \infty$.

A graph (shown in Figure 2.62) or a table of values suggests this limit. In fact, the values of e^x can be made arbitrarily large as x increases without bound.

Here are a few examples of infinite limits at infinity.

Example 11 Infinite Limit

Find $\lim_{x\to\infty}(x^2 - x)$.

Solution

As $x \to \infty$, the expression x^2 is increasing without bound, and the term x also increases without bound. This limit is in the indeterminate form $\infty - \infty$.

The strategy here is to factor the expression.

$$\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x(x-1) = \infty$$

As $x \to \infty$, both x and $x - 1$ also increase with bound.

Their product also increases without bound.

Therefore, this limit increases without bound. ■

Common Error

$$\lim_{x\to\infty}(x^2 - x)$$
$$= \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x$$
$$= \infty - \infty = 0$$

Correct Method

Limit Law 2 is valid only if the two limits $\lim_{x\to\infty} x^2$ and $\lim_{x\to\infty} x$ exist. And, we cannot evaluate ordinary arithmetic statements that include ∞.

Example 12 Limit of a Rational Function at Infinity

Find $\lim_{x\to\infty} \dfrac{x^2 + x}{3 - x}$.

Solution

Since the degree of the numerator, 2, is greater than the degree of the denominator, 1, we know this limit either increases or decreases without bound.

Divide the numerator and the denominator by x raised to the highest power in the denominator.

$$\lim_{x\to\infty} \frac{x^2 + x}{3 - x}$$

$$= \lim_{x\to\infty} \frac{\dfrac{x^2 + x}{x}}{\dfrac{3 - x}{x}} = \frac{x + 1}{\dfrac{3}{x} - 1}$$

Divide numerator and denominator by x; simplify.

$$= -\infty$$

As $x \to \infty$, $x + 1 \to \infty$ and $\dfrac{3}{x} - 1 \to 0 - 1 = -1$. Therefore, the fraction is decreasing without bound. ■

2.5 Exercises

1. Explain in your own words the meaning of each of the following.

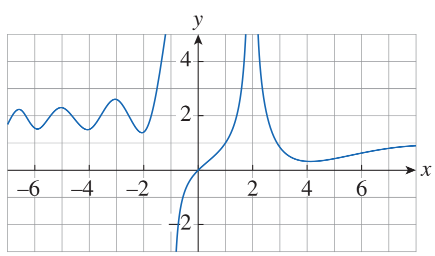
(a) $\lim_{x\to 2} f(x) = \infty$ (b) $\lim_{x\to 1^+} f(x) = -\infty$

(c) $\lim_{x\to \infty} f(x) = 5$ (d) $\lim_{x\to -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can the graph intersect a horizontal asymptote? Illustrate your answers by sketching graphs.

(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. The graph of the function f is shown.



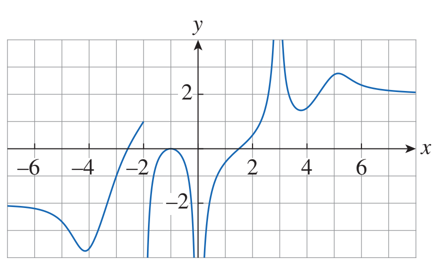
Use the graph to find the following.

(a) $\lim_{x\to 2} f(x)$ (b) $\lim_{x\to -1^-} f(x)$

(c) $\lim_{x\to -1^+} f(x)$ (d) $\lim_{x\to \infty} f(x)$

(e) $\lim_{x\to -\infty} f(x)$ (f) The equations of the asymptotes

4. The graph of the function g is shown.



Use the graph to find the following.

(a) $\lim_{x\to \infty} g(x)$ (b) $\lim_{x\to -\infty} g(x)$

(c) $\lim_{x\to 3} g(x)$ (d) $\lim_{x\to 0} g(x)$

(e) $\lim_{x\to -2^+} g(x)$ (f) The equations of the asymptotes

Sketch the graph of a function f that satisfies all of the given conditions.

5. $\lim_{x\to 0} f(x) = -\infty$, $\lim_{x\to -\infty} f(x) = 5$, $\lim_{x\to \infty} f(x) = -5$

6. $\lim_{x\to 2} f(x) = \infty$, $\lim_{x\to -2^+} f(x) = \infty$, $\lim_{x\to -2^-} f(x) = -\infty$, $\lim_{x\to -\infty} f(x) = 0$, $\lim_{x\to \infty} f(x) = 0$, $f(0) = 0$

7. $\lim_{x\to 2} f(x) = -\infty$, $\lim_{x\to \infty} f(x) = \infty$, $\lim_{x\to -\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$

8. $\lim_{x\to \infty} f(x) = 3$, $\lim_{x\to 2^-} f(x) = \infty$, $\lim_{x\to 2^+} f(x) = -\infty$, f is an odd function.

9. $f(0) = 3$, $\lim_{x\to 0^-} f(x) = 4$, $\lim_{x\to 0^+} f(x) = 2$, $\lim_{x\to -\infty} f(x) = -\infty$, $\lim_{x\to 4^-} f(x) = -\infty$, $\lim_{x\to 4^+} f(x) = \infty$, $\lim_{x\to \infty} f(x) = 3$

10. $\lim_{x\to 3} f(x) = -\infty$, $\lim_{x\to \infty} f(x) = 2$, $f(0) = 0$, f is an even function.

11. Use a table of values and a graph to estimate the value of

$$\lim_{x\to \infty} \frac{x^2}{2^x}$$

12. Use a table of values and a graph to estimate the value of

$$\lim_{x\to \infty} \left(1 - \frac{2}{x}\right)^x$$

13. Determine $\lim_{x\to 1^-} \dfrac{1}{x^3 - 1}$ and $\lim_{x\to 1^+} \dfrac{1}{x^3 - 1}$

(a) by evaluating $f(x) = \dfrac{1}{x^3 - 1}$ for values of x that approach 1 from the left and from the right.

(b) by reasoning as in Example 1.

(c) from a graph of f.

14. Use a graph to estimate all the vertical and horizontal asymptotes of the curve

$$y = \frac{x^3}{x^3 - 2x + 1}$$

Find the limit.

15. $\lim_{x\to 5^+} \dfrac{x+1}{x-5}$

16. $\lim_{x\to -3^-} \dfrac{x+2}{x+3}$

17. $\lim_{x\to 1} \dfrac{2-x}{(x-1)^2}$

18. $\lim_{x\to 3^-} \dfrac{\sqrt{x}}{(x-3)^2}$

19. $\lim_{x\to 3^+} \ln(x^2 - 9)$

20. $\lim_{x\to 2^+} e^{3/(2-x)}$

21. $\lim_{x\to 0^+} \ln(\sin x)$

22. $\lim_{x\to \frac{\pi}{2}^+} \dfrac{1}{x}\sec x$

23. $\lim_{x\to \pi^-} \cot x$

24. $\lim_{x\to 2\pi^-} x \csc x$

25. $\lim_{x\to 2^-} \dfrac{x^2 - 2x}{x^2 - 4x + 4}$

26. $\lim_{x\to 2^+} \dfrac{x^2 - 2x - 8}{x^2 - 5x + 6}$

27. $\lim_{x\to\infty} \dfrac{3x - 2}{2x + 1}$

28. $\lim_{x\to\infty} \dfrac{1 - x^2}{x^3 - x + 1}$

29. $\lim_{x\to\infty} \dfrac{x - 2}{x^2 + 1}$

30. $\lim_{x\to-\infty} \dfrac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5}$

31. $\lim_{t\to\infty} \dfrac{\sqrt{t} + t^2}{2t - t^2}$

32. $\lim_{t\to\infty} \dfrac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$

33. $\lim_{x\to\infty} \dfrac{(2x^2 + 1)^2}{(x - 1)^2(x^2 + x)}$

34. $\lim_{x\to\infty} \dfrac{x^2}{\sqrt{x^4 + 1}}$

35. $\lim_{x\to\infty} \dfrac{\sqrt{1 + 4x^6}}{2 - x^3}$

36. $\lim_{x\to-\infty} \dfrac{\sqrt{1 + 4x^6}}{2 - x^3}$

37. $\lim_{x\to\infty} \dfrac{\sqrt{x + 3x^2}}{4x - 1}$

38. $\lim_{x\to\infty} \dfrac{x + 3x^2}{4x - 1}$

39. $\lim_{x\to\infty} \left(\sqrt{9x^2 + x} - 3x\right)$

40. $\lim_{x\to-\infty} \left(\sqrt{4x^2 + 3x} + 2x\right)$

41. $\lim_{x\to\infty} \left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right)$

42. $\lim_{x\to\infty} e^{-x^2}$

43. $\lim_{x\to\infty} \sqrt{x^2 + 1}$

44. $\lim_{x\to\infty} \cos x$

45. $\lim_{x\to\infty} \dfrac{\sin^2 x}{x^2}$

46. $\lim_{x\to\infty} e^{-2x}\cos x$

47. $\lim_{x\to\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

48. $\lim_{x\to-\infty} (x^4 + x^5)$

49. $\lim_{x\to(\pi/2)^+} e^{\tan x}$

50. $\lim_{x\to\infty} \left[\ln(1 + x^2) - \ln(1 + x)\right]$

51. $\lim_{x\to\infty} \left[\ln(2 + x) - \ln(1 + x)\right]$

---

52. Let $f(x) = \dfrac{x}{\ln x}$.

(a) Find each of the following limits.
(i) $\lim_{x\to 0^+} f(x)$ (ii) $\lim_{x\to 1^-} f(x)$ (iii) $\lim_{x\to 1^+} f(x)$

(b) Use a table of values to estimate $\lim_{x\to\infty} f(x)$.

(c) Use the information from parts (a) and (b) to make a rough sketch of the graph of f.

53. Let $f(x) = \dfrac{2}{x} - \dfrac{1}{\ln x}$.

(a) Find each of the following limits.
(i) $\lim_{x\to\infty} f(x)$ (ii) $\lim_{x\to 0^+} f(x)$
(iii) $\lim_{x\to 1^-} f(x)$ (iv) $\lim_{x\to 1^+} f(x)$

(b) Use the information from part (a) to make a rough sketch of the graph of f.

54. Let $f(x) = \sqrt{x^2 + x + 1} + x$.

(a) Use a table of values and a graph to estimate the value of $\lim_{x\to-\infty} f(x)$.

(b) Use the properties of limits to confirm your guess in part (a).

55. Let $f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$.

(a) Use a table of values and a graph to estimate the value of $\lim_{x\to\infty} f(x)$.

(b) Find the exact value of the limit in part (a).

Find the horizontal and vertical asymptotes on the graph of the function. Use technology to graph the function and check your work.

56. $f(x) = \dfrac{5 + 4x}{x + 3}$

57. $f(x) = \dfrac{2x^2 + 1}{3x^2 + 2x - 1}$

58. $f(x) = \dfrac{2x^2 + x - 1}{x^2 + x - 2}$

59. $f(x) = \dfrac{1 + x^4}{x^2 - x^4}$

60. $f(x) = \dfrac{x^3 - x}{x^2 - 6x + 5}$

61. $f(x) = \dfrac{2e^x}{e^x - 5}$

---

62. Let f be the function defined by

$$f(x) = \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000}$$

(a) Use a graph of f for $-10 \le x \le 10$ to estimate the equation of the horizontal asymptote.

(b) Evaluate the limits

$$\lim_{x\to-\infty} f(x) \quad \text{and} \quad \lim_{x\to\infty} f(x)$$

to find the equation of the horizontal asymptote.

(c) How do your answers in parts (a) and (b) compare? Explain any discrepancies.

63. Let $f(x) = \dfrac{\sqrt{2x^2 + 1}}{3x - 5}$.

(a) Use a graph of f to estimate the equation of each horizontal and vertical asymptote on the graph of f and to estimate the values of the limits

$$\lim_{x\to-\infty} f(x) \quad \text{and} \quad \lim_{x\to\infty} f(x)$$

(b) Use a table of values to estimate the limits in part (a).

(c) Find the exact values of the limits in part (a). Explain any discrepancies.

64. Make a rough sketch of the graph of $f(x) = x^n$ (n is an integer) for the following five cases.
(i) $n = 0$ (ii) $n > 0$, n is odd
(iii) $n > 0$, n is even (iv) $n < 0$, n is odd
(v) $n < 0$, n is even

Use your graphs to find each of the following limits for each of the given cases.

(a) $\lim_{x\to 0^+} x^n$ (b) $\lim_{x\to 0^-} x^n$

(c) $\lim_{x\to \infty} x^n$ (d) $\lim_{x\to -\infty} x^n$

65. Find an expression for a function f that satisfies the following conditions.

$\lim_{x\to\infty} f(x) = 0,\ \lim_{x\to -\infty} f(x) = 0,\ \lim_{x\to 0} f(x) = -\infty$

$f(2) = 0,\ \lim_{x\to 3^-} f(x) = \infty,\ \lim_{x\to 3^+} f(x) = -\infty$

66. Find an expression for a function g whose graph has vertical asymptotes at $x = 1$ and $x = 3$ and horizontal asymptotes at $y = 1$.

67. Suppose the function f is the ratio of quadratic functions and that the graph of f has a vertical asymptote at $x = 4$ and one x-intercept at $x = 1$. In addition, f is not defined at $x = -1$ and $\lim_{x\to -1} f(x) = 2$. Find each value.

(a) $f(0)$ (b) $\lim_{x\to\infty} f(x)$

Find $\lim_{x\to\infty} f(x)$, $\lim_{x\to -\infty} f(x)$, and the x- and y-intercepts for the function. Use this information to construct a rough sketch of the graph of the function.

68. $f(x) = 2x^3 - x^4$

69. $f(x) = x^4 - x^6$

70. $f(x) = x^3(x + 2)^2(x - 1)$

71. $f(x) = x^2(x^2 - 1)^2(x + 2)$

72. $f(x) = (3 - x)(1 + x)^2(1 - x)^4$

73. Let $f(x) = \dfrac{\sin x}{x}$.

(a) Use the Squeeze Theorem to evaluate $\lim_{x\to\infty} f(x)$.

(b) Graph f in an appropriate viewing rectangle. Use your graph to determine how many times the graph of f crosses the asymptote.

74. The *end behavior* of a function means the behavior of the function values as $x \to \infty$ and as $x \to -\infty$.

(a) Describe and compare the end behavior of the functions

$$P(x) = 3x^5 - 5x^3 + 2x \quad \text{and} \quad Q(x) = 3x^5$$

by graphing both functions in the viewing rectangles $[-2, 2] \times [-2, 2]$ and $[-10, 10] \times [-10{,}000, 10{,}000]$.

(b) Two functions are said to have the *same end behavior* if their ratio approaches 1 as $x \to \infty$. Show that P and Q have the same end behavior.

75. Let $f(x) = e^x + \ln|x - 4|$.

(a) Graph f for $0 \le x \le 5$. Do you think this is an accurate representation of f?

(b) How would you use technology to obtain a more accurate representation of the graph of f?

76. Let P and Q be polynomials. Find

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

77. Find $\lim_{x\to\infty} f(x)$ if, for all $x > 1$,

$$\frac{10e^x - 21}{2e^x} < f(x) < \frac{5\sqrt{x}}{\sqrt{x - 1}}$$

78. A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min.

(a) Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200 + t}$$

(b) What happens to the concentration as $t \to \infty$?

79. In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. Explain what happens to the value of m as $v \to c^-$.

80. It can be shown that, under certain assumptions, the velocity $v(t)$ of a falling raindrop at time t is

$$v(t) = v_e(1 - e^{-gt/v_e})$$

where g is the acceleration due to gravity and v_e is the *terminal velocity* of the raindrop.

(a) Find $\lim_{t\to\infty} v(t)$.

(b) Graph the function v if $v_e = 1$ m/s and $g = 9.8$ m/s^2. How long does it take for the velocity of the raindrop to reach 99% of its terminal velocity?

81. Let $f(x) = e^{-x/10}$ and $g(x) = 0.1$.

(a) Graph f and g in an appropriate viewing rectangle to find a value N such that if $x > N$, then $e^{-x/10} < 0.1$.

(b) Try to solve part (a) analytically, that is, without the use of technology.

82. Let $f(x) = \dfrac{4x^2 - 5x}{2x^2 + 1}$.

(a) Show that $\lim_{x\to\infty} f(x) = 2$.

(b) Graph the function f and the line $y = 1.9$ in the same viewing rectangle. Find a number N such that $f(x) > 1.9$ whenever $x > N$.
What if 1.9 is replaced by 1.99?

2.6 Derivatives and Rates of Change

The problem of finding the tangent line to a curve at a point and the problem of finding the instantaneous velocity of a moving object both involve finding the same type of limit–a derivative. We will see that this special type of limit can be interpreted as a rate of change in any of the natural or social sciences or engineering fields.

Tangents

In Section 2.1 we discussed a method for finding the equation of the tangent line to the graph of a function f at a point $P(a, f(a))$. Start with a point nearby, $Q(a + h, f(a + h))$, where h is a (small) positive number, and compute the slope of the secant line PQ:

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}$$

Let the point Q approach P along the graph of f by letting $h \to 0$. If m_{sec} approaches a number m, then we define this to be the slope of the tangent line to the graph of f at P. The tangent line l is the *limiting position* of the secant line as Q approaches P. See Figures 2.63 and 2.64.

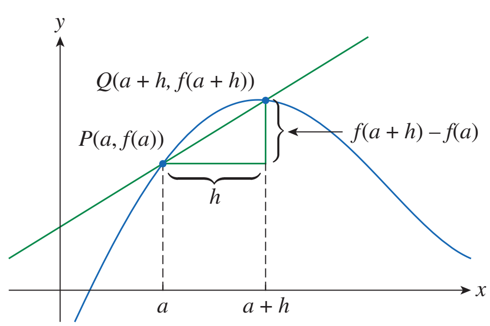


Figure 2.63
The slope of the secant line is $\dfrac{f(a+h) - f(a)}{h}$.

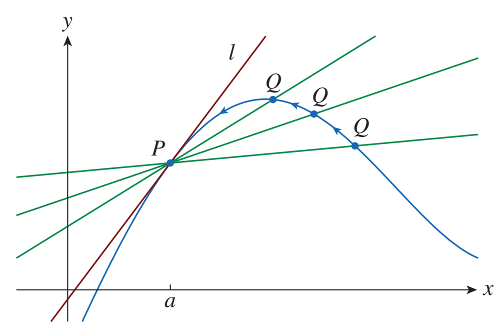


Figure 2.64
As $Q \to P$, the secant line approaches the tangent line l.

Definition • Tangent Line

The **tangent line** to the graph of $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

Example 1 Tangent Line Equation

Find an equation of the tangent line to the graph of $f(x) = x^2$ at the point $P(1, 1)$.

Solution

Use $a = 1$ and the definition above to find the slope of the tangent line.

Another Approach: Treat the numerator as the difference of two squares.

$$m = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{(1+h)^2-1}{h} \qquad \text{Use } a = 1 \text{ and } f(x) = x^2.$$

$$= \lim_{h\to 0}\frac{1+2h+h^2-1}{h} = \lim_{h\to 0}\frac{2h+h^2}{h} = \lim_{h\to 0}\frac{h(2+h)}{h} \qquad \text{Expand; simplify; factor.}$$

$$= \lim_{h\to 0}(2+h) = 2 \qquad \text{Cancel common factor; evaluate.}$$

An equation of the tangent line is

$$y - 1 = 2(x-1) \quad \text{or} \quad y = 2x - 1. \qquad \text{Point-slope form of the equation of a line.}$$

The tangent line is shown in Figure 2.65. ■

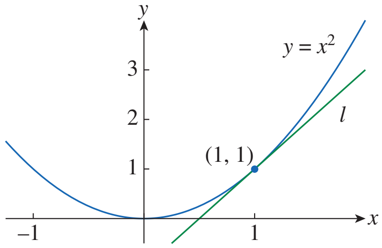
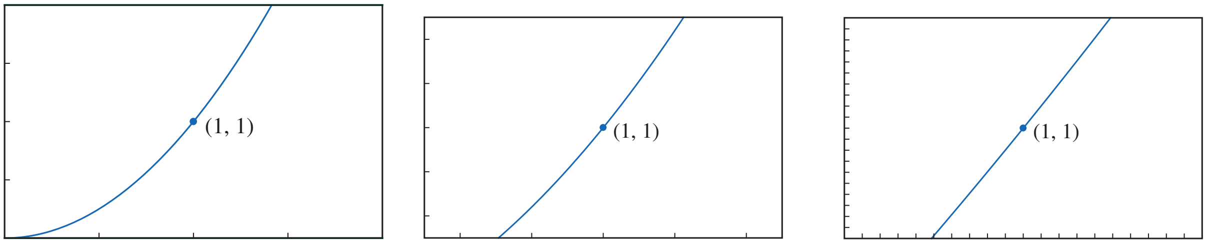


Figure 2.65
An equation of the tangent line to the graph of $y = x^2$ at the point (1, 1) is $y = 2x - 1$.

We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point. The idea is that if we zoom in far enough near the point, the curve looks like a straight line, or is locally linear. Figure 2.66 illustrates this procedure for the curve $y = x^2$ in Example 1. The more we zoom in, the more the parabola looks like a line. In other words, the curve becomes almost indistinguishable from its tangent line.



(a) $[0, 2] \times [0, 2]$ (b) $[0.5, 1.5] \times [0.5, 1.5]$ (c) $[0.9, 1.1] \times [0.9, 1.1]$

Figure 2.66
Calculator screenshots that illustrate the effect of zooming in near the point (1, 1) on the parabola $y = x^2$.

There is another common limit expression for the slope of a tangent line and, in certain cases, it may be easier to use this alternate expression to evaluate a limit. Again, let $P(a, f(a))$ be a point on the graph of the function $y = f(x)$. However, this time consider a point nearby, written as $Q(x, f(x))$, where $x \neq a$. Figure 2.67 illustrates the slope of the secant line at Q.

The slope of the secant line is

$$m_{\text{sec}} = \frac{f(x)-f(a)}{x-a}$$

As $Q \to P$, or equivalently, as $x \to a$, the secant line approaches the tangent line. Therefore, an equivalent expression for the slope of the tangent line at the point $(a, f(a))$ is

$$m = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \tag{1}$$

It takes practice and pattern recognition to make a choice between the two limit expressions to find the slope of the tangent line.

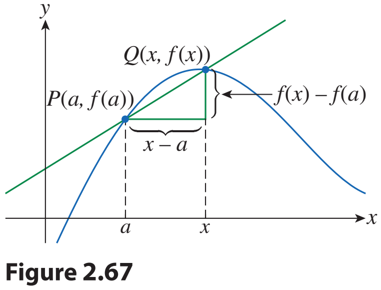


Figure 2.67
The slope of the secant line is $\dfrac{f(x)-f(a)}{x-a}$.

Example 2 Using the Alternate Limit Expression

Find an equation of the tangent line to the graph of $f(x) = \dfrac{3}{x}$ at the point (3, 1).

Solution

Use $a = 3$ and the alternate definition for the slope of the tangent line.

$$m = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{\frac{3}{x} - 1}{x - 3} \qquad \text{Use } a = 3 \text{ and } f(x) = \frac{3}{x}.$$

$$= \lim_{x \to 3} \frac{\frac{3 - x}{x}}{x - 3} = \lim_{x \to 3} \frac{3 - x}{x(x - 3)} \qquad \text{Common denominator; simplify.}$$

$$= \lim_{x \to 3} \frac{-(x - 3)}{x(x - 3)} = \lim_{x \to 3} \frac{-1}{x} = -\frac{1}{3} \qquad \text{Rewrite; cancel common factor; direct substitution.}$$

An equation of the tangent line is

$$y - 1 = -\tfrac{1}{3}(x - 3) \quad \text{or} \quad y = -\tfrac{1}{3}x + 2.$$

A graph of f and the tangent line are shown in Figure 2.68. ■

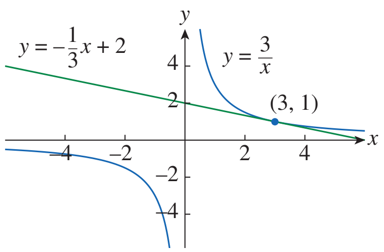


Figure 2.68
Graph of f and the tangent line at the point (3, 1).

Velocities

In Section 2.1 we considered the motion of a ball dropped from the CN Tower. The *instantaneous* velocity was defined to be the limiting value of average velocities over shorter and shorter time intervals.

Let's consider this problem in general. Suppose an object moves along a straight line so that its position relative to the origin at any time t is given by the function $s = f(t)$. The function f that describes the motion is called the **position function** of the object. For example, $f(3)$ is the position of the object at time $t = 3$; if $f(3)$ is less than 0, the object is to the left of the origin.

The change in position of the object over the time interval $t = a$ to $t = a + h$ is the difference $f(a + h) - f(a)$. See Figure 2.69.

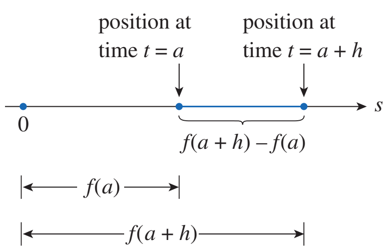


Figure 2.69
The change in position of the object over the time interval $t = a$ to $t = a + h$.

The *average velocity* over this time interval is

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}$$

Note that this is the same expression for the slope of the secant line through the points $(a, f(a))$ and $(a + h, f(a + h))$.

Suppose that we compute the average velocity over shorter and shorter time intervals $[a, a + h]$ by letting $h \to 0$. The **instantaneous velocity** (or just **velocity**) of the object at time $t = a$ is denoted $v(a)$ and is the limit of the average velocities.

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{2}$$

This limit expression for instantaneous velocity at time $t = a$ is the same as the limit expression for the slope of the tangent line to the graph of f at $x = a$.

Now that we know how to compute limits, let's reconsider the problem of the falling ball.

Example 3 Velocity and the CN Tower

See Example 3, Section 2.1, page 95.

Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) What is the velocity of the ball when it hits the ground?

Solution

Recall that the distance (in meters) the ball falls after t seconds is $4.9t^2$.

The ball is moving in a straight line (downward). Place the origin at the upper observation deck and assign downward as the positive direction.

Since we need to find the velocity at time $t = 5$ and also when the ball hits the ground, it is advantageous to find an expression for the velocity at a general time t.

$$v(t) = \lim_{h\to 0}\frac{f(t+h) - f(t)}{h} = \lim_{h\to 0}\frac{4.9(t+h)^2 - 4.9t^2}{h}$$ Use $s = f(t) = 4.9t^2$.

$$= \lim_{h\to 0}\frac{4.9(t^2 + 2th + h^2) - 4.9t^2}{h} = \lim_{h\to 0}\frac{4.9(2th + h^2)}{h}$$ Expand; simplify.

$$= \lim_{h\to 0}\frac{4.9h(2t+h)}{h} = \lim_{h\to 0} 4.9(2t+h) = 9.8t$$ Factor; cancel common factor; direct substitution.

Intermediate values, for example, $t = \sqrt{450/4.9} \approx 9.583$, are written with three digits to the right of the decimal but stored to the greatest degree of accuracy allowed by the technology. The stored value is used in subsequent calculations, for example, to find the velocity of the ball as it hits the ground.

(a) The velocity after 5 seconds is $v(5) = (9.8)(5) = 49$ m/s.

(b) The observation deck is 450 m above the ground.

The ball will hit the ground at time t when $s(t) = 450$.

$$4.9t^2 = 450 \;\Rightarrow\; t^2 = \frac{450}{4.9} \;\Rightarrow\; t = \sqrt{\frac{450}{4.9}} \approx 9.583 \text{ s}$$

The velocity of the ball as it hits the ground is $v(9.583) = 93.915$ m/s. ■

Derivatives

The same limit expression is used to find both the slope of a tangent line at a point on the graph of a function and the (instantaneous) velocity of an object. This limit expression

$$\lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

is used in many applications to calculate a rate of change. For example, we might want to find the rate of a reaction in chemistry, a marginal cost in economics, or the rate of change of the temperature in a kiln. This common limit expression has a special name and notation.

Definition • Derivative of a Function at a Number

The **derivative of a function f at a number a**, denoted $f'(a)$, is

$$f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

$f'(a)$ is read as "f prime of a."

Just as there was another way to find the slope of the tangent line, there is an alternate definition of the derivative of a function f at a number a. Let $h = x - a$. Then as $h \to 0$, $x \to a$ and we have

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(a + x - a) - f(a)}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad (3)$$

Example 4 Derivative of a Function at *a*

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

Solution

Problem-solving strategy: When finding the derivative of a function at a using the limit expression in the definition above, the variable h often cancels after some algebraic manipulation.

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} && \text{Use } f(x). \\ &= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} && \text{Expand; distribute terms.} \\ &= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} \frac{h(2a + h - 8)}{h} && \text{Simplify; factor.} \\ &= \lim_{h \to 0} (2a + h - 8) = 2a - 8 && \text{Cancel common factor; direct substitution.} \end{aligned}$$

■

We defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has slope m given in Definition 1. Since, by Definition 4, this is the same as the derivative $f'(a)$, we can now say the following.

> The tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

Example 5 Tangent Line Equation

Find an equation of the line tangent to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

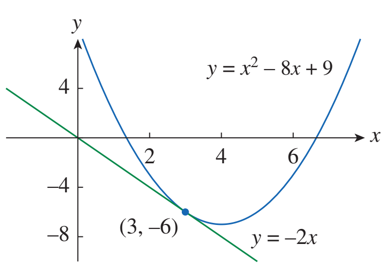


Figure 2.70
Graph of f and the tangent line at the point $(3, -6)$.

Solution

From Example 4:
the derivative of $f(x) = x^2 - 8x + 9$ at the number a is $f'(a) = 2a - 8$.

Therefore, the slope of the tangent line at $(3, -6)$ is $f'(3) = 2(3) - 8 = -2$.

An equation of the tangent line, shown in Figure 2.70, is

$y - (-6) = (-2)(x - 3)$ or $y = -2x$. ■

Rates of Change

Suppose the variable y represents a quantity that depends upon another quantity, represented by x. Then y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x or the *increment* of x, is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$. The difference quotient has many interpretations, depending on the context. Graphically, it can be interpreted as the slope of a secant line.

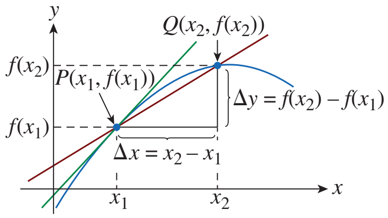


Figure 2.71
The slope of the secant line PQ is the average rate of change of y with respect to x over the interval $[x_1, x_2]$. The instantaneous rate of change is the slope of the tangent line to the graph of f at P.

Consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1, or equivalently, $\Delta x \to 0$. The limit of this average rate of change is called the **(instantaneous) rate of change of y with respect to x** at $x = x_1$. Graphically, this limit is interpreted as the slope of the tangent line to the graph of $y = f(x)$ at $P(x_1, f(x_1))$. See Figure 2.71.

$$\text{Instantaneous rate of change of } f \text{ at } x_1 = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad (4)$$

A Closer Look

1. Note that the limit Equation 4 is the derivative of f at x_1, $f'(x_1)$.
2. This limit also leads to a more general interpretation of the derivative $f'(a)$. The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.
3. Since $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, the units for the derivative are associated with the difference quotient.

   For example, suppose the change in y, Δy, is measured in feet, and the change in x, Δx, is measured in seconds.

   The units for $f'(x)$ are feet per second, often written as ft/s.
4. Interpretations:
   Given a function $y = f(x)$, the instantaneous rate of change at $x = a$, $f'(a)$, is the slope of the tangent line to the graph of f at the point where $x = a$. Therefore, when the derivative is large, the slope of the tangent line is steep near $x = a$. And when the derivative is small, the slope of the tangent line is relatively flat near $x = a$. See Figure 2.72.

   If $s = f(t)$ is a position function for a particle moving along a straight line, then $f'(t)$ is the rate of change of position s with respect to time t. Therefore, $f'(a)$ is the velocity of the particle at time $t = a$. The **speed** of the particle is defined to be the absolute value of the velocity, that is $|f'(a)|$.

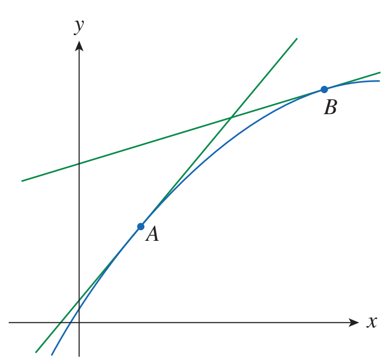


Figure 2.72
The slope of the graph near A is steep, and the slope of the graph near B is close to 0.

Note that velocity is a *vector,* that is, it has both magnitude and direction (velocity can be either positive or negative). However, speed is a *scalar,* that is, it has only magnitude (it is always a nonnegative number). ■

Example 6 Derivative of a Cost Function

A manufacturer produces bolts of fabric with a fixed width. The cost of producing x yards of fabric is $C = f(x)$ dollars.

(a) Explain the meaning of the derivative $f'(x)$ and indicate the units of measure.

(b) Explain the meaning of $f'(1000) = 9$ in the context of this problem.

(c) Which is likely to be greater, $f'(50)$ or $f'(500)$? Explain your reasoning.

Solution

(a) The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x. $f'(x)$ is the rate of change of production cost with respect to the number of yards produced. In economic theory, this is called the *marginal cost.*
The change in cost, ΔC, is measured in dollars.
The change in production, Δx, is measured in yards.
Therefore, the units for $f'(x)$ are dollars per yard, or dollars/yard.

(b) $f'(1000) = 9$ means that at the instant of manufacturing 1000 yards of fabric, the production cost is increasing at the rate of 9 dollars per yard.

(c) The economies of scale suggest that as production increases, the cost of producing each item decreases.
Because of this theory, we expect the rate at which production cost is increasing to be lower when $x = 500$ than when $x = 50$.
Therefore, $f'(50) > f'(500)$. ■

Example 7 The National Debt

Let $D(t)$ be the U.S. national debt, measured in billions of dollars, at time t, measured in years. Table 2.13 gives the approximate values of this function at selected times by providing year-end estimates. Estimate the value of $D'(2017)$ and indicate the units of measure.

t (year)	$D(t)$ (B dollars)
1985	1945.9
1990	3364.8
1995	4988.7
2000	5662.2
2005	8170.4
2010	14,025.2
2015	18,922.2
2020	27,747.8

Table 2.13
The U.S. national debt at the end of selected years.

Solution

We assume that the national debt did not fluctuate wildly between 2015 and 2020.

Therefore, we can estimate $D'(2017)$, the rate of change of the national debt in 2017, by finding the average rate of change close to 2017.

$$D'(2017) \approx \frac{D(2020) - D(2015)}{2020 - 2015} = \frac{27747.8 - 18922.2}{5} = 1765.12$$

The units are billions of dollars per year. ■

There are many other real-world applications of rates of change, and we will present several throughout this text. Here are a few examples: in physics, the rate of change of work with respect to time is called *power*. Chemists who study a chemical reaction are interested in the rate of change in the concentration of reactant with respect to time, called the *rate of reaction.* A biologist may be interested in the rate of change of the

population of a colony of bacteria with respect to time. A sociologist may be interested in the rate of cultural change with respect to new inventions, products, or discoveries. Certain rates of change are important in almost all of the natural sciences, in engineering, and in the social sciences. The important concept here is that the derivative is the instantaneous rate of change and there are many interpretations.

All of these rates of change are derivatives and can, therefore, be interpreted as slopes of tangent lines. This provides added significance to the solution of the tangent line problem. Whenever we solve a problem involving tangent lines, we are not just solving a problem in geometry. We are also implicitly solving a wide variety of problems involving rates of change in science and engineering.

2.6 Exercises

1. Consider the function $y = f(x)$.
(a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.
(b) Write an expression for the slope of the tangent line to the graph of f at P.

2. Let $f(x) = 4x - 3$. Find the limit of each difference quotient.

(a) $\lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1}$ (b) $\lim\limits_{x \to -2} \dfrac{f(x) - f(-2)}{x - (-2)}$
(c) $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

3. Graph the function $f(x) = e^x$ in the viewing rectangles $[-1, 1] \times [0, 2]$, $[-0.5, 0.5] \times [0.5, 1.5]$, and $[-0.1, 0.1] \times [0.9, 1.1]$. Explain how the graph changes as you zoom in near the point $(0, 1)$.

4. Let $f(x) = 4x - x^2$.
(a) Find the slope of the tangent line to the graph of f at the point $(1, 3)$ using Definition 2 and by using Equation 2.
(b) Use your results in part (a) to write an equation for the tangent line.
(c) Use technology to graph f and the tangent line in the same viewing rectangle. Explain the relationship between the graph of f and the tangent line as you zoom in near the point $(1, 3)$.

5. Let $f(x) = x - x^3$.
(a) Find the slope of the tangent line to the graph of f at the point $(1, 0)$ using Definition 2 and by using Equation 2.
(b) Use your results in part (a) to write an equation for the tangent line.
(c) Use technology to graph f and the tangent line in the same viewing rectangle. Explain the relationship between the graph of f and the tangent line as you zoom in near the point $(1, 0)$.

Find an equation of the tangent line to the graph of f at the given point.

6. $f(x) = 4x - 3x^2$, $(2, -4)$

7. $f(x) = x^3 - 3x + 1$, $(2, 3)$

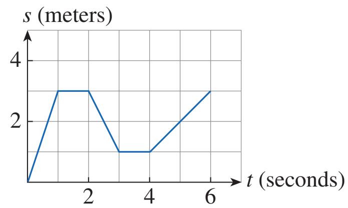
8. $f(x) = \sqrt{x}$, $(1, 1)$

9. $f(x) = \dfrac{2x + 1}{x + 2}$, $(1, 1)$

10. Let $f(x) = 3 + 4x^2 - 2x^3$.
(a) Find the slope of the tangent line to the graph of f at the point where $x = a$.
(b) Find an equation of the tangent line to the graph of f at the point $(1, 5)$.
(c) Find an equation of the tangent line to the graph of f at the point $(2, 3)$.
(d) Use technology to graph f and the two tangent lines in the same viewing rectangle.

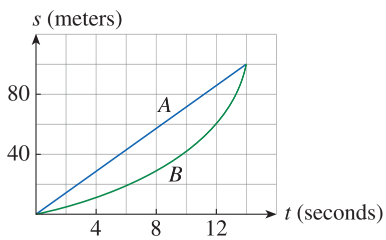
11. Let $f(x) = \dfrac{1}{\sqrt{x}}$.
(a) Find the slope of the tangent line to the graph of f at the point where $x = a$.
(b) Find an equation of the tangent line to the graph of f at the point $(1, 1)$.
(c) Find an equation of the tangent line to the graph of f at the point $\left(4, \dfrac{1}{2}\right)$.
(d) Use technology to graph f and the two tangent lines in the same viewing rectangle.

12. For $0 \le t \le 6$ a particle moves along a horizontal line. The particle initially moves to the right, and the graph of its position function is shown in the figure.



(a) When is the particle moving to the right? To the left? Standing still?
(b) Sketch a graph of the velocity function.

13. The graphs of the position functions for two runners, A and B, are shown in the figure. The runners competed in a 100-meter race and finished in a tie.



(a) Describe and compare how each person ran in this race.
(b) At what time was the distance between the runners the greatest? Explain your reasoning.
(c) At what time did the runners have the same velocity? Explain your reasoning.

14. A ball is thrown into the air with velocity of 40 ft/s. Its height, in feet, after t seconds is given by $s(t) = 40t - 16t^2$. Find the velocity of the ball at time $t = 2$ seconds.

15. A rock is thrown upward on the planet Mars with initial velocity of 10 m/s. Its height, in meters, after t seconds is given by $H(t) = 10t - 1.86t^2$.
(a) Find the velocity of the rock at time $t = 1$ second.
(b) Find the velocity of the rock at time $t = a$ seconds.
(c) When will the rock hit the surface?
(d) What is the velocity of the rock at the instant it hits the surface?

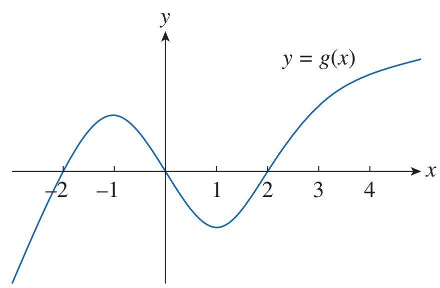
16. A particle is moving along a straight line. For $t > 1/2$, its position, in meters, is given by the equation $s = \dfrac{1}{t^2}$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

17. The position of a particle, in feet, moving along a straight line is given by $s(t) = \dfrac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.

(a) Find the average velocity over each time interval.
(i) [4, 8] (ii) [6, 8]
(iii) [8, 10] (iv) [8, 12]
(b) Find the instantaneous velocity of the particle when $t = 8$.
(c) Sketch the graph of s. Add to this graph the secant lines associated with the intervals in part (a). Add the tangent line with the slope equal to the instantaneous velocity in part (b).

18. The position of a particle, in meters, moving along a straight line is given by $s(t) = t + \sqrt{t}$, where $t \ge 0$ is measured in seconds.
(a) Find the average velocity over each time interval.
(i) [4, 5] (ii) [4, 4.5]
(iii) [4, 4.1] (iv) [4, 4.01]
(b) Use part (a) to estimate the instantaneous velocity of the particle at time $t = 4$.
(c) Find the instantaneous velocity of the particle when $t = 4$.

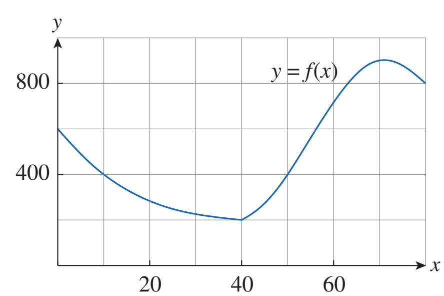
19. The graph of the function g is shown.



Arrange the following numbers in increasing order. Explain your reasoning.

$0 \quad g'(-2) \quad g'(0) \quad g'(2) \quad g'(4)$

20. The graph of a function f is shown.



(a) Find the average rate of change of f on the interval [20, 60].
(b) Find an interval on which the average rate of change of f is 0.
(c) Which interval gives a larger average rate of change, [40, 60] or [40, 70]?
(d) Find the value of $\dfrac{f(40) - f(10)}{40 - 10}$. What does this value represent graphically?

21. Refer to the graph of the function f in Exercise 20.
(a) Estimate the value of $f'(50)$.
(b) Is $f'(10) > f'(30)$? Explain your reasoning.
(c) Is $f'(60) > \dfrac{f(80) - f(40)}{80 - 40}$? Explain your reasoning.

---

22. Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

23. Let $g(x) = 3x^2 - 7x + 2$. Find the value of a such that

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = 11$$

24. Let $f(t) = t^4 - 2t^2$ for $t > 0$. Find the values of a such that

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = 0$$

25. Let k be a constant and n a positive integer. Use the alternate definition of the derivative of a function at a

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

to find $f'(a)$ for each function.
(a) $f(x) = kx$ (b) $f(x) = kx^2$ (c) $f(x) = kx^3$
(d) $f(x) = kx^4$ (e) $f(x) = kx^n$

26. An equation of the tangent line to the graph of $y = f(x)$ at the point where $x = 2$ is $y = 4x - 5$. Find $f(2)$ and $f'(2)$.

27. Suppose the tangent line to the graph of $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$. Find $f(4)$ and $f'(4)$.

28. Sketch the graph of a function f such that $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

29. Sketch the graph of a function g such that
$g(0) = g(2) = g(4) = 0, \quad g'(1) = g'(3) = 0,$
$g'(0) = g'(4) = 1, \quad g'(2) = -1,$
$\lim_{x \to \infty} g(x) = \infty, \quad \lim_{x \to -\infty} g(x) = -\infty.$

30. Sketch the graph of a function g that is continuous on its domain $(-5, 5)$ such that
$g'(0) = 1, \quad g'(-2) = 0,$
$\lim_{x \to -5^+} g(x) = \infty, \quad \lim_{x \to 5^-} g(x) = 3.$

31. Sketch the graph of a function f with domain $(-2, 2)$ such that $f'(0) = -2$, $\lim_{x \to 2^-} f(x) = \infty$, f is continuous at all numbers in its domain except ± 1, and f is an odd function.

32. Let $f(x) = 3x^2 - x^3$. Find $f'(1)$ and use it to write an equation of the tangent line to the graph of f at the point $(1, 2)$.

33. Let $g(x) = x^4 - 2$. Find $g'(1)$ and use it to write an equation of the tangent line to the graph of g at the point $(1, -1)$.

34. Let $F(x) = \dfrac{5x}{1 + x^2}$.
(a) Find $F'(2)$ and use it to write an equation of the tangent line to the graph of F at the point $(2, 2)$.
(b) Sketch the graph of F and the tangent line in the same viewing rectangle.

35. Let $G(x) = 4x^2 - x^3$.
(a) Find $G'(a)$ and use it to find equations of the tangent lines to the graph of G at the points $(2, 8)$ and $(3, 9)$.
(b) Sketch the graph of G and the two tangent lines in the same viewing rectangle.

Find $f'(a)$.

36. $f(x) = 3x^2 - 4x + 1$

37. $f(t) = 2t^3 + t$

38. $f(t) = \dfrac{2t + 1}{t + 3}$

39. $f(x) = x^{-2}$

40. $f(x) = \sqrt{1 - 2x}$

41. $f(x) = \dfrac{4}{\sqrt{1 - x}}$

42. $f(x) = (x - 2)(x + 2)$

43. $f(x) = (x - 2)(x - 2)$

The limit represents the derivative of a function $y = f(x)$ at a number a. Find the function f and the value of a.

44. $\displaystyle\lim_{h \to 0} \frac{\sqrt{9 + h} - 3}{h}$

45. $\displaystyle\lim_{h \to 0} \frac{e^{-2+h} - e^{-2}}{h}$

46. $\displaystyle\lim_{x \to 2} \frac{x^6 - 64}{x - 2}$

47. $\displaystyle\lim_{x \to 1/4} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}}$

48. $\displaystyle\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$

49. $\displaystyle\lim_{\theta \to \pi/6} \frac{\sin\theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$

50. $\displaystyle\lim_{h \to 0} \frac{\ln(2 + h) - \ln 2}{h}$

51. $\displaystyle\lim_{x \to 1} \frac{\tan^{-1} x - \frac{\pi}{4}}{x - 1}$

A particle moves along a straight line. For $t \geq 0$, the position of the particle is given by $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity and speed of the particle at time $t = 4$ seconds.

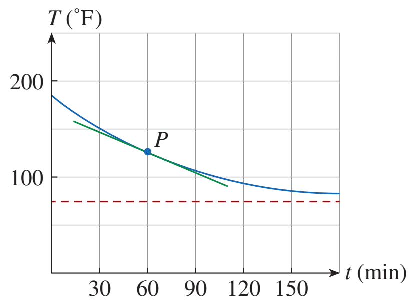
52. $f(t) = 80t - 6t^2$

53. $f(t) = 10 + \dfrac{45}{t + 1}$

---

54. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Compare the rate of change of temperature of the soda shortly after it is placed in the refrigerator to the rate of change of temperature 1 hour later. Explain your reasoning.

55. A roast turkey is taken from an oven when its internal temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. The graph shows how the temperature of the turkey decreases and gradually approaches room temperature. By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



56. Researchers measured the average amount of penicillin $P(t)$ in 12 patients being treated for a respiratory infection, starting 1 hour after the initial dose of 250 mg. Selected values of $P(t)$ are given in the table.

t (hours)	1	2	3	4	5	6
$P(t)$ (mg)	167	112	75	50	33	22

(a) Find the average rate of change of P with respect to t over each time interval. Indicate the units of measure.
(i) [1, 5] (ii) [1, 4]
(iii) [1, 3] (iv) [1, 2]
(b) Estimate the instantaneous rate of change at $t = 2$ and interpret your result in the context of the problem. Indicate the units of measure.

57. A certain coffeehouse chain has been adding stores all over the world for many years. The number of locations is given by $N(t)$, were t is measured in years. Selected values for $N(t)$ are given in the table.

t	2015	2016	2017	2018	2019
$N(t)$	23,043	25,085	27,339	29,324	31,256

(a) Find the average rate of growth
(i) from 2015 to 2017 (ii) from 2017 to 2019
Indicate the units of measure. What do your answers suggest about the growth of locations?
(b) Estimate the instantaneous growth in 2018. Indicate the units of measure.

58. The table shows the world average daily oil consumption from 1985 to 2020, measured in thousands of barrels per day.

Years since 1985	Thousands of barrels of oil per day
0	60,083
5	66,533
10	70,099
15	76,784
20	84,077
25	87,302
30	93,770
35	100,300

(a) Find the average rate of change from 2010 to 2015. Indicate the units of measure. Interpret this value in the context of the problem.
(b) Estimate the instantaneous rate of change in 2017. Indicate the units of measure.

59. The table gives selected values of $V(t)$, the value of a BMW 550i, measured in dollars, t years after the initial sale.

t	1	2	3	4	5
$V(t)$	60,000	54,000	48,600	43,740	39,366

(a) Find the average rate of change of V with respect to t over each time interval. Indicate the units of measure.
(i) [1, 3] (ii) [2, 3]
(iii) [3, 5] (iv) [4, 5]
(b) Estimate the value of $V'(5)$. Interpret your answer in the context of the problem.

60. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.
(a) Find the average rate of change of C with respect to x when the production level is changed
(i) from $x = 100$ to $x = 105$.
(ii) from $x = 100$ to $x = 101$.
(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. Indicate the units of measure.

61. Water stored in a cylindrical tank is being drained from the bottom. The amount of water in the tank at time t is given by the function V, where $V(t)$ is measured in gallons and t is measured in minutes.

$$V(t) = 100{,}000\left(1 - \frac{1}{60}t\right)^2 \qquad 0 \le t \le 60$$

At time $t = 0$ there are 100,000 gallons of water in the tank. Find the rate at which the water is flowing out of the tank as a function of t (the instantaneous rate of change of V with respect to t). Indicate the units of measure. For times $t = 0$, 10, 20, 30, 40, 50, and 60 minutes, find the flow rate and the amount of water remaining in the tank. Describe the rate at which water is flowing out of the tank over the interval [0, 60]. When is the flow rate the greatest? The least?

62. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.
(a) Interpret the derivative $f'(x)$ in the context of this problem. Indicate the units of measure.
(b) Interpret the expression $f'(800) = 17$.
(c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about in the long term? Explain your reasoning.

63. Bacteria are growing in a controlled laboratory experiment and the number of bacteria present after t hours is $n = f(t)$.
(a) Interpret the derivative $f'(5)$ in the context of this problem. Indicate the units of measure.
(b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f'(5)$ of $f'(10)$? If the supply of nutrients is limited, does this change your answer? Explain your reasoning.

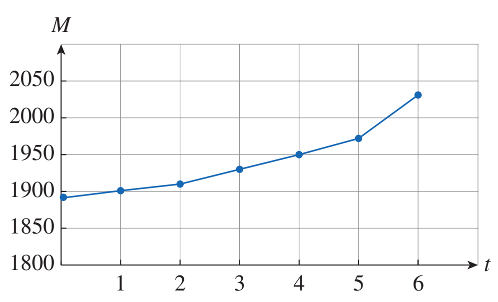
64. Let $H(t)$ be the daily cost (in dollars) to heat an office building when the outside temperature is t degrees Fahrenheit.
(a) Explain the meaning of $H'(58)$ in the context of this problem. Indicate the units of measure.
(b) Would you expect $H'(58)$ to be positive or negative? Explain your reasoning.

65. The quantity (in pounds) of a gourmet ground coffee that is sold by a certain company at a price p dollars per pound is $Q = f(p)$.
(a) Explain the meaning of $f'(8)$ in the context of this problem. Indicate the units of measure.
(b) Is $f'(8)$ positive or negative? Explain your reasoning.

66. Let $T(t)$ be the temperature (in °F) in Naples, Florida, t hours after noon on February 23, 2021. The table shows selected values of this function. Explain the meaning of $T'(4)$. Estimate its value.

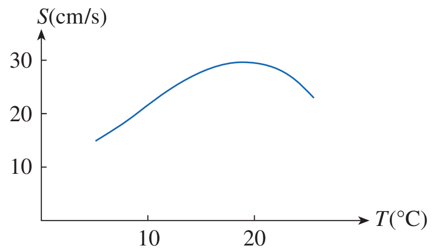
t	0	1	2	3	4	5	6
$T(t)$	78	79	80	80	78	77	75

67. Improved breeding has allowed farmers to get more milk from cows. The graph shows the average amount of milk produced per cow M as a function of time ($t = 0$ corresponds to 2014).



(a) Explain the meaning of $M'(t)$ in the context of this problem.
(b) Estimate the value of $M'(5)$ and interpret this value in the context of the problem.
Source: Based on information from USDA

68. The graph shows the influence of temperature on the maximum sustainable swimming speed S of Coho salmon.



(a) Explain the meaning of $S'(T)$ in the context of this problem. Indicate the units of measure.
(b) Estimate the values of $S'(15)$ and $S'(25)$ and interpret each in the context of this problem.

Determine whether $f'(0)$ exists.

69. $f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

70. $f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

---

71. Let $f(x) = \sin x - \dfrac{1}{1000}\sin(1000x)$.
(a) Graph the function f in the viewing rectangle $[-2\pi, 2\pi] \times [-4, 4]$. Describe the slope of the graph near the origin.
(b) Use the viewing rectangle $[-0.4, 0.4] \times [-0.25, 0.25]$ to estimate the value of $f'(0)$. Does this estimate agree with your description in part (a)?
(c) Graph the function f in the viewing rectangle $[-0.008, 0.008] \times [-0.005, 0.005]$ and estimate the value of $f'(0)$.
(d) Find a viewing rectangle in which you think the estimate of $f'(0)$ is best. Explain your reasoning.

Writing Project | Early Methods for Finding Tangents

The first person to formulate explicitly the ideas of limit and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that "If I have seen further than other men, it is because I have stood on the shoulders of giants." Two of those giants were Pierre Fermat (1601–1665) and Newton's mentor at Cambridge, Isaac Barrow (1630–1677). Newton was familiar with the methods that these men used to find tangent lines, and their methods played a role in Newton's eventual formulation of calculus.

The following references contain explanations of these methods. Read one or more of the references and write an essay comparing the methods of either Fermat or Barrow to modern methods. In particular, use the method of Section 2.6 to find an equation of the tangent line to the curve $y = x^3 + 2x$ at the point (1, 3) and show how either Fermat or Barrow would have solved the same problem. Although your solution involved derivatives and Fermat and Barrow used different approaches, discuss the similarities between the methods.

(1) Carl Boyer and Uta Merzbach, *A History of Mathematics* (New York: Wiley, 1989), pp. 389, 432.

(2) C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 124, 132.

(3) Howard Eves, *An Introduction to the History of Mathematics,* 6th ed. (New York: Saunders, 1990), pp. 391, 395.

(4) Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 344, 346.

2.7 The Derivative as a Function

In Section 2.6 we found the limit expression for the derivative of a function at a:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

We also discovered that it can be tedious to use this definition repeatedly to find the derivative at several values of a. So it seems reasonable to let a be any arbitrary value, that is, replace a by the variable x.

This definition is commonly called the **Limit Definition of the Derivative**.

$$f'(x) = \lim_{h \to x} \frac{f(x+h) - f(x)}{h} \tag{2}$$

A Closer Look

1. In words, Equation 2 says that for any number x for which the limit exists, we assign to x the number $f'(x)$. Conceptually, the word *assign* implies a function. Therefore, f' is a new function, called the **derivative of *f***.
2. We have already learned that for any x, the value $f'(x)$ can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.
3. The function f' is called the derivative of f because it has been *derived* from f via the limit expression in Equation 2.
4. The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$. Note that the domain of f' is always a subset of the domain of f and so may be smaller than the domain of f. ■

Example 1 Graph of f and f'

The graph of a function f is shown in Figure 2.73. Use this graph to sketch the graph of the derivative of f'.

Solution

We know that $f'(x)$ is the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Therefore, we can estimate the derivative at any value of x by drawing the tangent line at $(x, f(x))$ and estimating its slope.

For $x = 1$ the slope of the tangent line is approximately 1, so $f'(1) = 1$.

The point (1, 1) is on the graph of f'. In Figure 2.73 this point is located directly beneath P.

Note: On the graph of f', the y-axis now represents the slope of the graph of the function f at each x-value. Therefore, where the graph of f' is above the x-axis, the graph of f must have positive slope, and where the graph of f' is below the x-axis, the graph of f must have negative slope.

For $x = 3$ the slope of the tangent line is approximately -2.25, so $f'(3) = -2.25$.

The point $(3, -2.25)$ is on the graph of f'. In Figure 2.73 this point is located directly beneath Q.

For every value of x, the slope of the graph of f is the y-value on the graph of f'.

Continue in this manner, finding slopes and plotting points.

The graph of f' is shown beneath the graph of f in Figure 2.73.

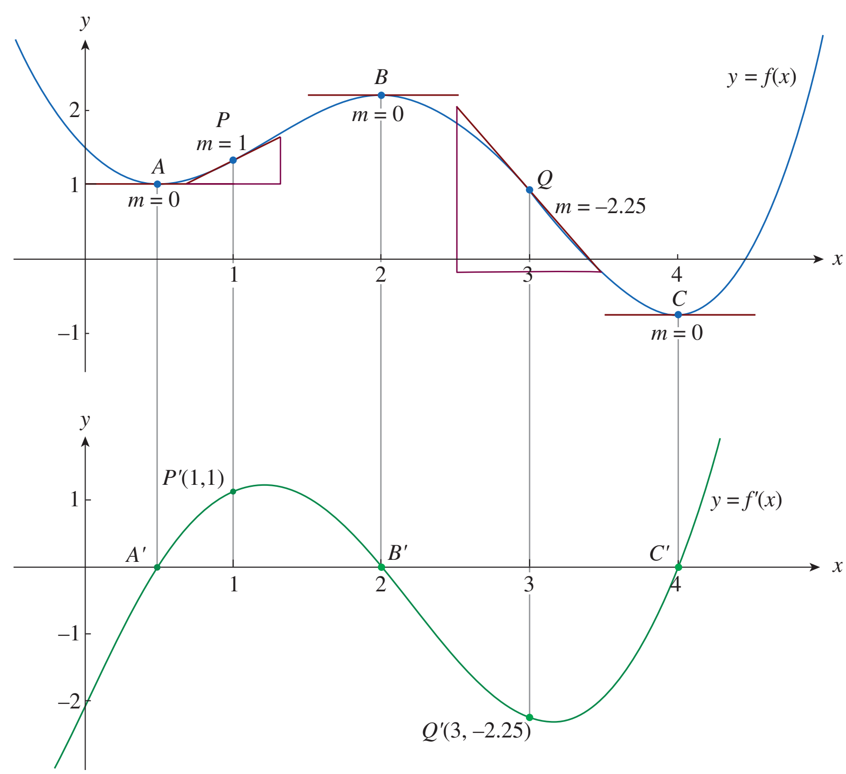


Figure 2.73
Graph of f and f'.

Note that the tangent lines at the points A, B, and C are horizontal.

The derivative is 0 whenever the tangent line is horizontal, and therefore the graph of f' crosses the x-axis (where $y = 0$) at the points A', B', and C'.

Between A and B, the slope of the tangent line is always positive.

Therefore, $f'(x)$ is positive in this interval, and the graph of f' is above the x-axis.

Between B and C, the slope of the tangent line is always negative.

Therefore, $f'(x)$ is negative in this interval, and the graph of f' is below the x-axis. ■

Example 2 Find $f'(x)$ Using the Definition

Let $f(x) = x^3 - x$.

(a) Find a formula, or analytic expression, for $f'(x)$.

(b) Illustrate the relationship between f and f' by graphing both functions on the same coordinate axes.

Solution

Note: Figure 2.74 shows the graphs of f and f' on the same coordinate axes, which is useful in order to understand the relationships between the two graphs. Remember, the y-axis represents the values associated with each function but has a different interpretation for each graph. For the graph of f, the y-axis represents the value of the function f at each value x. For the graph of f', the y-axis represents the value of f' at each x, the slope of f at x.

(a) Use Equation 2, the definition of the derivative of f.

Remember, the variable is h and x is held constant in this limit expression.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \text{Equation 2.}$$

$$= \lim_{h\to 0}\frac{[(x+h)^3-(x+h)]-[x^3-x]}{h} \qquad \text{Definition of } f.$$

$$= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x-h-x^3+x}{h} \qquad \text{Expand; distribute.}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-h}{h} = \lim_{h\to 0}\frac{h(3x^2+3xh+h^2-1)}{h} \qquad \text{Simplify; factor.}$$

$$= \lim_{h\to 0}(3x^2+3xh+h^2-1) = 3x^2-1 \qquad \text{Cancel } h\text{; direct substitution.}$$

(b) Figure 2.74 shows the graphs of f and f' on the same coordinate axes.

$f'(x) = 0$ wherever f has a horizontal tangent line.

$f'(x)$ is positive everywhere the slope of the tangent line to the graph of f is positive.

$f'(x)$ is negative everywhere the slope of the tangent line to the graph of f is negative.

The graphs illustrate the relationship between f and f' and helps confirm the expression for $f'(x)$. ■

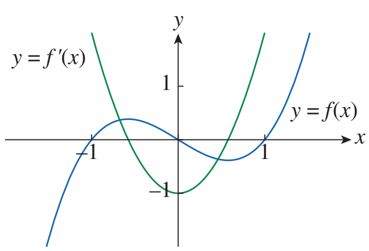


Figure 2.74
Graphs of f and f' on the same coordinate axes.

Example 3 Find $f'(x)$ and Its Domain

Let $f(x) = \sqrt{x}$. Find the derivative of f and the domain of f'.

Solution

Use the definition of the derivative of f.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \qquad \text{Equation 2; } f(x).$$

$$= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \qquad \text{Rationalize the numerator.}$$

$$= \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} \qquad \text{Simplify.}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \qquad \text{Cancel } h\text{; direct substitution.}$$

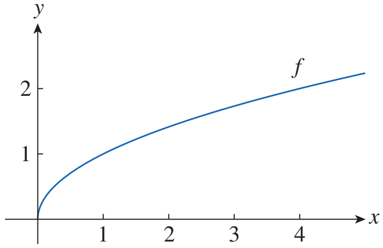


Figure 2.75
Graph of $f(x) = \sqrt{x}$.

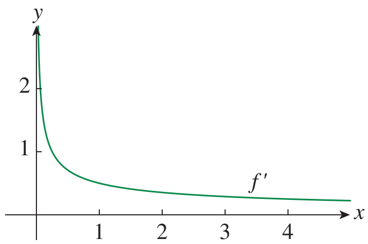


Figure 2.76
Graph of $f'(x) = \dfrac{1}{2\sqrt{x}}$.

The domain of f is $[0, \infty)$.

f' exists only if $x > 0$. Therefore, the domain of f' is $(0, \infty)$.

Note that the domain of f' is a subset of the domain of f and, in this case, contains one less value.

Let's check to make sure this result is reasonable by looking at the graphs of f and f' in Figures 2.75 and 2.76.

If x is close to 0 (and positive), then $\sqrt{x}$ is close to 0 and $\dfrac{1}{2\sqrt{x}}$ is very large.

This corresponds to steep tangent lines to the graph of f near (0, 0) and large values of $f'(x)$ just to the right of $x = 0$.

If x is large, then $\dfrac{1}{2\sqrt{x}}$ is small.

This corresponds to nearly flat tangent lines to the graph of f as x increases and the horizontal asymptote on the graph of f'.

Note that the function $f(x)$ is increasing everywhere it is defined (over its entire domain). Therefore, $f'(x)$ is always positive and its graph is above the x-axis over its domain. ■

Example 4 Derivative of a Rational Function

Find f' if $f(x) = \dfrac{1-x}{2+x}$.

Solution

Once again, use Equation 2, the definition of the derivative of f.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{1-(x+h)}{2+(x+h)} - \dfrac{1-x}{2+x}}{h} && \text{Equation 2; } f(x).\\
&= \lim_{h\to 0}\frac{(1-x-h)(2+x)-(1-x)(2+x+h)}{h(2+x+h)(2+x)} && \text{Common denominator in the numerator; write as one fraction.}\\
&= \lim_{h\to 0}\frac{(2-x-2h-x^2-xh)-(2-x+h-x^2-xh)}{h(2+x+h)(2+x)} && \text{Expand products in the numerator; simplify.}\\
&= \lim_{h\to 0}\frac{-3h}{h(2+x+h)(2+x)} && \text{Simplify the expression in the numerator.}\\
&= \lim_{h\to 0}\frac{-3}{(2+x+h)(2+x)} = -\frac{3}{(2+x)^2} && \text{Cancel } h\text{; direct substitution.}
\end{aligned}$$

■

■ Other Notations

If we use the traditional notation $y = f(x)$ to indicate that y is a function of x, or that the independent variable is x and the dependent variable is y, we can denote the derivative in several ways:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and $\dfrac{d}{dx}$ are called **differential operators** because they indicate an operation, namely **differentiation**: they *operate* on a function.

Remember, the symbol $\dfrac{dy}{dx}$ is not a fraction. This notation was introduced by Leibniz and is simply a synonym for $f'(x)$. However, it is very useful and conveys meaning, especially when used in conjunction with increment notation. We can rewrite the definition of the derivative using Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

In order to indicate the value of a derivative at a specific number a using Leibniz notation, we write

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{or} \quad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$. We often read the vertical bar or right bracket as *evaluated at.*

Differentiability on an Interval

In the same way that we applied the concept of continuity to intervals, we now extend the definition of the derivative of a function to intervals.

Definition • Differentiability of a Function

A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, b)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example 5 The Absolute Value Function and Differentiability

Where is the function $f(x) = |x|$ differentiable?

Solution

Start with the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

In order to evaluate an absolute value expression, we need to know whether the argument is less than, or greater than, or equal to 0. Consider three cases.

Case I If $x > 0$, then we can choose h small enough so that $x + h > 0$. Then use the definition of absolute value:

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

Therefore, f is differentiable for any $x > 0$.

Case II If $x < 0$, then we can choose h small enough so that $x + h < 0$. Then use the definition of absolute value again:

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{-(x+h) - (-x)}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} -1 = -1$$

Therefore, f is differentiable for any $x < 0$.

Case III For $x = 0$, we need to consider

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h} \text{ if the limit exists.}$$

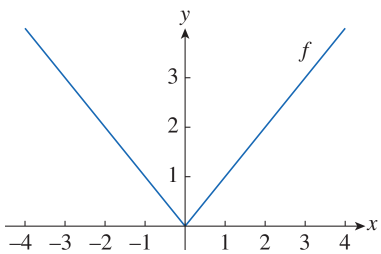


Figure 2.77
Graph of $f(x) = |x|$.

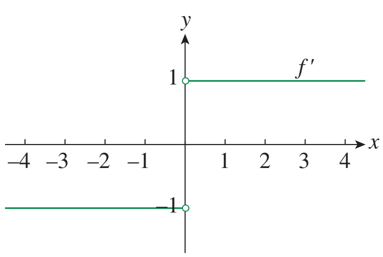


Figure 2.78
Graph of $y = f'(x)$.

To evaluate this limit, consider left and right limits separately.

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$$

Since these two one-sided limits are different, $f'(0)$ does not exist.

f is differentiable at all real numbers x except 0, even though this function is continuous for all real numbers.

Here is an expression for f', $f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

The graphs of f and f' are shown in Figures 2.77 and 2.78.

Geometrically, the graph of f does not have a tangent line at the point (0, 0). Therefore, the derivative of f does not exist at $x = 0$. ■

We now connect the two very important properties of continuity and differentiability in a theorem.

Theorem • Differentiability Implies Continuity

If f is differentiable at a, then f is continuous at a.

Proof

To prove that f is continuous at a, we must show that $\lim\limits_{x\to a} f(x) = f(a)$.

Start this proof by showing that the difference $f(x) - f(a)$ approaches 0 as x approaches a.

Since f is differentiable at a,

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$ Alternate definition of the derivative at a.

We need to connect $f(x) - f(a)$ to this limit.

$$\begin{aligned} \lim_{x\to a} [f(x) - f(a)] &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a) && \text{Multiply by 1 in a convenient form.} \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a}(x - a) && \text{Product Law (for limits).} \\ &= f'(a) \cdot 0 = 0 \end{aligned}$$

Use this limit result in the definition of continuity.

$$\begin{aligned} \lim_{x\to a} f(x) &= \lim_{x\to a} [f(a) + (f(x) - f(a))] && \text{Add 0 in a convenient form.} \\ &= \lim_{x\to a} f(a) + \lim_{x\to a} [f(x) - f(a)] && \text{Limit of a sum.} \\ &= f(a) + 0 = f(a) \end{aligned}$$

Therefore, f is continuous at a. ■

Note: The converse of this theorem is not true.

If f is continuous at a, we cannot say for certain whether f is differentiable at a. The function f may or may not be differentiable at a.

For example, the function $f(x) = |x|$ is continuous at $x = 0$, but we showed in Example 5 that f is not differentiable at $x = 0$.

Nondifferentiable Functions

We have shown (analytically) that the function $f(x) = |x|$ is not differentiable at $x = 0$. Geometrically, the graph of f (Figure 2.77) changes direction sharply, or abruptly, at $x = 0$. In general, if the graph of f has a sharp corner or cusp, or isn't *smooth,* at $(a, f(a))$, then the graph of f has no tangent line at this point, and f is not differentiable there. Analytically, if we try to compute $f'(a)$ by definition, then the limit does not exist.

The previous theorem provides another example in which a function is not differentiable. The *contrapositive* of the theorem states that if f is not continuous at $a,$ then f is not differentiable at a. Therefore, a function f is not differentiable at any discontinuity, for example, a jump discontinuity.

Remember that a vertical line has no slope.

Another case of a function that is *not differentiable* occurs if the graph of the function has a **vertical tangent line** at $x = a$; that is, suppose f is continuous at $x = a$ and $\lim\limits_{x \to a} |f'(x)| = \infty$.

This limit result means that the tangent lines become steeper and steeper as $x \to a$. Figure 2.79 shows one way this can happen.

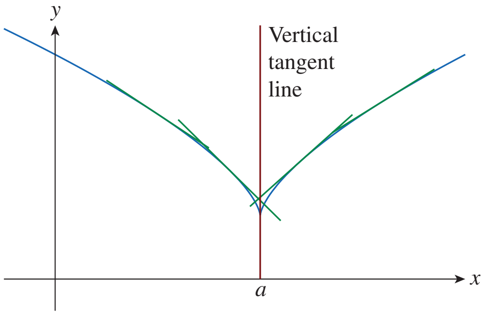
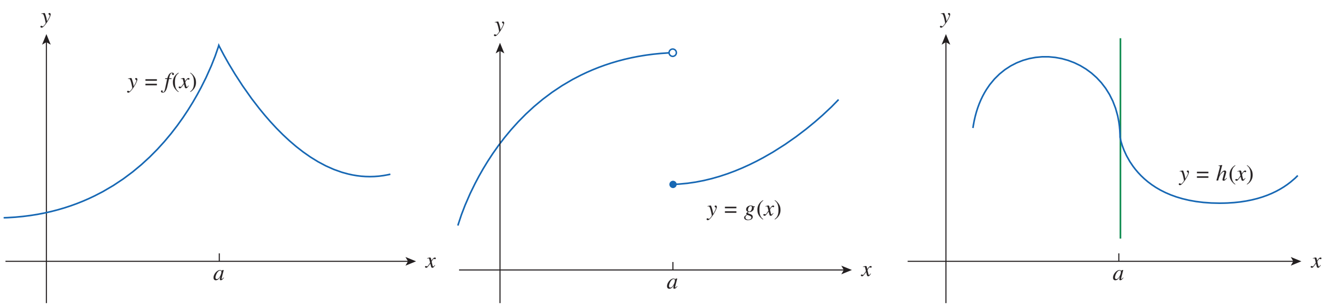


Figure 2.79
The tangent lines (green) become steeper and steeper as x approaches a.

Figures 2.80(a)–2.80(c) illustrate the three ways in which a function is not differentiable at a.



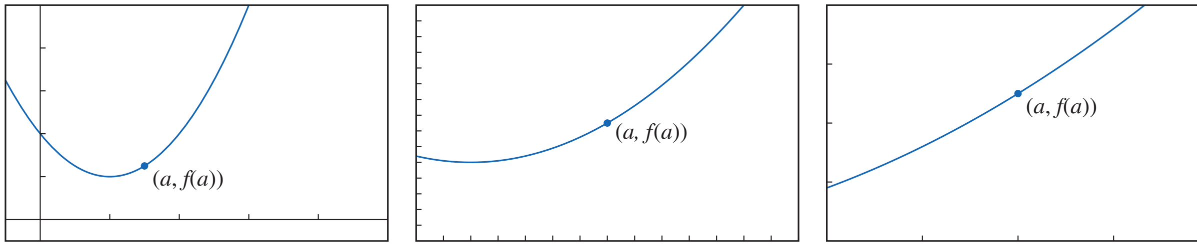
(a) f is not differentiable at a because the graph of f has a sharp comer at a.

(b) g is not differentiable at a because g is discontinuous at a.

(c) h is not differentiable at a because the graph of h has a vertical tangent at a.

Figure 2.80
Three ways in which a function is not differentiable at a.

Recall that technology provides a method to visualize differentiability. If f is differentiable at $a,$ then as we zoom in near the point $(a, f(a))$, the graph straightens out and appears more and more like a line. See Figures 2.81(a)–2.81(c).



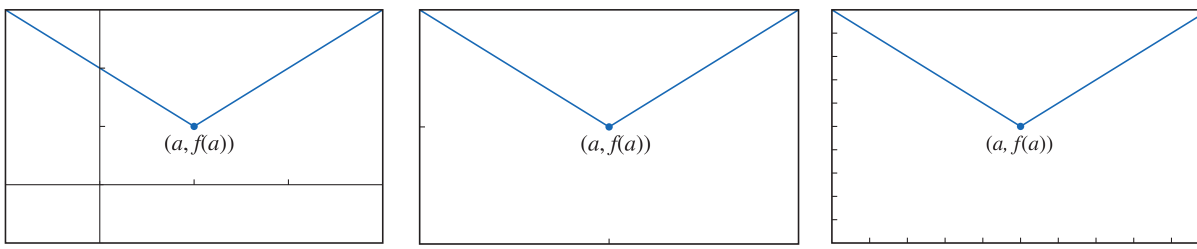
(a) $[-0.5, 5] \times [-0.5, 5]$; Graph of $y = f(x)$ and the point $(a, f(a))$

(b) $[0.8, 2.2] \times [0.8, 2.2] \times [0.5, 2]$; Start to zoom in near $(a, f(a))$ and the graph begins to look linear.

(c) $[1.3, 1.7] \times [1, 1.4]$; As we zoom in more, the graph of $y = f(x)$ looks like a straight line.

Figure 2.81
As we zoom in near $(a, f(a))$, the graph of f straightens out. The function f is differentiable at a.

However, no matter how much we zoom in near a point like the ones in Figures 2.79 and 2.80(a), the graph always has a sharp corner. See Figures 2.82(a)–2.82(c).



(a) $[-1, 3] \times [-1, 3]$; Graph of $y = f(x)$ and the point $(a, f(a))$

(b) $[0.5, 1.5] \times [0.5, 1.5]$; Start to zoom in near $(a, f(a))$ and the graph still has a sharp edge or corner at $(a, f(a))$.

(c) $[0.875, 1.125] \times [0.875, 1.125]$; As we zoom in more, the graph of $y = f(x)$ still has a sharp edge.

Figure 2.82
As we zoom in near $(a, f(a))$, we cannot eliminate the sharp point or corner. The function f is not differentiable at a.

Higher Order Derivatives

If f is a differentiable function, then its derivative f' is also a function. Therefore, we can consider the derivative of f' denoted by $(f')' = f''$. This new function, f'' is called the **second derivative** of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\underbrace{\frac{d}{dx}}_{\text{derivative of}} \underbrace{\left(\frac{dy}{dx}\right)}_{\text{first derivative}} = \underbrace{\frac{d^2y}{dx^2}}_{\text{second derivative}}$$

Example 6 Second Derivative

Let $f(x) = x^3 - x$. Find $f''(x)$ and sketch the graphs of f, f', and f'' on the same coordinate axes.

Solution

In Example 2, we found the first derivative: $f'(x) = 3x^2 - 1$.

Use the definition of the derivative to find $f''(x)$.

$$f''(x) = (f')'(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$ Definition of the derivative.

$$= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h}$$ Evaluate $f'(x)$.

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h}$$ Expand; distribute.

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h} = \lim_{h \to 0} \frac{h(6x + 3h)}{h}$$ Simplify; factor.

$$= \lim_{h \to 0} (6x + 3h) = 6x$$ Cancel h; direct substitution.

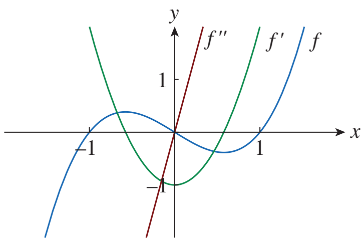


Figure 2.83
Graph of f, f', and f''.

The graphs of f, f', and f'' are shown in Figure 2.83.

We can interpret $f''(x)$ as the slope of the graph of the function $y = f'(x)$ at the point $(x, f'(x))$. It is the rate of change of the slope of $y = f(x)$.

Notice that $f''(x)$ is negative whenever $y = f'(x)$ has a negative slope,

and $f''(x)$ is positive whenever $y = f'(x)$ has a positive slope.

The sign of $f''(x)$ is also related to the *concavity* of the graph of f.

We will examine this characteristic in Section 4.3. ■

In general, we can interpret a second derivative as a rate of change of a rate of change. A common example of this is *acceleration,* defined as follows.

Suppose $s = s(t)$ is the position function of an object moving along a straight line. We have learned that the first derivative represents the velocity $v(t)$ of the object as a function of time.

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** of the object. So the acceleration function $a(t)$ is the derivative of the velocity function and is therefore the second derivative of the position function.

$$a(t) = v'(t) = s''(t)$$

or, using Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Acceleration is the change in velocity you experience when speeding up or slowing down in a moving vehicle.

Example 7 Graphing Velocity and Acceleration

A car starts from rest and the graph of its position function is shown in Figure 2.84, where s is measured in feet and t in seconds, Use this figure to graph the velocity and acceleration of the car. What is the acceleration at time $t = 2$ seconds?

Solution

Measure the slope of the graph of $s = f(t)$ at $t = 0, 1, 2, 3, 4,$ and 5, and use the method of Example 1 to plot the velocity function $v = f'(t)$, shown in Figure 2.85.

The acceleration at time $t = 2$ seconds is $a = f''(2)$, the slope of the tangent line to the graph of f' when $t = 2$.

Estimate the slope of this tangent line as shown in Figure 2.85:

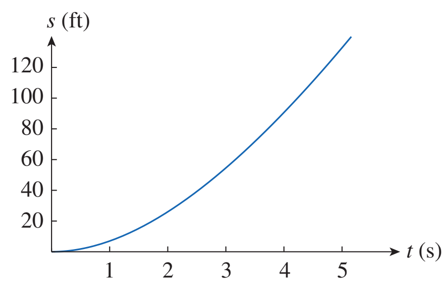


Figure 2.84
Graph of the position function of the car.

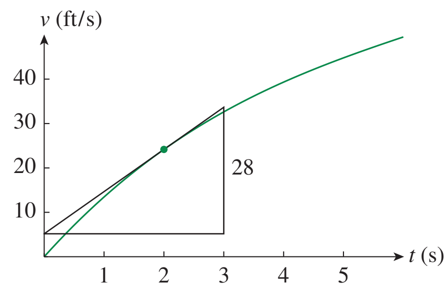


Figure 2.85
Graph of the velocity function of the car.

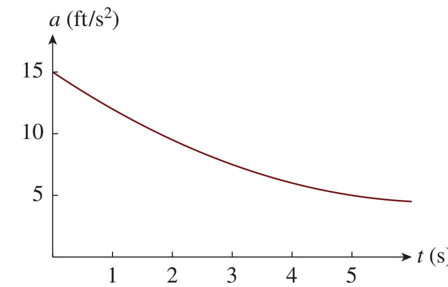


Figure 2.86
Graph of the acceleration function of the car.

$$a(2) = f''(2) = v'(2) \approx \frac{28}{3}\ \text{ft/s}^2.$$

Similar measurements enable us to graph the acceleration function, shown in Figure 2.86. ■

The *third derivative*, f''', is the derivative of the second derivative: $f''' = (f'')'$. The function $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$. If $y = f(x)$, then we have several ways to denote the third derivative:

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Suppose $s = s(t)$ is the position function of an object moving along a straight line. The third derivative, $s''' = (s'')' = a'$, is called the **jerk**, j, and is the rate of change of acceleration.

$$j = \frac{da}{dt} = \frac{d^2v}{dt^2} = \frac{d^3s}{dt^3}$$

This is an appropriate name because a large jerk represents a sudden change in acceleration, which causes a sharp, abrupt movement in the object.

The differentiation process can go on and on. The fourth derivative, f'''', is more often denoted by $f^{(4)}$. In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained by differentiating the function f, n times. If $y = f(x)$, we usually write the nth derivative as

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$$

Example 8 Finding Higher Order Derivatives

Let $f(x) = x^3 - x$. Given that $f'(x) = 3x^2 - 1$ and $f''(x) = 6x$, find $f'''(x)$ and $f^{(4)}(x)$.

Solution

Use the definition of the derivative to find $f'''(x)$.

$$f'''(x) = (f'')'(x) = \lim_{h\to 0}\frac{f''(x+h) - f''(x)}{h} \qquad \text{Definition of derivative.}$$

$$= \lim_{h\to 0}\frac{6(x+h) - 6x}{h} = \lim_{h\to 0}\frac{6x + 6h - 6x}{h} \qquad \text{Evaluate } f''(x)\text{; distribute.}$$

$$= \lim_{h\to 0}\frac{6h}{h} = \lim_{h\to 0} 6 = 6 \qquad \text{Simplify; limit law.}$$

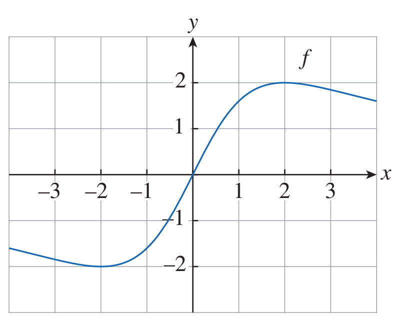
Here is another way to determine the third derivative in this example.

The graph of the second derivative, shown in Figure 2.83, is a straight line with equation $y = 6x$. The slope of this line is a constant, 6. Since the derivative $f'''(x)$ is the slope of $f''(x)$, it must be true that $f'''(x) = 6$.

Similarly, the graph of f''' is a horizontal line. Since this line has slope 0, $f^{(4)}(x) = 0$. ■

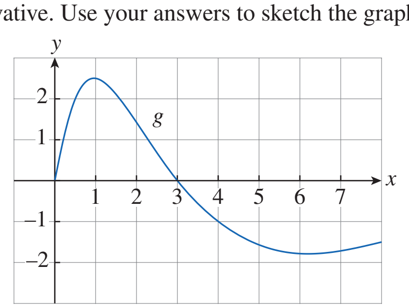
2.7 Exercises

1. Use the graph of f shown in the figure to estimate the value of each derivative. Use your answers to sketch the graph of f'.



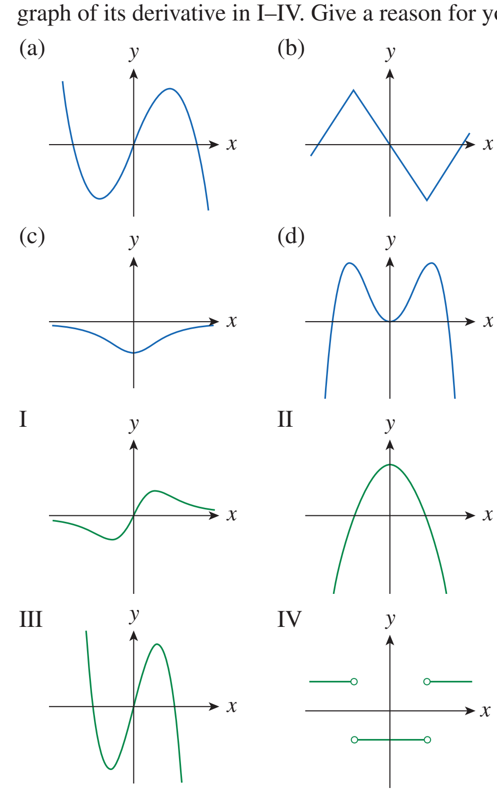
(a) $f'(-3)$ (b) $f'(-2)$ (c) $f'(-1)$ (d) $f'(0)$
(e) $f'(1)$ (f) $f'(2)$ (g) $f'(3)$

2. Use the graph of g shown in the figure to estimate the value of each derivative. Use your answers to sketch the graph of g'.



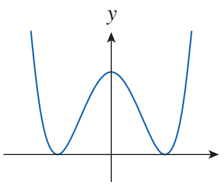
(a) $g'(0)$ (b) $g'(1)$ (c) $g'(2)$ (d) $g'(3)$
(e) $g'(4)$ (f) $g'(5)$ (g) $g'(6)$ (h) $g'(7)$

3. Match the graph of each function in parts (a)–(d) with the graph of its derivative in I–IV. Give a reason for your answer.



Trace or copy the graph of the given function *f*. Assume that the axes have equal scales. Use estimates of the derivative of *f* to sketch the graph of f' on the same coordinate axes.

4.



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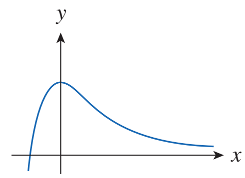

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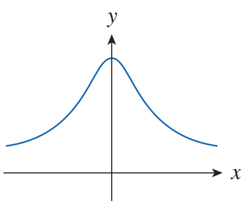

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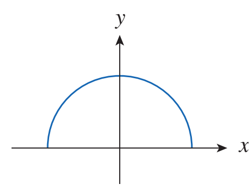

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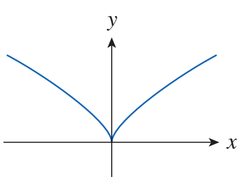

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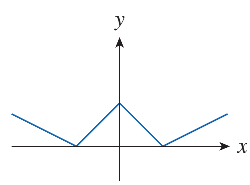

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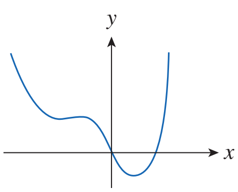

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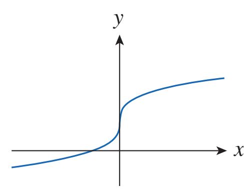
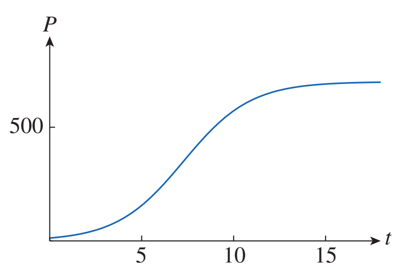
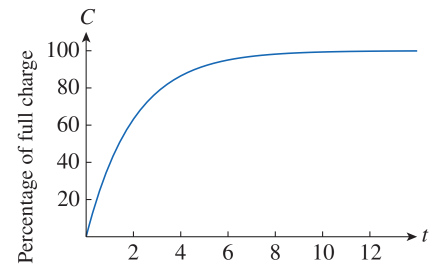


12. The number of yeast cells in a laboratory culture at time t is given by a function $P(t)$, where t is measured in hours. Use the given graph of P to estimate the derivative $P'(t)$ for various values of t and sketch the graph of P'.



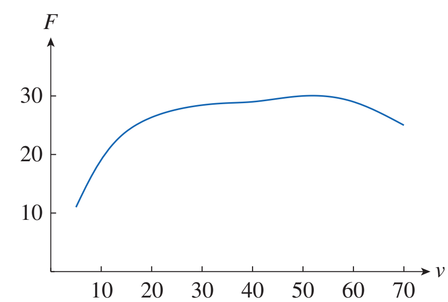
13. A rechargeable battery is plugged into a charger. The percentage of full capacity is given by a function $C(t)$, where t is measured in elapsed time, in hours. The graph of C is given in the figure.



(a) What is the meaning of the derivative $C'(t)$ in the context of this problem?

(b) Sketch the graph of $C'(t)$. Interpret the graph of C' in the context of this problem.

14. The graph shows how driving speed affects gas mileage. Fuel economy F is measured in miles per gallon and speed v is measured in miles per hour.



(a) What is the meaning of the derivative $F'(v)$ in the context of this problem?

(b) Sketch the graph of $F'(v)$.

(c) At what speed should you drive if you want the best fuel economy, that is, the maximum miles per gallon? Justify your answer.

15. Oil is leaking from a tank. For $0 \le t \le 24$, the number of gallons G in the tank at time t is given by $G(t) = 4000 - 3t^2$, where t is measured in hours past midnight.

(a) Use the definition of the derivative to find $G'(5)$.

(b) Interpret the meaning of $G'(5)$ in the context of this problem. Indicate the units of measure.

16. A rod of length 12 cm is heated at one end. The table gives values of the temperature $T(x)$ in degrees Celsius at selected values, x cm from the heated end.

x	0	2	5	7	9	12
$T(x)$	80	71	66	60	54	50

(a) Use the values in the table to estimate $T'(8)$.
(b) Interpret the meaning of $T'(8)$ in the context of this problem. Indicate the units of measure.

Sketch the graph of f and then the graph of f'. Guess a formula for $f'(x)$ from your graph.

17. $f(x) = \sin x$ **18.** $f(x) = e^x$ **19.** $f(x) = \ln x$

20. Let $f(x) = x^2$.
(a) Estimate the values of $f'(0)$, $f'\left(\frac{1}{2}\right)$, $f'(1)$, and $f'(2)$ by using technology to zoom in on the graph of f.
(b) Use symmetry to estimate the values of $f'\left(-\frac{1}{2}\right)$, $f'(-1)$, and $f'(-2)$.
(c) Use your results from parts (a) and (b) to guess a formula for $f'(x)$.
(d) Use the definition of the derivative to find $f'(x)$ and compare this with your guess.

21. Let $f(x) = x^3$.
(a) Estimate the values of $f'(0)$, $f'\left(\frac{1}{2}\right)$, $f'(1)$, $f'(2)$, and $f'(3)$ by using technology to zoom in on the graph of f.
(b) Use symmetry to estimate the values of $f'\left(-\frac{1}{2}\right)$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.
(c) Use your results from parts (a) and (b) to guess a formula for $f'(x)$.
(d) Use the definition of the derivative to find $f'(x)$ and compare this with your guess.

Find the derivative of the function using the definition of the derivative. State the domain of the function and the domain of its derivative.

22. $f(x) = 3x - 8$
23. $f(x) = mx + b$
24. $f(t) = 2.5t^2 + 6t$
25. $f(x) = 4 + 8x - 5x^2$
26. $f(x) = x^2 - 2x^3$
27. $g(t) = \dfrac{1}{\sqrt{t}}$
28. $g(x) = \sqrt{9 - x}$
29. $f(x) = \dfrac{x^2 - 1}{2x - 3}$
30. $G(t) = \dfrac{1 - 2t}{3 + t}$
31. $f(x) = x^{3/2}$
32. $f(x) = x^4$

33. Let $f(x) = 2x^3 - 3x^2$. Find $f''(x)$.

34. Let $f(x) = x^3$. Find $f^{(4)}(x)$.

35. Let $f(x) = \sqrt{6 - x}$.
(a) Start with the graph of $y = \sqrt{x}$ and use transformations to sketch the graph of f.
(b) Use the graph of f to sketch the graph of f'.
(c) Use the definition of the derivative to find $f'(x)$. Find the domain of f and the domain of f'.
(d) Use technology to graph f and f' in the same viewing rectangle. Describe the relationship between the two graphs.

36. Let $f(x) = x^4 + 2x$.
(a) Find $f'(x)$.
(b) Use technology to graph f and f' in the same viewing rectangle. Describe the relationship between the two graphs.

37. Let $f(x) = x + \dfrac{1}{x}$.
(a) Find $f'(x)$.
(b) Use technology to graph f and f' in the same viewing rectangle. Describe the relationship between the two graphs.

38. Let $f(x) = |x|$ and $g(x) = \sqrt{x^2 + 0.0001} - 0.01$.
(a) Graph f and g in the viewing rectangle $[-5, 5] \times [-3, 3]$. Describe the relationship between these two graphs.
(b) Construct a table of values for f and g for various values of x in the interval [0, 1]. Describe the difference between f and g at these values?
(c) Zoom in on the graphs of f and g at several times. What do you notice about the two graphs?
(d) Evaluate $f'(0)$ or explain why it does not exist.
(e) Using the limit definition, find an equation for $g'(x)$ and evaluate $g'(0)$ if it exists.

39. The unemployment rate in the United States varies with time. The table gives the percentage of unemployed in the U.S. labor force from 2004 to 2020.

t	$U(t)$	t	$U(t)$
2004	6.0	2013	7.4
2005	5.1	2014	6.2
2006	4.6	2015	5.3
2007	4.6	2016	4.9
2008	5.8	2017	4.3
2009	9.3	2018	3.9
2010	9.6	2019	3.7
2011	8.9	2020	6.7
2012	8.1		

(a) Explain the meaning of $U'(t)$ in the context of this problem. Indicate the units of measure.
(b) Construct a table of estimated values for $U'(t)$.

Source: U.S. Bureau of Labor Statistics

40. The table gives the number $N(t)$, measured in thousands, of minimally invasive cosmetic surgery procedures performed in the United States for selected years t.

t	$N(t)$ (thousands)	t	$N(t)$ (thousands)
2000	5500	2010	11,561
2002	4897	2012	13,035
2004	7470	2014	13,945
2006	9128	2016	15,412
2008	10,897	2018	17,700

(a) Explain the meaning of $N'(t)$ in the context of this problem. Indicate the units of measure.
(b) Construct a table of estimated value for $N'(t)$.
(c) Graph N and N' on the same coordinate axes.
(d) Explain a method to obtain more accurate estimates of $N'(t)$.

41. Trees grown for lumber are carefully managed and harvested. The table gives the height, in feet, of a typical pine tree at various ages.

Tree age (years)	14	21	28	35	42	49
Height (feet)	41	54	64	72	78	83

Let $H(t)$ be the height of the tree after t years. Construct a table of estimated values of H' and sketch the graph of H'.

42. Water temperature affects the growth rate of brook trout. The table shows the amount of weight gained by brook trout after 24 hours in various water temperatures.

Temperature (°C)	15.5	17.7	20.0	22.4	24.4
Weight gained (g)	37.2	31.0	19.8	9.7	−9.8

Let $W(x)$ be the weight gain at temperature x. Construct a table of estimated values for W' and sketch the graph of W'. What are the units for $W'(x)$?

43. Let P represent the percentage of a city's electrical power that is produced by solar panels, t years after January 1, 2000.

(a) Explain the meaning of $\dfrac{dP}{dt}$ in the context of this problem. Indicate the units of measure.
(b) Interpret the statement

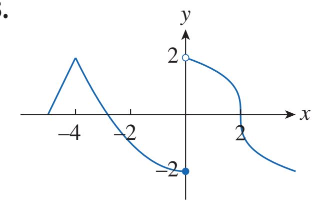
$$\left.\frac{dP}{dt}\right|_{t=2} = 3.5$$

in the context of this problem.

44. Suppose $N(p)$ is the number of people in the United States who travel by car to another state for a vacation this year, where p is the price of gasoline in dollars per gallon. Do you expect $\dfrac{dN}{dp}$ to be positive or negative? Explain you reasoning.

The graph of a function f is given. Find the numbers at which f is not differentiable. Explain your reasoning.

45.



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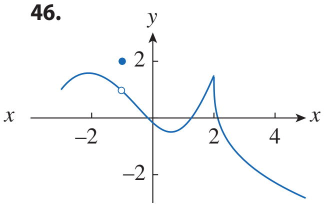

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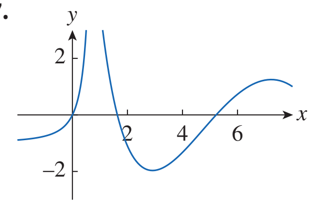

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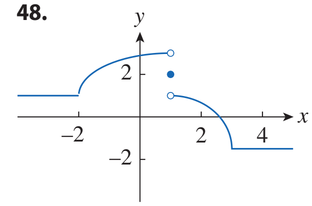
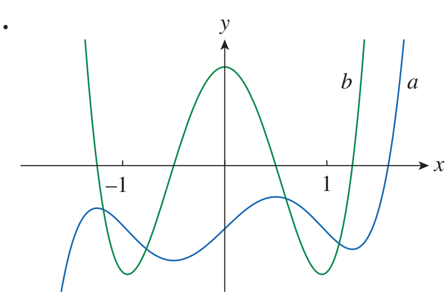


49. Graph the function $f(x) = x + \sqrt{|x|}$. Zoom in repeatedly, first near the point $(-1, 0)$ and then near the origin. What is different about the behavior of the graph of f near these two points? What does this suggest about the differentiability of f at $x = -1$ and at $x = 0$?

50. Graph the function $g(x) = (x^2 - 1)^{2/3}$. Zoom in near the points $(1, 0)$, $(0, 1)$, and $(-1, 0)$. Describe the behavior of the graph of g near each point. What does this suggest about the differentiability of g at $x = 1$, $x = 0$, and $x = -1$?

The graphs of the function f and its derivative f' are shown. Identify each graph and determine which value is greater, $f'(-1)$ or $f''(1)$.

51.



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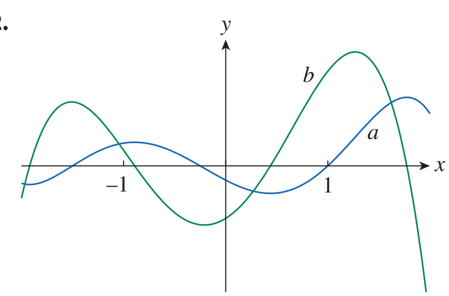
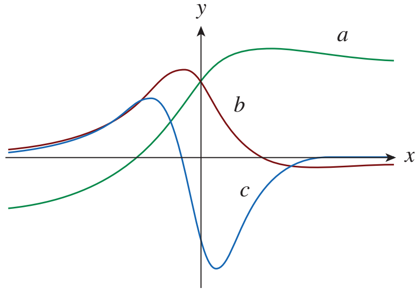
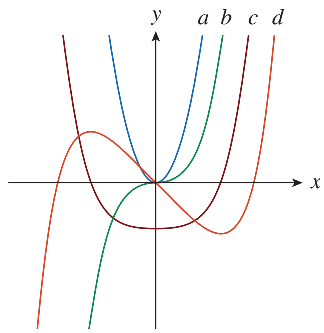


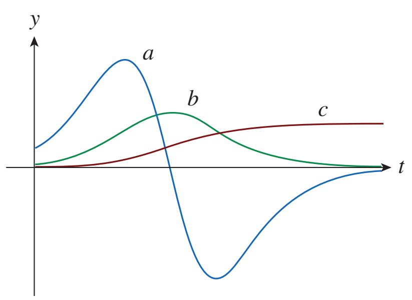
53. The figure shows the graphs of f, f', and f''. Identify each curve and justify your answer.



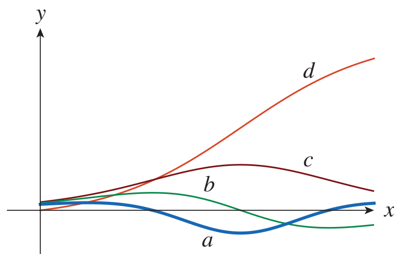
54. The figure shows the graphs of f, f', f'', and f'''. Identify each curve and justify your answer.



55. The figure shows the graphs of three functions. One is the graph of the position function of a car, one is the graph of the velocity of the car, and one is the graph of its acceleration. Identify each curve and justify your answer.



56. The figure shows the graphs of four functions: the graph of the position function of a car, the graph of the velocity of the car, the graph of its acceleration, and the graph of its jerk. Identify each curve and justify your answer.



Use the definition of the derivative to find $f'(x)$ and $f''(x)$. Graph f, f', and f'' in the same viewing rectangle and verify your answers are reasonable.

57. $f(x) = 3x^2 + 2x + 1$ **58.** $f(x) = x^3 - 3x$

59. Suppose the line $x + y = k$ is tangent to the graph of $y = f(x)$ at the point where $x = 2$. Find the value of $\lim_{x\to 2} \dfrac{f(x) - f(2)}{x - 2}$.

60. Let $f(x) = |x + 5| - |3x - 6|$. Find the value of $f'(-1)$.

61. Suppose the point $(5, 7)$ lies on the graph of f and

$$\lim_{h\to 0} \left| \frac{f(5 + h) - f(5)}{h} \right| = \infty$$

Determine whether each statement is true or false. Explain your reasoning.

(a) The graph of f has a vertical tangent line at $x = 5$.
(b) The graph of f has a horizontal tangent line at $x = 5$.
(c) $f'(5) = 7$
(d) $f'(5) = 0$

62. The line $2x + y = 13$ is tangent to the graph of $y = f(x)$ at the point where $x = 3$. Find the values of $f(3)$ and $f'(3)$.

63. (a) Let $f(x) = \dfrac{ax + 1}{bx + 1}$. Use the limit definition of the derivative to find an expression for $f'(x)$ in terms of a and b.

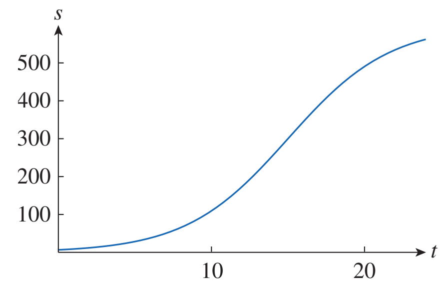
(b) Let $g(x) = \dfrac{3x + 1}{5x + 1}$. Use your answer in part (a) to write the derivative of g (without using the limit definition of the derivative).

(c) Let $f(x) = \dfrac{ax + c}{bx + c}$. Use the limit definition of the derivative to find an expression for $f'(x)$ in terms of a, b, and c.

(d) Let $g(x) = \dfrac{3x + 5}{4x + 5}$. Use your answer in part (c) to write the derivative of g.

64. Let $f(x) = 2x^2 - x^3$. Find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. Graph f, f', f'', and f''' in the same viewing rectangle. Are the graphs consistent with the geometric interpretations of these derivatives? Explain your reasoning.

65. The figure shows the graph of a position function of a car, where s is measured in feet and t in seconds.



(a) Use this graph to construct graphs of the velocity and acceleration of the car. What is the acceleration of the car at $t = 15$ s?
(b) Use the graph of the acceleration to estimate the jerk at $t = 15$ s. Indicate the units of measure.

66. Let $f(x) = \sqrt[3]{x}$.
(a) For $a \neq 0$, find $f'(a)$.
(b) Show that $f'(0)$ does not exist.
(c) Show that the graph of f has a vertical tangent line at $x = 0$.

67. Let $g(x) = x^{2/3}$.
(a) Show that $g'(0)$ does not exist.
(b) For $a \neq 0$, find $g'(a)$.
(c) Show that the graph of g has a vertical tangent line at $x = 0$.
(d) Sketch the graph of g and illustrate the vertical tangent line in part (c).

68. Show that the function $f(x) = |x - 6|$ is not differentiable at $x = 6$. Find a formula for f' and sketch its graph.

69. Let $f(x) = [\![x]\!]$, the greatest integer function.
(a) Where is f not differentiable?
(b) Find a formula for f' and sketch its graph.

70. Let $f(x) = x|x|$.
(a) Sketch the graph of f.
(b) For what values of x is f differentiable?
(c) Find a formula for f'.

71. Let $g(x) = x + |x|$.
(a) Sketch the graph of g.
(b) For what values of x is g differentiable?
(c) Find a formula for g'.

72. A function is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all x in its domain.
(a) Show that the derivative of an even function is an odd function.
(b) Show that the derivative of an odd function is an even function.

73. The **left-hand** and **right-hand derivatives** of f at a are defined by

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a + h) - f(a)}{h}$$

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}$$

if these limits exist. The derivative $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.

Let the function f be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \dfrac{1}{5 - x} & \text{if } x \geq 4, x \neq 5 \end{cases}$$

(a) Find $f'_-(4)$ and $f'_+(4)$.
(b) Sketch the graph of f.
(c) Where is f discontinuous?
(d) Where is f not differentiable?

74. A runner starts jogging and runs faster and faster for 3 minutes, then walks for 5 minutes. They stop at an intersection for 2 minutes, run fairly quickly for 5 minutes, then walk for 4 minutes.
(a) Sketch a possible graph of the distance s that the runner has covered after t minutes.
(b) Sketch a graph of $\dfrac{ds}{dt}$.

75. When a hot-water faucet is turned on, the temperature T of the water depends on how long the water has been running from the hot-water tank.
(a) Sketch a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
(b) Describe how the rate of change of T with respect to t varies as t increases.
(c) Sketch a graph of the derivative of T.
(d) A tankless water heater supplies hot water on demand, rather than storing hot water. Let $D(t)$ be the temperature of the water at a faucet connected to a tankless water heater, where t is the elapsed time since the faucet was turned on. Sketch a possible graph of D and compare this with the graph of T. Describe how the rate of change of D with respect to t varies as t increases, and compare this with $T'(t)$.

76. Let l be the tangent line to the graph of $f(x) = x^2$ at the point $(1, 1)$. The *angle of inclination* of l is the angle ϕ that l makes with the positive direction of the x-axis. Calculate ϕ for this tangent line.

2 Review

Concepts and Vocabulary

1. Explain the meaning of each limit expression and illustrate with a sketch.

(a) $\lim_{x \to a} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

(c) $\lim_{x \to a^-} f(x) = L$ (d) $\lim_{x \to a} f(x) = \infty$

(e) $\lim_{x \to \infty} f(x) = L$

2. Describe several ways in which a limit can fail to exist. Illustrate each case with a sketch.

3. Give an example of a function such that $\lim_{x \to a} f(x) \neq f(a)$.

4. Explain the limit statement

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

in your own words.

5. State the following Limit Laws.

(a) Sum Law (b) Difference Law

(c) Constant Multiple Law (d) Product Law

(e) Quotient Law (f) Power Law

(g) Root Law

6. Explain the Squeeze Theorem in your own words.

7. (a) What does it mean to say that the line $x = a$ is a vertical asymptote on the graph of $y = f(x)$? Sketch an example of the various possibilities.

(b) What does it mean to say that the line $y = L$ is a horizontal asymptote on the graph of $y = f(x)$? Sketch an example of the various possibilities.

8. Consider the graph of each of the following functions. Which graphs have vertical asymptotes? Which have horizontal asymptotes?

(a) $f(x) = x^4$ (b) $f(x) = \sin x$

(c) $f(x) = \tan x$ (d) $f(x) = e^x$

(e) $f(x) = \ln x$ (f) $f(x) = \dfrac{1}{x}$

(g) $f(x) = \sqrt{x}$

9. (a) What does it mean for f to be continuous at a?

(b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?

10. (a) Give several examples of functions that are continuous on the interval $[-1, 1]$.

(b) Give an example of a function that is not continuous on the interval $[0, 1]$.

11. Explain the Intermediate Value Theorem in your own words.

12. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

13. Suppose an object moves along a straight line with position $f(t)$ at time t. Write an expression for the instantaneous velocity of the object at time $t = a$. How can you interpret this velocity in terms of the graph of f?

14. If $y = f(x)$ and x changes from x_1 to x_2, write expressions for the following.

(a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$.

(b) The instantaneous rate of change of y with respect to x at $x = x_1$.

15. Write a definition for the derivative $f'(a)$. Explain two ways of interpreting this value.

16. Define the second derivative of f. If $f(t)$ is the position function of a particle moving along a line, how can you interpret the second derivative?

17. (a) What does it mean for f to be differentiable at a?

(b) What is the relation between the differentiability and continuity of a function?

(c) Sketch the graph of a function that is continuous but not differentiable at $a = 2$.

18. Describe several ways in which a function can fail to be differentiable. Illustrate each example with a sketch.

True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. $\displaystyle \lim_{x \to 4}\left(\frac{2}{x-4} - \frac{8}{x-4}\right) = \lim_{x \to 4}\frac{2x}{x-4} - \lim_{x \to 4}\frac{8}{x-4}$

2. $\displaystyle \lim_{x \to 1}\frac{x^2 + 6x - 7}{x^2 + 5x - 6} = \frac{\lim_{x \to 1}(x^2 + 6x - 7)}{\lim_{x \to 1}(x^2 + 5x - 6)}$

3. $\displaystyle \lim_{x \to 1}\frac{x - 3}{x^2 + 2x - 4} = \frac{\lim_{x \to 1}(x - 3)}{\lim_{x \to 1}(x^2 + 2x - 4)}$

4. $\displaystyle \frac{x^2 - 9}{x - 3} = x + 3$

5. $\displaystyle \lim_{x \to 3}\frac{x^2 - 9}{x - 3} = \lim_{x \to 3}(x + 3)$

6. $\displaystyle \lim_{x \to 4} 4|x - 3| = \pm 4$

7. $\lim_{x\to 0} \dfrac{\sin x}{6x} = 0$

8. $\lim_{h\to 0} \dfrac{4(x+h) - 4x}{h} = -4$

9. $\lim_{x\to 0} \dfrac{\sin 3x}{\sin 2x} = \dfrac{3}{2}$

10. If $\lim_{x\to 5} f(x) = 2$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5} \dfrac{f(x)}{g(x)}$ does not exist.

11. If $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$ then $\lim_{x\to 5} \dfrac{f(x)}{g(x)}$ does not exist.

12. If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a} [f(x) + g(x)]$ does not exist.

13. If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} [f(x) + g(x)]$ does not exist.

14. If $\lim_{x\to 6} [f(x)g(x)]$ exists, then the limit must be $f(6) \cdot g(6)$.

15. If p is a polynomial, then $\lim_{x\to b} p(x) = p(b)$.

16. If $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$, then $\lim_{x\to 0} [f(x) - g(x)] = 0$.

17. There can be two different horizontal asymptotes on the graph of a function.

18. If f has domain $[0, \infty)$ and the graph of f has no horizontal asymptote, then $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = -\infty$.

19. If the line $x = 1$ is a vertical asymptote on the graph of a function f, then f is not defined at 1.

20. If $f(1) > 0$ and $f(3) < 0$, then there exists a number c between 1 and 3 such that $f(c) = 0$.

21. If f is continuous at 5 and $f(5) = 2$ and $f(4) = 3$, then $\lim_{x\to 2} f(4x^2 - 11) = 2$.

22. If f is continuous on $[-1, 1]$ and $f(-1) = 4$ and $f(1) = 3$, then there exists a number r such that $|r| < 1$ and $f(r) = \pi$.

23. If $f(x) > 1$ for all x and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 1$.

24. The equation $x^{10} - 10x^2 + 5 = 0$ has a solution in the interval $(0, 2)$.

25. If $f(x) = \dfrac{x^2 + 3x - 10}{x + 5}$, then for any real number r, there exists a value c such that $f(c) = r$.

26. If f is continuous at a, so is $|f|$.

27. If $|f|$ is continuous at a, so is f.

28. Let $f(x) = \cos\left(\dfrac{x^2 + x}{x^3}\right)$.

(a) $\lim_{x\to 0} f(x)$ exists.

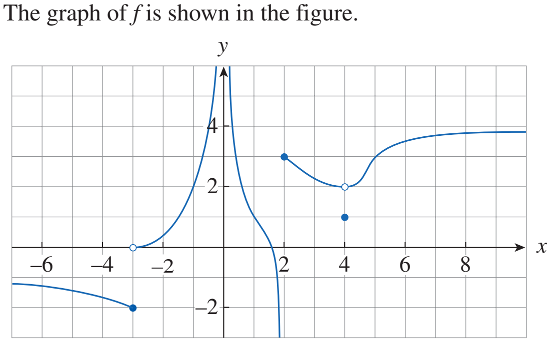
(b) The line $y = 1$ is a horizontal asymptote on the graph of f.

(c) The line $x = 0$ is a vertical asymptote on the graph of f.

29. $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

Exercises

1. The graph of f is shown in the figure.



(a) Find each limit, or explain why it does not exist.

(i) $\lim_{x\to 2^+} f(x)$ (ii) $\lim_{x\to -3^+} f(x)$ (iii) $\lim_{x\to -3} f(x)$

(iv) $\lim_{x\to 4} f(x)$ (v) $\lim_{x\to 0} f(x)$ (vi) $\lim_{x\to 2^-} f(x)$

(vii) $\lim_{x\to\infty} f(x)$ (viii) $\lim_{x\to -\infty} f(x)$

(b) Find the equation of each horizontal asymptote.

(c) Find the equation of each vertical asymptote.

(d) For what values is f discontinuous? Explain your reasoning.

2. Sketch the graph of a function that satisfies all of the following conditions.

$\lim_{x\to -\infty} f(x) = -2$, $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to -3} f(x) = \infty$,

$\lim_{x\to 3^-} f(x) = -\infty$, $\lim_{x\to 3^+} f(x) = 2$,

f is continuous from the right at 3.

3. Find, if possible, a rational function $f(x)$ that satisfies all of the following conditions.

$\lim_{x\to\infty} f(x) = 3$, $\lim_{x\to -4} f(x) = \infty$, $\lim_{x\to 2} f(x) = 3$

f is discontinuous at $x = 2$.

4. Find, if possible, a rational function $f(x)$ that satisfies all of the following conditions.

$\lim_{x\to -3} |f(x)| = \infty$, $\lim_{x\to\infty} f(x) = 2$, $\lim_{x\to 1} f(x) = 2$

f is discontinuous at $x = 1$.

5. Find, if possible, a rational function $f(x)$ that satisfies all of the following conditions.

$\lim_{x\to 3} f(x) = 10$, $\lim_{x\to\infty} f(x) = 2$, $\lim_{x\to -\infty} f(x) = 2$

$f(0) = -2$

6. Find a function $f(x)$ such that $\lim_{x \to 1} f(x) = \infty$ even though the limit corresponds to the indeterminate form $\frac{0}{0}$, or explain why this is not possible.

Find the limit.

7. $\lim_{x \to 1} e^{x^3 - x}$

8. $\lim_{x \to 3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

9. $\lim_{x \to -3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

10. $\lim_{x \to 1^+} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

11. $\lim_{h \to 0} \dfrac{(h - 1)^3 + 1}{h}$

12. $\lim_{t \to 2} \dfrac{t^2 - 4}{t^3 - 8}$

13. $\lim_{r \to 0} \dfrac{\sqrt{r}}{(r - 9)^2}$

14. $\lim_{v \to 4^+} \dfrac{4 - v}{|4 - v|}$

15. $\lim_{u \to 1} \dfrac{u^4 - 1}{u^3 + 5u^2 - 6u}$

16. $\lim_{x \to 3} \dfrac{\sqrt{x + 6} - x}{x^3 - 3x^2}$

17. $\lim_{x \to 0} \dfrac{\tan x}{x}$

18. $\lim_{x \to 1} \dfrac{\ln x}{x^2}$

19. $\lim_{x \to 0} \dfrac{|x|}{2x}$

20. $\lim_{x \to \infty} \dfrac{\sqrt{x^2 - 9}}{2x - 6}$

21. $\lim_{x \to -\infty} \dfrac{\sqrt{x^2 - 9}}{2x - 6}$

22. $\lim_{x \to \pi^-} \ln(\sin x)$

23. $\lim_{x \to -\infty} \dfrac{1 - 2x^2 - x^4}{5 + x - 3x^4}$

24. $\lim_{x \to \infty} \left(\sqrt{x^2 + 4x + 1} - x\right)$

25. $\lim_{x \to \infty} e^{x - x^2}$

26. $\lim_{x \to 1} \left(\dfrac{1}{x - 1} + \dfrac{1}{x^2 - 3x + 2}\right)$

Use technology to make a reasonable guess for the equations of the asymptotes on the graph of the function. Find each asymptote analytically.

27. $f(x) = \dfrac{\cos^2 x}{x^2}$

28. $g(x) = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}$

29. If $2x - 1 \leq f(x) \leq x^2$ for $0 < x < 3$, find $\lim_{x \to 1} f(x)$.

30. Prove that $\lim_{x \to 0} x^2 \cos\left(\dfrac{1}{x^2}\right) = 0$.

31. Find $\lim_{x \to \infty} \dfrac{10 \sin\left(\dfrac{4}{x}\right) - 6x^2 \cos\left(\dfrac{1}{x}\right)}{2x^2 + 3x + 2}$.

32. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leq x < 3 \\ (x - 3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit.

(i) $\lim_{x \to 0^+} f(x)$ (ii) $\lim_{x \to 0^-} f(x)$ (iii) $\lim_{x \to 0} f(x)$

(iv) $\lim_{x \to 3^-} f(x)$ (v) $\lim_{x \to 3^+} f(x)$ (vi) $\lim_{x \to 3} f(x)$

(b) Where is f discontinuous?

(c) Sketch the graph of f.

33. Let

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leq x \leq 2 \\ 2 - x & \text{if } 2 < x \leq 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \geq 4 \end{cases}$$

(a) Determine whether g is continuous from the left, continuous from the right, or continuous at $x = 2$, 3, and 4.

(b) Sketch the graph of g.

Find the domain and show that the function is continuous on its domain.

34. $h(x) = x\, e^{\sin x}$

35. $g(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$

36. The function

$$f(x) = \frac{\dfrac{6}{x} - 1}{x - 6}$$

is discontinuous at two different x-values. Find these two values. Explain why one of these discontinuities is removable and the other is not.

37. Find two different values of k such that the line $y = 10$ is a horizontal asymptote on the graph of

$$f(x) = \frac{kx + k \cdot 5^{-x}}{2x + 5^{-x}}$$

38. Let $f(x) = \dfrac{2 + 3^{1/x}}{7 + 3^{1/x}}$.

Find each limit.

(a) $\lim_{x \to \infty} f(x)$ (b) $\lim_{x \to 0^+} f(x)$ (c) $\lim_{x \to 0^-} f(x)$

39. Let $f(x) = \dfrac{x^3 - x^2 - 2x}{x^2 - 2x^2 + x}$.

Find the coordinates of the point where the graph of f crosses its horizontal asymptote.

Use the Intermediate Value Theorem to show that there is a solution to the equation in the given interval.

40. $x^5 - x^3 + 3x - 5 = 0, \quad (1, 2)$

41. $\cos\sqrt{x} = e^x - 2, \quad (0, 1)$

42. The position of a particle, in meters, moving along a straight line is given by $s = 1 + 2t + \frac{1}{4}t^2$, where t is measured in seconds.

(a) Find the average velocity of the particle over each time interval.

(i) [1, 3] (ii) [1, 2]
(iii) [1, 1.5] (iv) [1.1.1]

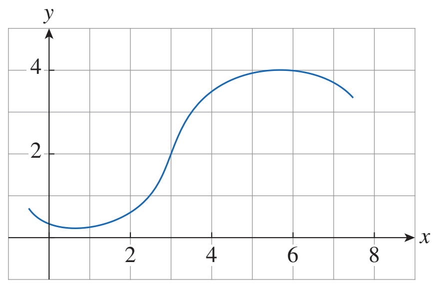
(b) Find the instantaneous velocity of the particle at time $t = 1$.

43. According to Boyle's Law, if the temperature of a confined gas is held fixed, then the product of the pressure P and the volume V is a constant. Suppose that, for a certain gas, $PV = 800$, where P is measured in pounds per square inch and V is measured in cubic inches.

(a) Find the average rate of change of P as V increases from 200 in^3 to 250 in^3.

(b) Express V as a function of P and show that the instantaneous rate of change of V with respect to P is inversely proportional to the square of P.

44. The graph of the function f is shown in the figure.



Arrange the following numbers in increasing order.

$$0 \quad 1 \quad f'(2) \quad f'(3) \quad f'(5) \quad f''(5)$$

45. (a) Let $f(x) = x^3 - 2x$. Use the definition of the derivative to find $f'(2)$.

(b) Find an equation of the line tangent to the graph of $y = x^3 - 2x$ at the point (2, 4).

(c) Graph the curve and the tangent line in the same viewing rectangle.

46. (a) Let $f(x) = e^{-x^2}$. Estimate the value of $f'(1)$ graphically and numerically.

(b) Find an approximate equation of the line tangent to the graph of $y = e^{-x^2}$ at the point where $x = 1$.

(c) Graph the curve and the tangent line in the same viewing rectangle.

47. Find a function f and a number a such that

$$\lim_{h\to 0}\frac{(2+h)^6 - 64}{h} = f'(a)$$

48. Suppose the total cost of repaying a student loan at an interest rate of r% per year is given by $C = f(r)$.

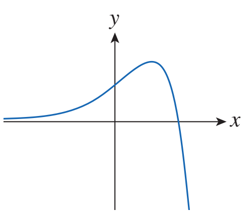
(a) Explain the meaning of the derivative $f'(r)$ in the context of this problem. Indicate the units.

(b) Explain the meaning of the expression $f'(10) = 1200$ in the context of this problem.

(c) Is $f'(r)$ always positive or does it change sign?

Trace or copy the graph of the function and sketch a graph of its derivative on the same coordinate axes.

49.



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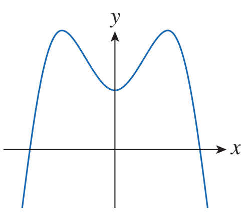
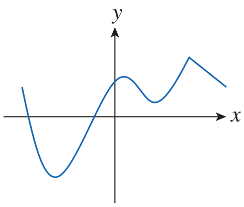


51.



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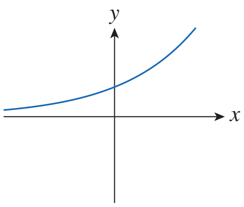


53. (a) Let $f(x) = \sqrt{3 - 5x}$. Use the definition of a derivative to find $f'(x)$.

(b) Find the domain of f and the domain of f'.

(c) Graph f and f' in the same viewing rectangle. Compare the graphs to determine if your answer to part (a) is reasonable.

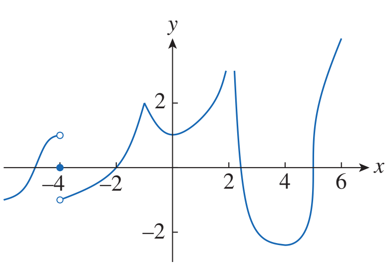
54. (a) Find equations for the asymptotes on the graph of $f(x) = \dfrac{4 - x}{3 + x}$ and use them to sketch the graph.

(b) Use your graph from part (a) to sketch the graph of f'.

(c) Use the definition of a derivative to find $f'(x)$.

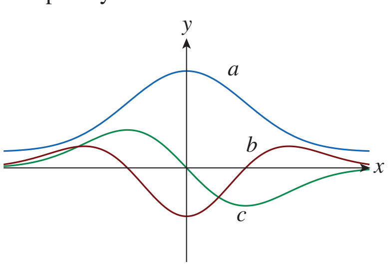
(d) Use technology to graph f' and compare with your sketch in part (b).

55. The graph of f is shown in the figure.



Find the values at which f is not differentiable. Justify your answers.

56. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.

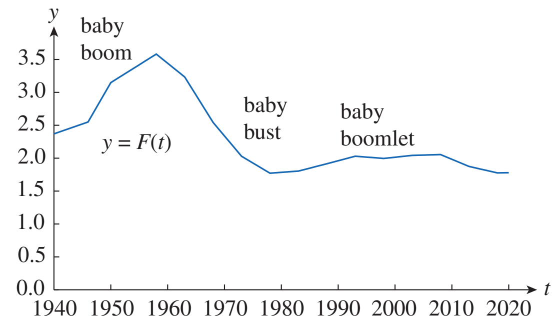


57. Let $C(t)$ be the total value of U.S. currency (coins and banknotes) in circulation at time t. Selected values of $C(t)$, as of December 31 each year, in billions of dollars, are given in the table.

t	2010	2012	2014	2016	2018
$C(t)$	942.0	1127.1	1299.1	1463.4	1671.9

Estimate the value of $C'(2017)$ and interpret this value in the context of this problem.

58. The *total fertility rate* at time t, denoted by $F(t)$, is an estimate of the average number of children born to each woman (assuming that current birth rates remain constant). The graph of the total fertility rate in the United States shows the fluctuations from 1940 to 2020.



(a) Estimate values of $F'(2000)$, $F'(2010)$, and $F'(2015)$.

(b) Explain the meaning of these derivatives in the context of this problem.

(c) Can you suggest any reasons for the values of these derivatives?

Focus on Problem Solving

In the presentation of the principles of problem solving at the end of Chapter 1, we considered the problem-solving strategy of *introducing something extra.* In the following example, we show how this technique is sometimes useful when we evaluate limits. The general idea is to change the variable, to introduce a new variable that is related to the original variable, and to use the relationship to rewrite the original problem so that it is simpler. Later in this text, we will use this general idea more extensively.

Example From x to t

Evaluate $\displaystyle\lim_{x\to 0}\frac{\sqrt[3]{1+cx}-1}{x}$, where c is a constant.

Solution

$$\lim_{x\to 0}(\sqrt[3]{1+cx}-1)=0 \quad \text{and} \quad \lim_{x\to 0}x=0$$

Therefore, the limit is in the indeterminate form $\frac{0}{0}$.

Algebraic manipulation might lead to a simpler expression, but the steps aren't clear here.

Introduce a new variable t, related to x by the equation $t=\sqrt[3]{1+cx}$.

Looking ahead, we also need an expression for x in terms of t.

$$t^3 = 1 + cx \quad \Rightarrow \quad x = \frac{t^3-1}{c} \quad (\text{if } c \neq 0)$$

Notice that $x\to 0$ is equivalent to $t\to 1$.

This allows us to convert the given limit into one involving the variable t.

$$\lim_{x\to 0}\frac{\sqrt[3]{1+cx}-1}{x}=\lim_{t\to 1}\frac{t-1}{(t^3-1)/c}$$

Use equivalent limit statements; convert from x's to t's.

$$=\lim_{t\to 1}\frac{c(t-1)}{t^3-1}$$

Simplify.

The resulting limit is certainly simpler; we have seen one like this before.

Factor the denominator as a difference of cubes.

$$\lim_{t\to 1}\frac{c(t-1)}{t^3-1}=\lim_{t\to 1}\frac{c(t-1)}{(t-1)(t^2+t+1)}$$

Factor the denominator.

$$=\lim_{t\to 1}\frac{c}{t^2+t+1}=\frac{c}{3}$$

Cancel common factor; use direct substitution.

Note that in making this change of variable, we had to rule out the case $c=0$. But, if $c=0$, the function is 0 for all nonzero x and so its limit is 0. Therefore, in all cases, the limit is $c/3$. ■

Example Two Tangent Lines

How many lines are tangent to both of the parabolas $y=-1-x^2$ and $y=1+x^2$? Find the coordinates of the points at which these tangent lines touch the parabolas.

Solution

Figure 2.87 shows a graph of the parabola $y = 1 + x^2$, which is the standard parabola $y = x^2$ shifted 1 unit upward, and a graph of $y = -1 - x^2$, which is obtained by reflecting the first parabola about the x-axis.

If we try to draw a line tangent to both parabolas, it becomes apparent that there are only two possibilities, as illustrated in Figure 2.87.

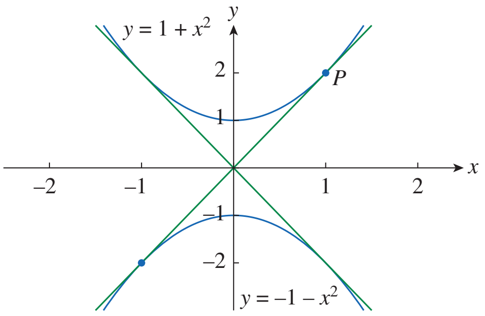


Figure 2.87
Graph of the two parabolas and the tangent lines.

Let P be a point at which one of these tangent lines touches the upper parabola and let a be its x-coordinate.

(Note that the choice of notation for the unknown is important. We could have used b or c or x_0 or x_1 instead of a. However, it would not be a good idea to use x in place of a because that x could be confused with the variable x in the equation of the parabola.)

Since P lies on the parabola $y = 1 + x^2$, its y-coordinate is $1 + a^2$.

Because of the symmetry shown in Figure 2.87, the coordinates of the point Q where the tangent line touches the lower parabola must be $(-a, -(1 + a^2))$.

Use the given information that the line is a tangent: equate the slope of the line PQ to the slope of the tangent line at P.

$$m_{PQ} = \frac{1 + a^2 - (-1 - a^2)}{a - (-a)} = \frac{1 + a^2}{a}$$

If $f(x) = 1 + x^2$, then the slope of the line tangent at P is $f'(a)$.

Using the definition of the derivative, $f'(a) = 2a$.

Therefore, equating the two expressions for the slope of the tangent line:

$$\frac{1 + a^2}{a} = 2a \quad \Rightarrow \quad 1 + a^2 = 2a^2 \quad \Rightarrow \quad a^2 = 1 \quad \Rightarrow \quad a = \pm 1$$

Therefore, the points are $(1, 2)$ and $(-1, -2)$.

By symmetry, the two remaining points are $(-1, 2)$ and $(1, -2)$. ■

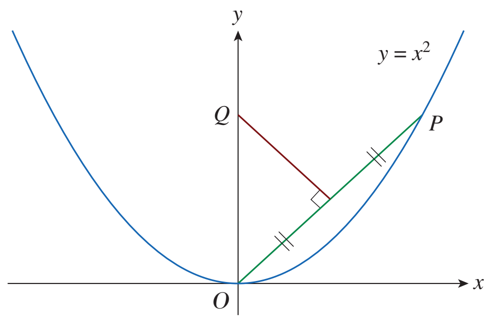
Problems

1. Evaluate $\lim_{x \to 1} \dfrac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$.

2. Find numbers a and b such that $\lim_{x \to 0} \dfrac{\sqrt{ax + b} - 2}{x} = 1$.

3. Evaluate $\lim_{x \to 0} \dfrac{|2x - 1| - |2x + 1|}{x}$.

4. The figure shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, explain what happens to Q. Does the point Q have a limiting position? If so, find it.



5. If $[\![x]\!]$ denotes the greatest integer function, find $\lim_{x \to \infty} \dfrac{x}{[\![x]\!]}$.

6. Sketch the region in the plane defined by each of the following equations.
   (a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$ (b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$
   (c) $[\![x + y]\!]^2 = 1$ (d) $[\![x]\!] + [\![y]\!] = 1$

7. Find all values of a such that f is continuous on $\mathbb{R}$.

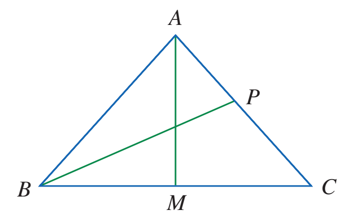
$$f(x) = \begin{cases} x + 1 & \text{if } x \le a \\ x^2 & \text{if } x > a \end{cases}$$

8. A **fixed point** of a function f is a number c in the domain of f such that $f(c) = c$. So, the function f doesn't move c; it, c, remains fixed.
   (a) Sketch a graph of a continuous function with domain $[0, 1]$ and range $[0, 1]$. Locate a fixed point on the graph of f.
   (b) Try to draw the graph of a continuous function with domain $[0, 1]$ and range $[0, 1]$ such that there is no fixed point. Explain the issues in trying to draw this graph.
   (c) Use the Intermediate Value Theorem to prove that any continuous function with domain $[0, 1]$ and range $[0, 1]$ must have a fixed point.

9. (a) Suppose we stand at 0° latitude (at a point on the equator) and proceed in a westerly direction. Let $T(x)$ denote the temperature at a point x units from the starting point at any given time on the equator. Assuming that T is a continuous function of x, show that at any fixed time there are at least two diametrically opposite points on the equator that have exactly the same temperature.
   (b) Does the result in part (a) hold for points lying on any circle on Earth's surface?
   (c) Does the result in part (a) hold for barometric pressure and for altitude above sea level?

10. (a) The figure shows an isosceles triangle ABC with $\angle B \cong \angle C$. The bisector of angle B intersects the side AC at the point P. Suppose that the base BC remains fixed but the altitude $|AM|$ of the triangle approaches 0, so A approaches the midpoint M of BC. What happens to P as A approaches M? Does the point A have a limiting position? If so, find it.

(b) Try to sketch the path traced out by P during this process; use technology if possible. Then, find an equation of this curve and use this equation to sketch the curve.



11. Find points P and Q on the parabola $y = 1 - x^2$ such that the triangle ABC formed by the x-axis and the tangent lines at P and A is an equilateral triangle (see Figure 2.88).

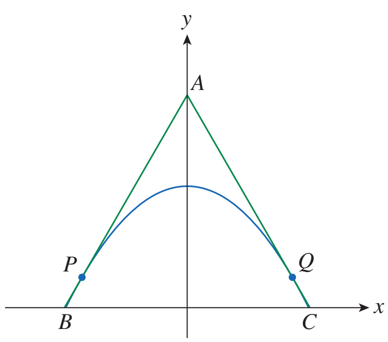


Figure 2.88
The parabola and the equilateral triangle ABC.

12. Water is flowing at a constant rate into a spherical tank. Let $V(t)$ be the volume of the water in the tank and $H(t)$ be the height of the water in the tank at time t.

(a) Explain the meanings of $V'(t)$ and $H'(t)$ in the context of this problem. Are these derivatives positive, negative, or zero?

(b) Is $V''(t)$ positive, negative, or zero? Explain your reasoning.

(c) Let t_1, t_2, and t_3 be the times when the tank is one-quarter full, half full, and three-quarters full, respectively. Are the values $H''(t_1)$, $H''(t_2)$, and $H''(t_3)$ positive, negative, or zero? Justify your answer.

13. Suppose f is a function that satisfies the equation

$$f(x + y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose also that

$$\lim_{x \to 0} \frac{f(x)}{x} = 1$$

(a) Find $f(0)$ (b) Find $f'(0)$ (c) Find $f'(x)$

14. A car is traveling at night along a highway shaped like a parabola with its vertex at the origin. The car starts at a point 100 m west and 100 m north of the origin and travels in an easterly direction. There is a statue located 100 m east and 50 m north of the origin. At what point on the highway will the car's headlights illuminate the statue (see Figure 2.89)?

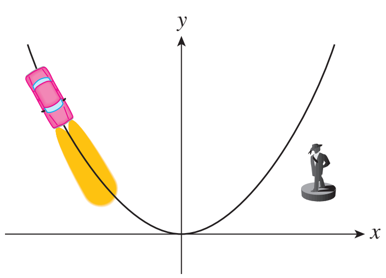


Figure 2.89
Car traveling along a parabola-shaped highway.

15. If f is a differentiable function and $g(x) = xf(x)$, use the definition of a derivative to show that $g'(x) = xf'(x) + f(x)$.

16. Suppose f is a function with the property that $|f(x)| \leq x^2$ for all x. Show that $f(0) = 0$. Then show that $f'(0) = 0$.

17. If $\lim_{x \to a} [f(x) + g(x)] = 2$ and $\lim_{x \to a} [f(x) - g(x)] = 1$, find $\lim_{x \to a} [f(x)g(x)]$.

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Master artisans mold various types of clay, including porcelain, stoneware, and earthenware, into unique pieces, and mechanical devices are used to mass-produce items, for example, dinnerware. Some clay is more workable than others, and the color of clay can be affected by the materials used in the blending process. All clay items are fired, that is, heated in a kiln that can reach temperatures close to 2000°F so that the item is transformed into a hard ceramic. The most common clay defect is cracking while cooling, often caused by the water content in the clay. However, we can use calculus to determine the rate of change in the temperature. This information would help the clay to dry slowly and evenly, avoid drying cracks, and save money and resources.

3 Differentiation Rules

Contents

The derivative is all about change and instantaneous change, and there are countless real-world applications. We have already seen how to interpret the derivative as the slope of a tangent line and how to estimate the derivative of a function numerically. We also learned how to sketch the derivative of a function defined graphically. We discovered and used the definition of a derivative to calculate the derivatives of functions defined analytically or by formulas. But we also realized that it can be very tedious to use the definition to compute a derivative.

In this chapter we develop rules for finding derivatives without having to use the definition directly. These differentiation rules enable us to calculate the derivatives of polynomials, rational functions, algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. We then use these rules to solve problems involving rates of change, tangents to parametric curves, and the approximations of functions.

3.1 Derivatives of Polynomials and Exponential Functions

In this section we learn how to differentiate constant functions, power functions, polynomials, and exponential functions.

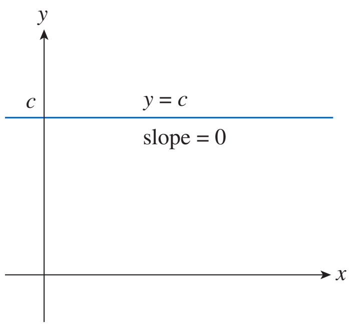


Figure 3.1
The graph of $f(x) = c$ is a horizontal line with slope 0.

Consider one of the simplest functions, the constant function $f(x) = c$. The graph of this function is a horizontal line at $y = c$ with slope 0, as shown in Figure 3.1. Therefore, using the geometric interpretation of the derivative, $f'(x) = 0$.

We can show that this is true for every constant function by using the definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0$$

Using Leibniz notation, we can write this rule as follows.

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Power Functions

Consider functions of the form $f(x) = x^n$, where n is a positive integer. If $n = 1$, the graph of $f(x) = x$ is a straight line through the origin with slope 1. See Figure 3.2. Therefore, using the geometric interpretation of the derivative,

$$\frac{d}{dx}(x) = 1 \tag{1}$$

We can also verify this equation using the limit definition of the derivative.

In Exercises 2.7.20 and 2.7.21, we considered the cases $n = 2$ and $n = 3$. Using the definition of the derivative,

$$\frac{d}{dx}\left(x^2\right) = 2x \qquad \frac{d}{dx}\left(x^3\right) = 3x^2 \tag{2}$$

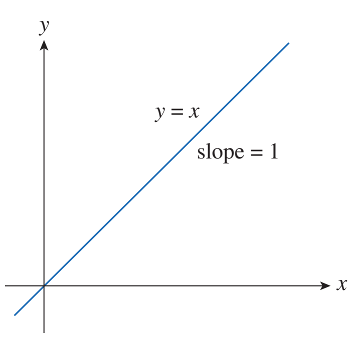


Figure 3.2
The graph of $f(x) = x$ is a straight line with slope 1.

We are searching for a pattern in the derivative of $f(x) = x^n$. Try one more: let $f(x) = x^4$ and use the definition to find $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} && \text{Definition; use } f(x) = x^4. \\ &= \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} && \text{Expand.} \end{aligned}$$

$$= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \qquad \text{Simplify.}$$

$$= \lim_{h\to 0} \frac{h\left(4x^3 + 6x^2h + 4xh^2 + h^3\right)}{h} \qquad \text{Factor out } h \text{ in the numerator.}$$

$$= \lim_{h\to 0} \left(4x^3 + 6x^2h + 4xh^2 + h^3\right) = 4x^3 \qquad \text{Cancel } h\text{; direct substitution.}$$

Therefore,

$$\frac{d}{dx}\left(x^4\right) = 4x^3 \tag{3}$$

Equations 1, 2, and 3 suggest a definitive pattern. It seems reasonable to guess that, when n is a positive integer, $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$. This differentiation rule is indeed true.

The Power Rule

If n is a positive integer, then

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Proof

Use the definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \qquad \text{Definition; use } f(x) = x^n.$$

$$= \lim_{h\to 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h} \qquad \text{Binomial Theorem.}$$

$$= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \qquad \text{Simplify; cancel } x^n.$$

$$= \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right] \qquad \text{Factor out } h \text{ in the numerator; cancel } h.$$

$$= nx^{n-1} \qquad \text{Direct substitution.}$$

■

A Closer Look

1. The Power Rule can also be verified by using the alternate definition of the derivative.
2. The Power Rule in words: if n is a positive integer, to find the derivative of x^n, write the exponent times x raised to the (original) exponent minus 1.
3. The Power Rule works for any dependent and independent variable combination. For example, if $q = h^3$, then $\frac{dq}{dh} = 3h^2$.

4. Remember, the Power Rule is used only when the variable, for example, *x,* is in the base. Later we will learn how to find the derivative of general exponential functions with the variable in the exponent. ■

Example 1 shows several basic applications of the Power Rule combined with a variety of notations.

Example 1 The Power Rule Applied

(a) If $f(x) = x^6$, then $f'(x) = 6x^{6-1} = 6x^5$.

(b) If $g(a) = a^{1000}$, then $g'(a) = 1000a^{999}$.

(c) If $y = t^4$, then $\dfrac{dy}{dt} = 4t^3$.

(d) $\dfrac{d}{dr}\left(r^3\right) = 3r^2$ ■

It seems reasonable to think that the Power Rule works for all real numbers, not just for positive integers. Using the definition of the derivative (Exercise 71), we can show that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Here is a way to write this result using exponents:

$$\frac{d}{dx}\left(x^{-1}\right) = (-1)\,x^{-2}$$

Therefore, the Power Rule works for $n = -1$. In fact, we will show in Section 3.2 that the Power Rule is true for all negative integers.

In addition, using the definition of the derivative (Example 3 in Section 2.7), we showed that

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

This result can also be written using exponents:

$$\frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}$$

So the Power Rule works for $n = \dfrac{1}{2}$. In fact, the Power Rule is true for *any real number,* and we will prove this in Section 3.7.

> **The Power Rule (General Version)**
>
> If n is any real number, then
>
> $$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Examples 2 and 3 demonstrate applications of this General Version.

Example 2 The Power Rule for Negative and Fractional Exponents

Differentiate each function.

(a) $f(x) = \dfrac{1}{x^2}, \quad x \neq 0$ (b) $g(x) = \sqrt[3]{x^2}$

Solution

In each case, rewrite the function as x raised to a power.

(a) $f(x) = \dfrac{1}{x^2} = x^{-2}$

$$f'(x) = \frac{d}{dx}\left(x^{-2}\right) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

(b) $g(x) = \sqrt[3]{x^2} = x^{2/3}$

$$g'(x) = \frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$$

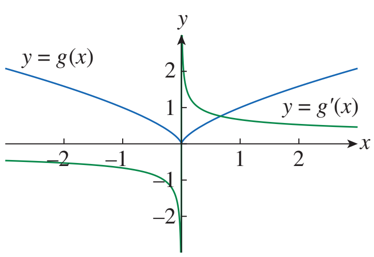


Figure 3.3
Graph of $g(x) = x^{2/3}$ and the derivative $g'(x) = \dfrac{2}{3}x^{-1/3}$.

Note: Remember, if the graph of a function f has a sharp corner at $x = a$, then the derivative does not exist at $x = a$. There may be a vertical tangent line to the graph of f at $x = a$, or there may be no tangent line at $x = a$.

A Closer Look

Figure 3.3 shows the graphs of $g(x) = x^{2/3}$ and $g'(x) = \dfrac{2}{3}x^{-1/3}$ from Example 2.

We can make some important observations from this illustration.

1. The function g is not differentiable at $x = 0$, so $g'(x)$ is not defined at 0. The graph of g has a vertical tangent line at $x = 0$, and we can see that the graph of g has a sharp corner, or cusp, at $x = 0$.
2. The function $g'(x)$ is positive whenever the graph of g is increasing and negative whenever the graph of g is decreasing. In Chapter 4 we will show that, in general, a function is increasing when its derivative is positive and decreasing when its derivative is negative.

The Power Rule allows us to find the derivative of certain functions very quickly and easily. Geometrically, we can now find the slope (and an equation) of the tangent line to the graph of a power function without having to use the definition of the derivative. In addition, we can also find an equation of the *normal line*. The **normal line** to a curve C at a point P is the line through P that is perpendicular to the tangent line at P. The normal line is often used in physics as well as in applications of calculus.

Example 3 Tangent Line and Normal Line Equations

Let $f(x) = x\sqrt{x}$. Find equations of the tangent line and the normal line to the graph of f at the point $(1, 1)$. Illustrate your results by graphing the function and the two lines.

Solution

Rewrite the function: $f(x) = x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$.

Use the Power Rule: $f'(x) = \dfrac{3}{2}x^{(3/2)-1} = \dfrac{3}{2}x^{1/2} = \dfrac{3}{2}\sqrt{x}$.

The slope of the tangent line to the graph of f at $(1, 1)$ is $f'(1) = \dfrac{3}{2}\sqrt{1} = \dfrac{3}{2}$.

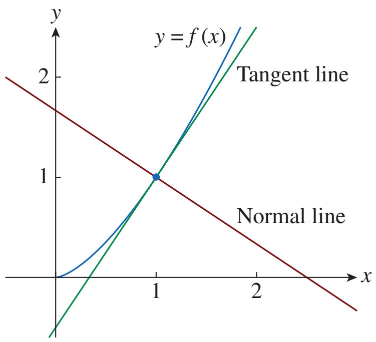


Figure 3.4
Graph of f, the tangent line, and the normal line.

An equation of the tangent line is

$$y - 1 = \frac{3}{2}(x - 1) \quad \text{or} \quad y = \frac{3}{2}x - \frac{1}{2}.$$

The normal line is perpendicular to the tangent line.

Therefore, its slope is the negative reciprocal of $\frac{3}{2}$, which is $-\frac{2}{3}$.

An equation of the normal line is

$$y - 1 = -\frac{2}{3}(x - 1) \quad \text{or} \quad y = -\frac{2}{3}x + \frac{5}{3}.$$

Figure 3.4 shows a graph of f, the tangent line, and the normal line. ■

New Derivatives from Old

We frequently form new functions from existing functions by addition, subtraction, or multiplication by a constant. It seems reasonable that the derivative of the new function is related to the derivative of the old function. The next rule involves the derivative of a constant times a function.

The Constant Multiple Rule

If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\left[\frac{d}{dx}f(x)\right]$$

Proof

Let $g(x) = cf(x)$ and use the definition of the derivative.

$$g'(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h) - cf(x)}{h}$$ Definition; use $g(x) = cf(x)$.

$$= \lim_{h\to 0} c\left[\frac{f(x+h) - f(x)}{h}\right]$$ Factor out c in the numerator.

$$= c\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$ Limit Law 3: constants pass freely through limit symbols.

$$= cf'(x)$$ Definition of the derivative of $f(x)$.

■

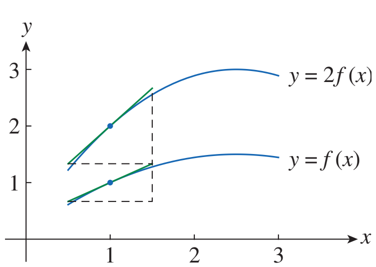


Figure 3.5
Multiplying by $c = 2$ stretches the graph of f vertically by a factor of 2. All *rises* have been doubled, but the *runs* remain the same. Therefore, the slopes are doubled also.

A Closer Look

1. In words, this rule says: the derivative of a constant times a function is the constant times the derivative of the function.

   Another way to think about this: constants pass freely through the differentiation operator.

2. Figure 3.5 provides a geometric illustration of the Constant Multiple Rule. ■

Example 4 provides illustrations of the Constant Multiple Rule.

Example 4 Constant Multiple Rule

Differentiate each function.

(a) $f(x) = 3x^4$ (b) $g(x) = -x^8$

Solution

Use the Constant Multiple Rule (and the Power Rule).

(a) $\dfrac{d}{dx}(3x^4) = 3\dfrac{d}{dx}(x^4) = 3(4x^3) = 12x^3$

(b) $\dfrac{d}{dx}(-x^8) = \dfrac{d}{dx}[(-1)x^8] = (-1)\dfrac{d}{dx}(x^8) = (-1)(8x^7) = -8x^7$ ■

We can also rewrite the derivative of a sum of two functions.

The Sum Rule

If f and g are differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof

Let $F(x) = f(x) + g(x)$, and use the definition of the derivative.

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} \qquad \text{Definition of the derivative.}$$

$$= \lim_{h\to 0}\frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \qquad \text{Use } F(x) = f(x) + g(x).$$

$$= \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right] \qquad \text{Rearrange terms.}$$

$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} \qquad \text{Limit Law 1: limit of a sum is the sum of the limits.}$$

$$= f'(x) + g'(x) \qquad \text{Definition of the derivative.}$$

■

A Closer Look

1. In words, the Sum Rule says: the derivative of a sum is the sum of the derivatives.
2. Using prime notation, we can write the Sum Rule as $(f + g)' = f' + g'$.
3. The Sum Rule can be extended to any number of functions. For example, using this rule twice, we get

$$(f + g + h)' = [(f + g) + h]' = (f + g)' + h' = f' + g' + h'$$ ■

If we write $f - g = f + (-1)g$, then we can use the Sum Rule and the Constant Multiple Rule to show the following result.

The Difference Rule

If f and g are differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be used along with the Power Rule to differentiate any polynomial. Examples 5–7 illustrate this concept.

Example 5 Differentiate a Polynomial

Let $f(x) = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$. Find $f'(x)$.

Solution

$$\begin{aligned}
&\frac{d}{dx}\left(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5\right)\\
&\quad = \frac{d}{dx}\left(x^8\right) + 12\frac{d}{dx}\left(x^5\right) - 4\frac{d}{dx}\left(x^4\right) + 10\frac{d}{dx}\left(x^3\right) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5)\\
&\quad = 8x^7 + 12\left(5x^4\right) - 4\left(4x^3\right) + 10\left(3x^2\right) - 6(1) + 0\\
&\quad = 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6
\end{aligned}$$

Example 6 Horizontal Tangent Lines

Find the points on the graph of $f(x) = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Solution

Horizontal tangent lines occur where the derivative is zero.

Start by finding the derivative, $f'(x)$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(x^4 - 6x^2 + 4\right)\\
&= \frac{d}{dx}\left(x^4\right) - 6\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(4)\\
&= 4x^3 - 12x + 0 = 4x\left(x^2 - 3\right)
\end{aligned}$$

Determine where the derivative of f is zero.

$$\begin{aligned}
f'(x) = 0 \;&\Rightarrow\; 4x\left(x^2 - 3\right) = 0\\
&\Rightarrow\; 4x\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right) = 0 && \text{Difference of two squares.}\\
&\Rightarrow\; x = 0, -\sqrt{3}, \sqrt{3} && \text{Principle of Zero Products.}
\end{aligned}$$

The graph of f has horizontal tangent lines when $x = 0, -\sqrt{3}$, and $\sqrt{3}$.

The corresponding points on the graph of f are

$\left(0, f(0)\right) = (0, 4)$, $\left(-\sqrt{3}, f\left(-\sqrt{3}\right)\right) = \left(-\sqrt{3}, -5\right)$, and $\left(\sqrt{3}, f\left(\sqrt{3}\right)\right) = \left(\sqrt{3}, -5\right)$.

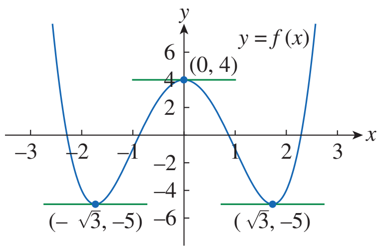


Figure 3.6
Graph of $f(x) = x^4 - 6x^2 + 4$ and its horizontal tangent lines.

Figure 3.6 shows the graph of f and the horizontal lines.

Notice that the horizontal lines occur at the values where f has a *relative maximum* or *relative minimum* value.

Example 7 Particle Motion and Acceleration

The position of a particle moving along a line is given by $s(t) = 2t^3 - 5t^2 + 3t + 4$, where $s(t)$ is measured in centimeters and t in seconds. Find the acceleration of the particle as a function of time. What is the acceleration at $t = 2$ seconds?

Solution

Find the velocity and then the acceleration functions.

$$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 3$$

$$a(t) = \frac{dv}{dt} = 12t - 10$$

The acceleration at $t = 2$ is $a(2) = 12(2) - 10 = 14\ \text{cm/s}^2$.

Exponential Functions

The next rule involves a formula for the derivative of an exponential function. Let $f(x) = b^x$ and start by using the definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} \qquad \text{Definition; use } f(x) = b^x.$$

$$= \lim_{h\to 0} \frac{b^x b^h - b^x}{h} = \lim_{h\to 0} \frac{b^x(b^h - 1)}{h} \qquad \text{Properties of exponents; factor out } b^x.$$

$$= b^x \lim_{h\to 0} \frac{b^h - 1}{h} \qquad b^x \text{ does not depend on } h.$$

The limit in this expression is the value of the derivative of f at 0. That is,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{b^h - 1}{h}.$$

This means that if the exponential function $f(x) = b^x$ is differentiable at $x = 0$, then it is differentiable everywhere, and

$$f'(x) = f'(0)b^x \tag{4}$$

Here is an important interpretation of this equation: the rate of change of any exponential function is proportional to the function itself. Another way to think about this graphically is: the slope of the graph of f is proportional to the height.

We need to determine whether $f'(0)$ exists. Table 3.1 provides some numerical evidence for the cases $b = 2$ and $b = 3$.

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

Table 3.1
Table of values to help estimate the limit $\lim_{h\to 0} \frac{b^h - 1}{h}$.

This table suggests that the limits exist.

For $b = 2$, $\quad f'(0) = \lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.69.$

For $b = 3$, $\quad f'(0) = \lim_{h\to 0} \frac{3^h - 1}{h} \approx 1.10.$

In fact, it can be proved that these limits exist and the values, correct to six decimal places, are

$$\frac{d}{dx}(2^x)\bigg|_{x=0} = 0.693147 \qquad \frac{d}{dx}(3^x)\bigg|_{x=0} = 1.098612$$

Using these results with Equation 4, we now have

Remember, for $f(x) = b^x$, $f'(x) = f'(0)b^x$.

$$\frac{d}{dx}(2^x) = (0.69)2^x \qquad \frac{d}{dx}(3^x) = (1.10)3^x \tag{5}$$

Of all the possible choices for b in Equation 4, the simplest differentiation formula occurs when $f'(0) = 1$. Given our estimates of $f'(0)$ when $b = 2$ and $b = 3$, it seems reasonable that there is a number b between 2 and 3 such that $f'(0) = 1$. There is! This number is traditionally denoted by the letter e, and this leads to the following definition.

The Number *e*

e is the number such that $\displaystyle\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$

Geometrically, this means that the slope of the tangent line to the graph of $f(x) = e^x$ at $(0, 1)$ is exactly 1. Figures 3.7 and 3.8 illustrate this concept.

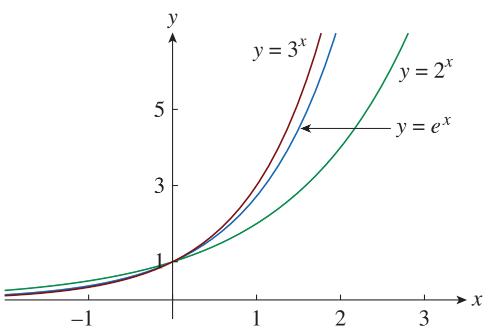


Figure 3.7
The graph of $y = e^x$ is between the graphs of $y = 2^x$ and $y = 3^x$.

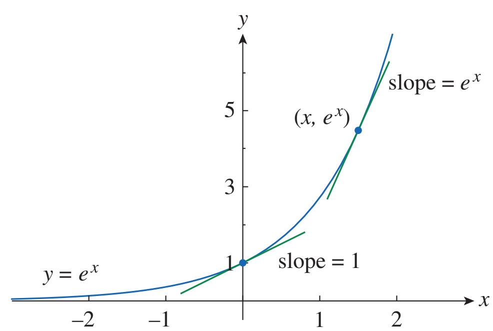


Figure 3.8
The slope of the tangent line to the graph of $f(x) = e^x$ at the point $(0, 1)$ is 1.

Now, let $b = e$ so that $f(x) = e^x$. Similar to the preceding argument,

$$f'(x) = e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$$

The function $f(x) = e^x$ is the natural exponential function because the number e arises naturally in mathematics and the physical sciences and because the derivative is the same function.

This leads to the following important rule for the derivative of the *natural* exponential function.

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

A Closer Look

1. The exponential function $f(x) = e^x$ has the unique property that it is its own derivative.
2. The graphical implications: the slope of a tangent line to the graph of $f(x) = e^x$ at $(a, f(a))$ is $f(a)$, the y-coordinate of the point.

Example 8 Derivatives Involving e^x

Let $f(x) = e^x - x$.

(a) Find f' and f''.

(b) Sketch the graphs of f and f' on the same coordinate axes. Discuss the relationship between these two graphs.

Solution

(a) $f'(x) = \frac{d}{dx}\left(e^x - x\right) = \frac{d}{dx}\left(e^x\right) - \frac{d}{dx}\left(x\right)$ Difference Rule.

$= e^x - 1$ Differentiation rules.

Similarly,

$$f''(x) = \frac{d}{dx}\left(e^x - 1\right) = \frac{d}{dx}\left(e^x\right) - \frac{d}{dx}\left(1\right) = e^x - 0 = e^x.$$

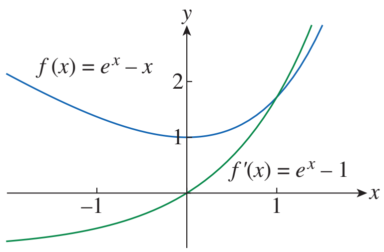


Figure 3.9
Graph of $f(x) = e^x - x$ and $f'(x) = e^x - 1$.

(b) The graph of f and f' are shown in Figure 3.9.

Here are some observations.

(1) The graph of f has a horizontal tangent when $x = 0$. This corresponds to a zero for f', that is $f'(0) = 0$.

(2) For $x > 0$, f is increasing and $f'(x)$ is positive.

(3) For $x < 0$, f is decreasing and $f'(x)$ is negative.

Example 9 A Point on a Graph with Specified Slope

Find a point on the graph of $f(x) = e^x$ such that the slope of the tangent line is parallel to the line $y = 2x$.

Solution

$f(x) = e^x \;\Rightarrow\; f'(x) = e^x$

Let $(a, f(a))$ denote the unknown point.

The slope of the tangent line at that point is $f'(a) = e^a$.

The slope of the line $y = 2x$ is 2.

These two lines are parallel if their slopes are equal.

$e^a = 2 \;\Rightarrow\; a = \ln 2$

The point is $\left(a, e^a\right) = \left(\ln 2, e^{\ln 2}\right) = \left(\ln 2, 2\right)$.

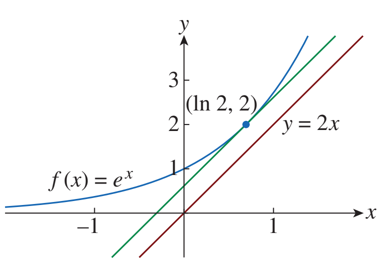


Figure 3.10
Graphs of $f(x) = e^x$, $y = 2x$, and the tangent line.

Figure 3.10 shows the graph of $f(x) = e^x$ and the tangent line that is parallel to the given line, $y = 2x$.

3.1 Exercises

1. (a) Write the definition of the number e.
(b) Use technology to estimate the values of the limits

$$\lim_{h\to 0}\frac{2.7^h - 1}{h} \quad \text{and} \quad \lim_{h\to 0}\frac{2.8^h - 1}{h}$$

What can you conclude about the value of e?

2. Let $f(x) = e^x$.
(a) Sketch the graph of f with special attention to how the graph crosses the y-axis. What is the slope of the tangent line to the graph of f where it crosses the y-axis?
(b) What type of function is $f(x) = e^x$? What type of function is $g(x) = x^e$? Find the derivatives, f' and g'.
(c) Which function, f or g, grows faster as x increases without bound? Use the derivatives, f' and g', to justify your answer.

Differentiate the function.

3. $f(x) = 2^{40}$

4. $f(x) = e^5$

5. $f(x) = 5.2x + 2.3$

6. $f(x) = \frac{7}{4}x^2 - 3x + 12$

7. $f(t) = 2t^3 - 3t^2 - 4t$

8. $f(t) = 1.4t^5 - 2.5t^2 + 6.7$

9. $g(x) = x^2(1 - 2x)$

10. $H(u) = (3u - 1)(u + 2)$

11. $g(t) = 2t^{-3/4}$

12. $B(y) = cy^{-6}$

13. $F(r) = \dfrac{5}{r^3}$

14. $y = x^{5/3} - x^{2/3}$

15. $R(a) = (3a + 1)^2$

16. $h(t) = \sqrt[4]{t} - 4e^t$

17. $S(p) = \sqrt{p} - p$

18. $y = \sqrt[3]{x}\,(2 + x)$

19. $y = 3e^x + \dfrac{4}{\sqrt[3]{x}}$

20. $S(R) = 4\pi R^2$

21. $h(u) = Au^3 + Bu^2 + Cu$

22. $y = \dfrac{\sqrt{x} + x}{x^2}$

23. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$

24. $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t}$

25. $f(x) = x^{2.4} + e^{2.4}$

26. $k(r) = e^r + r^e$

27. $G(q) = (1 + q^{-1})^2$

28. $F(z) = \dfrac{A + Bz + Cz^2}{z^2}$

29. $f(v) = \dfrac{\sqrt[3]{v} - 2ve^v}{v}$

30. $D(t) = \dfrac{1 + 16t^2}{(4t)^3}$

31. $z = \dfrac{A}{y^{10}} + Be^y$

32. $y = e^{x+1} + 1$

Find an equation of the tangent line to the graph of the function at the given point.

33. $f(x) = 2x^3 - x^2 + 2, \quad (1, 3)$

34. $f(x) = 2e^x + x, \quad (0, 2)$

35. $f(x) = x + \dfrac{2}{x}, \quad (2, 3)$

36. $f(x) = \sqrt[4]{x} - x, \quad (1, 0)$

Find equations of the tangent line and the normal line to the graph of the function at the given point.

37. $f(x) = x^4 + 2e^x, \quad (0, 2)$

38. $f(x) = x^{3/2}, \quad (1, 1)$

Find an equation of the tangent line to the graph of the function at the given point. Graph the function and the tangent line in the same viewing rectangle.

39. $f(x) = 3x^2 - x^3, \quad (1, 2)$

40. $f(x) = x - \sqrt{x}, \quad (1, 0)$

Find $f'(x)$. Graph f and f' in the same viewing rectangle and explain the relationship between the two graphs.

41. $f(x) = x^4 - 2x^3 + x^2$

42. $f(x) = x^5 - 2x^3 + x - 1$

Estimate the value of $f'(a)$ by zooming in on the graph of f. Then differentiate f to find the exact value of $f'(a)$ and compare with your estimate.

43. $f(x) = 3x^2 - x^3, \quad a = 1$

44. $f(x) = \dfrac{1}{\sqrt{x}}, \quad a = 4$

45. Let $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$.
(a) Graph f in the viewing rectangle $[-3, 5] \times [-10, 50]$.
(b) Use the graph in part (a) to estimate the slope of the graph of f at various points. Use your estimates to make a rough sketch of the graph of f'.
(c) Find $f'(x)$ and sketch the graph of f' using technology. Compare this result with your sketch in part (b).

46. Let $g(x) = e^x - 3x^2$.
(a) Graph g in the viewing rectangle $[-1, 4] \times [-8, 8]$.
(b) Use the graph in part (a) to estimate the slope of the graph of g at various points. Use your estimates to make a rough sketch of the graph of g'.
(c) Find $g'(x)$ and sketch the graph of g' using technology. Compare this result with your sketch in part (b).

Find the first and the second derivatives of the function.

47. $f(x) = 0.001x^5 - 0.002x^3$

48. $G(r) = \sqrt{r} + \sqrt[3]{r}$

Find the first and second derivatives of the function. Check your answers by graphing f, f', and f'' in the same viewing rectangle.

49. $f(x) = 2x - 5x^{3/4}$

50. $f(x) = e^x - x^3$

51. A particle moves along a straight line. For $t \geq 0$, the position of the particle is given by $s(t) = t^3 - 3t$, where $s(t)$ is measured in meters and t in seconds.
(a) Find the velocity function and the acceleration function.
(b) Find the acceleration at $t = 2$ seconds. Indicate the units of measure.

(c) Find the acceleration when the velocity of the particle is 0 m/s.

52. A particle moves along a straight line. For $t \geq 0$, the position of the particle is given by $s(t) = t^4 - 2t^3 + t^2 - t$, where $s(t)$ is measured in meters and t in seconds.
(a) Find the velocity function and the acceleration function.
(b) Find the acceleration at $t = 1$ second.
(c) Graph the position, velocity, and acceleration functions in the same viewing rectangle.

53. A particle moves along a straight line. For $t \geq 0$, the position of the particle is given by $s(t) = t^3 - 6t^2 + 9t + 5$, where $s(t)$ is measured in inches and t in seconds. Find the velocity of the particle when the acceleration is 0.

54. A particle moves along a straight line. For $t \geq 0$, the position of the particle is given by $s(t) = 2t^3 - t^2 + 4t + 3$, where $s(t)$ is measured in centimeters and t in seconds. Find the acceleration at the time when the velocity is zero. Indicate the units of measure.

55. Let $f(x) = 3e^x - 1$. Find an equation of the tangent line to the graph of f at the point where $x = 0$.

56. A group of research biologists have proposed a cubic polynomial to model the length L of Alaskan rockfish at age A:

$$L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21$$

where L is measured in inches and A in years. Find

$$\left.\frac{dL}{dA}\right|_{A=12}$$

and explain the meaning of your answer in the context of this problem.

57. Suppose the length L, measured in mm, of a bluegill fish in a Minnesota lake is related to its age A, in years. A proposed model for L in terms of A is

$$L(A) = -4.7A^2 + 54A + 13.1$$

Find $L'(5)$ and explain the meaning of your answer in the context of this problem.

58. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P of the gas is inversely proportional to the volume V of the gas.
(a) Suppose that the pressure of a sample of air that occupies $0.106\ \text{m}^3$ at 25°C is 50 kPa. Write V as a function of P.
(b) Calculate $\dfrac{dV}{dP}$ when $P = 50$ kPa. Explain the meaning of your answer in the context of this problem. Indicate the units of measure.

59. Car tires need to be inflated properly because overinflation or underinflation can cause premature tread wear. The data in the table show tire life L (in thousands of miles) for a certain type of tire at various pressures P (in lb/in^2).

P	26	28	31	35	38	42	45
L	50	66	78	81	74	70	59

(a) Use technology to construct a quadratic model for tire life as a function of pressure.
(b) Use your model to estimate $\dfrac{dL}{dP}$ when $P = 30$ and when $P = 40$. Explain the meaning of this derivative in the context of this problem. Indicate the units of measure. What is the significance of the sign of the derivative in this problem?

60. Find the points on the graph of $f(x) = 2x^3 + 3x^2 - 12x + 1$ where the tangent line is horizontal.

61. For what values of x does the graph of $f(x) = e^x - 2x$ have a horizontal tangent line?

62. Show that the graph of $f(x) = 2e^x + 3x + 5x^3$ has no tangent line with slope 2.

63. Find an equation of the tangent line to the graph of $f(x) = x^4 + 1$ that is parallel to the line $32x - y = 15$.

64. Find equations of both lines that are tangent to the graph of $f(x) = x^3 - 3x^2 + 3x - 3$ and parallel to the line $3x - y = 15$.

65. Find the point on the graph of $f(x) = 1 + 2e^x - 3x$ where the tangent line is parallel to the line $3x - y = 5$. Graph f and both lines in the same viewing rectangle.

66. Find an equation of the normal line to the graph of $f(x) = \sqrt{x}$ that is parallel to the line $2x + y = 1$.

67. Find an equation of the normal line to the graph of $f(x) = x^2 + 1$ that also passes through the point $(3, 1)$.

68. Find an equation of the normal line to the graph of $f(x) = x^2 - 1$ at the point $(-1, 0)$. Find the point where the normal line intersects the graph of f a second time. Graph f and the normal line, and indicate the two points of intersection.

69. Draw a graph to show that there are two tangent lines to the graph of $f(x) = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines intersect the graph of f.

70. (a) Find equations of both lines through the point $(2, -3)$ that are tangent to the graph of $f(x) = x^2 + x$.
(b) Show that there is no line through the point $(2, 7)$ that is tangent to the graph of f. Draw a graph to illustrate this result.

71. Use the definition of the derivative to show that if $f(x) = \dfrac{1}{x}$, then $f'(x) = -\dfrac{1}{x^2}$.

(This proves the Power Rule for the case $n = -1$.)

72. Find the nth derivative of each function by calculating the first few derivatives and observing the pattern that occurs.

(a) $f(x) = x^n$ (b) $f(x) = \dfrac{1}{x}$

73. Find a second-degree polynomial P such that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.

74. Let k be the y-intercept of the tangent line to the graph of $f(x) = x^2 + x$ passing through the point $(1, -2)$. Find the value of k.

75. A tangent line to the graph of $f(x) = 2\sqrt{x}$ has an x-intercept of -9. Find the y-intercept of this tangent line.

76. The graph of $f(x) = \dfrac{p}{\sqrt{x}} + q\sqrt{x}$ has a horizontal tangent line at the point $(4, 12)$. Find the values of p and q.

77. Determine the values of a and b such that the line $y = 5x + 6$ is tangent to the graph of $f(x) = ax - \dfrac{b}{x}$ at the point where $x = 1$.

---

78. The equation $y'' + y' - 2y = x^2$ is called a **differential equation** because it involves an unknown function y and some of its derivatives, in this case, y' and y''. Find constants A, B, and C such that the function $y = Ax^2 + Bx + C$ satisfies this equation.

79. Find the parabola with equation $y = ax^2 + bx$ whose tangent line at $(1, 1)$ has equation $y = 3x - 2$.

80. Suppose $f(x) = x^4 + ax^3 + bx^2 + cx + d$. The tangent line to the graph of f at $x = 0$ has equation $y = 2x + 1$, and at $x = 1$ has equation $y = 2 - 3x$. Find the values of a, b, c, and d.

81. For what values of a and b is the line $2x + y = b$ tangent to the graph of $f(x) = ax^2$ at the point where $x = 2$?

82. Find the value of c such that the line $y = \dfrac{3}{2}x + 6$ is tangent to the graph of $f(x) = c\sqrt{x}$.

83. Find the value of c such that the line $y = 2x + 3$ is tangent to the graph of $f(x) = cx^2$.

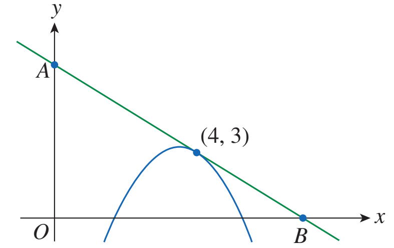
84. Find, if it exists, a value c in the interval $[1, 4]$ such that the instantaneous rate of change of $f(x) = \dfrac{6}{\sqrt{x}}$ at c is the same as the average rate of change of f over the interval $[1, 4]$.

85. Suppose a tangent line to the graph of $f(x) = \sqrt{x}$ passes through the point $(8, 3)$. Find the slope of each possible tangent line, or explain why no such tangent line exists.

86. A line has positive slope, passes through the origin, and is tangent to the graph of $f(x) = x^2 + 5$. Find the slope of this line.

87. Let l be any tangent line to the graph of $f(x) = \dfrac{6}{x}$ in the first quadrant. Show that the area of the triangle formed by l and the coordinate axes is the same for any tangent line l.

88. The graph of f and the tangent line to the graph of f at the point $(4, 3)$ are shown in the figure.



It is known that $f'(x) = -2x + 7$. The tangent line forms a triangle with the coordinate axes. Find the area of ΔABO.

89. The graph of any quadratic function $f(x) = ax^2 + bx + c$ is a parabola. Show that the average of the slopes of the tangent lines to the parabola at the endpoints of any interval $[p, q]$ equals the slope of the tangent line at the midpoint of the interval.

90. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ mx + b & \text{if } x > 2 \end{cases}$$

Find the values of m and b such that f is differentiable everywhere.

91. Evaluate $\displaystyle\lim_{x\to 1} \frac{x^{1000} - 1}{x - 1}$.

92. Draw a diagram showing two perpendicular lines that intersect on the y-axis and are both tangent to the parabola $y = x^2$. Where do these lines intersect?

93. If $c > \dfrac{1}{2}$, how many lines through the point $(0, c)$ are normal lines to the graph of the parabola $y = x^2$? What if $c \leq \dfrac{1}{2}$?

94. Sketch the parabolas $y = x^2$ and $y = x^2 - 2x + 2$. Do you think there is a line that is tangent to both curves? If so, find an equation of this line. If not, why not?

Applied Project | Building a Better Roller Coaster

Suppose you are asked to design the first ascent and drop (the first hill) for a new roller coaster. Research of your favorite coasters suggests that the slope of the ascent should be 0.8 and the slope of the drop should be -1.6. The two straight stretches modeled by $y = L_1(x)$ and $y = L_2(x)$ will be connected with part of a parabola $y = f(x) = ax^2 + bx + c$, where x and $f(x)$ are measured in feet.

In order for the track to be smooth, there cannot be an abrupt change in direction, so the linear segments L_1 and L_2 must be tangent to the parabola at the transition points P and Q. See Figure 3.11. To simplify the equations, place the origin at the point P.

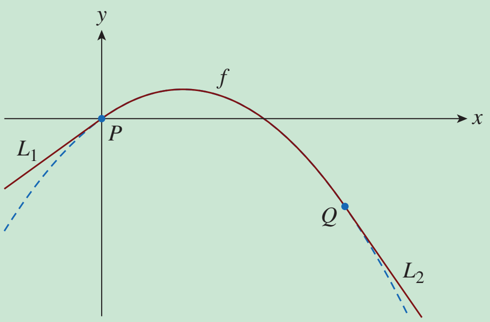


Figure 3.11
The design for the first ascent and drop.

1. (a) Suppose the horizontal distance between P and Q is 100 ft. Write equations in a, b, and c that will ensure that the track is smooth at the transition points.

(b) Solve the equations in part (a) for a, b, and c to find a formula for $f(x)$.

(c) Plot L_1, f, and L_2 to verify graphically that the transitions are smooth.

(d) Find the difference in elevation between P and Q.

2. The solution in Problem 1 might *look* smooth, but, riding the roller coaster, it might not *feel* smooth because the piecewise defined function [consisting of $L_1(x)$ for $x < 0$, $f(x)$ for $0 \le x \le 100$, and $L_2(x)$ for $x > 100$] doesn't have a continuous second derivative.

In order to improve the design, use a quadratic function $q(x) = ax^2 + bx + c$ only on the interval $10 \le x \le 90$ and connect it to the linear functions by using two cubic functions:

$$g(x) = kx^3 + lx^2 + mx + b \qquad 0 \le x < 10$$
$$h(x) = px^3 + qx^2 + rx + s \qquad 90 < x \le 100$$

(a) Write a system of equations in 11 unknowns that ensure that the functions and their first two derivatives agree at the transition points.

(b) Solve the equations in part (a) using technology (a computer algebra system) to find formulas for $q(x)$, $g(x)$, and $h(x)$.

(c) Plot L_1, g, q, h, and L_2, and compare with the plot in Problem 1(c).

3.2 The Product and Quotient Rules

The formulas in this section enable us to differentiate functions that are products or quotients without resorting to the limit definition.

The Product Rule

Suppose a new function h is a product of two functions: $h(x) = f(x)g(x)$. Analogous to the Sum and Difference Rules, we might be tempted to guess that the derivative of a product is the product of the derivatives. However, this guess is incorrect. For example, let $f(x) = x$ and $g(x) = x^2$.

$$h(x) = f(x) \cdot g(x) = x \cdot x^2 = x^3$$

$$h'(x) = 3x^2 \neq f'(x) \cdot g'(x) = 1 \cdot 2x = 2x$$

Therefore, $(fg)' \neq f'g'$. The correct formula was discovered by Leibniz (after a false start) and is called the Product Rule.

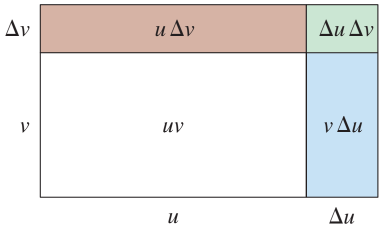


Figure 3.12
The geometry of the Product Rule.

Before we state and apply the Product Rule, let's consider how we might discover the explicit formula. Assume that $u = f(x)$ and $v = g(x)$ are differentiable functions. The product uv can be interpreted as the area of a rectangle of length u and width v (see Figure 3.12). If x changes by an amount Δx, then the corresponding changes in u and v are

$$\Delta u = f(x + \Delta x) - f(x) \qquad \Delta v = g(x + \Delta x) - g(x)$$

The value of the product $(u + \Delta u)(v + \Delta v)$ can be interpreted as the area of the large rectangle of length $u + \Delta u$ and width $v + \Delta v$ in Figure 3.12, provided that Δu and Δv are positive.

The change in the area is

$$\begin{aligned}\Delta(uv) &= (u + \Delta u)(v + \Delta v) - uv \\ &= uv + u\Delta v + v\Delta u + \Delta u\Delta v - uv \\ &= u\Delta v + v\Delta u + \Delta u\Delta v \qquad (1) \\ &= \text{the sum of the three shaded areas}\end{aligned}$$

Divide both sides of this equation by Δx.

$$\frac{\Delta(uv)}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}$$

As $\Delta x \to 0$, this leads to the definition of the derivative of uv.

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{\Delta x \to 0}\frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x \to 0}\left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}\right) \\ &= u\lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} + v\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \to 0}\Delta u\right)\left(\lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x}\right) \\ &= u\frac{dv}{dx} + v\frac{du}{dx} + 0 \cdot \frac{dv}{dx}\end{aligned}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad (2)$$

Notice that $\Delta u \to 0$ as $\Delta x \to 0$ because f is differentiable and therefore continuous.

For this geometric interpretation, we assumed that all quantities are positive. However, Equation 1 is always true. And the algebra is valid for positive or negative values of u, v, Δu, and Δv. Therefore, Equation 2 is always true and is known as the Product Rule.

The Product Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

A Closer Look

1. In words, the Product Rule says: the derivative of the product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

   Or, one could say, the derivative of the product of two functions is the second function times the derivative of the first function plus the first function times the derivative of the second function.

2. Here is the Product Rule using prime notation: $(fg)' = fg' + gf'$.
3. The Product Rule can be extended to the product of three functions. If f, g, and h are differentiable functions, then

$$(fgh)' = fgh' + fg'h + f'gh$$

Note: Consider the pattern of the three expressions on the right of the equation in the third note: 3 functions; 3 terms summed; each term consists of the product of 3 functions – 2 of the given functions and the derivative of the remaining function such that the derivative of each given function occurs once.

Example 1 The Product Rule and the *n*th Derivative

Let $f(x) = xe^x$.

(a) Find $f'(x)$.

(b) Find the nth derivative, $f^{(n)}(x)$.

Solution

(a) Use the Product Rule.

$$f'(x) = \frac{d}{dx}\left(xe^x\right)$$

$$= x\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}(x) \qquad \text{Product Rule.}$$

$$= xe^x + e^x \cdot 1 = (x+1)e^x \qquad \text{Derivatives; simplify.}$$

(b) Start by using the Product Rule a second time.

$$f''(x) = (x+1)\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}(x+1)$$

$$= (x+1)e^x + e^x \cdot 1 = (x+2)e^x$$

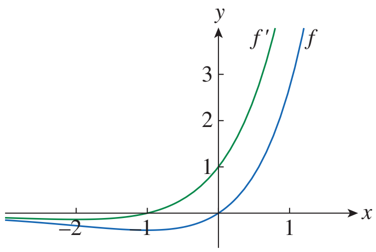


Figure 3.13
Graphs of f and f'.

Further applications of the Product Rule show

$$f'''(x) = (x+3)e^x \qquad \text{and} \qquad f^{(4)}(x) = (x+4)e^x.$$

You can probably recognize the pattern: each successive differentiation adds another e^x term.

Therefore, $f^{(n)}(x) = (x+n)e^x$.

Figure 3.13 shows the graph of f and f'. Notice that $f'(x)$ is positive when f is increasing and negative when f is decreasing. ■

Example 2 The Product Rule, or Expand First

Let $f(t) = \sqrt{t}\,(a + bt)$, where a and b are constants. Find $f'(t)$.

Solution 1

Use the Product Rule.

$$f'(t) = \sqrt{t}\,\frac{d}{dt}(a+bt) + (a+bt)\frac{d}{dt}(\sqrt{t}) \qquad \text{Product Rule.}$$

$$= \sqrt{t}\cdot b + (a+bt)\cdot \frac{1}{2}t^{-1/2} \qquad \text{Basic differentiation rules.}$$

$$= b\sqrt{t} + \frac{a+bt}{2\sqrt{t}} = \frac{a+3bt}{2\sqrt{t}} \qquad \text{Simplify.}$$

Solution 2

If we multiply out, or expand, $f(t)$ first, then we can find $f'(t)$ without the Product Rule.

$$f(t) = a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2}$$

$$f'(t) = \frac{1}{2}at^{-1/2} + \frac{3}{2}bt^{1/2} = \frac{a}{2\sqrt{t}} + \frac{3b\sqrt{t}}{2} = \frac{a+3bt}{2\sqrt{t}}$$

This is the same answer as we found in Solution 1. ■

Example 2 shows that sometimes it is easier to simplify a product of functions before differentiating than to use the Product Rule. In Example 1, the Product Rule is the only method possible.

Example 3 The Product Rule and an Evaluation

Let $f(x) = \sqrt{x}\cdot g(x)$, where $g(4) = 2$ and $g'(4) = 3$. Find $f'(4)$.

Solution

$$f'(x) = \frac{d}{dx}\left[\sqrt{x}\cdot g(x)\right] = \sqrt{x}\,\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[\sqrt{x}\right] \qquad \text{Product Rule.}$$

$$= \sqrt{x}\cdot g'(x) + g(x)\cdot\frac{1}{2}t^{-1/2} \qquad \text{Power Rule.}$$

$$= \sqrt{x}\cdot g'(x) + \frac{g(x)}{2\sqrt{x}} \qquad \text{Simplify.}$$

We can now evaluate $f'(4)$.

$$f'(4) = \sqrt{4}\, g'(4) + \frac{g(4)}{2\sqrt{4}} = 2 \cdot 3 + \frac{2}{2 \cdot 2} = \frac{13}{2}$$ ■

Example 4 Interpreting the Terms in the Product Rule

A regional satellite TV provider would like to estimate the number of new home television line installations that it will need to install during the upcoming month. At the beginning of January, the company had 100,000 subscribers, each of whom had 2.3 televisions, on average. The company estimated that its subscribership was increasing at the rate of 1000 monthly. A survey of existing subscribers suggested that each intended to install an average of 0.02 new televisions by the end of January.

Estimate the number of new television installations the company will have to install in January by finding the rate of increase of lines at the beginning of the month.

Solution

Let $s(t)$ be the number of subscribers and let $n(t)$ be the number of television lines per subscriber at time t, where t is measured in months and $t = 0$ corresponds to the beginning of January.

The total number of television lines is $L(t) = s(t)\, n(t)$.

We need to find $L'(0)$. Use the Product Rule to find $L'(t)$.

$$L'(t) = \frac{d}{dt}\big[s(t)\, n(t)\big] = s(t) \cdot \frac{d}{dt} n(t) + n(t) \cdot \frac{d}{dt} s(t)$$

We are given $s(0) = 100{,}000$ and $n(0) = 2.3$.

The company's estimates concerning rates are $s'(0) \approx 1000$ and $n'(0) \approx 0.02$.

$$\begin{aligned} L'(0) &= s(0)\, n'(0) + s'(0)\, n(0) \\ &\approx 100{,}000 \cdot 0.02 + 1000 \cdot 2.3 = 4300 \end{aligned}$$

The company will need to install approximately 4300 new television lines in homes in January.

Notice that the two terms arising from the Product Rule come from two different sources: old subscribers and new subscribers. One contribution to L' is the number of existing subscribers (100,000) times the rate at which they order new television lines (about 0.02 per subscriber monthly). A second contribution is the average number of television lines per subscriber (2.3 at the beginning of the month) times the rate of increase of subscribers (1000 monthly). ■

■ The Quotient Rule

We can find a rule for differentiating the quotient of two differentiable functions $u = f(x)$ and $v = g(x)$ using a method similar to the derivation of the Product Rule.

Suppose x, u, and v change by amounts Δx, Δu, and Δv, respectively. The corresponding change in the quotient $\dfrac{u}{v}$ is

$$\Delta\left(\frac{u}{v}\right) = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{(u + \Delta u)v - u(v + \Delta v)}{v(v + \Delta v)} = \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}.$$

Consider the definition of the derivative.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{\Delta x \to 0} \frac{\Delta\left(\dfrac{u}{v}\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

As $\Delta x \to 0$, $\Delta v \to 0$ also, because $v = g(x)$ is differentiable and therefore continuous. Using Limit Laws,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \displaystyle\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} - u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}}{v \displaystyle\lim_{\Delta x \to 0} (v + \Delta v)} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

This method leads to the following rule.

Common Error

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f(x)\dfrac{d}{dx}[g(x)] - g(x)\dfrac{d}{dx}[f(x)]}{[g(x)]^2}$$

Reversing the order of functions in the numerator results in the derivative with an incorrect sign.

Correct Method

Because the terms in the numerator are subtracted, the order matters in this rule.

The Quotient Rule

If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

A Closer Look

1. In words, the Quotient Rule says: the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.

2. Here is the Quotient Rule using prime notation: $\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$.

3. Remember, the derivative of a quotient is **not** the quotient of the derivatives.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}$$

Example 5 Using the Quotient Rule

Let $f(x) = \dfrac{x^2 + x - 2}{x^3 + 6}$. Find $f'(x)$.

Solution

$$f'(x) = \frac{(x^3 + 6)\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2} \qquad \text{Quotient Rule.}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2} \qquad \text{Derivative of polynomial.}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2} \qquad \text{Expand in the numerator.}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2} \qquad \text{Simplify.}$$

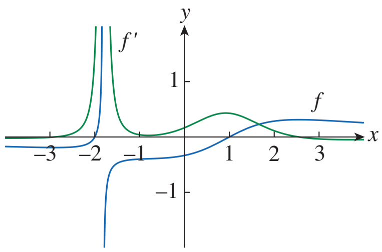


Figure 3.14
Graphs of f and f'. Notice the relationship between the two graphs.

Figure 3.14 shows graphs of f and f'. Notice that when f grows rapidly (near -2), the values of f' are large. And when f grows slowly, the values of f' are near 0.

Example 6 The Quotient Rule and a Tangent Line

Find an equation of the tangent line to the graph of $f(x) = \dfrac{e^x}{1+x^2}$ at the point $\left(1, \dfrac{e}{2}\right)$.

Solution

Use the Quotient Rule to find $f'(x)$.

$$f'(x) = \frac{\left(1+x^2\right)\dfrac{d}{dx}\left(e^x\right) - e^x\dfrac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2} \qquad \text{Quotient Rule.}$$

$$= \frac{\left(1+x^2\right)e^x - e^x(2x)}{\left(1+x^2\right)^2} = \frac{e^x\left(1-2x+x^2\right)}{\left(1+x^2\right)^2} \qquad \text{Derivatives in the numerator; factor out } e^x.$$

$$= \frac{e^x(1-x)^2}{\left(1+x^2\right)^2} \qquad \text{Simplify; quadratic is a perfect square.}$$

The slope of the tangent line at $\left(1, \dfrac{e}{2}\right)$ is

$$f'(1) = \frac{e^1(1-1)^2}{\left(1+1^2\right)^2} = 0.$$

Therefore, the tangent line at this point is horizontal.

The equation is $y = \dfrac{e}{2}$.

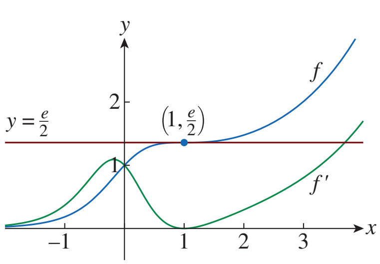


Figure 3.15
Graphs of f, f' and the tangent line. Notice the value of f' where the graph of f has a horizontal tangent line.

Figure 3.15 shows the graphs of f, f' and the tangent line. Notice that the graph of f is always increasing.

Note: It isn't always necessary to use the Quotient Rule to find the derivative of a quotient. Often we can rewrite the original quotient in a simpler form before differentiating. For example, consider the function

$$f(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

It is certainly possible to find $f'(x)$ using the Quotient Rule. However, we can first rewrite the function in a simpler form using division.

$$f(x) = \frac{3x^2}{x} + \frac{2\sqrt{x}}{x} = 3x + 2x^{-1/2}$$

Now, differentiate term by term.

$$f'(x) = 3 + 2\left(-\frac{1}{2}\right)x^{-3/2} = 3 - x^{-3/2}$$

The following table is a summary of the differentiation rules we have learned so far.

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}\left(x^n\right) = nx^{n-1} \qquad \frac{d}{dx}\left(e^x\right) = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

3.2 Exercises

1. Let $f(x) = \left(1 + 2x^2\right)\left(x - x^2\right)$. Find the derivative using the Product Rule. Find the derivative by multiplying first to obtain a polynomial. Do your answers agree?

2. Let $g(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2}$.

Find the derivative using the Quotient Rule. Find the derivative by simplifying first. Show that your answers are equivalent. Which method do you prefer in this case and why?

3. Let $h(x) = \dfrac{x^2}{x-1}$.

(a) Find the derivative using the Quotient Rule.
(b) Rewrite $h(x)$ using negative exponents and find the derivative using the Product Rule.
(c) Show that your answers in parts (a) and (b) are equivalent.

Differentiate the function.

4. $f(x) = \left(2x^2 - 5x\right)e^x$

5. $g(x) = \left(2 + 2\sqrt{x}\right)e^x$

6. $y = \dfrac{x}{e^x}$

7. $y = \dfrac{e^x}{1 - e^x}$

8. $g(x) = \dfrac{1+2}{3-4x}$

9. $G(x) = \dfrac{x^2 - 2}{2x + 1}$

10. $H(u) = \left(u - \sqrt{u}\right)\left(u + \sqrt{u}\right)$

11. $J(v) = \left(v^3 - 2v\right)\left(v^{-4} + v^{-2}\right)$

12. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)\left(y + 5y^3\right)$

13. $f(z) = \left(1 - e^z\right)\left(z + e^z\right)$

14. $y = \dfrac{x^2 + 1}{x^3 - 1}$

15. $y = \dfrac{\sqrt{x}}{2 + x}$

16. $y = \dfrac{t^3 + 3t}{t^2 - 4t + 3}$

17. $y = \dfrac{1}{t^3 + 2t^2 - 1}$

18. $y = e^p\left(p + p\sqrt{p}\right)$

19. $h(r) = \dfrac{ae^r}{b + e^r}$

20. $y = \dfrac{s - \sqrt{s}}{s^2}$

21. $y = \left(z^2 + e^z\right)\sqrt{z}$

22. $f(t) = \dfrac{\sqrt[3]{t}}{t - 3}$

23. $V(t) = \dfrac{4 + t}{te^t}$

24. $f(x) = \dfrac{x^2 e^x}{x^2 + e^x}$

25. $F(t) = \dfrac{At}{Bt^2 + Ct^3}$

26. $f(x) = \dfrac{x}{x + \dfrac{c}{x}}$

27. $f(x) = \dfrac{ax + b}{cx + d}$

Find $f'(x)$ and $f''(x)$.

28. $f(x) = \left(x^3 + 1\right)e^x$

29. $f(x) = \sqrt{x}e^x$

30. $f(x) = \dfrac{x^2}{1 + e^x}$

31. $f(x) = \dfrac{x}{x^2 - 1}$

Find an equation of the tangent line to the graph of the given function at the specified point.

32. $f(x) = \dfrac{x^2 - 1}{x^2 + x + 1}$, $(1, 0)$

33. $f(x) = \dfrac{1+x}{1+e^x}$, $\left(0, \dfrac{1}{2}\right)$

Find equations of the tangent line and the normal line to the graph of the given function at the specified point.

34. $f(x) = 2xe^x$, $(0, 0)$

35. $f(x) = \dfrac{2x}{x^2+1}$, $(1, 1)$

36. Let $f(x) = xe^x$. Find the value(s) of x where the graph of f has a horizontal tangent line.

37. Let $f(x) = \dfrac{4x}{1+x^2}$. Find the value(s) of x where the graph of f has a horizontal tangent line.

38. Let $f(x) = \dfrac{x^2}{4x+1}$. Find the value(s) of x where the graph of f has a horizontal tangent line.

39. A particle moves along a line so that its position at time t, $t \geq 0$, is given by $s(t) = \dfrac{t}{t^2+5}$. What is the position of the particle when it is at rest?

---

40. The graph of the function $f(x) = \dfrac{1}{1+x^2}$ is called a **witch of Maria Agnesi**.

(a) Find an equation of the tangent line to the graph of this function at the point $\left(-1, \dfrac{1}{2}\right)$.

(b) Illustrate your solution by graphing the function and the tangent line in the same viewing rectangle.

41. The graph of the function $f(x) = \dfrac{x}{1+x^2}$ is called a **serpentine**.

(a) Find an equation of the tangent line to the graph of this function at the point (3, 0.3).

(b) Illustrate your solution by graphing the function and the tangent line in the same viewing rectangle.

42. Let $f(x) = \left(x^3 - x\right)e^x$.

(a) Find $f'(x)$.

(b) Graph f and f' in the same viewing rectangle. Explain the relationship between the two graphs.

43. Let $f(x) = \dfrac{e^x}{2x^2+x+1}$.

(a) Find $f'(x)$.

(b) Graph f and f' in the same viewing rectangle. Explain the relationship between the two graphs.

44. Let $f(x) = \dfrac{x^2-1}{x^2+1}$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Graph f, f', and f'' in the same viewing rectangle. Explain the relationship between the three graphs.

45. Let $f(x) = (x^2-1)e^x$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Graph f, f', and f'' in the same viewing rectangle. Explain the relationship between the three graphs.

46. Let $f(x) = \dfrac{x^2}{1+x}$. Find $f''(1)$.

47. Let $g(x) = \dfrac{x}{e^x}$. Find $g^{(n)}(x)$.

48. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find each of the following values.

(a) $(fg)'(5)$ (b) $\left(\dfrac{f}{g}\right)'(5)$ (c) $\left(\dfrac{g}{f}\right)'(5)$

49. Suppose that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$. Find $h'(4)$ for each of the following functions.

(a) $h(x) = 3f(x) + 8g(x)$ (b) $h(x) = f(x)g(x)$

(c) $h(x) = \dfrac{f(x)}{g(x)}$ (d) $h(x) = \dfrac{g(x)}{f(x)+g(x)}$

50. The differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-4	10	3	3
2	3	-5	16	1
3	1	8	-4	-4
4	16	-4	1	2

Find each value.

(a) $\dfrac{d}{dx}\left[\sqrt{x} \cdot f(x)\right]$ at $x = 4$

(b) $\dfrac{d}{dx}\left[\dfrac{g(x)}{f(x)}\right]$ at $x = 2$

51. The differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-8	5	-3	1
4	1	6	-1	8

Find the derivative of each function at the given value.

(a) $h'(1)$, where $h(x) = 2x + g(x)$

(b) $q'(1)$, where $q(x) = \dfrac{g(x)}{f(x)}$

52. Suppose h is a function that is differentiable for all real numbers x, $h(2) = 7$, and $h'(2) = -2$. Find

$$\frac{d}{dx}\left[\frac{h(x)}{x}\right]\Bigg|_{x=2}$$

53. If $h(2) = 4$ and $h'(2) = -3$, find

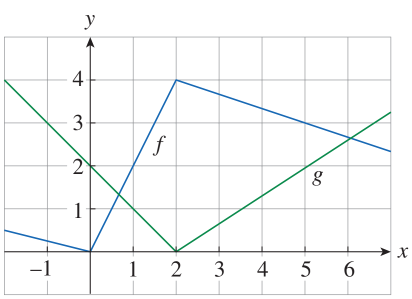
$$\frac{d}{dx}\left[\frac{h(x)}{x}\right]\Bigg|_{x=2}$$

54. Let $f(x) = e^x g(x)$ and suppose $g(0) = 2$ and $g'(0) = 5$. Find $f'(0)$.

55. Let $g(x) = xf(x)$, and suppose $f(3) = 4$ and $f'(3) = -2$. Find an equation of the tangent line to the graph of g at the point where $x = 3$.

56. Suppose $f(2) = 10$ and $f'(x) = x^2 f(x)$ for all x. Find $f''(2)$.

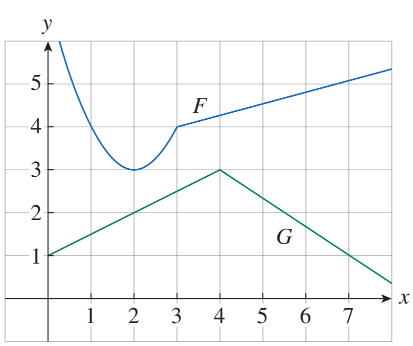
57. The graphs of f and g are shown in the figure.



Let $u(x) = f(x)g(x)$ and $v(x) = \dfrac{f(x)}{g(x)}$.

(a) Find $u'(1)$.
(b) Find $v'(5)$.

58. The graphs of F and G are shown in the figure.



Let $P(x) = F(x)G(x)$ and $Q(x) = \dfrac{F(x)}{G(x)}$.

(a) Find $P'(2)$.

(b) Find $Q'(7)$.

59. If g is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = xg(x)$ (b) $y = \dfrac{x}{g(x)}$

(c) $y = \dfrac{g(x)}{x}$ (d) $y = \dfrac{e^x}{g(x)}$

60. If f is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = x^2 f(x)$ (b) $y = \dfrac{f(x)}{x^2}$

(c) $y = \dfrac{x^2}{f(x)}$ (d) $y = \dfrac{1 + xf(x)}{\sqrt{x}}$

---

61. Let $f(x) = \dfrac{\sqrt{x} + 2}{x^2 + 4}$. Find a value x, $1 \le x \le 4$, such that $f'(x)$ is equal to the average rate of change of f over this interval.

62. How many tangent lines to the graph of $f(x) = \dfrac{x}{x + 1}$ pass through the point $(1, 2)$? For each tangent line, find the point of tangency on the graph of f.

63. Find an equation for each tangent line to the graph of

$$f(x) = \frac{x - 1}{x + 1}$$

that is parallel to the line $x - 2y = 2$.

64. Find $R'(0)$, where

$$R(x) = \frac{x - 3x^3 + 5x^5}{1 + 3x^3 + 6x^6 + 9x^9}$$

Hint: Instead of finding $R'(x)$ first, let $f(x)$ be the numerator and $g(x)$ the denominator of $R(x)$, and compute $R'(0)$ from $f(0)$, $f'(0)$, $g(0)$, and $g'(0)$.

65. Use the method described in Exercise 64 to compute $Q'(0)$, where

$$Q(x) = \frac{1 + x + x^2 + xe^x}{1 - x + x^2 - xe^x}$$

66. The Quotient Rule can be derived from the Product Rule in the following way.

(1) Let $h(x) = \dfrac{f(x)}{g(x)}$.
(2) Multiply both sides by $g(x)$.
(3) Differentiate both sides using the Product Rule.
(4) Replace $h(x)$ with $\dfrac{f(x)}{g(x)}$ and solve for $h'(x)$.

(a) Verify that this process works.

(b) Use this approach to find the derivative of $f(x) = \dfrac{3x + 2}{4x - 5}$.

67. In 2018, the population in Boston was 694,583, and the population was increasing by approximately 9624 people per year. The average annual income was \$41,794 per capita, and suppose this average was increasing by about \$1200 per year. Use the Product Rule and these figures to estimate the rate at which total personal income was rising in Boston in 2018. Explain the meaning of each term in the Product Rule in the context of this problem.

68. A manufacturer produces bolts of a fabric with a fixed width. The quantity q of this fabric (measured in yards) that is sold is a function of the selling price p (in dollars per yard), so we can write $q = f(p)$. Then the total revenue earned with selling price p is $R(p) = pf(p)$.
(a) What does it mean to say that $f(20) = 10{,}000$ and $f'(20) = -350$ in the context of this problem?
(b) Assuming the values in part (a), find $R'(20)$ and interpret your answer in the context of this problem.

69. The Michaelis–Menten equation for the digestive enzyme chymotrypsin is

$$v = \frac{0.24[S]}{0.015 + [S]}$$

where v is the rate of an enzymatic reaction and $[S]$ is the concentration of a substrate S. Calculate $\dfrac{dv}{d[S]}$ and interpret this expression in the context of this problem.

70. The biomass $B(t)$ of a fish population is the total mass of the members of the population at time t. It is the product of the number of individuals $N(t)$ in the population and the average mass $M(t)$ of a fish at time t. In the case of guppies, breeding occurs continuously. Suppose that at time $t = 4$ weeks, the population is 820 guppies and the population is growing at a rate of 50 guppies per week, while the average mass is 1.2 g and increasing at a rate of 0.14 g/week. At what rate is the biomass increasing when $t = 4$?

71. (a) Use the Product Rule twice to prove that if f, g, and h are differentiable, then $(fgh)' = f'gh + fg'h + fgh'$.
(b) Let $f = g = h$ in part (a). Show that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

(c) Use part (b) to differentiate $f(x) = e^{3x}$.

72. Let $F(x) = f(x)g(x)$, where f and g have derivatives of all orders.
(a) Show that $F'' = f''g + 2f'g' + fg''$.
(b) Find similar formulas for F''' and $F^{(4)}$.
(c) Try to recognize the pattern and guess a formula for $F^{(n)}$.

73. Find expressions for the first five derivatives of $f(x) = x^2e^x$. Try to determine the pattern in these expressions. Guess a formula for $f^{(n)}(x)$ and prove it using mathematical induction.

74. If g is differentiable, the **Reciprocal Rule** is given by

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{[g(x)]^2}$$

(a) Use the Quotient Rule to prove the Reciprocal Rule.
(b) Use the Reciprocal Rule to differentiate the function

$$f(x) = \frac{1}{x^3 + 2x^2 - 1}$$

(c) Use the Reciprocal Rule to verify that the Power Rule is valid for negative integers, that is

$$\frac{d}{dx}\left(x^{-n}\right) = -nx^{-n-1}$$

for all positive integers n.

3.3 Derivatives of Trigonometric Functions

Recall that the sine function, $f(x) = \sin x$, is defined for all real numbers x. It is understood that $\sin x$ means the sine of the angle whose radian measure is x.

A similar convention holds for the other trigonometric functions: cos, tan, csc, sec, and cot. For example, $\cos x$ means the cosine of the angle whose radian measure is x. Remember, all of the trigonometric functions are continuous at every number in their domains.

Remember, we always measure angles in radians.

To gain some insight into the derivative of the function $f(x) = \sin x$, first sketch the graph of f. Use the interpretation of $f'(x)$ as the slope of the tangent line to the graph of the sine curve in order to sketch the graph of f'. See Figure 3.16.

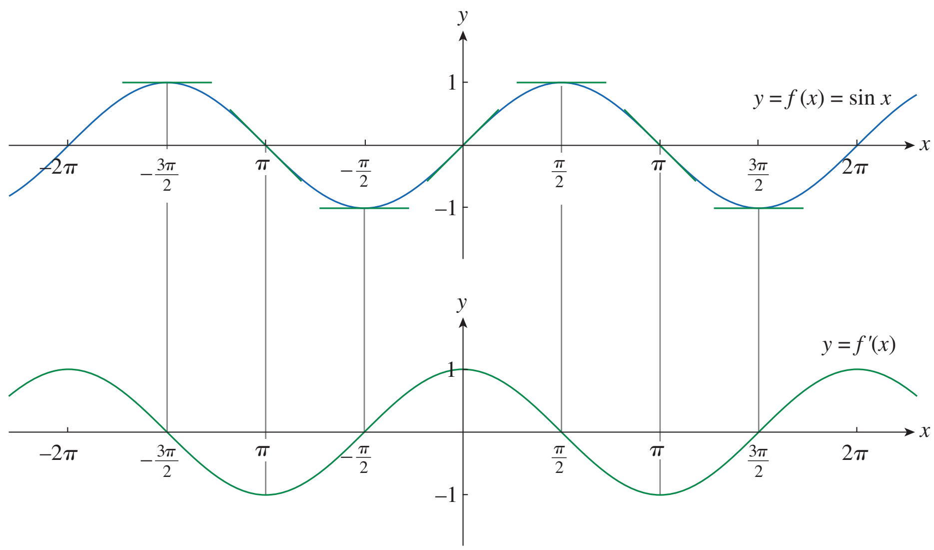


Figure 3.16
Graph of $y = f'(x)$ using the slope interpretation.

Notice that the graph of f' looks like the graph of the cosine function. We can try to confirm this guess, that $f'(x) = \cos x$, by using the definition of the derivative.

Let $f(x) = \sin x$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$ Definition; use $f(x) = \sin x$.

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}$$ Trigonometric identity: $\sin(a+b)$.

$$= \lim_{h\to 0}\left[\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right]$$ Write as two fractions.

$$= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right]$$ Factor each expression.

$$= \lim_{h\to 0}\sin x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} + \lim_{h\to 0}\cos x \cdot \lim_{h\to 0}\frac{\sin h}{h}$$ Limit Laws.

Two of these four limits are straightforward. As $h \to 0$, the variable x is treated as a constant. Therefore,

$$\lim_{h\to 0}\sin x = \sin x \qquad \text{and} \qquad \lim_{h\to 0}\cos x = \cos x.$$

The limit $\lim_{h\to 0}\frac{\sin h}{h}$ isn't so obvious. In Chapter 2, we investigated this limit and guessed the value using numerical and graphical evidence. This analysis suggested

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \tag{1}$$

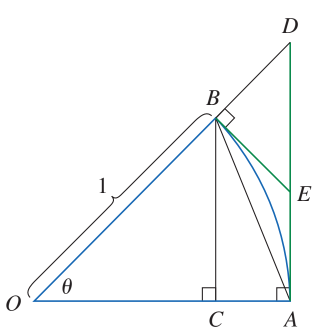


Figure 3.17
Sector of a circle with radius 1.

We can give a geometric argument to prove Equation 1. Assume that θ lies between 0 and $\pi/2$. Figure 3.17 shows a sector of a circle with center O, central angle θ, and radius 1. The segment BC is drawn perpendicular to OA. By definition of radian measure, we have arc $AB = \theta$. In addition, $|BC| = |OB|\sin\theta = \sin\theta$.
Using Figure 3.17,

$$|BC| < |AB| < \text{arc } AB$$

Therefore, substituting,

$$\sin\theta < \theta \qquad \text{so} \qquad \frac{\sin\theta}{\theta} < 1$$

Let the tangent lines at A and B intersect at E. Figure 3.18 shows that the circumference of a circle is less than the length of the circumscribed polygon. This means that

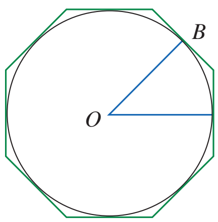


Figure 3.18
Circle with circumscribed polygon.

$$\text{arc } AB < |AE| + |EB|$$

Therefore,

$$\begin{aligned}
\theta = \text{arc } AB &< |AE| + |EB| \\
&< |AE| + |ED| && \text{Triangle geometry.} \\
&= |AD| = |OA| \tan\theta && \text{Definition of tangent.} \\
&= \tan\theta && \text{Circle of radius 1.}
\end{aligned}$$

The inequality $\theta \le \tan\theta$ can be proved directly from definition of the length of an arc, without resorting to geometric intuition. Therefore,

$$\theta < \frac{\sin\theta}{\cos\theta}$$

and multiplying both sides by $\dfrac{\cos\theta}{\theta} > 0$,

$$\frac{\cos\theta}{\theta}\cdot\theta < \frac{\cos\theta}{\theta}\cdot\frac{\sin\theta}{\cos\theta} \;\Rightarrow\; \cos\theta < \frac{\sin\theta}{\theta} < 1$$

θ approaches 0 from the right here because we assumed that θ is between 0 and $\pi/2$.

We know that $\lim\limits_{\theta\to 0} 1 = 1$ and $\lim\limits_{\theta\to 0}\cos\theta = 1$. By the Squeeze Theorem, we have

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$$

The function $\dfrac{\sin\theta}{\theta}$ is an even function. Therefore, the right and left limits at 0 must be equal. So,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

This proves Equation 1.

Here is how we find the remaining limit.

$$\begin{aligned}
\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} &= \lim_{\theta\to 0}\left[\frac{\cos\theta - 1}{\theta}\cdot\frac{\cos\theta + 1}{\cos\theta + 1}\right] = \lim_{\theta\to 0}\frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} && \text{Multiply by 1 in a convenient form.} \\
&= \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta + 1)} = -\lim_{\theta\to 0}\left[\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\cos\theta + 1}\right] && \text{Trigonometric identity; rewrite product.} \\
&= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta + 1} && \text{Limit Law.} \\
&= -1\cdot\left(\frac{0}{1+1}\right) = 0 && \text{Known limit; direct substitution.}
\end{aligned}$$

This shows

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0 \tag{2}$$

Putting all of these results together,

$$f'(x) = \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x \qquad \text{Use the limit results.}$$

This proves the rule for the derivative of the sine function,

$$\frac{d}{dx}(\sin x) = \cos x \tag{3}$$

Example 1 Derivative Involving sin *x*

Let $f(x) = x^2 \sin x$. Find $f'(x)$.

Solution

Use the Product Rule and Equation 3.

$$f'(x) = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2) \qquad \text{Product Rule.}$$

$$= x^2 \cos x + 2x \sin x \qquad \text{Equation 3; Power Rule.}$$

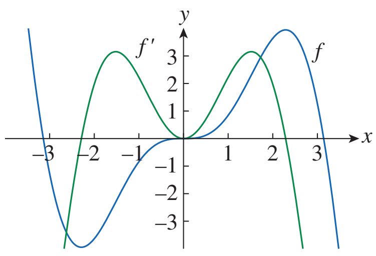


Figure 3.19
Graphs of f and f'. Notice the relationship between the two graphs.

Figure 3.19 shows the graphs of f and f'. Notice that whenever $f'(x) = 0$, the graph of f has a horizontal tangent line. ■

To find the derivative of $f(x) = \cos x$, we can use the same methods as in the proof of Equation 3 (see Exercise 22).

$$\frac{d}{dx}(\cos x) = -\sin x \tag{4}$$

We can use Equations 3 and 4 and the Quotient Rule to find the derivative of the tangent function.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \qquad \text{Definition of tangent function.}$$

$$= \frac{\cos x \dfrac{d}{dx}(\sin x) - \sin x \dfrac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{Quotient Rule.}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \qquad \text{Derivatives of } \sin x \text{ and } \cos x.$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \qquad \text{Simplify.}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \qquad \text{Trigonometric identity; definition of secant function.}$$

This calculation shows

$$\frac{d}{dx}(\tan x) = \sec^2 x \tag{5}$$

The derivatives of the remaining trigonometric functions – csc, sec, and cot – can also be found using the Quotient Rule. The differentiation rules for the six basic trigonometric functions are listed for reference. Remember that these are valid only when x is measured in radians.

Note that minus signs are associated with the derivatives of the *cofunctions,* that is, cosine, cosecant, and cotangent.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example 2 Trigonometric Derivatives

Let $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

Solution

$$f'(x) = \frac{(1+\tan x)\dfrac{d}{dx}(\sec x) - \sec x \dfrac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \qquad \text{Quotient Rule.}$$

$$= \frac{(1+\tan x)(\sec x \tan x) - \sec x \cdot \sec^2 x}{(1+\tan x)^2} \qquad \text{Derivatives of trigonometric functions.}$$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} \qquad \text{Distribute; factor out } \sec x.$$

$$= \frac{\sec x\,[\tan x + (\sec^2 x - 1) - \sec^2 x]}{(1+\tan x)^2} \qquad \text{Trigonometric identity.}$$

$$= \frac{\sec x\,(\tan x - 1)}{(1+\tan x)^2} \qquad \text{Simplify.}$$

The derivative is 0 where the graph of f has a horizontal tangent.

$$f'(x) = \frac{\sec x\,(\tan x - 1)}{(1+\tan x)^2} = 0$$

$$\Rightarrow \sec x\,(\tan x - 1) = 0 \qquad \text{A fraction that is 0 in the numerator and nonzero in the denominator equals zero.}$$

$$\Rightarrow \sec x = 0 \quad \text{or} \quad \tan x - 1 = 0 \qquad \text{Principle of Zero Products.}$$

sec x is never 0.

$$\tan x = 1 \qquad \text{when} \qquad x = n\pi + \frac{\pi}{4} \qquad \text{where } n \text{ is an integer.}$$

We need to check that the denominator is nonzero at these values of x. Otherwise, the derivative at these values would be indeterminate.

$$\text{For } x = n\pi + \frac{\pi}{4} \quad \Rightarrow \quad \tan x = 1 \quad \Rightarrow \quad (1 + \tan x)^2 = 4.$$

Figure 3.20 shows graphs of f and the horizontal tangent lines.

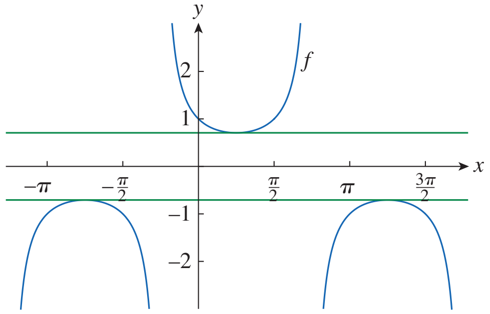


Figure 3.20
Graphs of f and the horizontal tangent lines.

Trigonometric functions are often used to model real-world phenomena. For example, vibrations, waves, elastic motions, and other quantities that vary in a periodic manner can be described using trigonometric functions. The next example involves simple harmonic motion.

Example 3 Analyzing the Motion of a Spring

An object at the end of a vertical spring is stretched downward 4 cm beyond its rest position and released at time $t = 0$. See Figure 3.21 and note that the downward direction is negative. Its position at time t is $s(t) = -4 \cos t$. Find the velocity and acceleration at time t and use these functions to analyze the motion of the object.

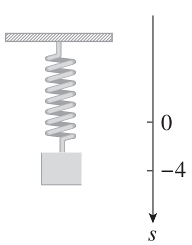


Figure 3.21
An object at the end of a vertical spring.

Solution

The velocity and acceleration functions are:

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}(-4 \cos t) = -4\frac{d}{dt}(\cos t) = (-4)(-\sin t) = 4 \sin t$$

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(4 \sin t) = 4\frac{d}{dt}(\sin t) = 4 \cos t$$

Figure 3.22 shows the graphs of the position, velocity, and acceleration of the object on the same coordinate axes.

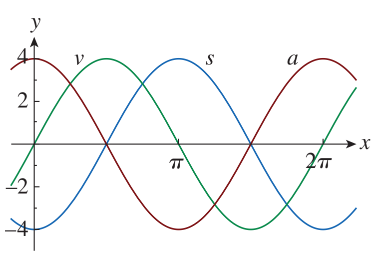


Figure 3.22
Graphs of s, v, and a.

The graph of s suggests that the object oscillates from the lowest point ($s = -4$ cm) to the highest point ($s = 4$ cm). The period of oscillation is 2π, the period of $\cos t$.

The speed of the object is $|v| = 4|\sin t|$. This expression is greatest when $|\sin t| = 1$, that is, when $t = \dfrac{(2n+1)\pi}{2}$, or when $\cos t = 0$. Therefore, the object moves fastest as it passes through its equilibrium position ($s = 0$). The speed of the object is 0 when $\sin t = 0$, at times when the position function is at its highest or lowest point.

The acceleration of the object, $a = 4\cos t = 0$ when $s = -4\cos t = 0$. This occurs where the speed of the object is greatest, at $t = \dfrac{(2n+1)\pi}{2}$. The acceleration is greatest when $t = n\pi$. ■

Example 4 Higher Order Derivatives

Find the 27th derivative of $\cos x$.

Solution

Example 4 illustrates the importance of pattern recognition in problem solving.

Consider the first few derivatives of $f(x) = \cos x$.

$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x$$
$$f^{(4)}(x) = \cos x$$
$$f^{(5)}(x) = -\sin x$$

The pattern here suggests that successive derivatives occur in a cycle of length 4.

For example, $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4.

Therefore, $f^{(24)}(x) = \cos x$, and differentiating three more times,

$f^{(27)}(x) = \sin x$. ■

3.3 Exercises

Find the derivative of the function.

1. $f(x) = x^2 \sin x$

2. $f(x) = x\cos x + 2\tan x$

3. $f(x) = e^x \cos x$

4. $f(x) = 2\sec x - \csc x$

5. $g(x) = \sec x \tan x$

6. $g(x) = e^x(\tan x - x)$

7. $f(t) = t\cos t + t^2 \sin t$

8. $f(t) = \dfrac{\cot t}{e^t}$

9. $f(x) = \dfrac{x}{2 - \tan x}$

10. $g(\theta) = \sin\theta\cos\theta$

11. $f(\theta) = \dfrac{\sin\theta}{1 + \cos\theta}$

12. $g(x) = \dfrac{\cos x}{1 - \sin x}$

13. $f(t) = \dfrac{t\sin t}{1 + t}$

14. $g(t) = \dfrac{\sin t}{1 + \tan t}$

15. $f(\theta) = \theta\cos\theta\sin\theta$

16. $f(t) = te^t \cot t$

17. $f(x) = \dfrac{2x}{\sin x \tan x}$

18. $f(x) = 2x\cos x - 3\sin x$

19. Prove that $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$.

20. Prove that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

21. Prove that $\dfrac{d}{dx}(\cot x) = -\csc^2 x$.

22. Prove, using the definition of derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

Find an equation of the tangent line to the graph of the given function at the specified point.

23. $f(x) = \sin x + \cos x, \quad (0, 1)$

24. $f(x) = e^x \cos x, \quad (0, 1)$

25. $f(x) = \cos x - \sin x, \quad (\pi, -1)$

26. $f(x) = x + \tan x, \quad (\pi, \pi)$

27. $f(x) = 2x\sin x, \quad \left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}\pi}{4}\right)$

28. Let $f(x) = 2x\sin x$.
(a) Find an equation of the tangent line to the graph of f at the point $\left(\dfrac{\pi}{2}, \pi\right)$.

(b) Illustrate part (a) by graphing f and the tangent line in the same viewing rectangle.

29. Let $f(x) = 3x + 6\cos x$.
(a) Find an equation of the tangent line to the graph of f at the point $\left(\frac{\pi}{3}, \pi + 3\right)$.
(b) Illustrate part (a) be graphing f and the tangent line in the same viewing rectangle.

30. Let $f(x) = \sec x - x$.
(a) Find $f'(x)$.
(b) Graph f and f' for $|x| < \frac{\pi}{2}$ and describe the relationship between the two graphs.

31. Let $f(x) = e^x \cos x$.
(a) Find $f'(x)$ and $f''(x)$.
(b) Graph f, f', and f'' in the same viewing rectangle and describe the relationship between the three graphs.

32. Let $g(x) = x \sin x$. Find $g'(x)$ and $g''(x)$.

33. Let $f(x) = \sin x \cos x$. Find $f'(x)$ and $f''(x)$.

34. Let $f(t) = \sec t$. Find $f''\left(\frac{\pi}{4}\right)$.

35. Let the function f be defined by

$$f(x) = \frac{\tan x - 1}{\sec x}$$

(a) Use the Quotient Rule to find $f'(x)$.
(b) Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.
(c) Show that your answers in parts (a) and (b) are equivalent.

36. Let $f(x) = \tan x$.
(a) Rewrite $f(x) = \frac{\sin x}{\cos x}$ and find the derivative using the Quotient Rule.
(b) Rewrite $f(x) = \sin x\,(\cos x)^{-1}$ and find the derivative using the Product Rule.
(c) Show that your answers in parts (a) and (b) are equivalent.

37. Suppose $f\left(\frac{\pi}{3}\right) = 4$ and $f'\left(\frac{\pi}{3}\right) = -2$.

Let $g(x) = f(x)\sin x$ and $h(x) = \frac{\cos x}{f(x)}$. Find

(a) $g'\left(\frac{\pi}{3}\right)$ (b) $h'\left(\frac{\pi}{3}\right)$

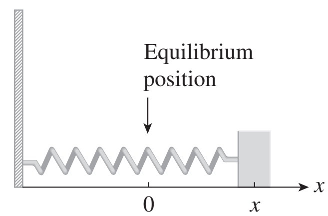
For what values of x does the graph of f have a horizontal tangent?

38. $f(x) = x + 2\sin x$ **39.** $f(x) = e^x \cos x$

40. A particle moves along a line so that its position at time t, $t \geq 0$, is given by $s(t) = 4\cos t \sin t - 4\sin t$. Find the first time t for which the particle is at rest.

41. A particle moves along a line so that its position at time t, $t \geq 0$, is given by $s(t) = \frac{\cos t}{2 + \sin t}$. Find the first time t for which the particle is at rest.

42. A mass on a spring vibrates horizontally on a smooth, level surface as shown in the figure. Its equation of motion is $x(t) = 8\sin t$, where t is measured in seconds and x in centimeters.



(a) Find the velocity and acceleration at time t.
(b) Find the position, velocity, and acceleration of the mass at time $t = \frac{2\pi}{3}$. In what direction is the mass moving at that time? Justify your answer.

43. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s(t) = 2\cos t + 3\sin t$, $t \geq 0$, where s is measured in centimeters and t in seconds. (Assign the positive direction to be downward.)
(a) Find the velocity and acceleration at time t.
(b) Graph the velocity and acceleration functions.
(c) When does the mass pass through the equilibrium position (the position when the system is at rest) for the first time?
(d) How far from its equilibrium position does the mass travel?
(e) When is the speed the greatest?

44. A ladder 10 ft long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \frac{\pi}{3}$?

45. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

where μ is a constant called the *coefficient of friction.*

(a) Find the rate of change of f with respect to θ.
(b) When is this rate of change equal to 0?
(c) If $W = 50$ lb and $\mu = 0.6$, draw the graph of f as a function of θ and use it to locate the value of θ for which $\dfrac{dF}{d\theta} = 0$. Is the value consistent with your answer to part (b)?

Determine the indicated derivative by finding the first few derivatives and recognizing the pattern that occurs.

46. $\dfrac{d^{99}}{dx^{99}}(\sin x)$ **47.** $\dfrac{d^{35}}{dx^{35}}(x \sin x)$

48. Find constants A and B such that the function $y = A \sin x + B \cos x$ satisfies the differential equation $y'' + y' - 2y = \sin x$.

49. (a) Use the substitution $\theta = 5x$ to evaluate

$$\lim_{x \to 0} \frac{\sin 5x}{x}$$

(b) Use part (a) and the definition of a derivative to find

$$\frac{d}{dx}(\sin 5x)$$

50. (a) Evaluate $\displaystyle\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$.

(b) Evaluate $\displaystyle\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

(c) Illustrate parts (a) and (b) by graphing $y = x \sin\left(\dfrac{1}{x}\right)$.

51. For each trigonometric identity, differentiate both sides to obtain a new (or familiar) identity.

(a) $\tan x = \dfrac{\sin x}{\cos x}$

(b) $\sec x = \dfrac{1}{\cos x}$

(c) $\sin x + \cos x = \dfrac{1 + \cot x}{\csc x}$

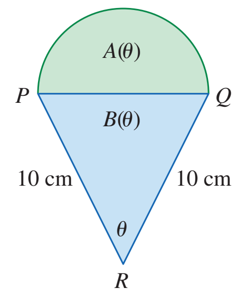
Use Equation 1, page 27, and trigonometric identities to evaluate the limit.

52. $\displaystyle\lim_{t \to 0} \frac{\tan 6t}{\sin 2t}$

53. $\displaystyle\lim_{x \to 0} \frac{\sin 3x \sin 5x}{x^2}$

54. $\displaystyle\lim_{\theta \to 0} \frac{\sin \theta}{\theta + \tan \theta}$

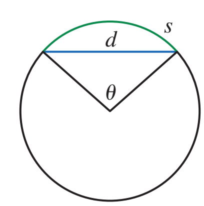
55. A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like a two-dimensional ice-cream cone, as shown in the figure.



If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find

$$\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}$$

56. The figure shows a circular arc of length s and a chord of length d, both subtended by a central angle θ.



Find $\displaystyle\lim_{\theta \to 0^+} \frac{s}{d}$.

57. Let $f(x) = \dfrac{x}{\sqrt{1 - \cos 2x}}$.

(a) Graph f. What type of discontinuity does it appear to have at 0?
(b) Calculate the left and right limits of f at 0. Do these values confirm your answer to part (a)? Explain your reasoning.

3.4 The Chain Rule

Although we already have many differentiation rules, we need to consider one more common combination of functions. Suppose

$$F(x) = (x^2 + 3x + 7)^{10}$$

To find $F'(x)$ using the differentiation rules we have learned so far, we would have to first expand $F(x)$, that is, multiply out $F(x)$ to obtain a polynomial of degree 20. We could then take the derivative term by term. There has to be a better way to find this derivative.

Notice that F is a composite function, the composition of a power and a polynomial function. If we let $y = f(u) = u^{10}$ and $u = g(x) = x^2 + 3x + 7$, then $y = F(x) = f(g(x))$, that is, $F = f \circ g$. We already know how to differentiate f and g, so it seems reasonable to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of f and g.

It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the *Chain Rule*. It seems plausible if we interpret derivatives as rates of change.

Think of $\dfrac{du}{dx}$ as the rate of change of u with respect to x, $\dfrac{dy}{du}$ as the rate of change of y with respect to u, and $\dfrac{dy}{dx}$ as the rate of change of y with respect to x. If u changes twice as fast as x, and y changes three times as fast as u, then it seems reasonable that y changes six times as fast as x, and so we expect that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Common Error

$F'(x) = f'(g(x))$

Correct Method

The derivative of a composite function is the product of the derivative of the outer function evaluated at the inner function and the derivative of the inner function.

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and $F'(x)$ is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Comments on the Proof of the Chain Rule

Let Δu be the change in u that corresponds to a change, Δx, in x.

$\Delta u = g(x + \Delta x) - g(x)$

The corresponding change in y is

$\Delta y = f(u + \Delta u) - f(u).$

Now, consider the derivative of y with respect to x.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$ Definition of the derivative.

$$= \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \right)$$ Multiply by 1 in a convenient form. Assume $\Delta u \neq 0$.

$$= \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \right) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \right)$$ Limit Laws.

$$= \left(\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \right) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \right)$$ As $\Delta x \to 0$, $\Delta u \to 0$ because g is continuous.

$$= \frac{dy}{du} \frac{du}{dx}$$ Definition of the derivative.

This is not a complete proof because Δu could be 0, even when $\Delta x \neq 0$, and we can't divide by 0. However, this argument does at least suggest that the Chain Rule is true. A full proof of the Chain Rule is given at the end of this section.

A Closer Look

1. In words, the derivative of the composition of two functions is the product of the derivative of the *outer* function evaluated at the *inner* function and of the derivative of the inner function.
2. Other notation: $(f \circ g)' = f'(g(x)) \cdot g'(x)$
3. And more notation: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$
4. Leibniz notation is easy to remember if the symbols representing the derivatives are treated as quotients.

$$\frac{dy}{du}\frac{du}{dx} = \frac{dy}{1} \cdot \frac{1}{dx} = \frac{dy}{dx}$$

However, du has not been defined, and $\frac{du}{dx}$ should not be considered an actual quotient.
5. The variable introduced in Leibniz notation is often referred to as an *intermediate* variable.
6. Given $(f \circ g)(x)$, here is a procedure for finding $(f \circ g)'(x)$ using the Chain Rule.

Step 1 Identify $f(x)$ and $g(x)$.

Step 2 Find $f'(x)$ and $g'(x)$.

Step 3 Write the final answer as $f'(g(x))\, g'(x)$.

Example 1 Using the Chain Rule

Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

Solution

$F(x) = (f \circ g)(x)$, where

$f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad g'(x) = 2x$$

$$F'(x) = \underbrace{\frac{1}{2\sqrt{x^2+1}}}_{f'(g(x))} \cdot \underbrace{2x}_{g'(x)} = \frac{x}{\sqrt{x^2+1}}$$

Using Leibniz notation,

let $y = f(u)$ and $u = g(x)$ with

$$y = \sqrt{u} \quad \text{and} \quad u = x^2 + 1$$

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \quad \text{and} \quad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{1}{2\sqrt{x^2+1}}(2x) = \frac{x}{\sqrt{x^2+1}}$$ ■

When using Leibniz notation, remember that $\frac{dy}{dx}$ means the derivative of y where y is considered a function of x (also read as the derivative of y with respect to x). The expression $\frac{dy}{du}$ refers to the derivative of y where y is considered as a function of u (the derivative of y with respect to u). In Example 1, notice that

$$\frac{dy}{dx} = F'(x) = \frac{x}{\sqrt{x^2+1}} \quad \text{and} \quad \frac{dy}{du} = f'(u) = \frac{1}{2\sqrt{u}}$$

Note: In applying the Chain Rule, we work from the outside to the inside. The Chain Rule says that we differentiate the outer function f [at the inner function $g(x)$] and then we multiply by the derivative of the inner function.

$$\frac{d}{dx} \underbrace{f}_{\substack{\text{outer}\\\text{function}}} \underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} = \underbrace{f'}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}} \underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

Another way to think about the Chain Rule: when we take the derivative of a composite function, we start at the outside and work our way in, taking derivatives along the way. After each derivative, we peel away that function and take the derivative of the next innermost function.

Example 2 The Chain Rule and Trigonometric Functions

Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

Solution

(a) If $y = \sin(x^2)$, the outer function is sine, and the inner function is the squaring function, x^2.

Use the Chain Rule.

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{\sin}_{\substack{\text{outer}\\\text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} = \underbrace{\cos}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \cdot \underbrace{2x}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

$$= 2x \cos(x^2)$$

(b) Remember, $\sin^2 x = (\sin x)^2$.

Here the outer function is the squaring function and the inner function is the sine function.

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{(\sin x)^2}_{\substack{\text{inner}\\\text{function}}} = \underbrace{2}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\underbrace{(\sin x)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \cdot \underbrace{\cos x}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

$$= 2 \sin x \cos x = \sin 2x$$ Trigonometric identity.

In Example 2(a), we used the Chain Rule and the rule for differentiating the sine function. In general, if $y = \sin u$, where u is a differentiable function of x, then, using the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos u \frac{du}{dx}$$

and

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

All the rules for differentiating trigonometric functions can be combined with the Chain Rule in a similar manner.

There is a common special case of the Chain Rule where the outer function is a power function. If $y = [g(x)]^n$, then we can write $y = f(u) = u^n$, where $u = g(x)$. Use the Chain Rule and the Power Rule to find the derivative of y with respect to x.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx} = n[g(x)]^{n-1}g'(x)$$

Here is this result stated more formally.

The Power Rule Combined with the Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \tag{1}$$

Alternatively, $$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Example 3 The Power Rule and Chain Rule

Differentiate $y = (x^3 - 1)^{100}$.

Solution

Let $u = g(x) = x^3 - 1$ and $n = 100$ and use Equation 1.

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 1)^{100} = 100(x^3 - 1)^{99} \frac{d}{dx}(x^3 - 1)$$
$$= 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}$$

■

Example 4 The Power Rule and Chain Rule; Rational Exponent

Let $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$. Find $f'(x)$.

Solution

Rewrite f: $f(x) = (x^2 + x + 1)^{-1/3}$.

$$f'(x) = -\frac{1}{3}(x^2 + x + 1)^{-4/3} \frac{d}{dx}(x^2 + x + 1)$$ Power Rule; Chain Rule.

$$= -\frac{1}{3}(x^2 + x + 1)^{-4/3}(2x + 1) = \frac{-(2x + 1)}{3\sqrt[3]{(x^2 + x + 1)^4}}$$ Derivative of the inner function; write as a fraction.

$$= \frac{-(2x + 1)}{3(x^2 + x + 1)\sqrt[3]{(x^2 + x + 1)}}$$ Simplify denominator.

■

Example 5 The Power Rule and Chain Rule; Quotient as the Inner Function

Find the derivative of the function $g(t) = \left(\dfrac{t - 2}{2t + 1}\right)^9$.

Solution

To find the derivative, we need to use the Power Rule, the Chain Rule, and the Quotient Rule, in that order.

$$g'(t) = 9\left(\frac{t - 2}{2t + 1}\right)^8 \frac{d}{dt}\left(\frac{t - 2}{2t + 1}\right)$$ Power Rule; Chain Rule.

$$= 9\left(\frac{t - 2}{2t + 1}\right)^8 \frac{(2t + 1) \cdot 1 - 2(t - 2)}{(2t + 1)^2}$$ Quotient Rule.

$$= \frac{45(t - 2)^8}{(2t + 1)^{10}}$$ Simplify.

■

Example 6 The Product Rule and the Chain Rule

Find the derivative of $f(x) = (2x + 1)^5(x^3 - x + 1)^4$.

Solution

To find $f'(x)$, we need to use the Product Rule before we use the Chain Rule.

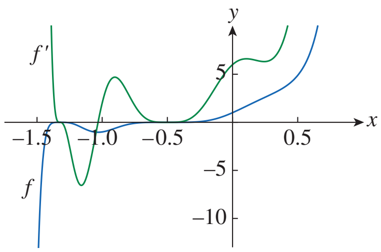


Figure 3.23
Graphs of f and f'.

$$f'(x) = (2x+1)^5 \frac{d}{dx}(x^3 - x + 1)^4 + (x^3 - x + 1)^4 \frac{d}{dx}(2x+1)^5 \qquad \text{Product Rule.}$$

$$= (2x+1)^5 \cdot 4(x^3 - x + 1)^3 \frac{d}{dx}(x^3 - x + 1)$$

$$+ (x^3 - x + 1)^4 \cdot 5(2x+1)^4 \frac{d}{dx}(2x+1) \qquad \text{Chain Rule.}$$

$$= 4(2x+1)^5(x^3 - x + 1)^3(3x^2 - 1) + (x^3 - x + 1)^4 \cdot 5(2x+1)^4 \cdot 2$$

$$= 2(2x+1)^4(x^3 - x + 1)^3(17x^3 + 6x^2 - 9x + 3) \qquad \text{Factor and simplify, if desired.}$$

The graphs of f and f' are shown in Figure 3.23. Notice that f' is large when f increases rapidly and $f'(x) = 0$ when the graph of f has a horizontal tangent line. ■

Example 7 Exponential Function and the Chain Rule

Find the derivative of $f(x) = e^{\sin x}$.

Solution

The outer function is the exponential function, e^x, and the inner function is the sine function, $\sin x$.

$$f'(x) = e^{\sin x} \frac{d}{dx}(\sin x) = e^{\sin x} \cos x$$ ■

We can also use the Chain Rule to differentiate any exponential function where the base is $b > 0$. Since the natural exponential function and the natural log function are inverses, we have $b = e^{\ln b}$. Therefore,

$$b^x = (e^{\ln b})^x = e^{(\ln b)x}$$

Here is a more general rule for the derivative of an exponential function for which the Chain Rule is necessary:

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

Be careful. In this formula, x is in the exponent. In the Power Rule, x is in the base:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Use the Chain Rule to find the derivative.

$$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{(\ln b)x}) = e^{(\ln b)x} \frac{d}{dx}[(\ln b)x] \qquad \text{Chain Rule.}$$

$$= e^{(\ln b)x} \cdot \ln b = b^x \ln b \qquad b \text{ is a constant.}$$

This leads to the following general result.

$$\frac{d}{dx}(b^x) = b^x \ln b \qquad (2)$$

Consider the special case with $b = 2$.

$$\frac{d}{dx}(2^x) = 2^x \ln 2 \qquad (3)$$

Earlier we found the following estimate.

$$\frac{d}{dx}(2^x) = (0.693)2^x$$

This is consistent with the exact formula in Equation 3 because $\ln 2 \approx 0.693147$.

The reason for the name *Chain Rule* becomes clear when we consider more complicated composite functions, or longer chains. Suppose $y = f(u)$, $u = g(x)$, and $x = h(t)$, where f, g, and h are differentiable functions. In order to find the derivative of y with respect to t, we use the Chain Rule twice.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}$$

Example 8 Use the Chain Rule Twice

Find the derivative of $f(x) = \sin(\cos(\tan x))$.

Solution

$$f'(x) = \cos(\cos(\tan x))\frac{d}{dx}[\cos(\tan x)] \qquad \text{The outermost function is the sine.}$$

$$= \cos(\cos(\tan x))[-\sin(\tan x)]\frac{d}{dx}(\tan x) \qquad \text{The next outer function is the cosine.}$$

$$= -\cos(\cos(\tan x))\sin(\tan x)\sec^2 x \qquad \text{The } \textit{inner} \text{ function is the tangent.}$$

■

Example 9 Use the Chain Rule Twice; Exponential Function

Find the derivative of $f(x) = e^{\sec 3x}$.

Solution

$$f'(x) = e^{\sec 3x}\frac{d}{dx}(\sec 3x) \qquad \text{Derivative of the exponential function.}$$

$$= e^{\sec 3x}\sec 3x \tan 3x \frac{d}{dx}(3x) \qquad \text{Derivative of the secant function.}$$

$$= 3e^{\sec 3x}\sec 3x \tan 3x \qquad \text{Derivative of } 3x.$$

■

Tangents to Parametric Curves

Suppose we want to find an equation of the tangent line at a point on the graph defined by the parametric equations $x = f(t)$ and $y = g(t)$. If f and g are differentiable functions and y is also a differentiable function of x, then we can use the Chain Rule to find the derivative of y with respect to t.

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

If $\dfrac{dx}{dt} \neq 0$, we can solve for $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \qquad \text{if } \frac{dx}{dt} \neq 0 \tag{4}$$

If you think about a particle moving along a curve, then $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ are the vertical and horizontal velocities of the particle. Equation 4 says that the slope of the tangent line is the ratio of these velocities.

A Closer Look

1. You can remember Equation 4 if you think of canceling the *dt*'s.
2. This equation enables us to find the slope $\frac{dy}{dx}$ of the tangent line at a point on the graph defined by parametric equations without having to eliminate the parameter t.
3. The graph has a horizontal tangent line when $\frac{dy}{dt} = 0$, provided that $\frac{dx}{dt} \neq 0$.

   The graph has a vertical tangent line when $\frac{dx}{dt} = 0$, provided that $\frac{dy}{dt} \neq 0$.

   This information is useful for sketching curves defined parametrically. ■

Example 10 Tangent Line to a Parametric Curve

Find an equation of the tangent line to the parametric curve

$$x = 2 \sin 2t \qquad y = 2 \sin t$$

at the point $(\sqrt{3}, 1)$. Find the points on the curve where the tangent line is horizontal or vertical.

Solution

Find the slope at the point with parameter t.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{d}{dt}(2 \sin t)}{\frac{d}{dt}(2 \sin 2t)} = \frac{2 \cos t}{2(\cos 2t)(2)} = \frac{\cos t}{2 \cos 2t}$$

The point $(\sqrt{3}, 1)$ corresponds to the parameter value $t = \frac{\pi}{6}$.

The slope of the tangent line at that point is

$$\left.\frac{dy}{dx}\right|_{t=\pi/6} = \frac{\cos \frac{\pi}{6}}{2 \cos \frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{2 \cdot \frac{1}{2}} = \frac{\sqrt{3}}{2}.$$

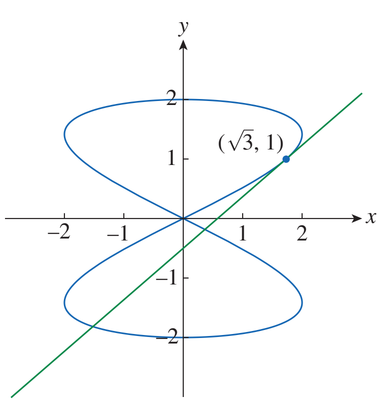


Figure 3.24
Graph of the curve and the tangent line.

An equation of the tangent line is

$$y - 1 = \frac{\sqrt{3}}{2}(x - \sqrt{3}) \qquad \text{or} \qquad y = \frac{\sqrt{3}}{2}x - \frac{1}{2}.$$

Figure 3.24 shows a graph of the curve and the tangent line.

The curve has a horizontal tangent line when $\frac{dy}{dx} = 0$, that is, when $\frac{dy}{dt} = \cos t = 0$ and $\frac{dx}{dt} = 2 \cos 2t \neq 0$.

Note that the entire curve is given by $0 \leq t \leq 2\pi$.

$$\frac{dy}{dt} = 2 \cos t = 0 \quad \Rightarrow \quad t = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\left.\frac{dx}{dt}\right|_{t=\pi/2} = 4\cos\pi = -4 \neq 0 \quad \text{and} \quad \left.\frac{dx}{dt}\right|_{t=3\pi/2} = 2\cos 3\pi = -4 \neq 0$$

Therefore, the curve has horizontal tangent lines at the points $(0, 2)$ and $(0, -2)$, as suggested by the graph in Figure 3.24.

The curve has a vertical tangent line when $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$.

$$\frac{dx}{dt} = 4\cos 2t = 0 \quad \Rightarrow \quad t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \quad \text{and} \quad \frac{dy}{dt} = 2\cos t \neq 0 \text{ for these values of } t.$$

The curve has vertical tangent lines at the four points $(\pm 2, \pm\sqrt{2})$.

Proving the Chain Rule

Recall, if $y = f(x)$ and x changes from a to $a + \Delta x$, we define the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

Using the definition of the derivative,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$$

Let ϵ denote the difference between the difference quotient and the derivative. Then

$$\lim_{\Delta x \to 0} \epsilon = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - f'(a) \right) = f'(a) - f'(a) = 0$$

But

$$\epsilon = \frac{\Delta y}{\Delta x} - f'(a) \qquad \Rightarrow \qquad \Delta y = f'(a)\Delta x + \epsilon \Delta x$$

If we define ϵ to be 0 when $\Delta x = 0$, then ϵ becomes a continuous function of Δx. Thus, for a differentiable function f, we can write

$$\Delta y = f'(a)\Delta x + \epsilon\, \Delta x \quad \text{where} \quad \epsilon \to 0 \text{ as } \Delta x \to 0 \tag{5}$$

This property of differentiable functions is what enables us to prove the Chain Rule.

Chain Rule Proof

Suppose $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. If Δx is a change in x and Δu and Δy are the corresponding changes in u and y, then we can use Equation 5 to write

$$\Delta u = g'(a)\Delta x + \epsilon_1\, \Delta x = [g'(a) + \epsilon_1]\, \Delta x \tag{6}$$

where $\epsilon_1 \to 0$ as $\Delta x \to 0$. Similarly,

$$\Delta y = f'(b)\Delta u + \epsilon_2 \Delta u = [f'(b) + \epsilon_2]\Delta u \tag{7}$$

where $\epsilon_2 \to 0$ as $\Delta u \to 0$.

Substitute the expression for Δu from Equation 6 into Equation 7.

$$\Delta y = [f'(b) + \epsilon_2][g'(a) + \epsilon_1]\Delta x$$

Divide both sides by Δx.

$$\frac{\Delta y}{\Delta x} = [f'(b) + \epsilon_2][g'(a) + \epsilon_1]$$

As $\Delta x \to 0$, Equation 6 shows that $\Delta u \to 0$. So both $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta x \to 0$. Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [f'(b) + \epsilon_2][g'(a) + \epsilon_1] \\ &= f'(b)g'(a) = f'(g(a))g'(a)\end{aligned}$$ $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$.

And this proves the Chain Rule! ■

3.4 Exercises

For the given composite function, identify the inner function, $u = g(x)$, and the outer function, $y = f(u)$. Then find the derivative, $\frac{dy}{dx}$.

1. $y = \sqrt[3]{1 + 4x}$

2. $y = (2x^3 + 5)^4$

3. $y = \tan \pi x$

4. $y = \sin(\cot x)$

5. $y = e^{\sqrt{x}}$

6. $y = \sqrt{2 - e^x}$

Find the derivative of the function.

7. $F(x) = (5x^6 + 2x^3)^4$

8. $F(x) = (1 + x + x^2)^{99}$

9. $f(x) = \sqrt{5x + 1}$

10. $f(x) = \dfrac{1}{\sqrt[3]{x^2 - 1}}$

11. $f(\theta) = \cos(\theta^2)$

12. $g(\theta) = \cos^2 \theta$

13. $y = x^2 e^{-3x}$

14. $f(t) = t \sin \pi t$

15. $f(t) = e^{at} \sin bt$

16. $g(x) = e^{x^2 - x}$

17. $f(x) = (2x - 3)^4 (x^2 + x + 1)^5$

18. $g(x) = (x^2 + 1)^3 (x^2 + 2)^6$

19. $h(t) = (t + 1)^{2/3} (2t^2 - 1)^3$

20. $F(t) = (3t - 1)^4 (2t + 1)^{-3}$

21. $y = \sqrt{\dfrac{x}{x + 1}}$

22. $y = \left(x + \dfrac{1}{x}\right)^5$

23. $y = e^{\tan \theta}$

24. $f(t) = 2^{t^3}$

25. $g(u) = \left(\dfrac{u^3 - 1}{u^3 + 1}\right)^8$

26. $s(t) = \sqrt{\dfrac{1 + \sin t}{1 + \cos t}}$

27. $r(t) = 10^{2\sqrt{t}}$

28. $f(z) = e^{z/(z - 1)}$

29. $H(r) = \dfrac{(r^2 - 1)^3}{(2r + 1)^5}$

30. $J(\theta) = \tan^2(n\theta)$

31. $F(t) = e^{t \sin 2t}$

32. $F(t) = \dfrac{t^2}{\sqrt{t^3 + 1}}$

33. $G(x) = 4^{c/x}$

34. $U(y) = \left(\dfrac{y^4 + 1}{y^2 + 1}\right)^5$

35. $y = \cos\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$

36. $y = x^2 e^{-1/x}$

37. $y = \cot^2(\sin \theta)$

38. $y = \sqrt{1 + xe^{-2x}}$

39. $f(t) = \tan(\sec(\cos t))$

40. $y = e^{\sin 2x} + \sin(e^{2x})$

41. $f(t) = \sin^2(e^{\sin^2 t})$

42. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

43. $g(x) = (2r a^{rx} + n)^p$

44. $y = 2^{3^{4^x}}$

45. $y = \cos\sqrt{\sin(\tan \pi x)}$

46. $y = [x + (x + \sin^2 x)^3]^4$

Find y' and y''.

47. $y = \cos(\sin 3\theta)$

48. $y = \dfrac{1}{(1 + \tan x)^2}$

49. $y = \sqrt{1 - \sec t}$

50. $y = e^{e^x}$

Find an equation of the tangent line to the graph of f at the given point.

51. $f(x) = 2^x$, $(0, 1)$

52. $f(x) = \sqrt{1 + x^3}$, $(2, 3)$

53. $f(x) = \sin(\sin x)$, $(\pi, 0)$

54. $f(x) = x e^{-x^2}$, $(0, 0)$

55. Let $f(x) = \dfrac{2}{1 + e^{-x}}$.

(a) Find an equation of the tangent line to the graph of f at the point (0, 1).
(b) Graph f and the tangent line in the same viewing rectangle.

56. The graph of $f(x) = \dfrac{|x|}{\sqrt{2 - x^2}}$ is called a *bullet-nose curve*.
(a) Find an equation of the tangent line to this curve at the point (1, 1).
(b) Graph f and the tangent line in the same viewing rectangle.

57. Let $f(x) = x\sqrt{2 - x^2}$.
(a) Find $f'(x)$.
(b) Graph f and f' in the same viewing rectangle and explain the relationship between the two graphs.

58. The function $f(x) = \sin(x + \sin 2x)$, $0 \le x \le \pi$, arises in applications to frequency modulation (FM) synthesis.
(a) Use technology to sketch the graph of f. Use the graph of f to sketch a graph of f'.
(b) Find $f'(x)$. Use technology to sketch the graph of f' and compare your results with your sketch in part (a). Explain any differences.

59. Find all the points on the graph of the function

$$f(x) = 2\sin x + \sin^2 x$$

at which the tangent line is horizontal.

60. Find the x-coordinates of all points on the curve $y = \sin 2x - 2\sin x$ at which the tangent line is horizontal.

61. Find the point on the graph of $f(x) = \sqrt{1 + 2x}$ such that the tangent line to the graph of f at that point is perpendicular to the line $6x + 2y = 1$.

62. If $F(x) = f(g(x))$, where $f(-1) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

63. If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

64. The differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	4	2	6
2	1	5	8	7
3	7	7	2	9

(a) If $h(x) = f(g(x))$, find $h'(1)$.
(b) If $H(x) = g(f(x))$, find $H'(1)$.

65. Let f and g be the functions in Exercise 64.
(a) If $F(x) = f(f(x))$, find $F'(2)$.
(b) If $G(x) = g(g(x))$, find $G'(3)$.

66. The differentiable functions p and q are defined for all real numbers x. Values of p, p', q, and q' for various values of x are given in the table.

x	$p(x)$	$p'(x)$	$q(x)$	$q'(x)$
4	10	8	4	2
5	4	9	16	7

(a) If $f(x) = p\left(\sqrt{q(x)}\right)$, find $f'(5)$.
(b) If $h(x) = \dfrac{q(x)}{x}$, find $h'(4)$.

67. The differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	−3	1	5	−2
5	4	7	−1	2

(a) If $h(x) = \dfrac{f(x)}{g(x)}$, find $h'(2)$.
(b) If $h(x) = f(g(x))$, find $h'(2)$.
(c) If $h(x) = \sqrt{f(x)}$, find $h'(5)$.

68. The differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
4	6	$\frac{1}{6}$	25	−5
5	4	10	7	2

If $D(x) = \dfrac{[f(x)]^2}{x}$, find $D'(4)$.

69. A particle moves along a line so that its position at time t is given by $s(t) = \cos t - \cos^2 t$ for $0 \le t \le \dfrac{3\pi}{2}$. Find the position of the particle when it changes direction from left to right.

70. A particle moves along a line so that its position at time t, for $2 \le t \le 6$, is given by

$$s(t) = \frac{3\sin\left(\dfrac{\pi t}{2}\right)}{t}$$

Find the time t at which the particle is at rest.

71. The line $y = 2x + k$ is tangent to the graph of $f(x) = \sqrt{6x^2 + 3}$. Find the coordinates of the point of tangency.

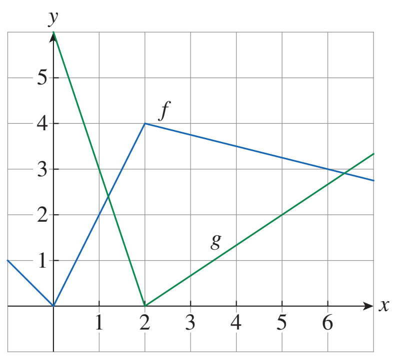
72. Let $f(x) = \sin(\pi\sqrt{x^2 + 3})$. Find the slope of the line tangent to the graph of f at the point where $x = 1$.

73. If $g'(x) = \sqrt{2x^2 + 1}$ and $y = g(\sec^2 x)$, find the value of $\dfrac{dy}{dx}$ when $x = \dfrac{\pi}{4}$.

74. Suppose f and g are differentiable functions such that $f(1) = 2$, $f'(1) = 1$, $g(1) = -1$, and $g'(1) = 0$. If $h(x) = x^2 f(x)\, g(x)$, find the value of $h'(1)$.

75. Let $f(x) = (ax + b)^n$, where $a, b \neq 0$ and n is a positive integer. Find a general expression for $f^{(n)}(x)$ in terms of a, b, and n.

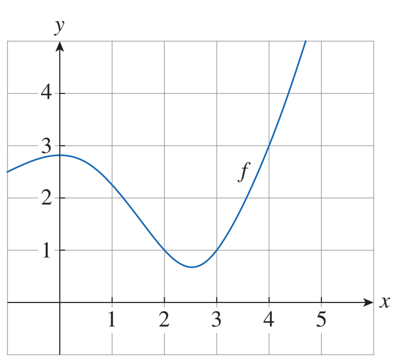
76. The graphs of the functions f and g are shown in the figure.



Let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$. Find each derivative, if it exists. If it does not exist, explain why.

(a) $u'(1)$ (b) $v'(1)$ (c) $w'(1)$

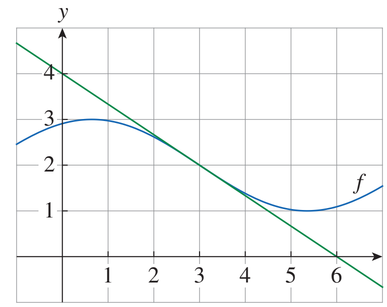
77. The graph of the function f is shown in the figure.



Let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each derivative.

(a) $h'(2)$ (b) $g'(2)$

78. The graph of the function f and the tangent line to the graph of f at the point where $x = 3$ are shown in the figure.



Let $g(x) = \sqrt{f(x)}$. Find $g'(3)$.

79. Use the table to estimate the value of $h'(0.5)$, where $h(x) = f(g(x))$.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	12.6	14.8	18.4	23.0	25.9	27.5	29.1
$g(x)$	0.58	0.40	0.37	0.26	0.17	0.10	0.05

80. If $g(x) = f(f(x))$, use the table to estimate the value of $g'(1)$.

x	0.0	0.5	1.0	1.5	2.0	2.5
$f(x)$	1.7	1.8	2.0	2.4	3.1	4.4

81. Suppose f is differentiable on $\mathbb{R}$ and α is a real number. Let $F(x) = f(x^\alpha)$ and $G(x) = [f(x)]^\alpha$. Find an expression for each derivative.
(a) $F'(x)$ (b) $G'(x)$

82. Suppose f is differentiable on $\mathbb{R}$. Let $F(x) = f(e^x)$ and $G(x) = e^{f(x)}$. Find an expression for each derivative.
(a) $F'(x)$ (b) $G'(x)$

83. Let $g(x) = e^{cx} + f(x)$ and $h(x) = e^{kx}f(x)$, where $f(0) = 3$, $f'(0) = 5$, and $f''(0) = -2$.
(a) Find $g'(0)$ and $g''(0)$ in terms of c.
(b) Find an equation of the tangent line to the graph of h at the point where $x = 0$, in terms of k.

84. Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

85. Suppose g is a twice differentiable function (the first derivative g' and the second derivative g'' exist) and $f(x) = x\,g(x^2)$. Find f'' in terms of g, g', and g''.

86. Suppose $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$. Find $F'(0)$.

87. Suppose $F(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$. Find $F'(1)$.

88. Show that the function $y = e^{2x}(A\cos 3x + B\sin 3x)$ satisfies the differential equation $y'' - 4y' + 13y = 0$.

89. Find the values of r such that the function $y = e^{rx}$ satisfies the differential equation $y'' - 4y' + y = 0$.

90. Find the 50th derivative of $y = \cos 2x$.

91. Find the 1000th derivative of $f(x) = xe^{-x}$.

92. The line tangent to the graph of $f(x) = e^{x^2}$, at the point where $x = 1$, forms a triangle in the fourth quadrant with the x- and y-axes. Find the area of this triangle.

93. The position of a particle on a vibrating string is given by the equation $s(t) = 10 + \frac{1}{4}\sin(10\pi t)$, where s is measured in centimeters and t in seconds. Find the velocity of the particle at t seconds.

94. If the equation of motion of a particle is given by $s(t) = A\cos(\omega t + \delta)$, the particle is said to undergo *simple harmonic motion*.
(a) Find the velocity of the particle at time t.
(b) When is the velocity 0?

95. A Cepheid variable star is a star whose brightness alternately increases and decreases. The most easily visible such star is Delta Cephei, for which the interval between times of maximum brightness is 5.4 days. The average brightness of this star is 4.0, and its brightness changes by ± 0.35. Using this information, the brightness of Delta Cephei at time t, where t is measured in days, has been modeled by the function

$$B(t) = 4.0 + 0.35\sin\left(\frac{2\pi t}{5.4}\right)$$

(a) Find the rate of change of the brightness at time t days.
(b) Find the rate of change of brightness at time $t = 1$ day and interpret your answer in the context of this problem.

96. In Chapter 1, we modeled the length of daylight (in hours) in Philadelphia on the tth day of the year:

$$L(t) = 12 + 2.8\sin\left[\frac{2\pi}{365}(t - 80)\right]$$

Use this model to compare the rate of change of daylight in Philadelphia on March 21 and May 21.

97. The motion of a spring that is subject to a frictional force or a damping force (such as a shock absorber in a car) is often modeled by the product of an exponential function and a sine or cosine function. Suppose the equation of motion (position) of a point on such a spring is

$$s(t) = 2e^{-1.5t}\sin 2\pi t$$

where s is measured in centimeters and t in seconds. Find the velocity at t seconds and graph both the position and velocity functions for $0 \le t \le 2$.

98. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that has heard the rumor at time t and a and k are positive constants.
(a) Find $\lim_{t\to\infty} p(t)$.
(b) Find the rate of spread of the rumor at time t.
(c) Graph p for the case $a = 10$, $k = 0.5$, with t measured in hours. Use the graph to estimate the length of time for 80% of the population to hear the rumor.

99. The average blood alcohol concentration (BAC) of eight subjects was measured after consumption of 15 mL of ethanol (corresponding to one alcoholic drink). The resulting data were used to model the concentration function

$$C(t) = 0.0225te^{-0.0467t}$$

where t is measured in minutes after consumption and C is measured in mg/mL.
(a) Find the rate of change of the BAC at time $t = 10$ minutes. Interpret your answer in the context of this problem.
(b) Find the rate of change of the BAC half an hour later. Interpret your answer in the context of this problem.

100. In Chapter 1, we modeled the world population from 1900 to 2020 with the exponential function

$$P(t) = 1368.59 \cdot (1.01471)^t$$

where $t = 0$ corresponds to the year 1990 and $P(t)$ is measured in millions. Use this model to find the rate of change in the world population in 1950, 1990, and 2020.

101. A particle moves along a line with position at time t given by $s(t)$, velocity at time t, $v(t)$, and acceleration at time t, $a(t)$. Show that

$$a(t) = v(t)\frac{dv}{ds}$$

Explain the meaning of the derivatives $\frac{dv}{dt}$ and $\frac{dv}{ds}$ in the context of this problem.

102. Air is being pumped into a spherical weather balloon. At any time t, the volume of the balloon is $V(t)$ and its radius is $r(t)$.
(a) Interpret the derivatives $\frac{dV}{dr}$ and $\frac{dV}{dt}$ in the context of this problem.
(b) Express $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$.

103. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data given in the table describe the charge Q remaining on the capacitor (measured in microcoulombs, μC) at time t (measured in seconds).

t	0.00	0.02	0.04	0.06	0.08	0.10
Q	100.00	81.87	67.03	54.88	44.93	36.76

(a) Use technology to find an exponential model for the charge.
(b) The derivative $Q'(t)$ represents the electric current (measured in microamperes, μA) flowing from the capacitor to the flash bulb. Use your answer in part (a) to estimate the current when $t = 0.04$ s.

104. The table gives the population of Texas in millions from 1900 to 2020.

Year	Population	Year	Population
1900	3.05	1970	11.20
1910	3.90	1980	14.23
1920	4.66	1990	16.99
1930	5.82	2000	20.85
1940	6.41	2010	25.24
1950	7.71	2020	29.47
1960	9.58		

(a) Use technology to fit an exponential function to the data. Graph the data points and the exponential model. Use this graph to explain whether this model provides a good fit to the data.
(b) Use the table to estimate the rate of population growth in 1950 and in 2000.
(c) Use the exponential model in part (a) to estimate the rates of growth in 1950 and in 2000. Compare these estimates with your answers in part (b).
(d) Use the exponential model to predict the population in 2021. Compare this answer to the actual population. Try to explain the discrepancy.

Find an equation of the tangent line to the curve at the point corresponding to the given value of the parameter.

105. $x = t^4 + 1,\ y = t^3 + t;\quad t = -1$

106. $x = \cos\theta + \sin 2\theta,\quad y = \sin\theta + \cos 2\theta;\quad \theta = 0$

107. $x = e^{\sqrt{t}},\quad y = t - \ln t^2;\quad t = 1$

Find the points on the curve where the tangent line is horizontal or vertical. Use technology to sketch the curve to check your answer.

108. $x = 2t^3 + 3t^2 - 12t,\ y = 2t^3 + 3t^2 + 1$

109. $x = 10 - t^2,\ y = t^3 - 12t$

110. Show that the curve with parametric equations

$$x = \sin t \qquad y = \sin(t + \sin t)$$

has two tangent lines at the origin and find their equations. Illustrate this result by graphing the curve and these tangent lines in the same viewing rectangle.

111. The curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.
(a) Show that C has two tangents at the point (3, 0) and find their equations.
(b) Find the points on C where the tangent line is horizontal or vertical.
(c) Illustrate parts (a) and (b) by graphing C and the tangent lines in the same viewing rectangle.

112. The cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ was discussed in Section 1.6.
(a) Find an equation of the tangent line to the cycloid at the point where $\theta = \pi/3$.
(b) At what points is the tangent line to the cycloid horizontal? Where is it vertical?
(c) Graph the cycloid and its tangent lines for the case $r = 1$.

113. Let $f(x) = \sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}$.
(a) Use a CAS to find $f'(x)$ and simplify the result.
(b) Find the values of x where the graph of f has horizontal tangent lines.
(c) Graph f and f' in the same viewing rectangle. Are these graphs consistent with your answer to part (b)? Explain any discrepancies.

114. Use the Chain Rule to prove the following.
(a) The derivative of an even function is an odd function.
(b) The derivative of an odd function is an even function.

115. Use the Chain Rule and the Product Rule to give an alternative proof of the Quotient Rule.

Hint: Write $\dfrac{f(x)}{g(x)} = f(x)[g(x)]^{-1}$.

116. Suppose n is a positive integer.
(a) Show that

$$\frac{d}{dx}(\sin^n x \cos nx) = n \sin^{n-1} x \cos(n + 1)x$$

(b) Find a formula for the derivative of $y = \cos^n x \cos nx$ that is similar to the one in part (a).

117. Suppose the function f is differentiable everywhere, the graph of $y = f(x)$ always lies above the x-axis and never has a horizontal tangent. For what value of y is the rate of change of y^5 with respect to x eighty times the rate of change of y with respect to x?

118. Use the Chain Rule to show that if θ is measured in degrees, then

$$\frac{d}{d\theta}(\sin\theta) = \frac{\pi}{180}\cos\theta$$

This is one reason for the convention that radian measure is always used when dealing with trigonometric functions in calculus. The differentiation rules that we have developed are not true for degree measure.

119. Recall that an alternate definition for the absolute value function is $|x| = \sqrt{x^2}$.

(a) Use the Chain Rule to show that

$$\frac{d}{dx}(|x|) = \frac{x}{|x|}$$

(b) If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of f and f'. Where is f not differentiable?

(c) If $g(x) = \sin|x|$, find $g'(x)$ and sketch the graphs of g and g'. Where is g not differentiable?

120. If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions (the first derivatives, f' and g', and the second derivatives, f'' and g'', exist), show that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\,\frac{d^2u}{dx^2}$$

121. If $y = f(u)$ and $u = g(x)$, where f and g have third derivatives, find a formula for $\dfrac{d^3y}{dx^3}$ similar to the one given in Exercise 120.

Laboratory Project | Bézier Curves

Bézier curves are used in computer-aided design and, in general, computer graphics to construct smooth curves. They are named after the French mathematician Pierre Bézier (1910–1999), who worked in the automotive industry for Renault cars.

A cubic Bézier curve is determined by four *control points*, $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$, and is defined by the parametric equations

$$x = x_0(1-t)^3 + 3x_1t(1-t)^2 + 3x_2t^2(1-t) + x_3t^3$$
$$y = y_0(1-t)^3 + 3y_1t(1-t)^2 + 3y_2t^2(1-t) + y_3t^3$$

where $0 \le t \le 1$. Notice that when $t = 0$, $(x, y) = (x_0, y_0)$, and when $t = 1$, $(x, y) = (x_3, y_3)$, so the curve starts at P_0 and ends at P_3.

1. Use technology to graph the Bézier curve with control points $P_0(4, 1)$, $P_1(28, 8)$, $P_2(50, 42)$, and $P_3(40, 5)$. In the same viewing rectangle, graph the line segments P_0P_1, P_1P_2, and P_2P_3. Notice that the middle control points, P_1 and P_2, do not lie on the curve, the curve starts at P_0, heads toward P_1 and P_2 without actually reaching them, and then ends at P_3.

2. The graph in Problem 1 suggests that the tangent line at P_0 passes through P_1 and the tangent line at P_3 passes through P_2. Prove these results about the tangent lines.

3. Try to produce a Bézier curve with a loop by changing the second control point in Problem 1.

4. Some laser printers use Bézier curves to represent letters and other symbols. Experiment with control points until you find a Bézier curve that produces a reasonable representation of the letter C.

5. More complicated shapes can be represented by joining two or more Bézier curves. Suppose the first Bézier curve has control points P_0, P_1, P_2, P_3, and the second one has control points P_3, P_4, P_5, P_6. If we want these two pieces to join together smoothly, then the tangents at P_3 should match. Therefore, the points P_2, P_3, and P_4 all have to lie on this common tangent line. Using this principle, find control points for a pair of Bézier curves that represent the letter S.

Applied Project | Where Should a Pilot Start Descent?

An approach path for an aircraft landing is shown in Figure 3.25 and satisfies the following conditions:

(i) The cruising altitude is h when descent starts at a horizontal distance l from touchdown at the origin.

(ii) The pilot must maintain a constant horizontal speed v throughout descent.

(iii) The absolute value of the vertical acceleration should not exceed a constant k (which is much less than the acceleration due to gravity).

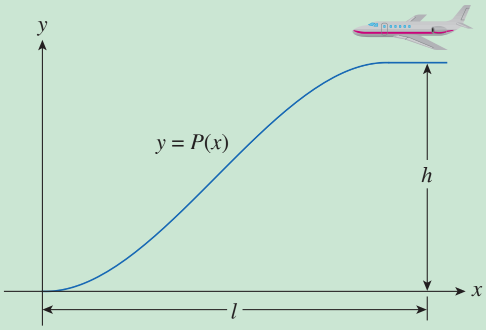


Figure 3.25
The path of an airplane's descent.

1. Find a cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ that satisfies condition (i) by imposing suitable conditions on $P(x)$ and $P'(x)$ at the start of descent and at touchdown.

2. Use conditions (ii) and (iii) to show that

$$\frac{6hv^2}{l^2} \leq k$$

3. Suppose that an airline decides not to allow vertical acceleration of a plane to exceed $k = 860 \text{ mi/h}^2$. If the cruising altitude of a plane is 35,000 ft and the speed is 300 mi/h, how far away from the airport should the pilot start descent?

4. Graph the approach path for the conditions stated in Problem 3.

3.5 | Implicit Differentiation

In the functions we have been working with so far, the variable y has been expressed explicitly as a function of x. For example,

$$y = \sqrt{x^3 + 1} \qquad \text{or} \qquad y = x \sin x$$

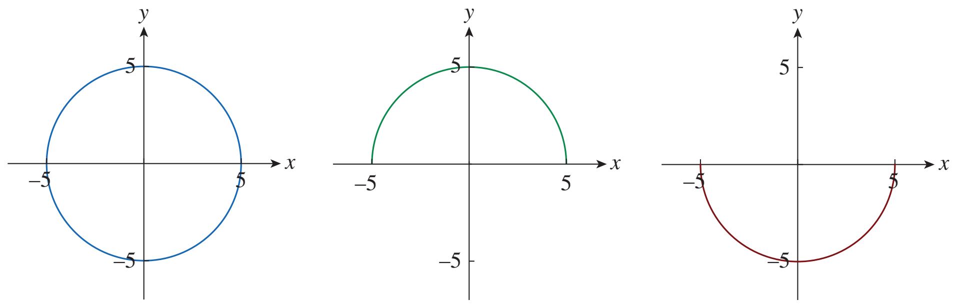
or, in general, $y = f(x)$. However, some functions are defined *implicitly* by an equation involving the variables x and y, such as

$$x^2 + y^2 = 25 \tag{1}$$

or

$$x^3 + y^3 = 6xy \tag{2}$$

In some cases, it is possible to solve such an equation explicitly for y as a function of x. For example, if we solve Equation 1 for y, we get $y = \pm\sqrt{25 - x^2}$. Therefore, there are actually two functions defined implicitly by the expression in Equation 1: $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$. The graphs of f and g are the upper and lower semicircles of the circle defined by $x^2 + y^2 = 25$. See Figure 3.26.



(a) Graph of the circle defined by $x^2 + y^2 = 25$

(b) Graph of $f(x) = \sqrt{25 - x^2}$

(c) Graph of $g(x) = -\sqrt{25 - x^2}$

Figure 3.26
The equation $x^2 + y^2 = 25$ *implicitly* defines y as two functions of x.

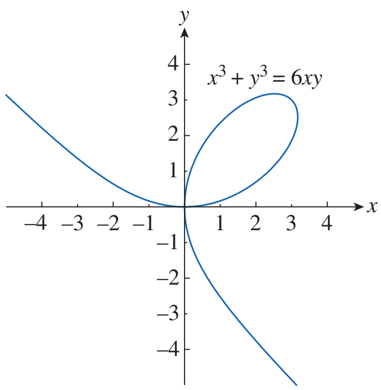


Figure 3.27
Graph of the folium of Descartes.

Remember, a curve is the graph of a function if it passes the Vertical Line Test.

It's not easy to solve Equation 2 for y explicitly as a function of x by hand. Here is an expression for y obtained using a computer algebra system.

$$y = \frac{2 \cdot 2^{1/3}x}{\left(-x^3 + \sqrt{-32x^3 + x^6}\right)^{1/3}} + \frac{\left(-x^3 + \sqrt{-32x^3 + x^6}\right)^{1/3}}{2^{1/3}}$$

So it can be done, but it's complicated. The graph of Equation 2 is called the **folium of Descartes** and is shown in Figure 3.27.

There are actually three functions defined implicitly by Equation 2. The graphs of these three functions are shown in Figure 3.28.

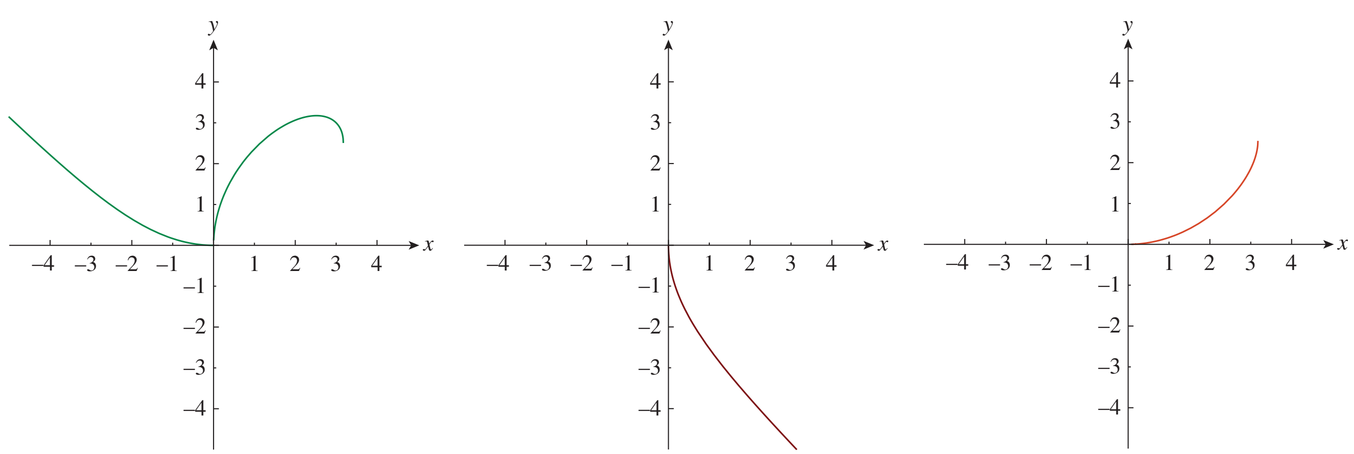


Figure 3.28
Graphs of three functions defined by the folium of Descartes.

It is important to understand that when we say that f is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)]^3 = 6xf(x)$$

is true for all values of x in the domain of f.

Fortunately, we do not need to solve an equation for y in terms of x in order to find the derivative of y. Instead, we can use the method of **implicit differentiation**. This method consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

Procedure for Implicit Differentiation

(1) Let $y = f(x)$, that is, consider y as a function of x.

(2) Differentiate implicitly: differentiate both sides of the equation with respect to x. Since y is a function of x, use the Chain Rule to find the derivative of an expression involving y.

(3) Solve for $\dfrac{dy}{dx}$; this expression may be in terms of both x and y.

In the examples and exercises of this section, we assume that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be used.

Example 1 Finding a Tangent Line Implicitly

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point (3, 4).

Solution

(a) Use the implicit differentiation procedure.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \qquad \text{Differentiate both sides of the equation with respect to } x.$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0 \qquad \text{Derivative of a sum; derivative of a constant.}$$

$$2x + 2y\frac{dy}{dx} = 0 \qquad y \text{ is a function of } x\text{; use the Chain Rule.}$$

$$\frac{dy}{dx} = -\frac{x}{y} \qquad \text{Solve for } \frac{dy}{dx}.$$

(b) At the point (3, 4): $x = 3$ and $y = 4$.

When using implicit differentiation to find the derivative at a point, we almost always need both the x- and y-coordinates.

The slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{(x,\,y)=(3,\,4)} = -\frac{3}{4}.$$

An equation of the tangent line to the circle at (3, 4) is

$$y - 4 = -\frac{3}{4}(x - 3) \quad \text{or} \quad y = -\frac{3}{4}x + \frac{25}{4}.$$

Here is an alternate solution to this problem.

Solve for y in terms of x in the equation $x^2 + y^2 = 25 \Rightarrow y = \pm\sqrt{25 - x^2}$.

The point (3, 4) lies on the upper semicircle, $y = \sqrt{25 - x^2}$.

Therefore, we use the function $f(x) = \sqrt{25 - x^2}$.

This example shows that even when it is possible to solve an equation explicitly for y in terms of x, it may be easier to use implicit differentiation.

Find the derivative $f'(x)$.

$$f'(x) = \frac{1}{2}(25 - x^2)^{-1/2}\frac{d}{dx}(25 - x^2)$$ Chain Rule.

$$= \frac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}}$$ Derivative of the inner function; simplify.

The slope of the tangent line is

$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}.$$

Therefore, we obtain the same equation for the tangent line, $y - 4 = -\frac{3}{4}(x - 3)$.

Figure 3.29 shows a graph of the circle and the tangent line. ■

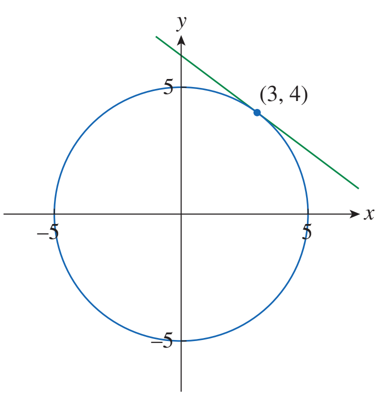


Figure 3.29
Graph of the circle and the tangent line.

Note: The expression $\frac{dy}{dx} = -\frac{x}{y}$ in Example 1 gives the derivative in terms of both x and y. It is correct for *every* function y that is implicitly defined by the equation (a remarkable result).

For example, if $y = f(x) = \sqrt{25 - x^2}$,

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{25 - x^2}},$$

and for $y = g(x) = -\sqrt{25 - x^2}$,

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{-\sqrt{25 - x^2}} = \frac{x}{\sqrt{25 - x^2}}.$$

Example 2 Implicit Differentiation and the Folium of Descartes

(a) Find y' if $x^3 + y^3 = 6xy$.
(b) Find an equation of the tangent line to the folium of Descartes at the point (3, 3).
(c) Find the point in the first quadrant at which the tangent line to the folium of Descartes is horizontal.

Solution

Note: Here we are using the symbol y' to represent the derivative of y with respect to x.

(a) Use the implicit differentiation procedure.

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy)$$ Differentiate both sides with respect to x.

$$3x^2 + 3y^2y' = 6xy' + 6y(1)$$ Chain Rule; Product Rule.

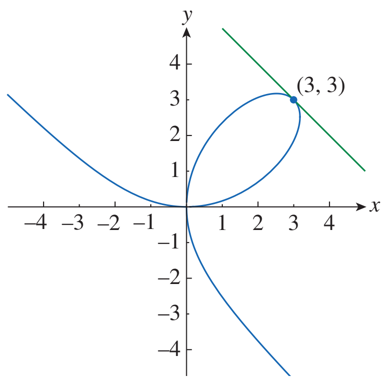


Figure 3.30
The folium of Descartes and the tangent line to the graph at the point (3, 3).

$$x^2 + y^2y' = 2xy' + 2y$$ Divide both sides by 3.

$$y^2y' - 2xy' = 2y - x^2$$ Isolate terms involving y'.

$$(y^2 - 2x)y' = 2y - x^2$$ Factor out y'.

$$y' = \frac{2y - x^2}{y^2 - 2x}$$ Solve for y'; divide both sides by $y^2 - 2x$.

(b) Find the slope of the tangent line at the point (3, 3).

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

An equation of the tangent line is

$$y - 3 = -1(x - 3) \quad \text{or} \quad y = -x + 6.$$

Figure 3.30 shows the graphs of the folium of Descartes and the tangent line.

(c) The tangent line is horizontal if $y' = 0$.

$$y' = \frac{2y - x^2}{y^2 - 2x} = 0 \quad \Rightarrow \quad 2y - x^2 = 0 \quad \Rightarrow \quad y = \frac{1}{2}x^2 \text{ (provided } y^2 - 2x \neq 0)$$

Substitute into the original equation to obtain one equation in one unknown.

$$x^3 + \left(\frac{1}{2}x^2\right)^3 = 6x\left(\frac{1}{2}x^2\right)$$ Use the expression $y = \frac{1}{2}x^2$.

$$x^3 + \frac{1}{8}x^6 = 3x^3$$ Expand; simplify.

$$x^6 - 16x^3 = 0$$ Collect terms on one side; clear fractions: multiply both sides by 8.

$$x^3(x^3 - 16) = 0$$ Factor.

$$x^3 = 0 \quad \text{or} \quad x^3 - 16 = 0$$ Principle of Zero Products.

$x^3 = 0 \quad \Rightarrow \quad x = 0$ However, $x \neq 0$ in the first quadrant.

$$x^3 - 16 = 0 \quad \Rightarrow \quad x = 16^{1/3} = 2^{4/3} \quad \text{and} \quad y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$$

Therefore, the tangent line is horizontal at the point $(2^{4/3}, 2^{5/3}) \approx (2.520, 3.175)$. See Figure 3.31.

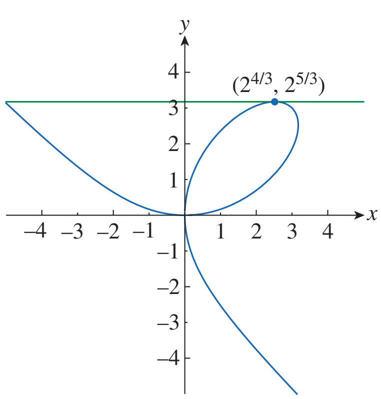


Figure 3.31
The folium of Descartes and the horizontal tangent line to the graph in the first quadrant.

Example 3 Implicit Differentiation Involving Trigonometric Functions

Find y' if $\sin(x + y) = y^2 \cos x$.

Solution

Differentiate implicitly with respect to x.

Remember, we consider y as a function of x.

$$\cos(x + y) \cdot (1 + y') = y^2(-\sin x) + (\cos x)(2yy')$$ Chain Rule; Product Rule.

$$\cos(x+y)\cdot y' - (2y\cos x)\cdot y' = -y^2\sin x - \cos(x+y) \quad \text{Collect terms that involve } y'.$$

$$y' = \frac{-y^2\sin x - \cos(x+y)}{\cos(x+y) - 2y\cos x} \quad \text{Solve for } y'.$$

Figure 3.32 shows part of the graph defined by $\sin(x+y) = y^2\cos x$. Notice that $y' = -1$ when $x = y = 0$ and it also appears from the graph that the slope is approximately -1 at the origin. ■

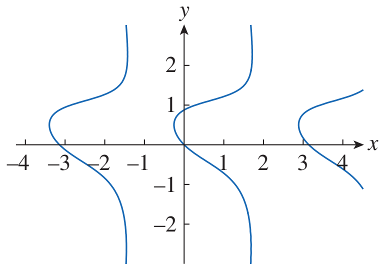


Figure 3.32
A portion of the graph defined by $\sin(x+y) = y^2\cos x$.

Figures 3.33–3.35 show three curves defined by complex equations. You will have the opportunity to create and examine some unusual curves like these in the exercises.

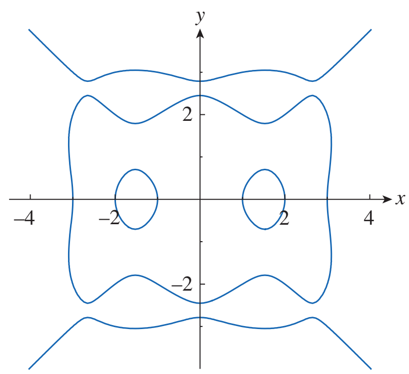


Figure 3.33
$(x^2-1)(x^2-4)(x^2-9) = y^2(y^2-4)(y^2-9)$

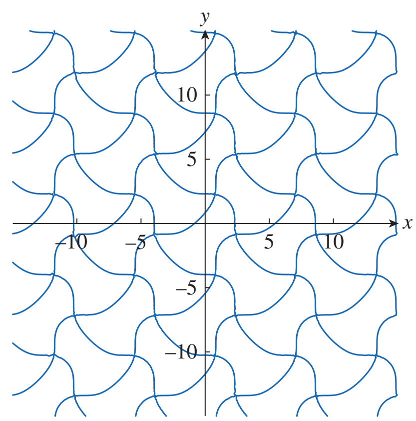


Figure 3.34
$\cos(x - \sin y) = \sin(y - \sin x)$

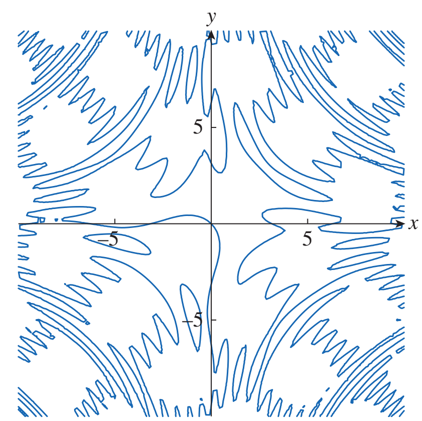


Figure 3.35
$\sin(xy) = \sin x + \sin y$

Example 4 Finding a Second Derivative Implicitly

Find y'' if $x^4 + y^4 = 16$.

Solution

Differentiate the equation implicitly with respect to x, and solve for y'.

$$4x^3 + 4y^3y' = 0$$

$$y' = -\frac{x^3}{y^3}$$

To find y'', differentiate this expression using the Quotient Rule.

Remember, y is a function of x.

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3\dfrac{d}{dx}(x^3) - x^3\dfrac{d}{dx}(y^3)}{(y^3)^2} \quad \text{Quotient Rule.}$$

$$= -\frac{y^3\cdot 3x^2 - x^3(3y^2y')}{y^6} \quad \text{Chain Rule.}$$

$$= -\frac{3x^2y^3 - 3x^3y^2\left(-\dfrac{x^3}{y^3}\right)}{y^6} \quad \text{Use the expression for } y'.$$

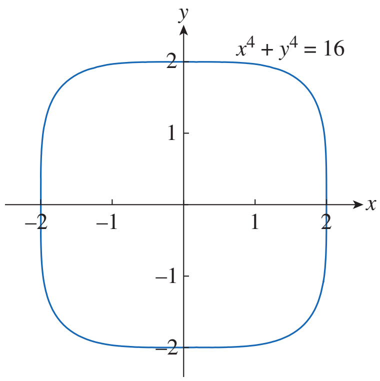


Figure 3.36
Graph of a fat circle.

$$= -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4 + x^4)}{y^7} \qquad \text{Simplify.}$$

$$= -\frac{3x^2(16)}{y^7} = -48\frac{x^2}{y^7} \qquad \text{Use the original equation } x^4 + y^4 = 16.$$

Figure 3.36 shows the graph of the equation $x^4 + y^4 = 16$. Notice that it is a stretched and flattened version of the circle defined by $x^2 + y^2 = 16$. Therefore, this graph is often called a *fat circle*. The graph is very steep on the far left and far right but quite flat in between. This characteristic is supported by the expression

$$y' = -\frac{x^3}{y^3} = -\left(\frac{x}{y}\right)^3.$$

For small values of y, this expression is large in absolute value. For small values of x, this expression is close to 0. ■

3.5 Exercises

For each expression,

(a) Find y' by implicit differentiation.

(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.

(c) Check that your solutions to parts (a) and (b) are consistent. Substitute the expression for y into your solution for part (a) and simplify if necessary.

1. $9x^2 - y^2 = 1$ **2.** $2x^2 + x + xy = 1$

3. $\sqrt{x} + \sqrt{y} = 1$ **4.** $\cos x + \sqrt{y} = 5$

5. Find the slope of the tangent line to $y^2 + x = 5$ at the point where $y = 2$.

6. Find the slope of the tangent line to the circle with equation $x^2 + y^2 = a^2$, where $a \neq 0$.

7. Find y'' if $x^2 - y^2 = 25$.

Find $\frac{dy}{dx}$ by implicit differentiation.

8. $x^2 - 4xy + y^2 = 4$ **9.** $2x^2 + xy - y^2 = 2$

10. $x^4 + x^2y^2 + y^3 = 5$ **11.** $x^3 - xy^2 + y^3 = 1$

12. $\frac{x^2}{x + y} = y^2 + 1$ **13.** $x e^y = x - y$

14. $y \cos x = x^2 + y^2$ **15.** $\cos(xy) = 1 + \sin y$

16. $\sqrt{x + y} = x^4 + y^4$ **17.** $e^y \sin x = x + xy$

18. $e^{x/y} = x - y$ **19.** $xy = \sqrt{x^2 + y^2}$

20. $y \sin(x^2) = x \sin(y^2)$ **21.** $x \sin y + y \sin x = 1$

22. $\sin(xy) = \cos(x + y)$ **23.** $\tan(x - y) = \frac{y}{1 + x^2}$

24. $\cos(xy) + \sin y = xy$ **25.** $2x e^x + 2y e^y = 4$

26. If $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$, find $f'(1)$.

27. If $g(x) + x \sin g(x) = x^2$, find $g'(0)$.

Assume y is the independent variable and x is the dependent variable and use implicit differentiation to find $\frac{dx}{dy}$.

28. $x^4y^2 - x^3y + 2xy^3 = 0$

29. $y \sec x = x \tan y$

Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

30. $x^2y + 2x = 15$, $(3, 1)$

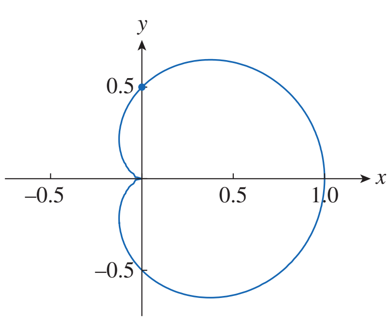
31. $y \sin 2x = x \cos 2y$, $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

32. $\sin(x + y) = 2x - 2y$, (π, π)

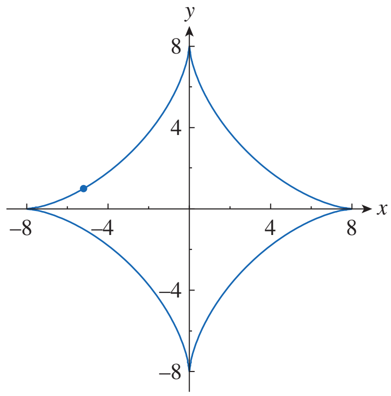
33. $x^2 - xy - y^2 = 1$, $(2, 1)$

34. $x^2 + 2xy + 4y^2 = 12$, $(2, 1)$

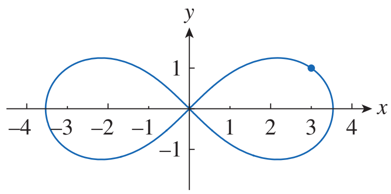
35. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$, $\left(0, \frac{1}{2}\right)$ (cardioid)



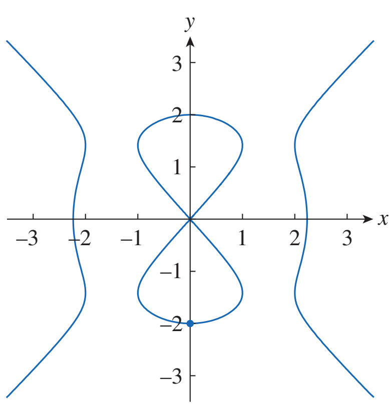
36. $x^{2/3} + y^{2/3} = 4$, $(-3\sqrt{3}, 1)$ (astroid)



37. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, $(3, 1)$ (lemniscate)



38. $y^2(y^2 - 4) = x^2(x^2 - 5)$, $(0, -2)$ (devil's curve)



39. Consider the equation $\tan(xy) = x$. Find $\dfrac{dy}{dx}$.

40. Find an equation of the tangent line to the graph of $x^2 - 3xy + y^2 = -20$ at the point (6, 4), or explain why no such tangent line exists.

41. Find an equation of the tangent line to the graph of $x^2 - 12x^3y + y^3 = -15$ at the point (1, 2), or explain why no such tangent line exists.

42. Find the slope of the normal line to the curve $3x + xy + y^2 = 9$ at the point where $y = 2$.

43. Find the coordinates of the point on the curve $x^2 + xy + y^2 = 7$ where the slope is the same as at the point (2, 1).

44. Consider the equation $e^y = \sin x$. Find $\dfrac{dy}{dx}$.

45. If $\dfrac{dy}{dx} = \sqrt{25 - y^2}$, find the value of $\dfrac{d^2y}{dx^2}$ at the point where $y = 3$.

46. The graph of the equation $y^2 = 5x^4 - x^2$ is called a **kampyle of Eudoxus**.
(a) Find an equation of the tangent line to this graph at the point (1, 2).
(b) Illustrate your solution to part (a) by sketching the graph and the tangent line in the same viewing rectangle.

47. The graph of the equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**.
(a) Find an equation of the tangent line to this graph at the point $(1, -2)$.
(b) At what points does this graph have horizontal tangents?
(c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines in the same viewing rectangle.

Find y'' by implicit differentiation.

48. $x^2 + 4y^2 = 4$

49. $x^2 + xy + y^2 = 3$

50. $\sin y + \cos x = 1$

51. $x^3 - y^3 = 7$

52. If $xy + e^y = e$, find the value of y'' at the point where $x = 0$.

53. If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$.

54. Consider the equation $y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$.
(a) Sketch a graph of this equation. At how many points does this graph have horizontal tangents? Estimate the x-coordinates of these points?
(b) Find equations of the tangent lines to the graph at the points (0, 1) and (0, 2).
(c) Find the exact x-coordinates of the points in part (a).

55. Consider the equation $2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$.
(a) The graph of this equation has been referred to as a bouncing wagon. Sketch a graph of this equation and explain why.
(b) At how many points does this graph have horizontal tangent lines? Find the x-coordinates of these points.

56. (a) If $x^2 + y^2 = 16$, find $\dfrac{dy}{dx}$.
(b) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.
(c) Compare the derivatives in parts (a) and (b). Make a conjecture about the derivative for any equation for a circle.

57. Find the points on the lemniscate in Exercise 37 where the tangent line is horizontal.

58. Use implicit differentiation to show that the tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

59. Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0).

60. Show that the sum of the x- and y-intercepts of any tangent line to the graph of $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

61. Use implicit differentiation to show that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

62. The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n = \frac{p}{q}$, and $y = f(x) = x^n$ is assumed to be a differentiable function. If $y = x^{p/q}$, then $y^q = x^p$. Use implicit differentiation to show that

$$y' = \frac{p}{q} x^{(p/q)-1}$$

Two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are **orthogonal trajectories** of each other; that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same coordinate axes.

63. $x^2 + y^2 = r^2, \quad ax + by = 0$

64. $x^2 + y^2 = ax, \quad x^2 + y^2 = by$

65. $y = cx^2, \quad x^2 + 2y^2 = k$

66. $y = ax^3, \quad x^2 + 3y^2 = b$

---

67. Show that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$ are orthogonal trajectories if $A^2 < a^2$ and $a^2 - b^2 = A^2 + B^2$ (so the ellipse and the hyperbola have the same foci).

68. Find the value of the number a such that the families of curves $y = (x + c)^{-1}$ and $y = a(x + k)^{1/3}$ are orthogonal trajectories as defined in Exercises 63–66.

69. The *van der Waals equation* for n moles of a gas is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas.

(a) If T remains constant, use implicit differentiation to find $\frac{dV}{dP}$.

(b) Find the rate of change of volume with respect to pressure of 1 mole of carbon dioxide at a volume of $V = 10$ L and a pressure of $P = 2.5$ atm. Use $a = 3.592\ \text{L}^2\ \text{atm}/\text{mole}^2$ and $b = 0.04267$ L/mole.

70. Consider the equation $x^2 + xy + y^2 + 1 = 0$.

(a) Use implicit differentiation to find y'.

(b) Sketch the graph of this equation. What do you see? Try to prove that what you see is correct.

71. The equation $x^2 - xy + y^2 = 3$ represents a *rotated ellipse*, that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x-axis and show that the tangent lines at these points are parallel.

72. Consider the equation of the rotated ellipse $x^2 - xy + y^2 = 3$.

(a) Where does the normal line to this ellipse at the point $(-1, 1)$ intersect the ellipse a second time?

(b) Graph the ellipse and the normal line in the same viewing rectangle.

73. Find all points on the graph of the equation $x^2y^2 + xy = 2$ where the slope of the tangent line is -1.

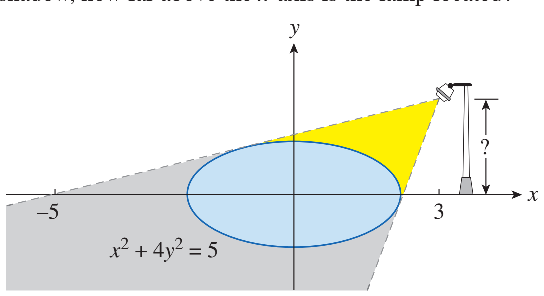
74. Find equations of both tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point (12, 3).

75. The **Bessel function** of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x, and its value at 0 is $J(0) = 1$.

(a) Find $J'(0)$.

(b) Use implicit differentiation to find $J''(0)$.

76. The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 \le 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?



3.6 Inverse Trigonometric Functions and Their Derivatives

Recall from Section 1.5 that the only functions that have inverse functions are one-to-one functions. The six basic trigonometric functions do not have inverse functions because they are not one-to-one; each graph fails the Horizontal Line Test. However, if we restrict the domains, these new functions are one-to-one and, therefore, have inverses.

Figure 3.37 shows graphically, using the Horizontal Line Test, that the sine function, $y = \sin x$, is not one-to-one. However, the new function, $f(x) = \sin x$, $-\pi/2 \le x \le \pi/2$, is one-to-one (see Figure 3.38). The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or **arcsine function**.

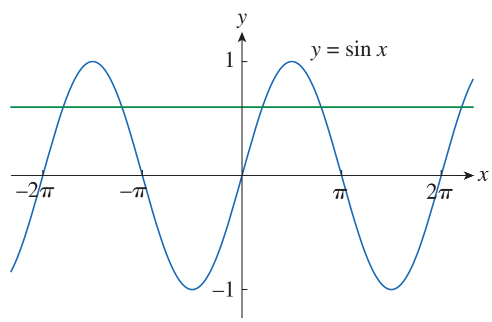


Figure 3.37
The graph of $y = \sin x$ is not one-to-one.

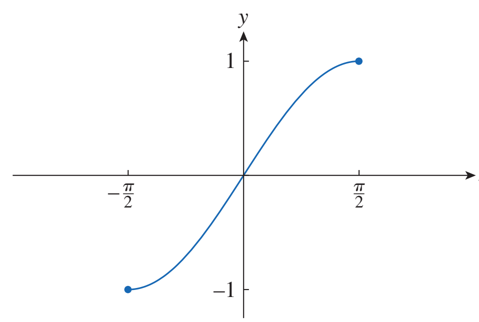


Figure 3.38
The graph of $y = \sin x$, $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$.

Note: $\sin^{-1} x \ne \dfrac{1}{\sin x}$.

Apply the definition of an inverse function to obtain

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Therefore, if $-1 \le x \le 1$, then $\sin^{-1} x$ is a number (angle) between $-\pi/2$ and $\pi/2$ whose sine is x.

Example 1 Inverse Trigonometric Evaluations

Evaluate (a) $\sin^{-1}\left(\dfrac{1}{2}\right)$ and (b) $\tan\left(\arcsin \dfrac{1}{3}\right)$.

Solution

(a) $\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$ because $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$ and $\dfrac{\pi}{6}$ lies between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$.

(b) Let $\theta = \arcsin \dfrac{1}{3}$, therefore, $\sin \theta = \dfrac{1}{3}$.

Draw a right triangle with angle θ as in Figure 3.39.

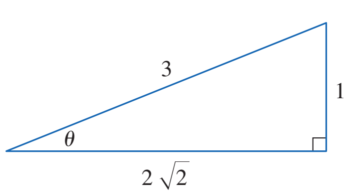


Figure 3.39
Right triangle with angle θ.

The third side has length $\sqrt{9-1} = 2\sqrt{2}$. The Pythagorean Theorem.

$$\tan\left(\arcsin \frac{1}{3}\right) = \tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{2\sqrt{2}}$$ Read from the triangle.

The cancellation equations associated with sine and arcsine are

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$
$$\sin(\sin^{-1} x) = x \quad \text{for } -1 \le x \le 1$$

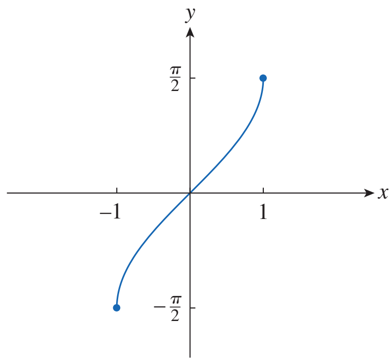


Figure 3.40
Graph of $y = \sin^{-1} x = \arcsin x$.

Remember, $\cos y$ could be positive or negative but is positive here because of the range of $y = \sin^{-1} x$.

The same method can be used to find an expression for the derivative of *any* inverse function.

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$. The graph of the inverse sine function is shown in Figure 3.40. This graph may be obtained by reflecting the graph of the restricted sine function across the line $y = x$.

We can use implicit differentiation to find the derivative of the inverse sine function, assuming that it is differentiable. The differentiability certainly seems reasonable from the graph in Figure 3.40.

Let $y = \sin^{-1} x$. Then $\sin y = x$ and $-\pi/2 \le y \le \pi/2$. Differentiate the expression $\sin y = x$ implicitly with respect to x.

$$\cos y \frac{dy}{dx} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, then $\cos y \ge 0$, and

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Therefore,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

and

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1 \tag{1}$$

Example 2 Derivative of an Inverse Trigonometric Function

If $f(x) = \sin^{-1}(x^2 - 1)$, find (a) the domain of f, (b) $f'(x)$, and (c) the domain of f'.

Solution

(a) Since the domain of the inverse sine function is $[-1, 1]$, the domain of f is

$$\{x \mid -1 \le x^2 - 1 \le 1\} = \{x \mid 0 \le x^2 \le 2\}$$
$$= \{x \mid |x| \le \sqrt{2}\} = [-\sqrt{2}, \sqrt{2}]$$

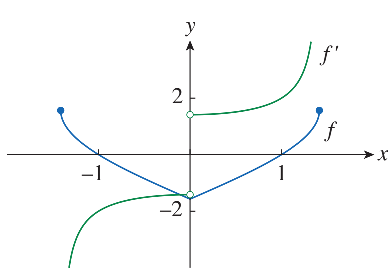


Figure 3.41
Graphs of f and f'.

(b) Use Equation 1 and the Chain Rule to find $f'(x)$.

$$f'(x) = \frac{1}{\sqrt{1 - (x^2 - 1)^2}} \frac{d}{dx}(x^2 - 1) \qquad \text{Equation 1; Chain Rule.}$$

$$= \frac{1}{\sqrt{1 - (x^4 - 2x^2 + 1)}} 2x = \frac{2x}{\sqrt{2x^2 - x^4}} \qquad \text{Derivative of the inner function; simplify.}$$

(c) Since the denominator of $f'(x)$ cannot equal 0, the domain of f' is

$$\{x \mid -1 < x^2 - 1 < 1\} = \{x \mid 0 < x^2 < 2\}$$
$$= \{x \mid 0 < |x| < \sqrt{2}\} = (-\sqrt{2}, 0) \cup (0, \sqrt{2})$$

The graphs of f and f' are shown in Figure 3.41. Notice that f is not differentiable at 0; there is a sharp edge, or corner, in the graph at the point where $x = 0$. This is consistent with the fact that the graph of f' makes a sudden jump at $x = 0$. ■

We can find the derivative of the **inverse cosine function** in a similar manner. The restricted cosine function $f(x) = \cos x$, $0 \leq x \leq \pi$, is one-to-one (see Figure 3.42) and so it has an inverse function denoted by $\cos^{-1}$ or arccos.

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x \quad \text{and} \quad 0 \leq y \leq \pi$$

The inverse cosine function, $\cos^{-1}$, has domain $[-1, 1]$ and range $[0, \pi]$ and is a continuous function whose graph is shown in Figure 3.43. Its derivative is given by

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1 \tag{2}$$

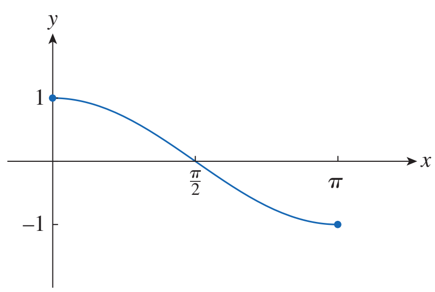


Figure 3.42
Graph of $y = \cos x$, $0 \leq x \leq \pi$.

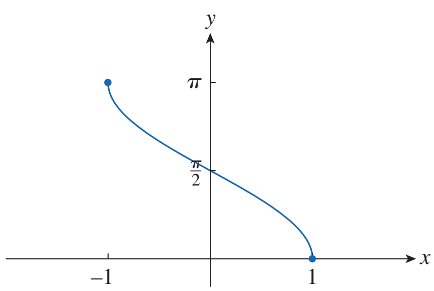


Figure 3.43
Graph of $y = \cos^{-1} x = \arccos x$.

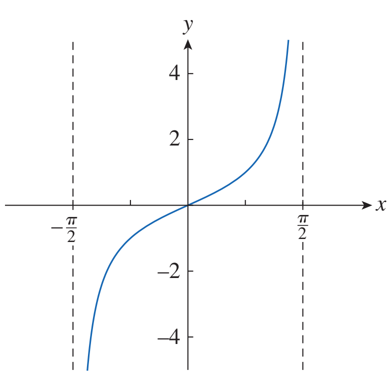


Figure 3.44
Graph of the restricted tangent function, $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Equation 2 can be proved using the same methods as for Equation 1, and is left as an exercise.

The tangent function can be made one-to-one by restricting the domain to the interval $(-\pi/2, \pi/2)$. Therefore, the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$, as shown in Figure 3.44. It is denoted by $\tan^{-1}$ or arctan.

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Example 3 Simplification

Simplify the expression $\cos(\tan^{-1} x)$.

Solution 1

Let $y = \tan^{-1} x$, then $\tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

We need $\cos y$, but since $\tan y$ is known, find $\sec y$ first.

$\sec^2 y = 1 + \tan^2 y = 1 + x^2$ Trigonometric identity.

$\sec y = \sqrt{1 + x^2}$ $\sec y > 0$ for $-\pi/2 < y < \pi/2$.

$\cos(\tan^{-1} x) = \cos y = \dfrac{1}{\sec y} = \dfrac{1}{\sqrt{1 + x^2}}$ Definition of cosine.

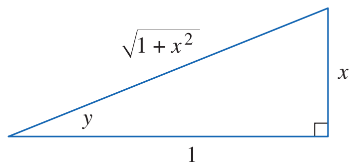


Figure 3.45
Right triangle with angle y.

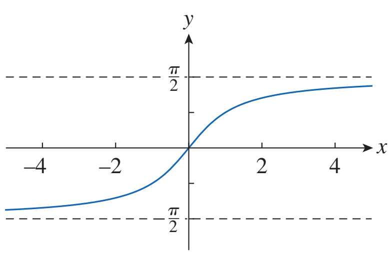


Figure 3.46
Graph of the inverse tangent function, $y = \tan^{-1} x = \arctan x$.

Solution 2

If $y = \tan^{-1} x$, then $\tan y = x$.

Draw a right triangle with angle y as in Figure 3.45.

Find the remaining side using the Pythagorean Theorem, and read the solution from the triangle.

$$\cos(\tan^{-1} x) = \cos y = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1+x^2}}$$ ■

The inverse tangent function has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. The graph is shown in Figure 3.46.

We know that

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \quad \text{and} \quad \lim_{x \to -(\pi/2)^+} \tan x = -\infty$$

and so the lines $x = \pi/2$ and $x = -\pi/2$ are vertical asymptotes of the graph of $y = \tan x$. Since the graph of $y = \tan^{-1} x$ is obtained by reflecting the graph of the restricted tangent function across the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $y = \tan^{-1} x$. This fact is expressed by the following limits:

$$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2} \qquad \lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2} \tag{3}$$

Example 4 Limit Involving the Inverse Tangent

Evaluate $\displaystyle\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right)$.

Solution

Let $t = \dfrac{1}{x-2}$. Then as $x \to 2^+$, $t \to \infty$.

Use the first limit in Equation 3.

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right) = \lim_{t \to \infty} \arctan t = \frac{\pi}{2}$$ ■

The formula for the derivative of the arctangent function is derived in a way that is similar to the method we used for arcsine. If $y = \tan^{-1} x$, then $\tan y = x$. Differentiate this later equation implicitly with respect to x:

$$\sec^2 y \frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \tag{4}$$

Example 5 Derivatives Involving the Inverse Tangent Function

Differentiate (a) $y = \dfrac{1}{\tan^{-1} x}$ and (b) $f(x) = x \arctan\sqrt{x}$.

Solution

(a) $$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}x)^{-1} = -(\tan^{-1}x)^{-2}\frac{d}{dx}(\tan^{-1}x) \qquad \text{Power Rule; Chain Rule.}$$

$$= -\frac{1}{(\tan^{-1}x)^2(1+x^2)} \qquad \text{Derivative of } \tan^{-1}x\text{; simplify.}$$

(b) $$f'(x) = x\left(\frac{1}{1+(\sqrt{x})^2}\right)\left(\frac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}\cdot 1 \qquad \text{Product Rule; Chain Rule.}$$

$$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x} \qquad \text{Simplify.}$$

The inverse sine, cosine, and tangent functions discussed in this section occur most frequently in applications. The differentiation formulas for the remaining inverse trigonometric functions are shown below and on Reference Page 5.

And include the remaining three formulas here. They should fit at the bottom of this page, on a single line. (They are numbers 22 - 24.)

3.6 Exercises

Find the exact value of each expression.

1. (a) $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ (b) $\cos^{-1}(-1)$

2. (a) $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ (b) $\sec^{-1} 2$

3. (a) $\arctan 1$ (b) $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$

4. (a) $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$ (b) $\cos\left(\arcsin\dfrac{1}{2}\right)$

5. $\tan\left(\sin^{-1}\left(\dfrac{2}{3}\right)\right)$

6. $\csc\left(\arccos\dfrac{3}{5}\right)$

7. $\sin(2\tan^{-1}\sqrt{2})$

8. $\cos(\tan^{-1}2 + \tan^{-1}3)$

9. Prove that $\cos(\sin^{-1}x) = \sqrt{1-x^2}$

Simplify each expression.

10. $\tan(\sin^{-1}x)$

11. $\sin(\tan^{-1}x)$

12. $\sin(2\arccos x)$

Graph the collection of functions on the same coordinate axes. Explain how these graphs are related.

13. $y = \sin x,\ -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2};\quad y = \sin^{-1}x;\quad y = x$

14. $y = \tan x,\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2};\quad y = \tan^{-1}x;\quad y = x$

15. Prove that $\dfrac{d}{dx}(\cos^{-1}x) = -\dfrac{1}{\sqrt{1-x^2}}$ using the same method as for Equation 1.

16. (a) Prove that $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$.

(b) Use part (a) to prove Equation 2.

Find the derivative of the function. Simplify where possible.

17. $y = (\tan^{-1}x)^2$

18. $y = \tan^{-1}(x^2)$

19. $y = \sin^{-1}(2x+1)$

20. $F(\theta) = \arcsin\sqrt{\sin\theta}$

21. $G(x) = \sqrt{1-x^2}\arccos x$

22. $y = \cos^{-1}(e^{2x})$

23. $y = \tan^{-1}\left(x - \sqrt{1+x^2}\right)$

24. $y = \arctan(\cos\theta)$

25. $y = \cos^{-1}(\sin^{-1} t)$

26. $y = \arcsin\left(\dfrac{1}{t}\right)$

27. $y = x \sin^{-1} x + \sqrt{1 - x^2}$

28. $y = \arctan\sqrt{\dfrac{1 - x}{1 + x}}$

29. $y = \arccos\left(\dfrac{b + a\cos x}{a + b\cos x}\right), \quad 0 \le x \le \pi, \quad a > b > 0$

Find the derivative of the function. Find the domain of the function and its derivative.

30. $f(x) = \arcsin(e^x)$

31. $g(x) = \cos^{-1}(3 - 2x)$

32. Find y' if $\tan^{-1}(xy) = 1 + x^2 y$.

33. If $g(x) = x \sin^{-1}(x/4) + \sqrt{16 - x^2}$, find $g'(2)$.

34. Find an equation of the line tangent to the curve $y = 3\arccos(x/2)$ at the point $(1, \pi)$.

Find $f'(x)$. Graph f and f' in the same viewing rectangle and explain the relationship between the two graphs.

35. $f(x) = \sqrt{1 - x^2} \arcsin x$

36. $f(x) = \arctan(x^2 - x)$

Find the limit.

37. $\lim\limits_{x \to -1^+} \sin^{-1} x$

38. $\lim\limits_{x \to \infty} \arccos\left(\dfrac{1 + x^2}{1 + 2x^2}\right)$

39. $\lim\limits_{x \to \infty} \arctan(e^x)$

40. $\lim\limits_{x \to 0^+} \tan^{-1}(\ln x)$

41. (a) Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

(b) If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.

42. The function $f(x) = 2x + \cos x$ is one-to-one.

(a) Find $f^{-1}(1)$.

(b) Use the formula from Exercise 41(a) to find $(f^{-1})'(1)$.

43. Use the formula from Exercise 41(a) to prove

(a) Equation 1 (b) Equation 4

44. (a) Sketch the graph of the function $f(x) = \sin(\sin^{-1} x)$.

(b) Sketch the graph of the function $g(x) = \sin^{-1}(\sin x)$, $x \in \mathbb{R}$.

(c) Show that $g'(x) = \dfrac{\cos x}{|\cos x|}$.

(d) Sketch the graph of $h(x) = \cos^{-1}(\sin x)$, $x \in \mathbb{R}$, and find its derivative.

3.7 Derivatives of Logarithmic Functions

It seems reasonable that logarithmic functions are differentiable – think about their graphs. In fact, it can be proved that they are differentiable and implicit differentiation can be used to find the derivatives of the logarithmic function $y = \log_b x$ and the natural logarithmic function $y = \ln x$.

The general differentiation rule for the logarithmic function is

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \qquad (1)$$

Proof

Let $y = \log_a x$, then $a^y = x$.

Recall: $\dfrac{d}{dx}(a^x) = a^x \ln a$.

Differentiate this expression implicitly with respect to x.

$$a^y (\ln a)\frac{dy}{dx} = 1$$ Derivative of an exponential function; Chain Rule.

$$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$ Solve for $\frac{dy}{dx}$. ■

If we let $a = e$ in Equation 1, then $\ln a = \ln e = 1$. This leads to the derivative of the natural logarithmic function, $\log_e x = \ln x$.

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \tag{2}$$

Equations 1 and 2 suggest one of the reasons that natural logarithms (logarithms with base e) are used in calculus: the log differentiation rule is simplest (most natural) when $a = e$ because $\ln e = 1$.

Example 1 Derivative of a Natural Logarithmic Function

Differentiate $y = \ln(x^3 + 1)$.

Solution

Use the Chain Rule: let $y = \ln u$ and $u = x^3 + 1$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} \qquad \text{Derivative of logarithmic function; Chain Rule.}$$

$$= \frac{1}{x^3+1}(3x^2) = \frac{3x^2}{x^3+1} \qquad \text{Use } u = x^3 + 1\text{; simplify.}$$

■

Example 1 suggests that we can combine Equation 2 and the Chain Rule to obtain the following, more general, rules.

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)} \tag{3}$$

Example 2 Chain Rule Application

Find $\frac{d}{dx}[\ln(\sin x)]$.

Solution

The argument of the natural log is a function of x: use Equation 3.

$$\frac{d}{dx}[\ln(\sin x)] = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$

■

Example 3 A Logarithm as the Inner Function

Let $f(x) = \sqrt{\ln x}$. Find $f'(x)$.

Solution

Use the Chain Rule: notice that the logarithm is the inner function.

Rewrite the original function: $f(x) = (\ln x)^{1/2}$.

$$f'(x) = \frac{1}{2}(\ln x)^{-1/2} \cdot \frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

■

Example 4 Derivative of a Common Logarithm

Let $f(x) = \log_{10}(2 + \sin x)$. Find $f'(x)$.

Solution

Use Equation 1 with $a = 10$.

$$f'(x) = \frac{d}{dx}[\log_{10}(2 + \sin x)]$$

$$= \frac{1}{(2 + \sin x)\ln 10} \cdot \frac{d}{dx}(2 + \sin x) \qquad \text{Equation 1; Chain Rule.}$$

$$= \frac{\cos x}{(2 + \sin x)\ln 10} \qquad \text{Derivative of the inner function.}$$

■

Example 5 Simplify Before Differentiating

Let $f(x) = \ln \dfrac{x + 1}{\sqrt{x - 2}}$. Find $f'(x)$.

Solution 1

$$f'(x) = \frac{1}{\left(\dfrac{x+1}{\sqrt{x-2}}\right)} \cdot \frac{d}{dx}\left[\frac{x+1}{\sqrt{x-2}}\right] \qquad \text{Equation 1; Chain Rule.}$$

$$= \frac{\sqrt{x-2}}{x+1} \cdot \frac{\sqrt{x-2}\cdot 1 - (x+1)\cdot \frac{1}{2}(x-2)^{-1/2}}{x-2} \qquad \text{Simplify fraction; Quotient Rule.}$$

$$= \frac{x - 2 - \dfrac{1}{2}(x+1)}{(x+1)(x-2)} \qquad \text{Multiply fractions.}$$

$$= \frac{x-5}{2(x+1)(x-2)} \qquad \text{Simplify.}$$

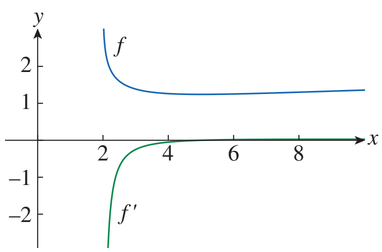


Figure 3.47
Graphs of f and f'. Notice the relationship between the two graphs. Can you find the domain of f and f'?

Figure 3.47 shows the graphs of the functions f and f'. This provides a cursory visual check of our calculation. Notice that $f'(x)$ is large negative when the graph of f is decreasing very quickly. In addition, $f'(x)$ is close to 0 as the graph of f flattens out.

Solution 2

Another way to find the derivative in this problem is to use the Laws of Logarithms first to write the function as a difference of two logarithms.

$$\frac{d}{dx}\left[\ln \frac{x+1}{\sqrt{x-2}}\right] = \frac{d}{dx}\left[\ln(x+1) - \frac{1}{2}\ln(x-2)\right] \qquad \text{Laws of Logarithms.}$$

$$= \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right) \qquad \text{Derivative of each term.}$$

The derivative can be left in this form. However, if we combined terms using a common denominator, we would obtain the same answer as in Solution 1. ■

Example 6 An Important Result Involving the Natural Logarithm

Let $f(x) = \ln|x|$. Find $f'(x)$.

Solution

Before we can differentiate, we need to find an equivalent expression for f without the absolute value notation. Use the definition of the absolute value function to write f as a piecewise defined function. Note that $x = 0$ is not in the domain of f.

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

To find the derivative of f, we consider each *piece*, or *branch*, of f.

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

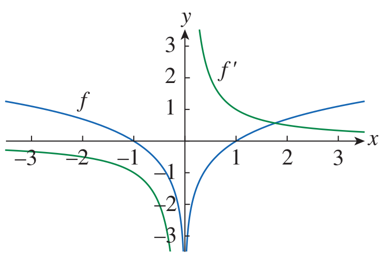


Figure 3.48
Graphs of f and f'. Notice the relationship between the two graphs.

Therefore, $f'(x) = \dfrac{1}{x}$ for all $x \neq 0$.

Figure 3.48 shows the graphs of f and f'. For x close to 0, the graph of f is steep and therefore, as $x \to 0^+$, $f'(x) \to \infty$, and as $x \to 0^-$, $f'(x) \to -\infty$. ■

This important result will be used throughout the rest of this text.

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \tag{4}$$

Logarithmic Differentiation

In order to find the derivative of a complicated function involving products, quotients, and/or powers, we often simplify first by taking logarithms. The method used in the next example is called **logarithmic differentiation**.

Example 7 Logarithmic Differentiation

Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Solution

Take the (natural) logarithm of both sides of the equation, and use the Laws of Logarithms to simplify the right side.

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiate both sides with respect to x.

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solve for $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Use the explicit expression for y to obtain a final answer in terms of x.

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Note: To find the derivative without using logarithmic differentiation, we would have to use the Quotient Rule and the Product Rule. The calculations using this method are a little longer. ■

Here is the procedure for logarithmic differentiation.

Procedure for in Logarithmic Differentiation

(1) Given $y = f(x)$, we want to find $y' = \frac{dy}{dx}$.

(2) Take the natural logarithm of both sides of the equation: $\ln y = \ln f(x)$. Use the Laws of Logarithms to simplify $\ln f(x)$.

(3) Differentiate implicitly with respect to x.

(4) Solve for y'. Then replace y by the given, original expression $f(x)$.

Note: If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined. However, we can write $|y| = |f(x)|$ and use the result in Example 6; the logarithmic differentiation procedure is the same. We use this procedure in the proof of the general version of the Power Rule.

Using this procedure, we can now prove and use the Power Rule for any real number in the exponent.

The Power Rule

If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of a derivative.

Proof

Let $y = x^n$ and use logarithmic differentiation.

$\ln|y| = \ln|x|^n = n\ln|x|, x \neq 0$ — Natural logarithm of both sides; Laws of Logarithms.

$\frac{y'}{y} = \frac{n}{x}$ — Implicit differentiation, on both sides with respect to x.

$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$ — Solve for y'; use the expression for y.

■

A Closer Look

Be careful to distinguish between the Power Rule $\frac{d}{dx}(x^n) = nx^{n-1}$, where the base is the variable and the exponent is a constant, and the rule for differentiating exponential functions $\frac{d}{dx}(a^x) = a^x \ln a$, where the base is constant and the exponent is the variable.

There are four cases for exponents and bases.

1. $\dfrac{d}{dx}(a^b) = 0$ (a and b are constants) — Constant base, constant exponent.

Example: $\dfrac{d}{dx}(3^\pi) = 0$

2. $\dfrac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$ — Variable base, constant exponent.

Example: $\dfrac{d}{dx}(x^3 + \sin x)^7 = 7(x^3 + \sin x)^6(3x^2 + \cos x)$

3. $\dfrac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)\; g'(x)$ — Constant base, variable exponent.

Example: $\dfrac{d}{dx}2^{\sin x} = 2^{\sin x}(\ln 2)\cos x$

4. $\dfrac{d}{dx}[f(x)]^{g(x)}$ (use logarithmic differentiation) — Variable base, variable exponent.

Example: $y = (\sin x)^x$

$\ln y = x \ln(\sin x)$

$$\frac{y'}{y} = x \cdot \frac{\cos x}{\sin x} + \ln(\sin x) \cdot 1$$

$$y' = (\sin x)^x(x \cot x + \ln(\sin x))$$

Example 8 Variable Base, Variable Exponent

Differentiate $y = x^{\sqrt{x}}$.

Solution 1

The base and the exponent both contain variables. Therefore, use logarithmic differentiation.

$\ln y = \ln(x^{\sqrt{x}}) = \sqrt{x}\ln x$ — Natural logarithm of both sides; Laws of Logarithms.

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + \ln x \cdot \frac{1}{2\sqrt{x}}$$ Implicit differentiation with respect to x; Chain Rule; Product Rule.

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$ Solve for y'; simplify.

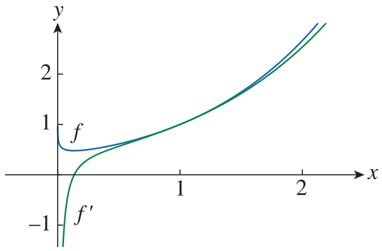


Figure 3.49
Graphs of $y = f(x) = x^{\sqrt{x}}$ and $y = f'(x)$. The graph of f is very steep near 0 and therefore as $x \to 0^+$, $f'(x) \to -\infty$.

Solution 2

An alternate solution method starts by rewriting the original expression.

$$x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}} = e^{\sqrt{x}\ln x}$$

$$\frac{d}{dx}(x^{\sqrt{x}}) = \frac{d}{dx}(e^{\sqrt{x}\ln x}) = e^{\sqrt{x}\ln x}\frac{d}{dx}(\sqrt{x}\ln x)$$ Derivative of an exponential function; Chain Rule.

$$= x^{\sqrt{x}}\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$ Product Rule; simplify.

Figure 3.49 shows graphs of $f(x) = x^{\sqrt{x}}$ and its derivative.

The Number *e* as a Limit

We have seen that if $f(x) = \ln x$, then $f'(x) = \dfrac{1}{x}$ and therefore $f'(1) = 1$. We can use these facts to produce a definition of the number e involving a limit.

Use the definition of the derivative to write $f'(1)$.

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{x\to 0}\frac{f(1+x) - f(1)}{x}$$ Definition of derivative; change variables.

$$= \lim_{x\to 0}\frac{\ln(1+x) - \ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x)$$ Use $f(x) = \ln x$, $\ln 1 = 0$.

$$= \lim_{x\to 0}\ln(1+x)^{1/x}$$ Properties of logarithms.

Because $f'(1) = 1$, we know the value of this limit.

$$\lim_{x\to 0}\ln(1+x)^{1/x} = 1$$

Since the (natural) exponential function is continuous,

$$e = e^1 = e^{\lim\limits_{x\to 0}\ln(1+x)^{1/x}} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = \lim_{x\to 0}(1+x)^{1/x}$$

This is the *limit definition* of e

$$e = \lim_{x\to 0}(1+x)^{1/x} \tag{5}$$

Figure 3.50 is a graph of the function $y = (1+x)^{1/x}$, and Table 3.2 presents values of this function for x close to 0. Both confirm that $e \approx 2.7182818$.

x	$(1+x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828180

Table 3.2
Table of values for x close to 0.

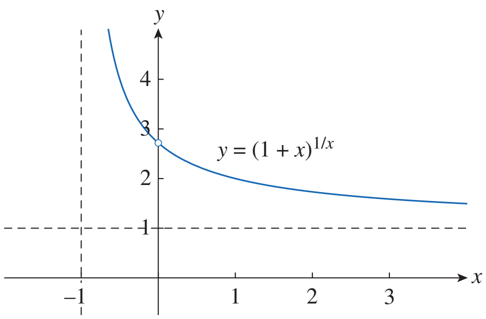


Figure 3.50
Graph of $y = (1+x)^{1/x}$.

If we let $n = \dfrac{1}{x}$ in Equation 5, then $n \to \infty$ as $x \to 0^+$. This leads to an alternate limit expression for e.

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \tag{6}$$

3.7 Exercises

1. Explain why the natural logarithm function, $y = \ln x$, is used more frequently in calculus than other logarithmic functions, $y = \log_a x$.

Differentiate the function.

2. $f(x) = x \ln x - x$

3. $f(x) = \dfrac{\ln x}{x^2}$

4. $f(x) = \sin(\ln x)$

5. $f(x) = \ln(\sin^2 x)$

6. $f(x) = \ln \dfrac{1}{x}$

7. $g(x) = \dfrac{1}{\ln x}$

8. $f(x) = \log_{10}(1 + \cos x)$

9. $f(x) = \log_{10}\sqrt{x}$

10. $g(x) = \ln(x e^{-2x})$

11. $g(t) = \sqrt{1 + \ln t}$

12. $F(t) = (\ln t)^2 \sin t$

13. $h(x) = \ln(x + \sqrt{x^2 - 1})$

14. $G(y) = \ln \dfrac{(2y+1)^5}{\sqrt{y^2 + 1}}$

15. $P(v) = \dfrac{\ln v}{1 - v}$

16. $F(s) = \ln(\ln s)$

17. $y = \ln|1 + t - t^3|$

18. $T(z) = 2^z \log_2 z$

19. $y = \ln(\csc x - \cot x)$

20. $y = \ln(e^{-x} + x e^{-x})$

21. $H(z) = \ln\sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

22. $y = \tan[\ln(ax + b)]$

23. $y = \log_2(x \log_5 x)$

24. $f(x) = 2^{\sin 2x}$

25. $f(x) = 3^x x^3$

26. $y = [\ln(1 + e^x)]^2$

27. $g(x) = e^{2x} \ln\sqrt{x^2 + 4}$

Use logarithmic differentiation to find the derivative of the function.

28. $y = (x^2 + 2)^2(x^4 + 4)^4$

29. $y = \dfrac{e^{-x}\cos^2 x}{x^2 + x + 1}$

30. $y = \sqrt{\dfrac{x - 1}{x^4 + 1}}$

31. $y = \sqrt{x}\, e^{x^2 - x}(x + 1)^{2/3}$

32. $y = x^x$

33. $y = x^{\cos x}$

34. $y = x^{\sin x}$

35. $y = (\sqrt{x})^x$

36. $y = (\cos x)^x$

37. $y = (\sin x)^{\ln x}$

38. $y = (\tan x)^{1/x}$

39. $y = (\ln x)^{\cos x}$

Find y' and y''.

40. $y = x^2 \ln(2x)$

41. $y = \dfrac{\ln x}{x^2}$

42. $y = \sqrt{x} \ln x$

43. $y = \dfrac{\ln x}{1 + \ln x}$

44. $y = \ln|\sec x|$

45. $y = \ln(1 + \ln x)$

Find the domain of f and the derivative $f'(x)$.

46. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$

47. $f(x) = \sqrt{2 + \ln x}$

48. $f(x) = \ln(x^2 - 2x)$

49. $f(x) = \ln(\ln(\ln x))$

50. If $f(x) = \ln(x + \ln x)$, find $f'(1)$.

51. If $f(x) = \cos(\ln x^2)$, find $f'(1)$.

Find an equation of the tangent line to the graph of the function at the given point.

52. $f(x) = \ln(x^2 - 3x + 1)$, $(3, 0)$

53. $f(x) = x^2 \ln x$, $(1, 0)$

54. $f(x) = \ln\left(xe^{x^2}\right)$, $(1, 1)$

55. If $f(x) = \sin x + \ln x$, find $f'(x)$. Graph f and f' in the same viewing rectangle and check that your answer is reasonable by explaining the relationship between the graphs.

56. Find an equation of the tangent lines to the graph of $f(x) = \dfrac{\ln x}{x}$ at the point $(1, 0)$ and at the point $\left(e, \dfrac{1}{e}\right)$. Graph f and the tangent lines in the same viewing rectangle.

57. A line tangent to the graph of $f(x) = 2 \ln x$ has y-intercept 4. Find the slope of this line.

58. Let $f(x) = \dfrac{1}{x \ln x}$. For $0 < x < 1$, find the point on the graph of f at which the tangent line is horizontal.

59. Let $g(x) = \dfrac{\ln x}{x}$. Find the points on the graph of g at which the tangent line is horizontal.

60. Let $g(x) = \dfrac{(\ln x)^2}{x}$. Find the points on the graph of g at which the tangent line is horizontal.

61. Let $f(x) = cx + \ln(\cos x)$. Find the value of c such that $f'\left(\dfrac{\pi}{4}\right) = 6$.

62. Let $f(x) = \log_a(3x^2)$. Find the value of a such that $f'(1) = 3$.

63. A particle moves along a horizontal line with position at time t given by $s(t) = \ln(1 + t^2)$ for $t \geq 0$. Find the position of the particle at the first time the acceleration is zero.

64. A particle moves along a horizontal line with position at time t given by $s(t) = \dfrac{1}{t} + \ln t$ for $t > 0$. Find the value of t at which the acceleration of the particle is zero.

65. Find y' if $y = \ln(x^2 + y^2)$.

66. Find y' if $x^y = y^x$.

67. Find a general formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

68. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

69. Use the definition of the derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

70. Show that for any $x > 0$,

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

Discovery Project | Hyperbolic Functions

Certain combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and applications that they are given special names. This project explores the properties of the **hyperbolic functions**: the **hyperbolic sine**, **hyperbolic cosine**, **hyperbolic tangent**, and **hyperbolic secant** functions, defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

The reason for these function names is that they are related to the hyperbola in much the same way that trigonometric functions are related to the circle.

1. (a) Without using technology, sketch the graphs of the functions $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ on the same coordinate axes, and then use these graphs to draw the graph of $y = \cosh x$.

(b) Check the accuracy of your sketch in part (a) by using technology to graph $y = \cosh x$. What are the domain and range of this function?

2. One of the most common applications of the hyperbolic functions is the use of the hyperbolic cosine to describe the shape of a hanging wire. It can be proved that if a heavy flexible cable (for example, a power line or cable TV line) is suspended between two points at the same height, then it takes the shape of a curve with equation $y = a\cosh(x/a)$. This curve is called a *catenary*. The Latin word *catena* means "chain." Graph several members of the family of functions $y = a\cosh(x/a)$. Explain how the graph changes as a varies.

3. Graph the functions $y = \sinh x$ and $y = \tanh x$. Using their graphs, which of the functions $y = \sinh x$, $y = \cosh x$, and $y = \tanh x$ are even? Which are odd? Use the definitions of even and odd functions to prove your assertions.

4. Prove the identity $\cosh^2 x - \sinh^2 x = 1$.

5. Prove the identity $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$.

6. The identities in Problems 4 and 5 are similar to well-known trigonometric identities. Try to discover some other hyperbolic identities by using known trigonometric identities as motivation.

7. The differentiation formulas for the hyperbolic functions are analogous to those for the trigonometric functions, but the signs are sometimes different.

(a) Show that $\frac{d}{dx}(\sinh x) = \cosh x$.

(b) Find formulas for the derivatives of $y = \cosh x$ and $y = \tanh x$.

8. (a) Explain why $y = \sinh x$ is a one-to-one function.

(b) Find a formula for the derivative of the inverse hyperbolic sine function $y = \sinh^{-1} x$.

Hint: Consider the method used to find the derivative of $y = \sin^{-1} x$.

(c) Show that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

(d) Use the result in part (c) to find the derivative of $y = \sinh^{-1} x$. Compare this result with your answer in part (b).

9. (a) Explain why $y = \tanh x$ is a one-to-one function.

(b) Find a formula for the derivative of the inverse hyperbolic tangent function $y = \tanh^{-1} x$.

(c) Show that $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.

(d) Use the result in part (c) to find the derivative of $y = \tanh^{-1} x$. Compare this result with your answer in part (b).

10. Find the point on the graph of $y = \cosh x$ where the tangent line has slope 1.

3.8 Rates of Change in the Natural and Social Sciences

We have learned that if $y = f(x)$, then the derivative, $\frac{dy}{dx}$, can be interpreted as the rate of change of y with respect to x. In this section, we will apply this interpretation to many different areas of study.

Recall the basic concept of a rate of change: if x changes from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is **the average rate of change of y with respect to x** over the interval $[x_1, x_2]$. The difference quotient can be interpreted as the slope of the secant line PQ as in Figure 3.51.

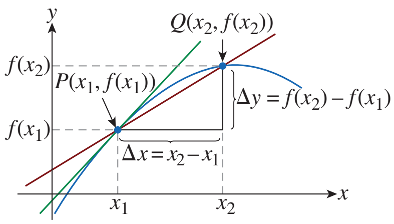


Figure 3.51
m_{PQ} = average rate of change.
$m = f'(x_1)$ = instantaneous rate of change of f at P.

The limit of the average rate of change, as $\Delta x \to 0$, is the derivative $f'(x_1)$, which can be interpreted as the **instantaneous rate of change of y with respect to x** or the slope of the tangent line to the graph of f at the point $P(x_1, f(x_1))$. Using Leibniz notation, we write this limit as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If the function $y = f(x)$ has a specific interpretation, then its derivative $f'(x)$ also has a specific interpretation as a rate of change in the context of the problem. Recall that the units for $\frac{dy}{dx}$ are the units for y divided by the units for x. Let's consider several real-world applications involving rates of change.

Physics

If $s(t)$ is the position function of a particle moving along a line, then $\frac{\Delta s}{\Delta t}$ represents the average velocity of the particle over a time interval of length Δt. The function $v = \frac{ds}{dt}$ represents the instantaneous **velocity** of the particle, the rate of change of position with respect to time. The instantaneous rate of change of velocity with respect to time is **acceleration**: $a(t) = v'(t) = s''(t)$. These formulas and notation were introduced earlier. However, using the many differentiation rules we have learned, we are now able to solve a wider variety of problems involving particle motion.

Example 1 Analyzing the Motion of a Particle

A particle moves along a straight line. For $t \geq 0$, the position of the particle is given by $s(t) = t^3 - 6t^2 + 9t$, where t is measured in seconds and s in meters.

(a) Find the velocity at any time t.
(b) Find the velocity at time $t = 2$ and $t = 4$.
(c) Find all times t when the particle is at rest.
(d) When is the particle moving in the positive direction?
(e) Draw a diagram to represent the motion of the particle.
(f) Find the total distance traveled by the particle over the time interval [0, 5].
(g) Graph the position, velocity and acceleration functions for $0 \leq t \leq 5$.
(h) When is the particle speeding up? When is it slowing down?

Solution

(a) The velocity function is the derivative of the position function.

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

The units are meters divided by seconds, or meters/second.

(b) Use the general expression $v(t)$ to find the (instantaneous) velocity at $t = 2$ and $t = 4$.

$$v(2) = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}$$

$$v(4) = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}$$

(c) The particle is at rest when the velocity is 0.

$v(t) = 3t^2 - 12t + 9$

$3(t-1)(t-3) = 0$ Factor completely.

$t - 1 = 0 \quad \text{or} \quad t - 3 = 0$ Principle of Zero Products.

$t = 1 \quad \text{or} \quad t = 3$ Solve each equation.

The particle is at rest at times $t = 1$ s and $t = 3$ s.

(d) The particle moves in the positive direction when $v(t) > 0$, that is, when

$3t^2 - 12t + 9 = 3(t-1)(t-3) > 0.$

This inequality is true when both factors are positive ($t > 3$) or when both factors are negative ($t < 1$). Therefore, the particle is moving in the positive direction when $0 \leq t < 1$ and when $t > 3$. It moves in the negative direction when $1 < t < 3$.

We can also use a sign chart to visualize the motion of the particle.

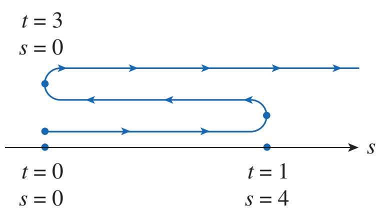


Figure 3.52
A diagram to represent the motion of the particle.

(e) Use the function s and the information about the velocity to sketch the motion of the particle as in Figure 3.52. Remember that the particle is moving back and forth along the straight line (the s-axis).

(f) The results in parts (d) and (e) indicate that we need to calculate the distance traveled during the time intervals [0, 1], [1, 3], and [3, 5] separately.

The distance traveled in the first second is

$|s(1) - s(0)| = |4 - 0| = 4$ m

From $t = 1$ to $t = 3$, the distance traveled is

$|s(3) - s(1)| = |0 - 4| = 4$ m

From $t = 3$ to $t = 5$, the distance traveled is

$|s(5) - s(3)| = |20 - 0| = 20$ m

The total distance traveled is $4 + 4 + 20 = 28$ m.

Common Error

Total distance traveled is the same as displacement.

Correct Method

The total distance traveled by a particle is a scalar quantity and is the total ground covered by the particle over a time interval. The displacement of a particle is a vector quantity that indicates the change in position of the particle over a time interval – by how much and in which direction.

(g) The acceleration is the derivative of the velocity function.

$$a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t - 12$$

The units for acceleration are meters/second divided by seconds, or m/s^2.

$a(4) = 6(4) - 12 = 12$ m/s^2

(h) Figure 3.53 shows the graphs of s, v, and a.

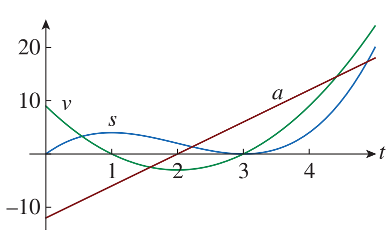


Figure 3.53
Graphs of s, v, and a.

(i) The speed of the particle is the absolute value of the velocity. That is, $|v(t)|$ is the speed at time t. Figure 3.54 shows a graph of $|v|$.

Using Figure 3.54, the particle is slowing down when $0 \le t < 1$ and when $2 < t < 3$. The particle speeds up when $1 < t < 2$ and when $t > 3$.

Alternatively, the particle is speeding up when the velocity is positive and increasing (v and a are both positive) and when the velocity is negative and decreasing (v and a are both negative).

In other words, the particle speeds up when the velocity and acceleration have the same sign. The velocity is pushed in the same direction the particle is moving.

Using Figure 3.53, the particle speeds up when $1 < t < 2$ and when $t > 3$.

The particle slows down when v and a have opposite signs – when $0 \le t < 1$ and when $2 < t < 3$. Figure 3.55 summarizes the motion of the particle.

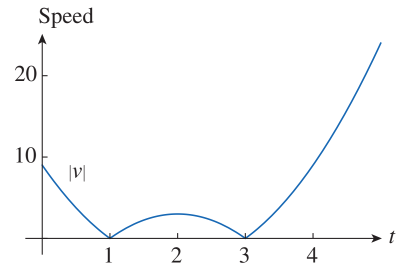


Figure 3.54
Graph of $|v|$, the speed of the particle.

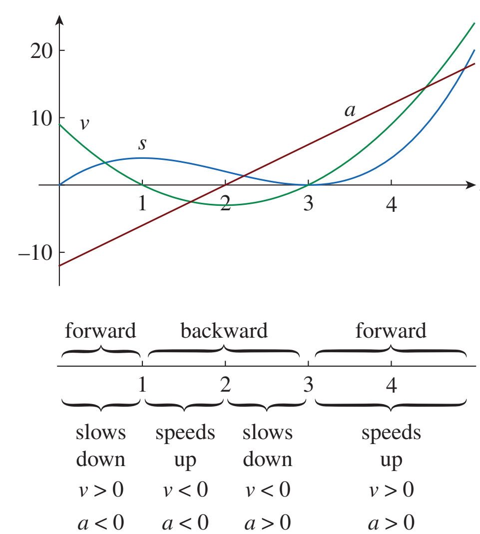


Figure 3.55
Summary of the motion of the particle.

Example 2 Linear Density

If the mass of a rod or a piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length $(\rho = m/l)$ and is measured in kilograms per meter.

Suppose, however, that the rod is not homogeneous and its mass, measured from the left end to a point x, is $m = f(x)$, as shown in Figure 3.56.

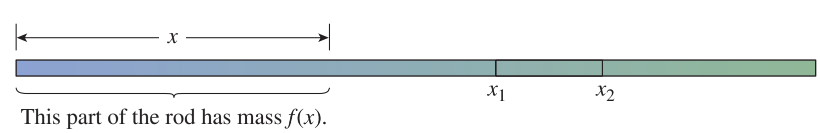


Figure 3.56
A nonhomogeneous rod.

The mass of the rod that lies between $x = x_1$ and $x = x_2$ is given by

$$\Delta m = f(x_2) - f(x_1).$$

The average density of that part of the rod is

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

As $\Delta x \to 0$, or equivalently, as $x_2 \to x_1$, we can compute the average density over smaller and smaller intervals.

The **linear density** ρ at x_1 is the limit of these average densities as $\Delta x \to 0$.

That is, the linear density is the rate of change of mass with respect to length:

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

Therefore, the linear density of the rod is the derivative of mass with respect to length.

For example, if $m = f(x) = \sqrt{x}$, where x is measured in meters and m in kilograms, then the average density of the part of the rod given by $1 \leq x \leq 1.2$ is

$$\frac{\Delta m}{\Delta x} = \frac{f(1.2) - f(1)}{1.2 - 1} = \frac{\sqrt{1.2} - 1}{0.2} = 0.477 \text{ kg/m}$$

The density of the rod at $x = 1$ is

$$\rho = \left.\frac{dm}{dx}\right|_{x=1} = \left.\frac{1}{2\sqrt{x}}\right|_{x=1} = 0.50 \text{ kg/m}$$ ■

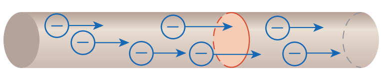


Figure 3.57
Electrons moving through a plane surface indicated by the red circle.

Example 3 Electrical Current

A current exists whenever electrical charges move. Figure 3.57 shows part of a wire and electrons moving through a plane surface, indicated by the red shaded circle.

If ΔQ is the net charge that passes through this surface during a time period Δt, then the average current during this time interval is defined as

$$\text{average current} = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

The limit of this average current over smaller and smaller time intervals is the **current** I at a given time t_1:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

Therefore, the current is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes, denoted A). ■

In addition to velocity, density, and current, many other important rates of change occur in physics. Others include power (the rate at which work is done), the rate of heat flow, temperature gradient (the rate of change of temperature with respect to position), and the rate of decay of a radioactive substance in nuclear physics.

Chemistry

Example 4 Rate of Reaction

A chemical reaction results in the formation of one or more substances, called *products*, from one or more starting materials, called *reactants*. For example, the equation

$$2H_2 + O_2 \to 2H_2O$$

indicates that two molecules of hydrogen and one molecule of oxygen combine to form two molecules of water.

Consider the reaction $A + B \rightarrow C$,

where A and B are reactants and C is the product. The **concentration** of a reactant A is the number of moles (1 mole $= 6.022 \times 10^{23}$ molecules) per liter and is denoted by [A]. The concentration varies during a reaction. Therefore, [A], [B], and [C] are all functions of time t.

The average rate of reaction of the product C over a time interval $t_1 \leq t \leq t_2$ is

$$\frac{\Delta[C]}{\Delta t} = \frac{[C](t_2) - [C](t_1)}{t_2 - t_2}.$$

Chemists are often interested in the **instantaneous rate of reaction**, obtained by taking the limit of the average rate of reaction as the time interval Δt approaches 0:

$$\text{rate of reaction} = \lim_{\Delta t \to 0} \frac{\Delta[C]}{\Delta t} = \frac{d[C]}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative $\frac{d[C]}{dt}$ will be positive, and so the rate of reaction of C is positive. The concentrations of the reactants, however, decrease during the reaction, so, to indicate positive rates of reaction of A and B, we can use negative signs in front of the derivatives $\frac{d[A]}{dt}$ and $\frac{d[B]}{dt}$. Since [A] and [B] each decrease at the same rate that [C] increases, we can write

$$\text{rate of reaction} = \frac{d[C]}{dt} = -\frac{d[A]}{dt} = -\frac{d[B]}{dt}$$

More generally, for any reaction of the form $aA + bB \rightarrow cC + dD$, we have

$$-\frac{1}{a}\frac{d[A]}{dt} = -\frac{1}{b}\frac{d[B]}{dt} = \frac{1}{c}\frac{d[C]}{dt} = \frac{1}{d}\frac{d[D]}{dt}$$

The rate of reaction can be determined from data and graphical methods. In some cases, explicit formulas for the concentrations as functions of time enable us to compute the rate of reaction. ■

Example 5 Compressibility

One quantity of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P. We can consider the rate of change of volume with respect to pressure, that is, the derivative $\frac{dV}{dP}$. As P increases, V decreases, so $\frac{dV}{dP} < 0$.

The **compressibility** is defined by changing the sign and dividing this derivative by the volume V:

$$\text{isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}$$

Therefore, β measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For example, suppose the volume V (in cubic meters) of a sample of air at 25°C is related to the pressure P (in kilopascals) by the equation $V = \dfrac{5.3}{P}$.

The rate of change of V with respect to P when $P = 50$ kPa is

$$\left.\frac{dV}{dP}\right|_{P=50} = -\left.\frac{5.3}{P^2}\right|_{P=50} = -\frac{5.3}{2500} = -0.00212 \text{ m}^3/\text{kPa}$$

The compressibility at that pressure is

$$\beta = -\left.\frac{1}{V}\frac{dV}{dP}\right|_{P=50} = \frac{0.00212}{\frac{5.3}{50}} = 0.02\ (\text{m}^3/\text{kPa})/\text{m}^3$$ ■

Biology

Example 6 Growth Rate

Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and the average rate of growth during the time period $t_1 \le t \le t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

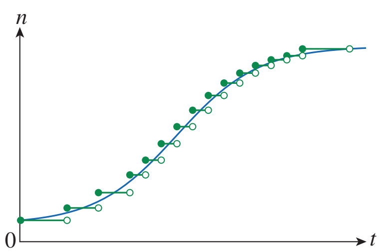


Figure 3.58
A smooth curve that approximates a growth function.

The **instantaneous rate of growth** is obtained from the average rate of growth by letting the time period Δt approach 0.

$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

This equation for growth rate is not quite accurate because the actual graph of a population function $n = f(t)$ is a step function that is discontinuous whenever a birth or death occurs and, therefore, n is not differentiable. However, for a large animal or plant population, we can approximate the graph by a smooth curve as shown in Figure 3.58.

Here is a more specific example: consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals, it is determined that the population doubles every hour. If the initial population is n_0 and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0,$$

$$f(2) = 2f(1) = 2^2 n_0,$$

$$f(3) = 2f(2) = 2^3 n_0,$$

and, in general,

$$f(t) = 2^t n_0.$$

fusebulb/Shutterstock.com

Figure 3.59
E. coli bacteria are about 2 micrometers (μm) long and 0.75 μm wide.

The population function is $n(t) = n_0 2^t$.

In Section 3.8, we learned that

$$\frac{d}{dx}(a^x) = a^x \ln a.$$

Therefore, the rate of growth of the bacteria population at time t is

$$\frac{dn}{dt} = \frac{d}{dt}(n_0\, 2^t) = n_0\, 2^t \ln 2.$$

For example, suppose that we start with an initial population of $n_0 = 100$ bacteria. Then the rate of growth at time $t = 4$ hours is

$$\left.\frac{dn}{dt}\right|_{t=4} = 100 \cdot 2^4 \cdot \ln 2 = 1600 \ln 2 \approx 1109.$$

In the context of this example, this means that at time $t = 4$ hours, the bacteria population is growing at a rate of approximately 1109 bacteria per hour. ■

Example 7 Blood Flow

Consider the flow of blood through a blood vessel, such as a vein or artery. Suppose we model the blood vessel by a cylindrical tube with radius R and length l, as illustrated in Figure 3.60.

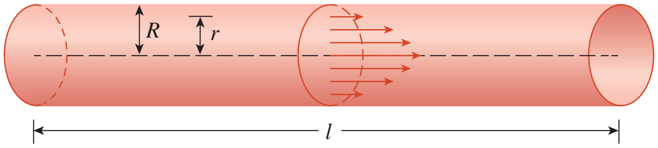


Figure 3.60
Blood flow through an artery.

For more detailed information, see W. Nichols and M. O'Rourke (eds.), McDonald's Blood Flow in Arteries: Theoretical, Experimental, and Clinical Principles, 6th ed. (Boca Raton, FL, 2011).

Because of friction at the walls of the tube (blood vessel), the velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall. The relationship between v and r is given by the **law of laminar flow** discovered by the French physician Jean-Louis-Marie Poiseuille in 1840. This law states that

$$v = \frac{P}{4\eta l}(R^2 - r^2) \tag{1}$$

where η is the viscosity of the blood and P is the pressure difference between the start and end of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$.

The average rate of change of the velocity as we move from $r = r_1$ outward to $r = r_2$ is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}.$$

and if we let $\Delta r \to 0$, we obtain the **velocity gradient**, that is, the instantaneous rate of change of velocity with respect to r:

$$\text{velocity gradient} = \lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

Use Equation 1 and find the derivative of v with respect to r.

$$\frac{dv}{dr} = \frac{P}{4\eta l}(0 - 2r) = -\frac{Pr}{2\eta l}$$

For one of the smaller human arteries, we can take $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm^2. The equation for v is

$$v = \frac{4000}{4(0.027)2}(0.000064 - r^2) \approx (1.85 \times 10^4)(6.4 \times 10^{-5} - r^2).$$

At $r = 0.002$ cm the blood is flowing at a speed of

$$v(0.002) \approx (1.85 \times 10^4)(64 \times 10^{-6} - 4 \times 10^{-6}) = 1.11 \text{ cm/s},$$

and the velocity gradient at that point is

$$\left.\frac{dv}{dr}\right|_{r=0.002} = -\frac{4000(0.002)}{2(0.027)2} \approx -74(\text{cm/s})/\text{cm}.$$

In order to better understand this last statement, change the units from centimeters to micrometers (1 cm = 10,000 μm). Then the radius of the artery is 80 μm. The velocity at the central axis is 11,850 μm/s, which decreases to 11,110 μm/s at a distance of $r = 20\ \mu$m. The derivative $\frac{dv}{dr} = -74\ (\mu\text{m/s})/\mu\text{m}$ means that when $r = 20\ \mu$m, the velocity is decreasing at a rate of approximately 74 μm/s for each micrometer that we move away from the center. ■

Economics

Example 8 Marginal Cost

Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2, then the additional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}.$$

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost**.

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

Since x usually takes on only integer values, it may not make practical sense to let Δx approach 0, but we could approximate $C(x)$ with a smooth curve (as in population growth).

If $\Delta x = 1$ and n is large (so that Δx is small compared to n), then

$$C'(n) \approx C(n + 1) - C(n).$$

Therefore, the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the $(n + 1)$st unit].

It is often reasonable to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3,$$

where a represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. Note that the cost of raw materials may be proportional to x, but labor costs might depend partly on higher powers of x because of the overtime costs and inefficiencies involved in large-scale operations.

For example, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10{,}000 + 5x + 0.01x^2.$$

Then the marginal cost function is

$C'(x) = 5 + 0.02x.$

The marginal cost at the production level of 500 items is

$C'(500) = 5 + 0.02(500) = \$15$ per unit.

This calculation is the rate at which costs are increasing with respect to the production level when $x = 500$, and it predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$
\begin{aligned}
&C(501) - C(500) \\
&\quad = [10{,}000 + 5(501) + 0.01(501)^2] - [10{,}000 + 5(500) + 0.01(500)^2] \\
&\quad = \$15.01
\end{aligned}
$$

Notice that $C'(500) \approx C(501) - C(500)$. ■

Economists also study marginal demand, marginal revenue, and marginal profit, which are derivatives of the demand, revenue, and profit functions. We will consider these concepts after we have developed techniques for finding the maximum and minimum values of functions.

■ Other Sciences

Rates of change occur in all the sciences. A geologist may be interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks. An engineer may need to know the rate at which water flows into or out of a reservoir. An urban geographer is often interested in the rate of change of the population density in a city as the distance from the city center increases. A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height.

In psychology, those interested in learning theory study the so-called learning curve, which graphs the performance $P(t)$ of someone learning a skill as a function of the training time t. Of particular interest is the rate at which performance improves as time passes, that is, $\dfrac{dP}{dt}$.

In sociology, differential calculus is used in analyzing the spread of rumors (or innovations, or fads, or fashions). If $p(t)$ denotes the proportion of a population that knows a rumor by time t, then the derivative $\dfrac{dp}{dt}$ represents the rate of spread of the rumor.

■ A Single Idea, Many Interpretations

Velocity, density, current, power, and temperature gradient in physics; rate of reaction and compressibility in chemistry; rate of growth and blood velocity gradient in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology; rate of spread of a rumor in sociology – these are all special cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as the derivative) can have different

interpretations in each of the sciences. When we develop the properties of a mathematical concept, we can then apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science. The French mathematician Joseph Fourier (1768–1830) put it succinctly: "Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them."

3.8 Exercises

A particle moves along a line. For $t \geq 0$, the position of the particle is given by $s = f(t)$, where t is measured in seconds and s in feet.

(a) Find the velocity at time t.
(b) Find the velocity at time $t = 1$ second.
(c) Find all times t when the particle is at rest.
(d) When is the particle moving in the positive direction? Negative direction?
(e) Find the total distance traveled by the particle during the first 6 seconds.
(f) Draw a diagram to represent the motion of the particle.
(g) Find the acceleration of the particle at time t and at time $t = 1$ second.
(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 6$.
(i) When is the particle speeding up? When is it slowing down?

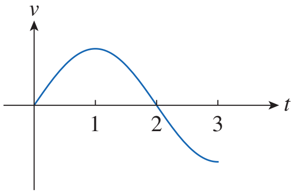
1. $f(t) = t^3 - 8t^2 + 24t$

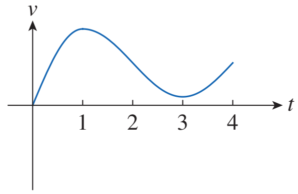
2. $f(t) = \dfrac{9t}{t^2 + 9}$

3. $f(t) = \sin\left(\dfrac{\pi t}{2}\right)$

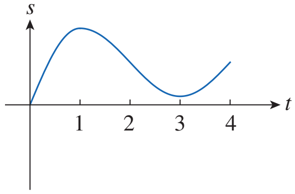
4. $f(t) = t^2 e^{-t}$

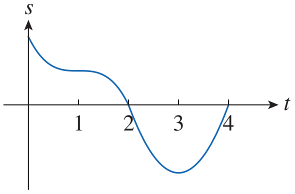
5. Graphs of the *velocity* functions of two particles are shown in the figures, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain your reasoning.





6. Graphs of the *position* functions of two particles are shown in the figures, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain your reasoning.





7. A projectile is shot vertically upward from a point 2 m above the ground with initial velocity of 24.5 m/s. The height (in meters) of the projectile at time t seconds is given by $h(t) = 2 + 24.5t - 4.9t^2$.
(a) Find the velocity of the projectile at times $t = 2$ seconds and $t = 4$ seconds.
(b) When does the projectile reach its maximum height?
(c) What is the maximum height?
(d) When does the projectile hit the ground?
(e) What is the velocity of the projectile when it hits the ground?

8. If a ball is thrown vertically upward with an initial velocity of 80 ft/s, then its height after t seconds is $s(t) = 80t - 16t^2$.
(a) What is the maximum height reached by the ball?
(b) What is the velocity of the ball when it is 96 feet above the ground on its way up? On its way down?

9. If a rock is thrown vertically upward on the surface of Mars with velocity 15 m/s, its height after t seconds is given by $h(t) = 15t - 1.86t^2$.
(a) What is the velocity of the rock at time $t = 2$ seconds?
(b) What is the velocity of the rock when its height is 25 m on its way up? On its way down?

10. A particle moves along a line. For $t \geq 0$, the position of the particle is given by

$$s(t) = t^4 - 4t^3 - 20t^2 + 20t$$

where t is measured in seconds and s in meters.
(a) At what time does the particle have a velocity of 20 m/s?
(b) At what time is the acceleration 0? Explain the significance of this value of t in the context of this problem.

11. A company makes computer chips from square wafers of silicon. It wants to keep the side length of a wafer very close to 15 mm, and it wants to know how the area $A(x)$ of a wafer changes when the side length x changes.
(a) Find $A'(15)$ and explain the meaning of your answer in the context of this problem.
(b) Show that the rate of change of the area of a square with respect to its side length is half its perimeter. Try to explain this result geometrically by drawing a square whose side length x is increased by an amount Δx. How can you approximate the resulting change in area, ΔA, if Δx is small?

12. Sodium chlorate crystals are easy to grow in the shape of cubes by allowing a solution of water and sodium chlorate to evaporate slowly.
(a) If V is the volume of a sodium chlorate crystal cube with side length x, find $\dfrac{dV}{dx}$ when $x = 3$ and explain the meaning of your answer in the context of this problem.
(b) Show that the rate of change of the volume of a cube with respect to edge length is equal to half the surface area of the cube. Explain this result geometrically using an argument similar to Exercise 11(b).

13. Recall the area of a circle with radius r is $A = \pi r^2$.
(a) Find the average rate of change of the area of a circle with respect to its radius r as r changes from
(i) 2 to 3 (ii) 2 to 2.5 (iii) 2 to 2.1
(b) Find the instantaneous rate of change when $r = 2$.
(c) Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain this result geometrically by drawing a circle whose radius is increased by an amount Δr. How can you approximate the resulting change in area ΔA if Δr is small?

14. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area of the circle is increasing at (a) 1 s, (b) 3 s, and (c) 5 s. What can you conclude about the rate of change of the area of the circle?

15. A spherical balloon is being inflated. Find the rate of increase of the surface area $(S = 4\pi r^2)$ with respect to the radius r when r is (a) 1 ft, (b) 2 ft, and (c) 3 ft. What conclusion can you make about the rate of increase of surface area?

16. The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers $(1\ \mu\text{m} = 10^{-6}\ \text{m})$.
(a) Find the average rate of change of V with respect to r when r changes from
(i) 5 to 8 μm (ii) 5 to 6 μm (iii) 5 to 5.1 μm
(b) Find the instantaneous rate of change of V with respect to r when $r = 5\ \mu$m.
(c) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Explain this result geometrically.

17. The mass of the part of a metal rod that lies between its left end and a point x meters to the right is $3x^2$ kg. Find the linear density when x is (a) 1 m, (b) 2 m, and (c) 3 m. Where is the density the highest? The lowest?

18. If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000\left(1 - \frac{1}{40}t\right)^2, \quad 0 \leq t \leq 40$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At what time is the water flowing out the fastest? The slowest?

19. The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. Find the current when (a) $t = 0.5$ s and (b) $t = 1$ s. At what time is the current lowest?

20. Newton's Law of Gravitation states that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.
(a) Find $\dfrac{dF}{dr}$ and explain the meaning of this expression in the context of this problem. What does the minus sign in this expression indicate?
(b) Suppose it is known that the earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20{,}000$ km. How fast does this force change when $r = 10{,}000$ km?

21. The force F acting on a body of mass m and velocity v is the rate of change of momentum: $F = \dfrac{d}{dt}(mv)$. If m is constant, this expression becomes $F = ma$, where $a = \dfrac{dv}{dt}$ is the

acceleration. However, in the theory of relativity, the mass of a particle varies with v and is given by

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. Show that

$$F = \frac{m_0\, a}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}$$

22. Some of the highest tides in the world occur in the Bay of Fundy on the Atlantic Coast of Canada. At Hopewell Cape, the water depth at low tide is about 2.0 m and at high tide it is about 12.0 m. The natural period of oscillation is a little more than 12 hours and on June 30 high tide occurs at 6:45 AM. This helps explain the following model for the water depth D (in meters) as a function of the time t (in hours after midnight) on that day:

$$D(t) = 7 + 5\cos[0.503(t - 6.75)]$$

How fast was the tide rising (or falling) at the following times?

(a) 3:00 AM (b) 6:00 AM (c) 9:00 AM (d) Noon

23. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the product of the pressure and the volume remains constant: $PV = C$.

(a) Find the rate of change of volume with respect to pressure.

(b) A sample of gas is in a container at low pressure and is steadily compressed at constant temperature for 10 minutes. Is the volume decreasing more rapidly at the beginning or the end of the 10 minutes? Explain your reasoning.

(c) Show that the isothermal compressibility is given by $\beta = \frac{1}{P}$.

24. If, as explained in Example 4, one molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentrations of A and B have a common value $[\mathrm{A}] = [\mathrm{B}] = a$ moles/L, then

$$[\mathrm{C}] = \frac{a^2kt}{akt + 1}$$

where k is a constant.

(a) Find the rate of the reaction at time t.

(b) Show that if $x = [\mathrm{C}]$, then

$$\frac{dx}{dt} = k(a - x)^2$$

(c) What happens to the concentration as $t \to \infty$?

(d) What happens to the rate of reaction as $t \to \infty$?

(e) Using correct units, explain your results in parts (c) and (d) in the context of this problem.

25. Suppose a population of bacteria triples every hour and that the initial population is 400 bacteria. Find an expression for the number n of bacteria at time t hours and use it to estimate the rate of growth of the bacteria population at 2.5 hours.

26. The number of yeast cells in a laboratory culture initially increases rapidly but eventually levels off. The population is modeled by the function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b. According to this model, what happens to the yeast population in the long run, that is, as $t \to \infty$?

27. The table gives the population of the world $P(t)$ in millions, where t is measured in years and $t = 0$ correspond to the year 1900.

t	(millions)	t	(millions)
0	1650	70	3710
10	1750	80	4450
20	1860	90	5280
30	2070	100	6080
40	2300	110	6870
50	2560	120	7755
60	3040		

(a) Estimate the rate of population growth in 1990 and in 2010. Indicate the units of measure.

(b) Use technology to find a cubic function (a third-degree polynomial) that models this data.

(c) Use your model in part (b) to derive a model for the rate of population growth.

(d) Use part (c) to estimate the rates of growth in 1990 and in 2010. Compare these results with your estimates in part (a).

(e) In Section 1.1, we modeled $P(t)$ with the exponential function

$$f(t) = (1358.03) \cdot (1.01478)^t$$

Use this model to derive a model for the rate of population growth.

(f) Use your model in part (e) to estimate the rate of growth in 1990 and in 2010. Compare these results with your estimates in parts (a) and (d).

(g) Estimate the rate of growth in 2015 using the model in part (b) and the model in part (e).

28. The table shows how the average annual rainfall (in inches) in a certain county in Nebraska for selected years.

t	$A(t)$	t	$A(t)$
1950	23.0	1990	25.9
1955	23.8	1995	26.3
1960	24.4	2000	27.0
1965	24.5	2005	28.0
1970	24.2	2010	28.8
1975	24.7	2013	29.3
1980	25.2	2015	29.4
1985	25.5	2018	29.4

(a) Use technology to find a fourth-degree polynomial to model these data.
(b) Use your answer in part (a) to derive a model for $A'(t)$.
(c) Estimate the rate of change of rainfall in 1990.
(d) Estimate the rate of change of rainfall in 2015. Does this estimate agree with the data? Why or why not?
(e) Construct a scatter plot of these data and graphs of A and A' in the same viewing rectangle.

29. Refer to the law of laminar flow presented in Example 7. Consider a blood vessel with radius 0.01 cm, length 3 cm, pressure difference 300 dynes/cm^2, and viscosity $\eta = 0.027$.
(a) Find the velocity of the blood along the centerline $r = 0$, at radius $r = 0.005$ cm, and at the wall $r = R = 0.01$ cm.
(b) Find the velocity gradient at $r = 0$, $r = 0.005$, and $r = 0.01$.
(c) Where is the velocity the greatest? Where is the greatest change in velocity?

30. The frequency of vibrations of a vibrating string is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

where L is the length of the sting, T is the tension, and ρ is its linear density. [See Chapter 11 in D. E. Hall, *Musical Acoustics*, 3rd ed. (Pacific Grove, CA: Brooks/ Cole, 2002).]
(a) Find the rate of change of the frequency with respect to
 (i) the length (when T and ρ are constant),
 (ii) the tension (when L and ρ are constant), and
 (iii) the linear density (when L and T are constant).
(b) The pitch of a note (how high or low the note sounds) is determined by the frequency f. (The higher the frequency, the higher the pitch.) Use the signs of the derivatives in part (a) to determine what happens to the pitch of a note
 (i) when the effective length of a string is decreased by placing a finger on the string so a shorter portion of the string vibrates,
 (ii) when the tension is increased by turning a tuning peg, and
 (iii) when the linear density is increased by switching to another string.

31. Suppose that the cost (in dollars) for a company to produce x pairs of a new line of jeans is

$$C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3$$

(a) Find the marginal cost function.
(b) Find $C'(100)$ and, using correct units, explain the meaning of your answer in the context of this problem.
(c) Compare $C'(100)$ with the cost of manufacturing the 101st pair of jeans.

32. The cost function for a certain commodity is

$$C(q) = 84 + 0.16q - 0.006q^2 + 0.000003q^3$$

(a) Find $C'(100)$ and interpret your answer in the context of this problem.
(b) Compare $C'(100)$ with the cost of producing the 101st item.

33. If $p(x)$ is the total value of the production when there are x workers in a manufacturing facility, then the average productivity of the workforce at the facility is

$$A(x) = \frac{p(x)}{x}$$

(a) Find $A'(x)$. Why would the company want to hire more workers if $A'(x) > 0$?
(b) Show that $A'(x) > 0$ if $p'(x)$ is greater than the average productivity.

34. If R denotes the reaction of the body to some stimulus of strength x, the *sensitivity* S is defined to be the rate of change of the reaction with respect to x. For example, when the brightness x of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}$$

has been used to model the dependence of R on x when R is measured in square millimeters and x is measured in appropriate units of brightness.
(a) Find the sensitivity.
(b) Illustrate part (a) by graphing both R and S as functions of x. Explain the values of R and S at low levels of brightness. Is this what you would expect?

35. Patients undergo dialysis treatment to remove urea from their blood when their kidneys are not functioning properly. Blood is diverted from the patient through a machine that filters out urea. Under certain conditions, the duration of dialysis required, given that the initial urea concentration is $c > 1$, is given by the equation

$$t = \ln\left(\frac{3c + \sqrt{9c^2 - 8c}}{2}\right)$$

Find the derivative of t with respect to c and interpret this expression in the context of this problem.

36. Invasive species often display a wave of advance as they colonize new areas. Mathematical models based on random dispersal and reproduction have demonstrated that the speed with which such waves move is given by the function $f(r) = 2\sqrt{Dr}$, where r is the reproductive rate of individuals and D is a parameter quantifying dispersal. Find the derivative of the wave speed with respect to the reproductive rate r and explain the meaning of this expression in the context of this problem.

37. The gas law for an ideal gas at absolute temperature T (in kelvins), pressure P (in atmospheres), and volume V (in liters) is $PV = nRT$, where n is the number of moles of gas and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8.0$ atm and is increasing at a rate of 0.10 atm/min and $V = 10$ L and is decreasing at a rate of 0.15 L/min. Find the rate of change of T with respect to time at that instant if $n = 10$ mol.

38. At a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$\frac{dP}{dt} = r_0\left(1 - \frac{P(t)}{P_c}\right)P(t) - \beta P(t)$$

where r_0 is the birth rate of the fish, P_c is the maximum population that the pond can sustain (called the *carrying capacity*), and β is the percentage of the population that is harvested.

(a) What value of $\dfrac{dP}{dt}$ corresponds to a stable population?

(b) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.

(c) What happens if β is raised to 5%?

39. In the study of ecosystems, *predator-prey* models are often used to study the interaction between species. Consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction has been modeled by the equations

$$\frac{dC}{dt} = aC - bcW \qquad \frac{dW}{dt} = -cW + dCW$$

(a) What values of $\dfrac{dC}{dt}$ and $\dfrac{dW}{dt}$ correspond to stable populations?

(b) How would the statement "The caribou go extinct" be represented mathematically?

(c) Suppose that $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$. Find all population pairs (C, W) that lead to stable populations. According to this model, is it possible for the two species to live in balance or will one or both species become extinct? Explain your reasoning.

3.9 Linear Approximations and Differentials

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, if f is differentiable at a and we zoom in near the point $(a, f(a))$, then the graph of $y = f(x)$ straightens out and looks more and more like its tangent line (see Figure 2.67 in Section 2.6). This observation is the basis for a method of finding approximate values of functions.

Linear Approximations

Suppose the function f is differentiable at $x = a$ and the value $f(a)$ is known. Let L be the linear function whose graph is the tangent line to the graph of f at the point $(a, f(a))$. See Figure 3.61.

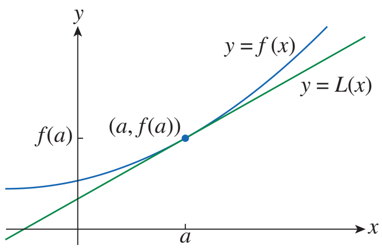


Figure 3.61
Graphs of f and L near $(a, f(a))$.

It seems very reasonable, and the graph suggests, that we can use the line tangent to the graph of f at $(a, f(a))$ as an approximation to the graph $y = f(x)$ when x is near a. An equation of the tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x - a) \tag{1}$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is the tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a) \tag{2}$$

is called the **linearization** of f at a.

The following example is typical of situations in which we use a linear approximation to predict the future behavior of a function given empirical data.

Example 1 Predicting from a Linear Approximation

Suppose a turkey is prepared for roasting and its temperature is 50°F. The turkey is then placed into an oven at temperature 325°F. After 1 hour the meat thermometer indicates that the temperature of the turkey is 93°F and after 2 hours it indicates 129°F. Predict the temperature of the turkey after 3 hours.

Solution

Let $T(t)$ represent the temperature of the turkey after t hours.

We are given: $T(0) = 50$, $T(1) = 93$, and $T(2) = 129$.

In order to construct a linear approximation with $a = 2$, we need an estimate for the derivative $T'(2)$.

Because $T'(2) = \lim_{t \to 2} \dfrac{T(t) - T(2)}{t - 2}$ we could estimate $T'(2)$ using a difference quotient with $t = 1$:

$$T'(2) \approx \frac{T(1) - T(2)}{1 - 2} = \frac{93 - 129}{-1} = 36$$

This corresponds to approximating the instantaneous rate of temperature change by the average rate of change between $t = 1$ and $t = 2$, which is 36°F/h.

With this estimate, the linear approximation for the temperature after 3 hours is

$$T(3) \approx T(2) + T'(2)(3 - 2) \approx 129 + 36 \cdot 1 = 165.$$

So the predicted temperature after 3 hours is 165°F.

We can obtain a more accurate estimate (in this case) for $T'(2)$ by plotting the given data, as in Figure 3.62, and estimating the slope of the tangent line at $t = 2$ to be

$$T'(2) \approx \frac{129 - 96}{2 - 1} = 33$$

Then our linear approximation becomes

$$T(3) \approx T(2) + T'(2) \cdot 1 \approx 129 + 33 = 162,$$

and our improved estimate for the temperature is 162°F.

Because the actual temperature curve lies below the tangent line, it appears that that the actual temperature after 3 hours will be somewhat less than 162°F, perhaps closer to 160°F. ■

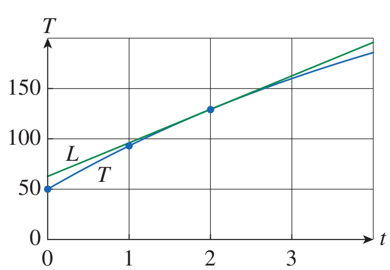


Figure 3.62
Graphs of the temperature curve and the tangent line at $t = 2$.

Example 2 Linear Approximation

Let $f(x) = \sqrt{x+3}$. Find an equation of the tangent line to the graph of f at the point where $x = 1$ and use this linearization to approximate $f(1.05)$ and $f(0.98)$.

Solution

$f(x) = \sqrt{x+3} = (x+3)^{1/2}$ Rewrite $f(x)$ using an exponent.

Find the derivative of f: $f'(x) = \dfrac{1}{2}(x+3)^{-1/2} = \dfrac{1}{2\sqrt{x+3}}$.

The tangent line passes through $(1, f(1)) = (1, \sqrt{3+1}) = (1, 2)$

and has slope $f'(1) = \dfrac{1}{2\sqrt{1+3}} = \dfrac{1}{4}$.

An equation of the tangent line is

$$y = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{7}{4} + \frac{x}{4}.$$

and the linearization is (the equation of this tangent line)

$$L(x) = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1).$$

When x is near 1, $f(x) \approx L(x)$. See Figure 3.63.

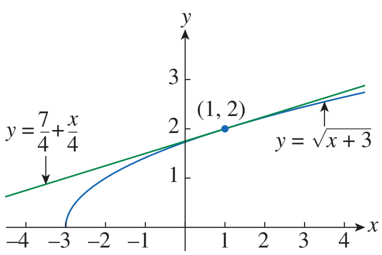


Figure 3.63
Graphs of f and L near (1, 2).

Therefore,

$$f(1.05) = \sqrt{1.05+3} = \sqrt{4.05} \approx L(1.05) = 2 + \frac{1}{4}(1.05-1) = 2.013,$$

$$f(0.98) = \sqrt{0.98+3} = \sqrt{3.98} \approx L(3.98) = 2 + \frac{1}{4}(3.98-1) = 1.995.$$

A Closer Look

Technology can certainly provide approximations for values like $\sqrt{3.98}$ and $\sqrt{4.05}$, but the linear approximation provides an approximation *over an entire interval.*

The accuracy of the tangent line approximation depends on three factors.

1. The distance between x and a
2. The steepness of the graph of f near a
3. The *concavity* of the graph of f: whether the graph of f is curved up like a cup or down like an umbrella. We will learn more about concavity in Chapter 4.

In Example 2 the graph of f is curved down like an umbrella near $x = 1$ (concave down). Therefore, the graph of the tangent line is above the graph of f, and the tangent line approximation is an *overestimate.*

Example 3 Accuracy of a Linear Approximation

For what values of x is the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5? What about accuracy to within 0.1?

Solution

Accuracy to within 0.5 means the functions should differ by less than 0.5:

$$\left|\sqrt{x+3}-\left(\frac{7}{4}+\frac{x}{4}\right)\right|<0.5$$

Using the definition of absolute value, an equivalent expression is

$$\sqrt{x+3}-0.5<\frac{7}{4}+\frac{x}{4}<\sqrt{x+3}+0.5.$$

This inequality can be interpreted as the linear approximation should lie between the curves obtained by shifting the curve $y=\sqrt{x+3}$ upward and downward by an amount 0.5. Figure 3.64 shows the tangent line $y=(7+x)/4$ intersecting the curve $y=\sqrt{x+3}+0.5$ at P and Q.

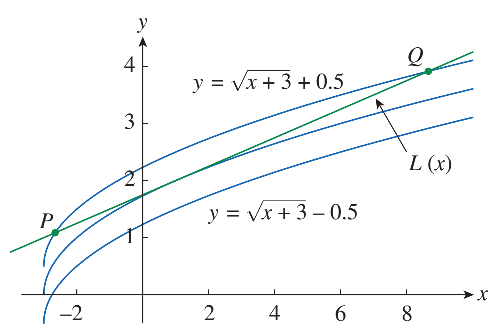


Figure 3.64
The graph of the tangent line intersects the curve $y=\sqrt{x+3}+0.5$ at P and Q.

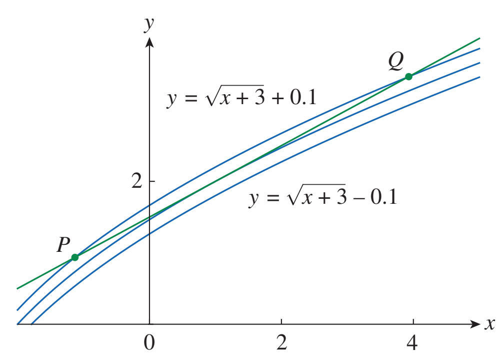


Figure 3.65
The graph of the tangent line intersects the curve $y=\sqrt{x+3}+0.1$ at P and Q.

If we use technology to zoom in, we can estimate the x-coordinate of P in Figure 3.64 is about -2.66 and the x-coordinate of Q is about 8.66.

Therefore, the graph suggests that the approximation $\sqrt{x+3}\approx\frac{7}{4}+\frac{x}{4}$ is accurate to within 0.5 when $-2.6<x<8.6$ (rounding to be safe).

Similarly, Figure 3.65 suggests that the approximation is accurate to within 0.1 when $-1.1<x<3.9$.

Applications to Physics

Linear approximations are often used in physics. In analyzing a particular model, a physicist may need to simplify a function by replacing it with its linear approximation. For example, in deriving a formula for the period of a pendulum, we obtain the expression $a_T=-g\sin\theta$ for tangential acceleration. Often we replace $\sin\theta$ by θ in this expression because $\sin\theta$ is close to θ if θ is small (close to 0).

We can actually verify this linearization of the function $f(x) = \sin x$ at $a = 0$.

$f(x) = \sin x \quad \Rightarrow \quad f'(x) = \cos x$

At $x = 0$: $f(0) = \sin 0 = 0$, $f'(0) = \cos 0 = 1$.

Therefore, $L(x) = f(0) + f'(0)(x - 0) = 0 + 1(x - 0) = x$.

So, for x close to 0, $\sin x \approx x$.

Another example occurs in the theory of optics, where light rays arriving at shallow angles relative to the optical axis are called *paraxial rays.* In paraxial (or Gaussian) optics, both $\sin\theta$ and $\cos\theta$ are replaced by their linearizations. The linear approximations

$$\sin\theta \approx \theta \qquad \cos\theta \approx 1$$

are used because θ is close to 0. The results of calculations made with these approximations became the basic theoretical tool used to design lenses. [See *Optics,* 5th ed., by Eugene Hecht, [Boston, 2017], p. 164.]

In Section 8.8 we will present several other applications of the idea of linear approximations.

Differentials

The concept of a linear approximation is often developed using the terminology and notation associated with *differentials.* If $y = f(x)$, where f is a differentiable function, then the **differential *dx*** is considered an independent variable; that is, dx can take on the value of any real number. The **differential *dy*** is then defined in terms of dx by the equation

$$dy = f'(x)\,dx \tag{3}$$

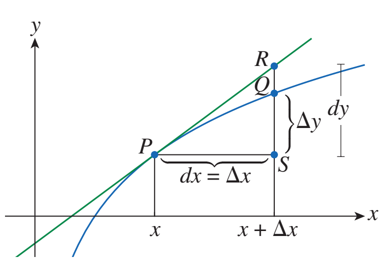


Figure 3.66
A geometric interpretation of differentials.

Therefore, dy is a dependent variable; it depends on the values of x and dx. If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy is determined (using Equation 3).

Here is a geometric interpretation of differentials. Consider Figure 3.66.

Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The tangent line to the graph of $y = f(x)$ at the point P and through the point R has slope $f'(x)$. This slope must be equal to rise over run, or $\dfrac{dy}{dx}$. Therefore,

$f'(x) = \dfrac{dy}{dx} \quad \Rightarrow \quad f'(x)\,dx = dy.$

The differential dy represents the amount that the tangent lines rises or falls (or the change in the linearization). However, Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes from x by an amount dx. Notice from Figure 3.66 that the approximation $\Delta y \approx dy$ becomes better as Δx becomes smaller.

Suppose we let $dx = x - a$, then $x = a + dx$ and we can rewrite the linear approximation (Equation 1) using differential notation:

$$f(a + dx) \approx f(a) + dy$$

For example, for the function $f(x) = \sqrt{x + 3}$ in Example 1, we have

$$dy = f'(x)\,dx = \frac{dx}{2\sqrt{x + 3}}$$

If $a = 1$ and $dx = \Delta x = 0.05$, then

$$dy = \frac{0.05}{2\sqrt{1 + 3}} = 0.013$$

and
$$\sqrt{4.05} = f(1.05) \approx f(1) + dy = 2.013$$

just as we found in Example 1.

Here is one more example that illustrates the use of differentials to estimate the errors that occur due to approximate measurements.

Example 4 Error in Approximating the Volume of a Sphere

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

Solution

If the radius of the sphere is r, then the volume is $V(r) = \frac{4}{3}\pi r^3$.

If the error in the measured value of r is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of V is ΔV. This value can be approximated by

$dV = V'(r)\,dr = 4\pi r^2\,dr.$

For $r = 21$, $dr = 0.05$: $dV = 4\pi(21)^2(0.05) \approx 277.$

The maximum error in the calculated volume is approximately 277 cm^3. ■

Note: The error in Example 4 seems pretty large. Often, a better way to describe this kind of inaccuracy is by using the **relative error**. This is computed by dividing the error (estimate) by the total volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r}$$

Therefore, in Example 4, the relative error in the volume is about three times the relative error in the radius.

Relative error in the radius is approximately $\frac{dr}{r} = \frac{0.05}{21} \approx 0.0024.$

Relative error in the volume is approximately $3(0.024) \approx 0.0072.$

These errors could also be expressed as **percentage errors**, for example, a 0.24% in the radius and 0.7% in the volume.

3.9 Exercises

Find the linearization $L(x)$ of the function at a.

1. $f(x) = x^3 - x^2 + 3, \quad a = -2$

2. $f(x) = \sin x, \quad a = \dfrac{\pi}{6}$

3. $f(x) = \sqrt{x}, \quad a = 4$

4. $f(x) = 2^x, \quad a = 0$

5. Find the linear approximation of the function $f(x) = \sqrt{1 - x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$. Use technology to graph f and the tangent line in the same viewing rectangle.

6. Find the linear approximation of the function $g(x) = \sqrt[3]{1 + x}$ at $a = 0$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. Use technology to graph g and the tangent line in the same viewing rectangle.

Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1.

7. $\sqrt[3]{1 - x} \approx 1 - \dfrac{1}{3}x$

8. $\tan x \approx x$

9. $\dfrac{1}{(1 + 2x)^4} \approx 1 - 8x$

10. $e^x \approx 1 + x$

Find the differential dy of each function.

11. $y = \dfrac{x + 1}{x - 1}$

12. $y = (1 + r^3)^{-2}$

13. $y = e^{\tan \pi x}$

14. $y = \sqrt{1 + \ln x}$

Use a linear approximation (or differentials) to estimate the given number.

15. $(1.999)^4$

16. $\dfrac{1}{4.002}$

17. $\sqrt[3]{1001}$

18. $\sqrt{100.5}$

19. $e^{0.1}$

20. $\cos\left(\dfrac{29\pi}{180}\right)$

Explain, in terms of linear approximations or differentials, why the approximation is reasonable.

21. $\sec 0.08 \approx 1$

22. $(1.01)^6 \approx 1.06$

23. $\ln 1.05 \approx 0.05$

24. Let $y = e^{x/10}$.
(a) Find the differential dy.
(b) Evaluate dy and Δy if $x = 0$ and $dx = 0.1$.

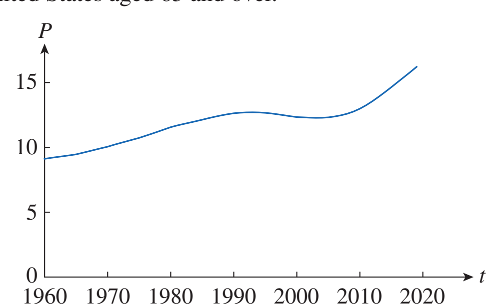
25. Let $y = \sqrt{x}$.
(a) Find the differential dy.
(b) Evaluate dy and Δy if $x = 1$ and $dx = \Delta x = 1$.
(c) Sketch a graph showing the line segments with lengths dx, dy, and Δy.

26. Let $f(x) = (x - 1)^2$, $g(x) = e^{-2x}$, and $h(x) = 1 + \ln(1 - 2x)$.
(a) Find the linearization of f, g, and h at $a = 0$. What do you notice about these expressions? How do you explain this result?
(b) Graph f, g, and h and their linear approximations. For which function is the linear approximation the best? Worst? Explain your reasoning.

27. Suppose the turkey in Example 1 is removed from the oven when its temperature reaches 185°F and is placed on a table in a room where the temperature is 75°F. After 10 minutes the temperature of the turkey is 172°F and after 20 minutes it is 160°F. Use a linear approximation to predict the temperature of the turkey after half an hour. Do you think your prediction is an overestimate or an underestimate? Why?

28. Atmospheric pressure P decreases as altitude h increases. At a temperature of 15°C, the pressure is 101.3 kilopascals (kPA) at sea level, 87.1 kPa at $h = 1$ km, and 74.9 kPa at $h = 2$ km. Use a linear approximation to estimate the atmospheric pressure at an altitude of 3 km.

29. The graph illustrates the percentage of the population in the United States aged 65 and over.



Use a linear approximation to predict the percentage of the population that will be 65 and over in the years 2030 and 2040. Do you think your predictions are too high or too low? Why?

30. The table shows the population of Canada (in millions) at the end of the given year.

t	2000	2005	2010	2015	2020
$N(t)$	30.59	32.16	34.15	36.03	37.74

Use a linear approximation to estimate the population on December 31, 2009. Use another linear approximation to predict the population in 2025.

31. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing (a) the volume of the cube and (b) the surface area of the cube.

32. The radius of a circular disk was measured as 24 cm with a maximum error in measurement of 0.2 cm.
(a) Use differentials to estimate the maximum error in the calculated area of the disk.
(b) What is the relative error? What is the percentage error?

33. The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm.
(a) Use differentials to estimate the maximum error in the calculated surface area. What is the relative error?
(b) Use differentials to estimate the maximum error in the calculated volume. What is the relative error?

34. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

35. (a) Use differentials to find a formula for the approximate volume of a thin cylindrical shell with height h, inner radius r, and thickness Δr.
(b) What is the error involved in using the formula from part (a)?

36. One side of a right triangle is known to be 20 cm long and the opposite angle is measured as 30°, with possible error of $\pm 1°$.
(a) Use differentials to estimate the error in computing the length of the hypotenuse.
(b) What is the percentage error?

37. When blood flows along a blood vessel, the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4$$

This is known as Poiseuille's Law. A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore the normal blood flow.

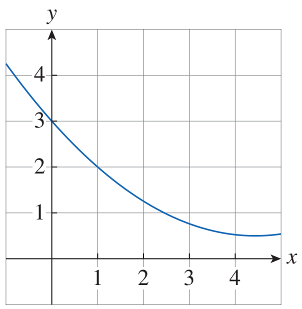
Show that the relative change in F is about four times the relative change in R. How will a 5% increase in the radius affect the flow of blood?

38. In his book *Physics: Calculus*, 2d ed., Eugene Hecht derives the formula $T = 2\pi\sqrt{L/g}$ for the period of a pendulum. In deriving this formula, he obtains the equation $a_T = -g \sin\theta$ for the tangential acceleration of the bob of the pendulum. He then says, "for small angles, the value of θ in radians is very nearly the value of $\sin\theta$; they differ by less than 2% out to about 20°."
(a) Verify the linear approximation at 0 for the sine function:

$$\sin x \approx x$$

(b) Use technology to determine the values of x for which $\sin x$ and x differ by less than 2%. Then verify Hecht's statement by converting from radians to degrees.

39. Suppose f is a functions such that $f(1) = 5$ and the graph of its derivative is shown in the figure.



(a) Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.
(b) Are the approximations in part (a) overestimates or underestimates? Explain your reasoning.

40. Suppose g is a function such that $g(2) = -4$ and $g'(x) = \sqrt{x^2 + 5}$.
(a) Use a linear approximation to estimate $g(1.95)$.
(b) Are the approximations in part (a) overestimates or underestimates? Explain your reasoning.

Laboratory Project | Taylor Polynomials

The tangent line approximation $L(x)$ is the best first-degree (linear) approximation to $f(x)$ near $x = a$ because $f(x)$ and $L(x)$ have the same rate of change (derivative) at a. For a better approximation, it seems reasonable to use a second-degree (quadratic) approximation $P(x)$. Geometrically this means that we approximate the graph of $y = f(x)$ near $x = a$ by a parabola instead of a line.

To make sure that the approximation is good, we impose the following conditions:

(i) $P(a) = f(a)$ (P and F should have the same value at a.)

(ii) $P'(a) = f'(a)$ (P and f should have the same rate of change at a.)

(iii) $P''(a) = f''(a)$ (The slopes of P and f should change at the same rate at a.)

1. Find the quadratic approximation $P(x) = A + Bx + Cx^2$ to the function $f(x) = \cos x$ that satisfies conditions (i), (ii), and (iii) for $a = 0$. Graph P, f, and the linear approximation $L(x) = 1$ in the same viewing rectangle. Use the graph to explain how well the functions
P and L approximate f.

2. Determine the values of x for which the quadratic approximation $f(x) \approx P(x)$ in Problem 1
is accurate to within 0.1. Hint: graph $y = P(x)$, $y = \cos x - 0.1$, and $y = \cos x + 0.1$ in the same viewing rectangle.

3. To approximate a function f by a quadratic function P near a number a, the best form for the function P is

$$P(x) = A + B(x - a) + C(x - a)^2$$

Show that the quadratic function that satisfies conditions (i), (ii), and (iii) is

$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

4. Find the quadratic approximation to $f(x) = \sqrt{x + 3}$ at $a = 1$. Graph f, the quadratic approximation, and the linear approximation from Example 2 in Section 3.9 in the same viewing rectangle. What can you conclude from these graphs?

5. Suppose we want an even better approximation to f at $x = a$. It seems reasonable to use higher-degree polynomials. Consider an nth-degree polynomial of the form

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots + c_n(x - a)^n$$

such that T_n and its first n derivatives have the same values at $x = a$ as f and its first n derivatives. Differentiate T_n repeatedly, and after each step let $x = a$. Show that if $c_0 = f(a)$, $c_1 = f'(a)$, $c_2 = \frac{1}{2}f''(a)$, and in general

$$c_k = \frac{f^{(k)}(a)}{k!}$$

where $k! = k \cdot (k - 1) \cdot (k - 2) \cdots 3 \cdot 2 \cdot 1$. The resulting polynomial

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

is called the ***n*th-degree Taylor polynomial of *f* centered at *a***.

6. Find the 8th-degree Taylor polynomial centered at $a = 0$ for the function $f(x) = \cos x$. Graph f and the Taylor polynomials T_2, T_4, T_6, and T_8 in the viewing rectangle $[-5, 5] \times [-1.4, 1.4]$ and comment on how well they approximate f (near $a = 0$).

3 Review

Concepts and Vocabulary

1. State each differentiation rule symbolically and in your own words.
(a) The Power Rule
(b) The Constant Multiple Rule
(c) The Sum Rule
(d) The Difference Rule
(e) The Product Rule
(f) The Quotient Rule
(g) The Chain Rule

2. State the derivative of each function.
(a) $y = x^n$
(b) $y = e^x$
(c) $y = b^x$
(d) $y = \ln x$
(e) $y = \log_b x$
(f) $y = \sin x$
(g) $y = \cos x$
(h) $y = \tan x$
(i) $y = \csc x$
(j) $y = \sec x$
(k) $y = \cot x$
(l) $y = \sin^{-1} x$
(m) $y = \cos^{-1} x$
(n) $y = \tan^{-1} x$

3. (a) How is the number e defined?
(b) Express e as a limit.
(c) Why is the natural exponential function $y = e^x$ used more often in calculus than the other exponential functions $y = a^x$?
(d) Why is the natural logarithmic function $y = \ln x$ used more often in calculus than the other logarithmic functions $y = \log_a x$?

4. (a) In your own words, explain the process of implicit differentiation. When should this process be used?
(b) In your own words, explain the process of logarithmic differentiation. When should this process be used?

5. Explain the method to find the slope of a tangent line to a parametric curve $x = f(t)$, $y = g(t)$.

6. Give several examples of how the derivative can be interpreted as a rate of change in physics, chemistry, biology, economics, and other sciences.

7. Write an expression for the linearization of f at a.

8. Write an expression for the differential dy. If $dx = \Delta x$, explain how dy could be an overestimate or an underestimate for Δy.

True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. If f and g are differentiable, then
$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

2. If f and g are differentiable, then
$$\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$$

3. If f and g are differentiable, then
$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

4. If f is differentiable, then
$$\frac{d}{dx}\left[\sqrt{f(x)}\right] = \frac{f'(x)}{2\sqrt{f(x)}}$$

5. If f is differentiable, then
$$\frac{d}{dx}\left[f(\sqrt{x})\right] = \frac{f'(x)}{2\sqrt{x}}$$

6. If $y = e^2$, then $y' = 2e$.

7. $\dfrac{d}{dx}(10^x) = x\,10^{x-1}$

8. $\dfrac{d}{dx}(\ln 10) = \dfrac{1}{10}$

9. $\dfrac{d}{dx}(\tan^2 x) = \dfrac{d}{dx}(\sec^2 x)$

10. $\dfrac{d}{dx}(|x^2 + x|) = |2x + 1|$

11. The derivative of a polynomial is a polynomial.

12. If $f(x) = (x^6 - x^4)^5$, then $f^{(31)}(x) = 0$.

13. The derivative of a rational function is a rational function.

14. An equation of the tangent line to the parabola $y = x^2$ at the point $(-2, 4)$ is $y - 4 = 2x(x + 2)$.

15. If $g(x) = x^5$, then $\displaystyle\lim_{x\to 2} \frac{g(x) - g(2)}{x - 2} = 80$.

16. If $\Delta x = dx$, then $\Delta y = dy$.

Exercises

Find y'.

1. $y = (x^2 + x^3)^4$

2. $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt[5]{x^3}}$

3. $y = \dfrac{x^2 - x + 2}{\sqrt{x}}$

4. $y = \dfrac{\tan x}{1 + \cos x}$

5. $y = x^2 \sin \pi x$

6. $y = x \cos^{-1} x$

7. $y = \dfrac{t^4 - 1}{t^4 + 1}$

8. $x e^y = y \sin x$

9. $y = \ln(x \ln x)$

10. $y = e^{mx} \cos nx$

11. $y = \sqrt{x} \cos \sqrt{x}$

12. $y = (\arcsin 2x)^2$

13. $y = \dfrac{e^{1/x}}{x^2}$

14. $y = \ln \sec x$

15. $y + x \cos y = x^2 y$

16. $y = \left(\dfrac{u - 1}{u^2 + u + 1}\right)^4$

17. $y = \sqrt{\arctan x}$

18. $y = \cot(\csc x)$

19. $y = \tan\left(\dfrac{t}{1 + t^2}\right)$

20. $y = e^{x \sec x}$

21. $y = 3^{x \ln x}$

22. $y = \sec(1 + x^2)$

23. $y = (1 - x^{-1})^{-1}$

24. $y = \dfrac{1}{\sqrt[3]{x + \sqrt{x}}}$

25. $\sin(xy) = x^2 - y$

26. $y = \sqrt{\sin \sqrt{x}}$

27. $y = \log_5(1 + 2x)$

28. $y = (\cos x)^x$

29. $y = \ln \sin x - \dfrac{1}{2} \sin^2 x$

30. $y = \dfrac{(x^2 + 1)^4}{(2x + 1)^3 (3x - 1)^5}$

31. $y = x \tan^{-1}(4x)$

32. $y = e^{\cos x} + \cos(e^x)$

33. $y = \ln|\sec 5x + \tan 5x|$

34. $y = 10^{\tan \pi\theta}$

35. $y = \cot(3x^2 + 5)$

36. $y = \sqrt{t \ln(t^4)}$

37. $y = \sin\left(\tan\sqrt{1 + x^3}\right)$

38. $y = \arctan(\arcsin \sqrt{x})$

39. $y = \tan^2(\sin \theta)$

40. $x e^y = y - 1$

41. $y = \dfrac{\sqrt{x + 1}(2 - x)^5}{(x + 3)^7}$

42. $y = \dfrac{(x + \lambda)^4}{x^4 + \lambda^4}$

43. $y = \dfrac{\sin mx}{x}$

44. $y = \ln\left|\dfrac{x^2 - 4}{2x + 5}\right|$

45. $y = \cos\left(e^{\sqrt{\tan 3x}}\right)$

46. $y = \sin^2(\cos \sqrt{\sin \pi x})$

47. If $f(t) = \sqrt{4t + 1}$, find $f''(2)$.

48. If $g(\theta) = \theta \sin \theta$, find $g''\left(\dfrac{\pi}{6}\right)$.

49. Find y'' if $x^6 + y^6 = 1$.

50. Find $f^{(n)}(x)$ if $f(x) = \dfrac{1}{2 - x}$.

51. If $f(x) = xe^x$, find an expression for $f^{(n)}(x)$.

Find an equation of the tangent line to the graph of the function at the given point.

52. $f(x) = 4 \sin^2 x$, $\left(\dfrac{\pi}{6}, 1\right)$

53. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$, $(0, -1)$

54. $y = \sqrt{1 + 4 \sin x}$, $(0, 1)$

55. $x = \ln t$, $y = t^2 + 1$, $(0, 2)$

56. $x = t^3 - 2t^2 + t + 1$, $y = t^2 + 1$, $(1, 0)$

Find equations of the tangent line and the normal line to the graph of each expression at the given point.

57. $x^2 + 4xy + y^2 = 13$, $(2, 1)$

58. $y = (2 + x)e^{-x}$, $(0, 2)$

59. $y = e^{-x/(x+1)}$, $(4, e^{-4/5})$

60. If $f(x) = x e^{\sin x}$, find $f'(x)$. Graph f and f' in the same viewing rectangle and describe the relationship between the two graphs.

61. Let $f(x) = x\sqrt{5 - x}$.
(a) Find $f'(x)$.
(b) Find equations of the tangent lines to the graph of f at the points $(1, 2)$ and $(4, 4)$.
(c) Illustrate part (b) by graphing f and the tangent lines in the same viewing rectangle.
(d) Graph f and f' in the same viewing rectangle and describe the relationship between the two graphs.

62. Let $f(x) = 4x - \tan x$, $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.
(a) Find $f'(x)$ and $f''(x)$.
(b) Graph f, f', and f'' in the same viewing rectangle and describe the relationship among the three graphs.

63. Let $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$. Find the points on the graph of f where the tangent line is horizontal.

64. Find the points on the ellipse described by $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

65. If $f(x) = (x - a)(x - b)(x - c)$, show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c}$$

66. (a) Differentiate both sides of the double-angle formula

$$\cos 2x = \cos^2 x - \sin^2 x$$

to obtain the double-angle formula for the sine function.

(b) Differentiate both sides of the addition formula

$$\sin(x + a) = \sin x \sin a + \cos x \sin a$$

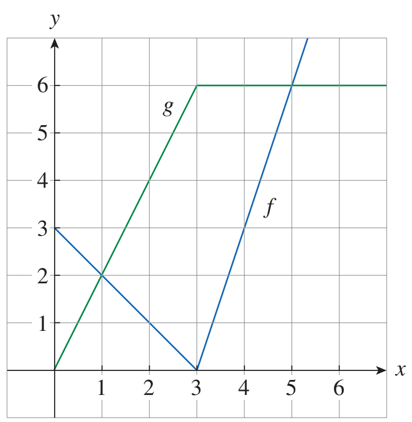
to obtain the addition formula for the cosine function.

67. Suppose that

$f(1) = 2$	$f'(1) = 3$	$f(2) = 1$	$f'(2) = 2$
$g(1) = 3$	$g'(1) = 1$	$g(2) = 1$	$g'(2) = 4$

(a) If $S(x) = f(x) + g(x)$, find $S'(1)$.
(b) If $P(x) = f(x)\,g(x)$, find $P'(2)$.
(c) If $Q(x) = \dfrac{f(x)}{g(x)}$, find $Q'(1)$.
(d) If $C(x) = f(g(x))$, find $C'(2)$.

68. The graphs of the functions f and g are shown in the figure.



Let $P(x) = f(x)g(x)$, $Q(x) = \dfrac{f(x)}{g(x)}$, and $C(x) = f(g(x))$. Find

(a) $P'(2)$, (b) $Q'(2)$, and (c) $C'(2)$.

Find f' in terms of g and g'.

69. $f(x) = x^2 g(x)$

70. $f(x) = g(x^2)$

71. $f(x) = [g(x)]^2$

72. $f(x) = g(g(x))$

73. $f(x) = g(e^x)$

74. $f(x) = e^{g(x)}$

75. $f(x) = \ln |g(x)|$

76. $f(x) = g(\ln x)$

77. $f(x) = g(\tan^{-1} x)$

78. $f(x) = \tan^{-1} g(x)$

Find h' in terms of f, f', g, and g'.

79. $h(x) = \dfrac{f(x)g(x)}{f(x) + g(x)}$

80. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}}$

81. $h(x) = f(g(\sin 4x))$

82. $h(x) = f(g(f(x) \cdot g(x)))$

83. Let $f(x) = x - 2 \sin x$.
(a) Graph f in the viewing rectangle $[0, 8] \times [-2, 8]$.
(b) On what interval is the average rate of change larger: $[1, 2]$ or $[2, 3]$?
(c) At which value of x is the instantaneous rate of change larger: $x = 2$ or $x = 5$?
(d) Check your visual estimates in part (c) by computing $f'(x)$ and comparing the numerical values of $f'(2)$ an $f'(5)$.

84. Let $f(x) = [\ln(x + 4)]^2$. Find the point(s) on the graph of f where the tangent line is horizontal.

85. Let $f(x) = e^x$.
(a) Find an equation of the tangent line to the graph of f that is parallel to the line $x - 4y = 1$.
(b) Find an equation of the tangent line to the graph of f that passes through the origin.

86. Find the slope of the line that passes through the point $(4, 0)$ and is tangent to the graph of $f(x) = e^{-x}$.

87. Find a parabola $y = ax^2 + bx + c$ that passes through the point $(1, 4)$ and whose tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2, respectively.

88. The function $C(t) = K(e^{-at} - e^{-bt})$, where a, b, and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.
(a) Show that $\lim_{t \to \infty} C(t) = 0$.
(b) Find $C'(t)$, the rate of change of drug concentration in the blood.
(c) When is the rate equal to 0?
(d) Let $a = 1$, $b = 2$, and $K = 10$. Graph C and C' in the same viewing rectangle. Explain the relationship between the two graphs in the context of this problem.

89. An equation of the form $s(t) = Ae^{-ct}\cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object at time t.

90. A particle moves along a horizontal line so that its position at time t, $t \geq 0$, is given by $s(t) = 3t^4 - 20t^3 - 48t^2 + 48t$, where t is measured in seconds and s in feet.
(a) Find the average velocity for the first 2 seconds.
(b) When is the particle at rest?
(c) When does the particle change direction?
(d) What is the total distance traveled by the particle for $0 \leq t \leq 4$? Show the work that leads to your answer.
(e) What is the acceleration when $t = 1$?
(f) Is the speed increasing or decreasing when $t = 0.5$? Justify your answer.

91. A particle moves along a horizontal line so that its position at time t is given by $s(t) = 10 + te^{-t/5}$ for $0 \leq t \leq 50$.
(a) Find the average velocity for $0 \leq t \leq 50$.
(b) For what values of t is the particle moving to the left? What is the particle's leftmost position? Show the analysis that leads to your conclusion.

(c) Find the acceleration when the particle is first at rest.
(d) What is the total distance traveled by the particle for $0 \le t \le 50$?
(e) Is the speed increasing or decreasing at $t = 20$?

92. A particle moves along a horizontal line so that its velocity at time t, $t \geq 0$, is given by $v(t) = e^{t \sin t} - 1$. The particle's position at time t, $s(t)$, is not explicitly given.
(a) For what values of t is the particle moving to the right?
(b) For $2 \leq t \leq 4$, find all values of t for which the speed of the particle is 0.5.
(c) What is the average acceleration of the particle for $1 \leq t \leq 3$? Show the computations that lead to your answer.
(d) Find a time t, $1 \leq t \leq 3$, such that the instantaneous acceleration equals the average acceleration.
(e) Is the speed increasing or decreasing at time $t = 6$?

93. A particle moves along a horizontal line so that its position at time t, for $t \geq 0$, is given by $s(t) = t^3 - 2t^2 - 4t + 5$. Find the total distance traveled by the particle for $0 \leq t \leq 5$.

94. A particle moves along the curve given by $y = x \ln x$ so that y increases at a rate of 12 units per second. How fast is x increasing when the particle is at the point $(e^2, 2e^2)$?

95. A particle moves along a horizontal line so that its position at time t is given by $s(t) = (\sin t) \cdot e^{\cos^2 t}$. How many times does the particle change direction for $0 \leq t \leq \pi$? Justify your answer.

96. A particle moves along a horizontal line so that its position at time t, for $0 < t < 1$, is given by $s(t) = \dfrac{1}{3t \ln t}$. Find the value of t in this interval when the particle is farthest to the right.

97. A particle moves along a horizontal line so that its position at time t, for $t \geq 0$, is given by $s(t) = \sqrt{b^2 + c^2 t}$, where b and c are positive constants.
(a) Find the velocity and acceleration functions.
(b) Show that the particle always moves in the positive direction.

98. A particle moves along a vertical line so that its y-coordinate at time t is given by

$$y(t) = t^3 - 12t + 3, \quad t \ge 0$$

(a) Find the velocity and acceleration functions.
(b) When is the particle moving upward and when is it moving downward?
(c) Find the total distance the particle travels in the time interval $0 \leq t \leq 3$.
(d) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 3$.
(e) When is the particle speeding up? When is it slowing down?

99. The mass of a wire is $x(1 + \sqrt{x})$ kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 4$ cm.

100. The volume of a right circular cone is $V = \dfrac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.
(a) Find the rate of change of the volume with respect to the height if the radius is constant.
(b) Find the rate of change of the volume with respect to the radius if the height is constant.

101. The cost, in dollars, of producing x units of a certain commodity is

$$C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3$$

(a) Find the marginal cost function.
(b) Find $C'(100)$ and explain its meaning.
(c) Compare $C'(100)$ with the cost of producing the 101st item.

102. Let $f(x) = \sqrt[3]{1 + 3x}$.
(a) Find the linearization of f at $a = 0$. Use it to find an approximate value for $\sqrt[3]{1.03}$.
(b) Determine the values of x for which the linear approximation given in part (a) is accurate to within 0.1.

103. A window has the shape of a square surmounted by a semicircle. The base of the window is measured as having width 60 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum error possible in computing the area of the window.

Express the limit as a derivative and evaluate.

104. $\displaystyle\lim_{x \to 1} \frac{x^{17} - 1}{x - 1}$

105. $\displaystyle\lim_{h \to 0} \frac{\sqrt[4]{16 + h} - 2}{h}$

106. $\displaystyle\lim_{\theta \to \pi/3} \frac{\cos \theta - 0.5}{\theta - \dfrac{\pi}{3}}$

107. $\displaystyle\lim_{h \to 0} \frac{e^{-3(3+h)} - e^{-9}}{h}$

108. Evaluate $\displaystyle\lim_{h \to 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$.

109. Suppose f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that

$$g'(x) = \frac{1}{1 + x^2}$$

110. Find $f'(x)$ if it is known that

$$\frac{d}{dx}[f(2x)] = x^2$$

111. Show that the length of the portion of any tangent line to the graph of the astroid described by $x^{2/3} + y^{2/3} = a^{2/3}$ cut off by the coordinate axes is constant.

Focus on Problem Solving

Before reading the solution to the example, consider solving the problem yourself. The principles of problem solving presented earlier might be helpful.

Example Unique c

For what values of c does the equation $\ln x = cx^2$ have exactly one solution?

Solution

One of the most important principles of problem solving is to draw a diagram to represent the given information, even if the problem doesn't explicitly involve geometry.

This problem can be restated as: for what values of c does the curve $y = \ln x$ intersect the curve $y = cx^2$ in exactly one point?

Consider the graph of $y = \ln x$ and the graph of $y = cx^2$ for various values of c.

For $c \neq 0$, $y = cx^2$ is a parabola that opens upward if $c > 0$ and downward if $c < 0$. Figure 3.67 shows graphs of the parabolas $y = cx^2$ for several positive values of c. Most of curves $y = cx^2$ do not intersect $y = \ln x$ at any point, and one curve intersects $y = \ln x$ at two points.

It seems reasonable that there must be a value c (the graphs suggest somewhere between 0.1 and 0.3) for which the curves intersect exactly once, as in Figure 3.68.

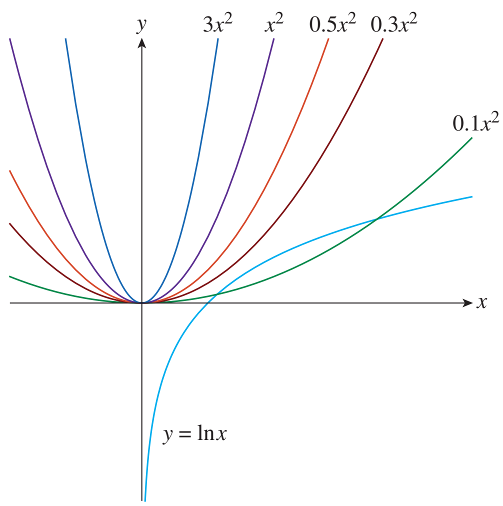


Figure 3.67
Graphs of $y = \ln x$ and $y = cx^2$ for various values of $c > 0$.

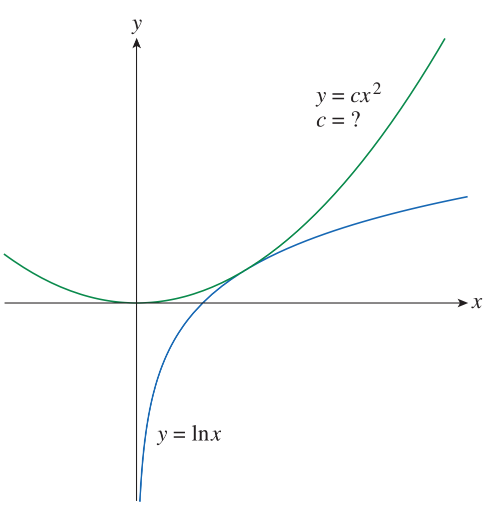


Figure 3.68
Graphs of $y = \ln x$ and $y = cx^2$.

To find that particular value of c, let a be the x-coordinate of the single point of intersection.

Therefore, a is the unique solution to the equation $\ln a = ca^2$.

Figure 3.68 suggests that the two curves just touch and, therefore, have a common tangent line at the point where $x = a$.

So the curves $y = \ln x$ and $y = cx^2$ have the same slope when $x = a$:

$$\frac{1}{a} = 2ca$$

Solve the two equations

$$\ln a = ca^2 \quad \text{and} \quad \frac{1}{a} = 2ca \quad \text{simultaneously}.$$

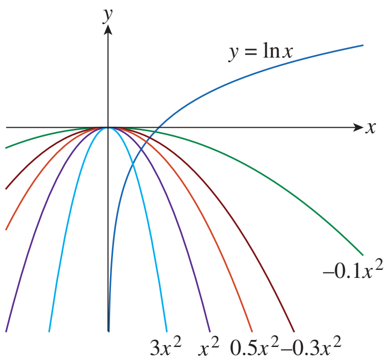


Figure 3.69
Graphs of $y = \ln x$ and $y = cx^2$ for various values of $c < 0$.

$$\frac{1}{a} = 2ca \quad \Rightarrow \quad a^2 = \frac{1}{2c}$$ Solve for a^2.

$$\ln a = ca^2 = c \cdot \frac{1}{2c} = \frac{1}{2}$$ Substitute for a^2.

$$a = e^{1/2}$$ Exponentiate both sides of the equation.

$$c = \frac{\ln a}{a^2} = \frac{\ln e^{1/2}}{e} = \frac{1}{2e}$$ Use the value of a to find c.

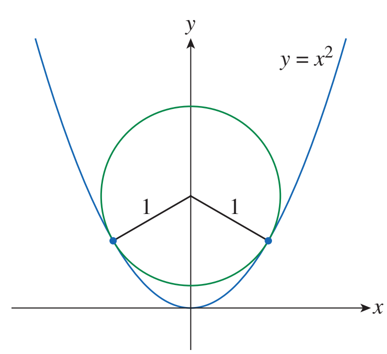
For negative values of c, all parabolas $y = cx^2$ intersect $y = \ln x$ exactly once. See Figure 3.69.

And of course the curve $y = 0x^2 = 0$ is just the x-axis, which intersects the curve $y = \ln x$ exactly once.

Finally, the values of c are $c = \dfrac{1}{2e}$ and any $c \leq 0$. ■

Problems

1. The figure shows a circle of radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.



2. Find the point where the curves $y = x^3 - 3x + 4$ and $y = 3(x^2 - x)$ are tangent to each other, that is, have a common tangent line. Illustrate your solution by sketching both curves and the common tangent.

3. Show that the tangent lines to the parabola $y = ax^2 + bx + c$ at any two points with x-coordinates p and q must intersect at a point whose y-coordinate is halfway between p and q.

4. Show that

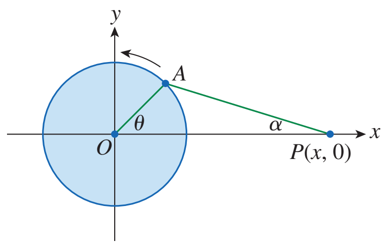
$$\frac{d}{dx}\left(\frac{\sin^2 x}{1+\cot x}+\frac{\cos^2 x}{1+\tan x}\right)=-\cos 2x$$

5. If $f(x)=\lim_{t\to x}\dfrac{\sec t-\sec x}{t-x}$, find the value of $f'(\pi/4)$.

6. If f is differentiable at a, where $a>0$, evaluate the following limit in terms of $f'(a)$:

$$\lim_{x\to a}\frac{f(x)-f(a)}{\sqrt{x}-\sqrt{a}}$$

7. The figure shows a rotating wheel with radius 40 cm and a connecting rod AP with length 1.2 m. The point P slides back and forth along the x-axis as the wheel rotates counterclockwise at a rate of 360 revolutions per minute.



(a) Find the angular velocity of the connecting rod, $\dfrac{d\alpha}{dt}$, in radians per second, when $\theta=\dfrac{\pi}{3}$.

(b) Express the distance $x=|OP|$ in terms of θ.

(c) Find an expression for the velocity of the point P in terms of θ.

8. Tangent lines T_1 and T_2 are drawn at two points P_1 and P_2 on the parabola $y=x^2$ and they intersect at a point P. Another tangent line T is drawn at a point between P_1 and P_2; it intersects T_1 at Q_1 and T_2 and Q_2. Show that

$$\frac{|PQ_1|}{|PP_1|}+\frac{|PQ_2|}{|PP_2|}=1$$

9. Show that

$$\frac{d^n}{dx^n}(e^{ax}\sin bx)=r^n e^{ax}\sin(bx+n\theta)$$

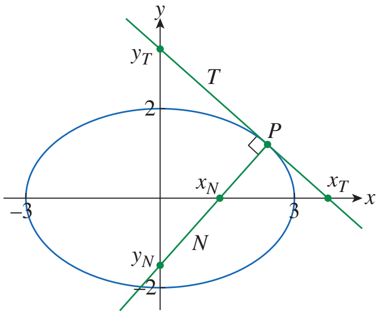
where a and b are positive constants, $r^2=a^2+b^2$, and $\theta=\tan^{-1}(b/a)$.

10. Find all the values of the constants a and b such that

$$\lim_{x\to 0}\frac{\sqrt[3]{ax+b}-2}{x}=\frac{5}{12}$$

11. Let T and N be the tangent and normal lines to the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ at any point P on the ellipse in the first quadrant. Let x_T and y_T be the x- and y-intercepts of T and x_N and y_N be the intercepts of N. As P moves along the ellipse in the first quadrant (but not on the axes), what values can x_T, y_T, x_N, and y_N take on?

Try to guess the answers by using the figure, and then use calculus to solve the problem.



12. If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$$

13. If

$$y = \frac{x}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \cdot \arctan\left[\frac{\sin x}{a + \sqrt{a^2 - 1} + \cos x}\right]$$

show that $y' = \dfrac{1}{a + \cos x}$.

14. For which positive numbers a is it true that $a^x \geq 1 + x$ for all x?

15. For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?

16. (a) The cubic function $f(x) = x(x - 2)(x - 6)$ has three distinct zeros: 0, 2, and 6. Graph f and the tangent lines to the graph of f at the arithmetic mean of each pair of zeros. What do you notice about these tangent lines?

(b) Suppose the cubic function $f(x) = (x - a)(x - b)(x - c)$ has three distinct zeros: a, b, and c. Prove that a tangent line drawn at the arithmetic mean of the zeros a and b intersects the graph of f at the third zero.

17. (a) Use the trigonometric identity for $\tan(x - y)$ to show that if two lines L_1 and L_2 intersect at an angle α, then

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$$

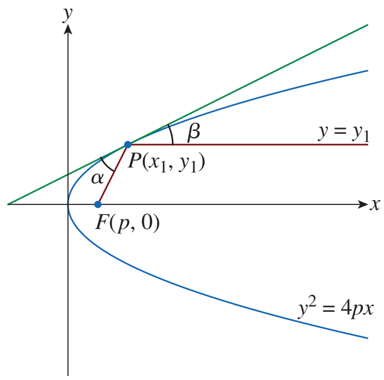
where m_1 and m_2 are the slopes of L_1 and L_2, respectively.

(b) The **angle between the curves** C_1 and C_2 at a point of intersection P is defined to be the angle between the tangent lines to C_1 and C_2 at P (if these tangent lines exist). Use part (a) to find, correct to the nearest degree, the angle between each pair of curves at each point of intersection.

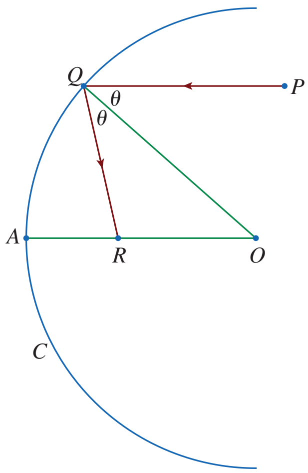
(i) $y = x^2$ and $y = (x - 2)^2$

(ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$

18. Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 4px$ with focus $F(p, 0)$. Let α be the angle between the parabola and the line segment FP, and let β be the angle between the horizontal line $y = y_1$ and the parabola as in the figure. Prove that $\alpha = \beta$. (Thus, by a principle of geometrical optics, light from a source placed at F will be reflected along a line parallel to the x-axis. This explains why *paraboloids*, the surfaces obtained by rotating parabolas about their axes, are used as the shape of some automobile headlights and mirrors for telescopes.)



19. Suppose instead of a parabolic mirror, as in Problem 18, we use a spherical mirror. Although the mirror has no focus, we can show the existence of an *approximate* focus. In the figure, C is a semicircle with center O. A ray of light coming in toward the mirror parallel to the axis along the line PQ will be reflected to the point R on the axis so that $\angle PQO = \angle OQR$ (the angle of incidence is equal to the angle of reflection). What happens to the point R as P moves closer and closer to the axis?



20. Given an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a \neq b$, find the equation of the set of all points from which there are two tangents to the curve whose slopes are (a) reciprocals and (b) negative reciprocals.

21. Find two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.

22. Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point. Show that the sum of the x-coordinates is 0.

23. A *lattice point* in the plane is a point with integer coordinates. Suppose that circles with radius r are drawn using all lattice points as centers. Find the smallest value of r such that any line with slope $\frac{2}{5}$ intersects some of these circles.

https://www.nasa.gov/centers/langley/news/factsheets/FS-2003-11-81-LaRC.html

The tube-and-wing design of conventional aircraft has been used for decades. In fact, this design can be easily modified, to be bigger or smaller, which helps to control development and construction costs. In addition, these conventional airplanes can be evacuated quickly, are easy to maintain because the engines and other components are accessible, and airports are built to accommodate this type of aircraft. However, new aircraft, including the Blended Wing Body, are designed to maximize lift and increase flight efficiency. Calculus can be used to optimize functions that model airplane lift, fuel cost, or even the number of passengers. Blended Wing Body aircraft aren't compatible with today's commercial airports, but this innovative design could be used for some applications in a few years.

4 Applications of Differentiation

Contents

We have already investigated some of the applications of derivatives, but now that we know the differentiation rules, we can consider applications of differentiation in greater depth. In this chapter, we will learn how derivatives can be used to describe the shape of the graph of a function and, in particular, how they help us locate maximum and minimum values of functions. Many real-world problems require us to find a maximum or a minimum value. For example, we may need to minimize cost or maximize an area. Derivatives will help us solve a variety of optimization problems.

4.1 Related Rates

Recall that when a quantity y depends upon time t, for example, $y = f(t)$, then $\frac{dy}{dt} = f'(t)$ represents the instantaneous rate of change of y with respect to t (time). If two or more quantities that depend on t are related by an equation, then an expression relating their rates of change with respect to time (called **related rates**) can be obtained by using the Chain Rule to differentiate both sides of the equation with respect to t. The general idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity.

Example 1 Inflating a Balloon

Air is being pumped into a spherical balloon so that its volume increases at a rate of $100\ \text{cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solution

According to the Principles of Problem Solving discussed on page 85, the first step is to understand the problem. This includes reading the problem carefully, identifying the given and the unknown, and introducing suitable notation.

Start by identifying two things:

The *given information*: the rate of increase of the volume of air is $100\ \text{cm}^3/\text{s}$.

The *unknown:* the rate of increase of the radius when the diameter is 50 cm.

To express and relate these quantities mathematically, introduce some reasonable *notation*.

Let V be the volume of the balloon and let r be its radius.

It is important to remember that rates of change are derivatives.

In this problem, the volume and the radius are both functions of time t.

The rate of increase of the volume with respect to time is the derivative $\frac{dV}{dt}$.

The rate of increase of the radius is $\frac{dr}{dt}$.

Using this notation, we can now translate the given information and unknown quantity into mathematics.

Given: $\frac{dV}{dt} = 100\ \text{cm}^3/\text{s}$

Unknown: $\frac{dr}{dt}$ when $r = 25\ \text{cm}$

The second stage of problem solving is to think of a plan for connecting the given and the unknown.

In order to connect $\frac{dV}{dt}$ and $\frac{dr}{dt}$, consider the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Differentiate each side of this expression with respect to t.

Use the Chain Rule where necessary.

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Notice that, although $\frac{dV}{dt}$ is constant, $\frac{dr}{dt}$ is *not* constant.

Solve for the unknown quantity.

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

Use $r = 25$ and $\frac{dV}{dt} = 100$ in this equation.

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}(100) = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $\frac{1}{25\pi} = 0.0127 \text{ cm/s}$. ■

Example 2 The Sliding Ladder Problem

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

djmilic/Shutterstock.com

No discussion of related rates would be complete without solving this classic problem.

Solution

Consider the diagram in Figure 4.1. Let x ft be the distance from the bottom of the ladder to the wall and y ft be the distance from the top of the ladder to the ground.

Remember, x and y are both functions of t (time, measured in seconds).

We are given $\frac{dx}{dt} = 1 \text{ ft/s}$.

And we are asked to find $\frac{dy}{dt}$ when $x = 6$. See Figure 4.2.

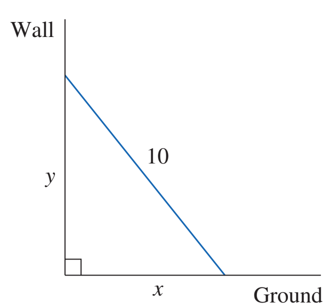


Figure 4.1
The ladder in its initial position.

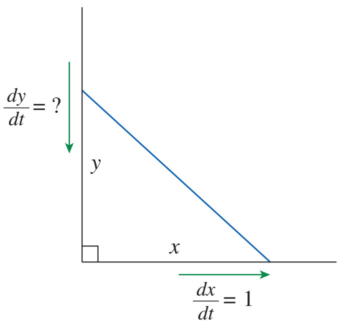


Figure 4.2
The bottom of the ladder begins to slide away from the wall.

An advantageous relationship between x and y is given by the Pythagorean Theorem: $x^2 + y^2 = 100$.

Differentiate each side of this equation with respect to t and use the Chain Rule.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Solve this equation for the unknown rate.

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dy}{dt}$$

When $x = 6 \Rightarrow (6)^2 + y^2 = 100 \Rightarrow y = 8$ Pythagorean Theorem.

$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4}\ \text{ft/s}$$

Since $\frac{dy}{dt} < 0$, the distance from the top of the ladder to the ground is *decreasing* at a rate of $\frac{3}{4}$ ft/s. Therefore, the top of the ladder is sliding down the wall at a rate of $\frac{3}{4}$ ft/s. ■

Example 3 Filling a Tank

A water tank has the shape of an inverted circular cone with base radius of 2 m and height of 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Solution

Consider the diagram in Figure 4.3. Let V, r, and h be the volume of the water, the radius of the surface, and the height of the water at time t, respectively, where t is measured in minutes.

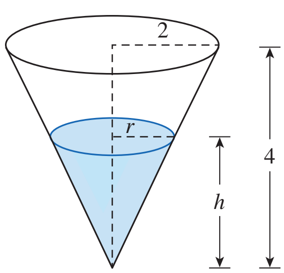


Figure 4.3
A water tank in the shape of an inverted circular cone.

We are given $\frac{dV}{dt} = 2\ \text{m}^3/\text{min}$.

We are asked to find $\frac{dh}{dt}$ when $h = 3$ m.

The quantities V and h are related by the equation $V = \frac{1}{3}\pi r^2 h$.

We need to rewrite V as a function of one variable, either r or h. Because we are asked to find $\frac{dh}{dt}$ when $h = 3$, express V as a function of h alone.

We can eliminate r by using similar triangles, illustrated in Figure 4.3.

$$\frac{r}{h} = \frac{2}{4} \Rightarrow r = \frac{h}{2}$$

The expression for V can now be written in terms of h.

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Differentiate each side with respect to t.

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$$

Use $h = 3$ m and $\frac{dV}{dt} = 2\ \text{m}^3/\text{min}$.

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2} \cdot 2 = \frac{8}{9\pi}$$

The water level is rising at a rate of $\frac{8}{9\pi} = 0.283\ \text{m}/\text{min}$. ■

A Closer Look

A common error in related rates problems is to substitute the given numerical information (for quantities that vary with time) too early. This substitution should be done only after the differentiation step. For example, in Example 3, we worked with general values of h until we finally substituted $h = 3$ at the last step. Note that if we had used $h = 3$ earlier (incorrectly), we would have found $\frac{dV}{dt} = 0$, which is certainly wrong. The volume of water is not constant, but increasing (in this example). ■

It is useful to adapt some of the problem-solving principles from page 85 to related rates given what we have learned in Examples 1–3.

Strategy for Solving Related Rates Problems

1. Read the problem carefully.
2. Sketch a figure if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities in the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

The following examples are further illustrations of this strategy.

Example 4 Four-Leg Intersection

Car A is traveling due west at 50 mi/h, and car B is traveling due north at 60 mi/h. Both are headed for the intersection of the two roads. Find the rate at which the distance between the cars is changing when car A is 0.3 mi and car B is 0.4 mi from the intersection.

Solution

Consider the diagram in Figure 4.4, where the point C is the intersection of the roads. At a given time t, let x be the distance from car A to C, let y be the distance from car B to C, and let z be the distance between the cars, where x, y, and z are measured in miles.

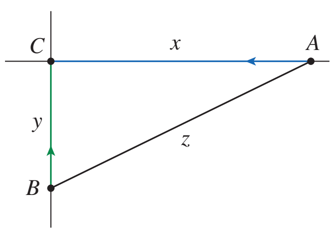


Figure 4.4
Two cars headed for the same intersection.

We are given $\dfrac{dx}{dt} = -50$ mi/h and $\dfrac{dy}{dt} = -60$ mi/h.

Note that the derivatives are negative because the distances x and y are decreasing.

We are asked to find $\dfrac{dz}{dt}$.

An equation that relates all three quantities is given by the Pythagorean Theorem.

$$z^2 = x^2 + y^2$$

Differentiate both sides of this equation with respect to t.

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \;\Rightarrow\; \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

For $x = 0.3$ mi and $y = 0.4$ mi, use the Pythagorean Theorem to find z.

$$z^2 = (0.3)^2 + (0.4)^2 \Rightarrow z = 0.5$$ Pythagorean Theorem.

Finally, $\dfrac{dz}{dt} = \dfrac{1}{0.5}[0.3(-50) + 0.4(-60)] = -78$ mi/h.

The cars are approaching each other at a rate of 78 mi/h. ■

Example 5 In the Spotlight

A person walks along a straight path at a speed of 4 ft/s. A spotlight is located on the ground 20 ft from the path and rotates to keep its focus on the person. At what rate is the spotlight rotating when the person is 15 ft from the point on the path closest to the spotlight?

Solution

Consider the diagram in Figure 4.5. Let x be the distance from the person to the point on the path closest to the spotlight, and let θ be the angle between the beam of the spotlight and the perpendicular to the path.

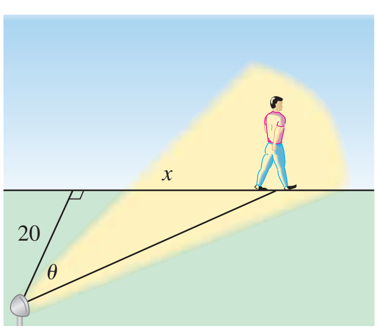


Figure 4.5
The spotlight is focused on the person's path.

We are given $\dfrac{dx}{dt} = 4$ ft/s.

We are asked to find $\dfrac{d\theta}{dt}$ when $x = 15$.

Use the definition of the tangent function to write an equation that relates x and θ.

$$\frac{x}{20} = \tan\theta \;\Rightarrow\; x = 20\tan\theta$$

Differentiate both sides of this equation with respect to t.

$$\frac{dx}{dt} = 20\sec^2\theta\,\frac{d\theta}{dt}$$

Solve for $\dfrac{d\theta}{dt}$.

$$\frac{d\theta}{dt} = \frac{1}{20}\cos^2\theta\,\frac{dx}{dt} = \frac{1}{20}\cos^2\theta(4) = \frac{1}{5}\cos^2\theta$$

When $x = 15$, the length of the beam is 25 (by the Pythagorean Theorem).

Therefore, $\cos\,\theta = \dfrac{20}{25} = \dfrac{4}{5}$ and $\dfrac{d\theta}{dt} = \dfrac{1}{5}\left(\dfrac{4}{5}\right)^2 = \dfrac{16}{125} = 0.128.$

When the person is 15 ft from the point on the path closest to the spotlight, the spotlight is rotating at a rate of 0.128 rad/s. ■

4.1 Exercises

1. If V is the volume of a cube with edge length x and the cube expands as time passes, find $\dfrac{dV}{dt}$ in terms of $\dfrac{dx}{dt}$.

2. Let A be the area of a circle with radius r and suppose the circle expands as time passes.

(a) Find $\dfrac{dA}{dt}$ in terms of $\dfrac{dr}{dt}$.

(b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?

3. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2?

4. The length of a rectangle is increasing at a rate of 8 cm/s, and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

5. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing?

6. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

7. The radius of a spherical ball is increasing at a rate of 2 cm/min. At what rate is the surface area of the ball increasing when the radius is 8 cm?

8. The area of a regular pentagon with sides of length s is

$$A = \frac{1}{4}\sqrt{5(5 + 2\sqrt{5})}\ s^2$$

(a) If s is increasing at a rate of 0.5 cm/min, how fast is the area of the pentagon increasing when $s = 2$?

(b) Suppose that the area is increasing at a rate of $\sqrt{5}$ cm^2/min when $s = 3$. How fast is a side increasing at that instant?

9. The area of a triangle with sides of lengths a and b and contained angle θ is $A = \dfrac{1}{2}ab\ \sin\ \theta$.

(a) If $a = 2$ cm, $b = 3$ cm, and θ increases at a rate of 0.2 rad/min, how fast is the area of the triangle increasing when $\theta = \dfrac{\pi}{3}$?

(b) If $a = 2$ cm, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area of the triangle increasing when $b = 3$ cm and $\theta = \dfrac{\pi}{3}$?

(c) If a increases at a rate of 2.5 cm/min, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area of the triangle increasing when $a = 2$ cm, $b = 3$ cm, and $\theta = \dfrac{\pi}{3}$?

10. Suppose $y = \sqrt{2x + 1}$, where x and y are functions of t.

(a) If $\dfrac{dx}{dt} = 3$, find $\dfrac{dy}{dt}$ when $x = 4$.

(b) If $\dfrac{dx}{dt} = 5$, find $\dfrac{dy}{dt}$ when $x = 12$.

11. Suppose $4x^2 + 9y^2 = 36$, where x and y are functions of t.

(a) If $\dfrac{dy}{dt} = \dfrac{1}{3}$, find $\dfrac{dx}{dt}$ when $x = 2$ and $y = \dfrac{2}{3}\sqrt{5}$.

(b) If $\dfrac{dx}{dt} = 3$, find $\dfrac{dy}{dt}$ when $x = -2$ and $y = \dfrac{2}{3}\sqrt{5}$.

12. Suppose $xy = 20$, where x and y are functions of t. Find the value of $\dfrac{dx}{dt}$ when $x = 4$ and $\dfrac{dy}{dt} = -2$.

13. Suppose $x^2 + y^2 = 45$ and $x = 2y$ for positive values of x and y. Find $\dfrac{dy}{dt}$ when $\dfrac{dx}{dt} = 2$.

14. If $x^2 + y^2 + z^2 = 9$, $\dfrac{dx}{dt} = 5$, and $\dfrac{dy}{dt} = 4$, find $\dfrac{dz}{dt}$ when $(x,\ y,\ z) = (2,\ 2,\ 1)$.

15. A particle is moving along a hyperbola given by the equation $xy = 8$. As it reaches the point (4, 2), the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?

16. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.

17. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

18. A street light is mounted at the top of a 15-ft-tall pole. A person 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of their shadow moving when they are 40 ft from the pole?

19. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4 PM?

20. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing 2 hours later?

21. A spotlight on the ground shines on a wall 12 m away. If a person 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of their shadow on the building decreasing when they are 4 m from the building?

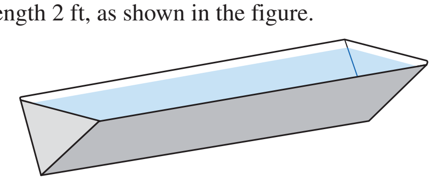
22. A person starts walking north at 4 ft/s from a point *P*. Five minutes later, another person starts walking south at 5 ft/s from a point 500 ft due east of *P*. At what rate are the people moving apart 15 minutes after the second person starts walking?

23. A backyard above-ground circular swimming pool has diameter of 14 ft and height of 4 ft. The pool is being filled with a garden hose at the rate of 9 gal/min.

(a) How fast is the height of the water in the pool increasing when the height is 2 ft and when the height is 3 ft?

(b) Explain why $\frac{dh}{dt}$ is the same for any height?

24. A water trough is constructed in the shape of a triangular prism with length 6 ft and the ends are equilateral triangles of sidelength 2 ft, as shown in the figure.



The volume of a triangular prism of length *l* and side *s* is

$$V = \frac{\sqrt{3}}{4}s^2 l$$

(a) How many cubic feet of water does the trough hold when full?

(b) What is the depth of the water in feet when the trough is half full?

(c) Water is leaking from the bottom of the trough at the rate of 0.5 ft^3/min. At what rate is the depth of the water changing, in feet per hour, when the trough is half full?

25. A balloon in the shape of a sphere is being inflated with air so that it maintains its spherical shape.

(a) When the volume of air in the balloon is 288π cubic inches, what is the radius in inches?

(b) At the instant when the volume is 288π cubic inches, the radius is increasing at the rate of 0.15 in/s. At what rate, in cubic inches per second, is the volume increasing?

(c) Suppose that when the volume is 288π cubic inches, the balloon is deflating at the rate of 3.6 in^3/s. At what rate is the radius decreasing, in inches per second, at this instant?

26. Two trains are traveling on perpendicular tracks. Train A is traveling east at the rate of 40 mi/h and train B is traveling north at the rate of 30 mi/h. At noon, train A is at the intersection and train B is 10 mi north of the intersection.

(a) How far apart are the two trains at 12:30 PM?

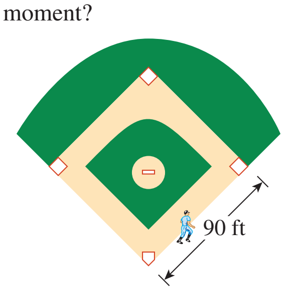
(b) At what rate, in miles per hour, are the two trains separating at 12:30 PM?

(c) At what time of day are the two trains at an equal distance from the intersection? At what rate, in miles per hour, is the distance between the two trains changing at this time?

27. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.

(a) At what rate is their distance from second base decreasing when they are halfway to first base?

(b) At what rate is their distance from third base increasing at the same moment?



28. The altitude of a triangle is increasing at a rate of 1 cm/min, while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

29. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

30. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4 PM?

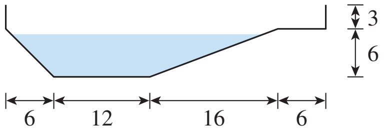
31. A particle moves along the curve $y = 2\sin\left(\dfrac{\pi x}{2}\right)$. As the particle passes through the point $\left(\dfrac{1}{3}, 1\right)$, its x-coordinate is increasing at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the particle to the origin changing at this instant?

32. Water is leaking out of an inverted conical tank at a rate of $10{,}000\ \text{cm}^3/\text{min}$ at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m, and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

33. A trough is 10 ft long, and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is filled with water at a rate of $12\ \text{ft}^3/\text{min}$, how fast is the water level rising when the water is 6-in deep?

34. A water trough is 10 m long, and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of $0.2\ \text{m}^3/\text{min}$, how fast is the water level rising when the water is 30 cm deep?

35. A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of $0.8\ \text{ft}^3/\text{min}$, how fast is the water level rising when the depth at the deepest point is 5 ft?



36. Gravel is being dumped from a conveyor belt at a rate of $30\ \text{ft}^3/\text{min}$, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

37. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?

38. The sides of an equilateral triangle are increasing at a rate of 10 cm/min. At what rate is the area of the triangle increasing when the sides are 30 cm long?

39. How fast is the angle between the ladder and the ground changing in Example 2 when the bottom of the ladder is 6 ft from the wall?

40. The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides away from the wall at a rate of 0.2 m/s. How long is the ladder?

41. According to the model used to solve Example 2, what happens as the top of the ladder approaches the ground? Is the model appropriate for small values of y? Explain your reasoning.

42. If the minute hand of a clock has length r (in centimeters), find the rate at which it sweeps out area as a function of r.

43. A faucet is filling a hemispherical basin of diameter 60 cm with water at a rate of 2 L/min. Find the rate at which the water is rising in the basin when it is half full. Use the following facts: 1 L is $1000\ \text{cm}^3$. The volume of the portion of a sphere with radius r from the bottom to a height h is $V = \pi\left(rh^2 - \dfrac{1}{3}h^3\right)$.

44. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is $600\ \text{cm}^3$, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

45. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is $400\ \text{cm}^3$ and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this instant?

46. If two resistors with resistances R_1 and R_2 are connected in parallel, then the total resistance R, measured in ohms (Ω), is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 and R_2 are increasing at rates of $0.3\ \Omega/\text{s}$ and $0.2\ \Omega/\text{s}$, respectively, how fast is R changing when $R_1 = 80\ \Omega$ and $R_2 = 100\ \Omega$?

47. Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast was this species' brain growing when its average length was 18 cm?

48. Two sides of a triangle have lengths 12 m and 15 m. The angle between them is increasing at a rate of $2°/\text{min}$. How fast is the length of the third side increasing when the angle between the sides of fixed length is $60°$?

49. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P. The point Q is on the floor, 12 ft directly beneath P, and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?

50. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.
(a) How fast is the distance from the television camera to the rocket changing at that moment?
(b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

51. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

52. A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\frac{\pi}{3}$, this angle is decreasing at a rate of $\frac{\pi}{6}$ rad/min. How fast is the plane traveling at that time?

53. A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when their seat is 16 m above ground level?

54. A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing a minute later?

55. Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?

56. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?

57. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

4.2 Maximum and Minimum Values

Some of the most important applications of differential calculus are contextual *optimization problems*, in which we need to find the optimal (best) way of doing something. Here are some examples of optimization problems.

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who must withstand the effects of acceleration.)
- How does a company allocate power along a grid to maximize profits?
- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?
- What is the speed of a tractor trailer that maximizes fuel efficiency?
- What is the best delivery route to minimize time, or cost?

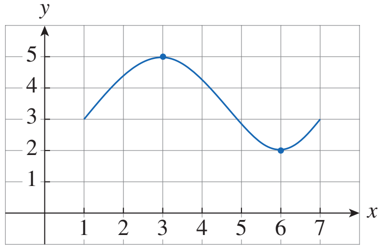


Figure 4.6
Graph of f. The domain is $1 \leq x \leq 7$.

These problems involve finding the maximum or minimum values of a function. To begin this study, we need to consider the precise definition of maximum and minimum values.

Consider the graph of the function f in Figure 4.6.

Using this figure, the highest point on the graph of f is the point (3, 5). That is, the largest value of f (over its entire domain) is $f(3) = 5$. Similarly, the smallest value of f is $f(6) = 2$. These observations can be expressed using mathematical terminology as $f(3) = 5$ is the *absolute maximum* (value) of f and $f(6) = 2$ is the *absolute minimum* (value) of f. Here are the formal mathematical definitions.

> **Definition • Absolute Maximum and Absolute Minimum**
>
> Let c be a number in the domain D of a function f.
>
> The function f has an **absolute maximum** at c if $f(c) \geq f(x)$ for all x in D, and the number $f(c)$ is called the **absolute maximum (value)** of f on D.
>
> The function f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in D, and the number $f(c)$ is called the **absolute minimum (value)** of f on D.

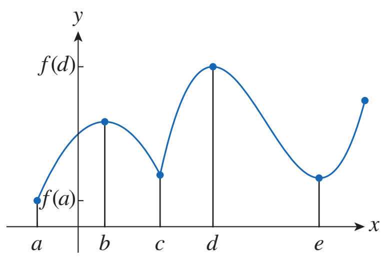


Figure 4.7
Graph of f: absolute minimum, $f(a)$; absolute maximum, $f(d)$; local minimums, $f(c)$, $f(e)$; local maximums, $f(b)$, $f(d)$.

An absolute maximum or minimum value is also often called a **global** maximum or minimum value. The maximum and minimum values of f are also called **extreme values** of f.

Figure 4.7 shows the graph of a function f with absolute maximum at d and absolute minimum at a. Note that $(d, f(d))$ is the highest point on the graph of f and $(a, f(a))$ is the lowest point.

In Figure 4.7, notice that $f(b)$ is the largest value of f in a *neighborhood* of b, that is, for values of x close to b. The value $f(b)$ is called a *local maximum value* of f. Similarly, $f(c)$ is called a *local minimum value* of f because $f(c) \leq f(x)$ for x near c (in a neighborhood of c). The graph also suggests that the function f has a local minimum at e. Here are the formal mathematical definitions for these concepts.

> **Definition • Local Maximum and Local Minimum**
>
> The function f has a **local (or relative) maximum** at c if there is an open interval I containing c such that $f(c) \geq f(x)$ for all x in I, that is, $f(c) \geq f(x)$ when x is near c. The number $f(c)$ is called a **local (or relative) maximum (value)**.
>
> The function f has a **local (or relative) minimum** at c if there is an open interval I containing c such that $f(c) \leq f(x)$ for all x in I, that is, $f(c) \leq f(x)$ when x is near c. The number $f(c)$ is called a **local (or relative) minimum (value)**.

A Closer Look

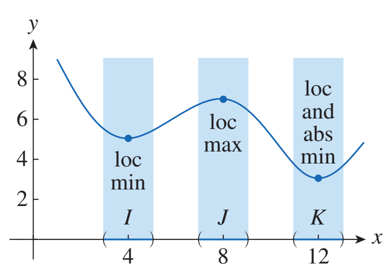


Figure 4.8
This graph shows that $f(4) = 5$ is a local minimum value because it is the smallest value of f in an interval containing 4. Similarly, $f(8) = 7$ is a local maximum value because it is the largest value of f in an interval containing 8.

1. The concept of a neighborhood of c, or near c, means an open interval containing c. For example, in Figure 4.8, $f(4) = 5$ is a local minimum because it is the smallest value of f on the interval I. It is not the absolute minimum because $f(x)$ is smaller for values of x near 12. The value $f(12) = 3$ is a local minimum because it is the smallest value of f on the interval K. Similarly, $f(8) = 7$ is a local maximum because it is the largest value of f on the interval J. It is not the absolute maximum because f takes on larger values near 1.
2. It is possible for a value $f(c)$ to be both an absolute and a local extreme value. For example, in Figure 4.8, the value $f(12) = 3$ is both an absolute and a local minimum value.
3. A function f cannot have a local maximum or local minimum at an endpoint of its domain. If c is an endpoint of the domain of f, then we cannot form an open interval I containing c such that all values in I, on both sides of c, are in the domain. However, it is possible for a function to have an absolute maximum or minimum value at an endpoint of its domain. ■

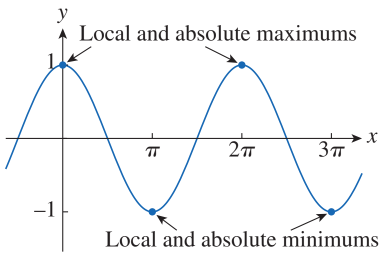


Figure 4.9
The function $f(x) = \cos x$ takes on its local and absolute maximum and minimum values infinitely many times.

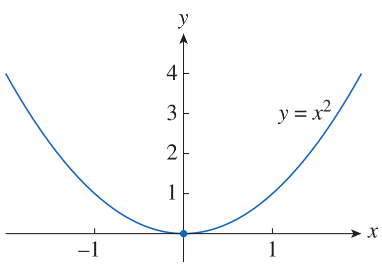


Figure 4.10
The function $f(x) = x^2$ has a minimum value, 0, but no maximum value.

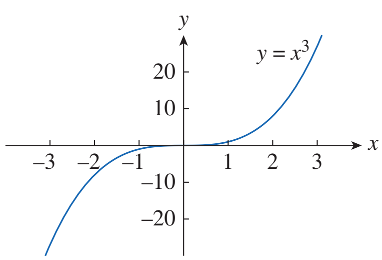


Figure 4.11
The function $f(x) = x^3$ has no minimum value and no maximum value.

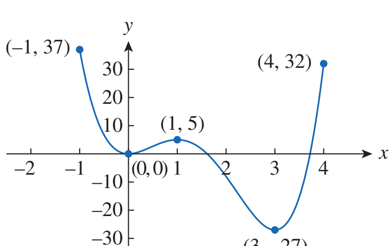


Figure 4.12
Graph of $f(x) = 3x^4 - 16x^3 + 18x^2$ for $-1 \leq x \leq 4$.

Since optimization problems involve finding extreme values, it is important to know when these values exist. Examples 1–4 will help us discover and develop an existence theorem for extreme values.

Example 1 Infinitely Many Extreme Values

The function $f(x) = \cos x$ takes on its local and absolute maximum value of 1 infinitely many times.

$\cos 2n\pi = 1 \quad \text{for any integer } n \quad \text{and} \quad -1 \leq \cos x \leq 1 \quad \text{for all } x$

Similarly, $\cos(2n + 1)\pi = -1$ is its minimum value, where n is any integer.
So, f takes on its minimum value infinitely many times.

Figure 4.9 illustrates this conclusion. ■

Example 2 One Minimum, No Maximum

If $f(x) = x^2$, then $f(x) \geq f(0) = 0$ because $x^2 \geq 0$.

Therefore, $f(0) = 0$ is the absolute (and a local) minimum value of f.

Graphically, the point (0, 0) is the lowest point on the graph of f, a parabola opening upward. See Figure 4.10.

There is no highest point on the parabola. This function has no maximum value on its entire domain. ■

Example 3 No Extreme Values

The graph of $f(x) = x^3$ is shown in Figure 4.11.

This function has neither an absolute maximum value nor an absolute minimum value.

There are no local extreme values either. ■

Example 4 Maximum at an Endpoint

Let $f(x) = 3x^4 - 16x^3 + 18x^2$, $-1 \leq x \leq 4$.

The graph of f is shown in Figure 4.12.

The graph suggests the following.

$f(1) = 5$ is a local maximum.

$f(-1) = 37$ is the absolute maximum.

It is not a local maximum because this value occurs at an endpoint of the domain.

$f(0) = 0$ is a local minimum.

$f(3) = -27$ is both a local and an absolute minimum.

f has neither an absolute minimum nor absolute maximum at $x = 4$. ■

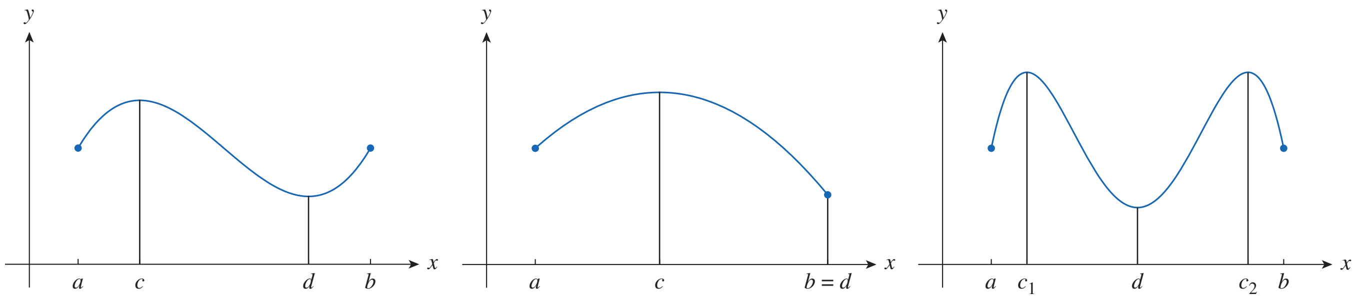
The preceding examples suggest that some functions have extreme values, while others do not. The Extreme Value Theorem gives specific conditions under which a function is guaranteed to have extreme values.

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

A Closer Look

1. Figure 4.13 illustrates some applications of the Extreme Value Theorem (EVT). Note that an extreme value can occur at more than one place in the closed interval.



(a) The absolute maximum occurs at c and the absolute minimum at d.

(b) The absolute minimum occurs at an endpoint b.

(c) The absolute maximum occurs at two values in the closed interval, $x = c_1$ and $x = c_2$.

Figure 4.13
A continuous function on a closed interval must attain extreme values.

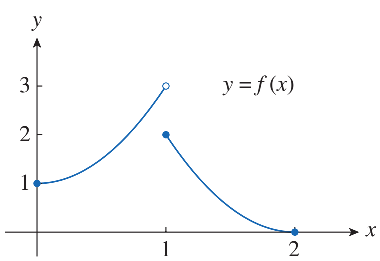


Figure 4.14
The function f has an absolute minimum value, $f(2) = 0$, but no maximum value.

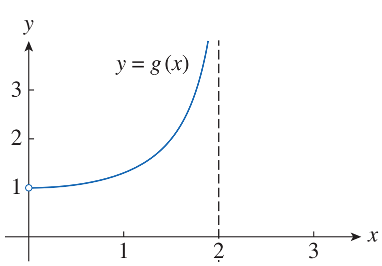


Figure 4.15
The continuous function g has no maximum or minimum value.

2. Both conditions, continuity and a closed interval, are necessary in the EVT. If either one is not satisfied, then absolute extrema are not guaranteed.

   The function f whose graph is shown in Figure 4.14 is defined on a closed interval, $[0, 2]$, but has no maximum value. The range of f is $[0, 3)$. The function takes on values arbitrarily close to 3, but never actually attains the value 3. This example does not contradict the EVT because f is not continuous on the closed interval. However, a discontinuous function *could* have maximum and minimum values.

   The graph of a function g is shown in Figure 4.15. This function is continuous on the open interval $(0, 2)$ but has neither a maximum nor a minimum value. The range of g is $(1, \infty)$. The function takes on arbitrarily large values. This example does not contradict the EVT either; in this case, the interval $(0, 2)$ is not closed.

3. The Extreme Value Theorem is an *existence* theorem. If f is continuous on the *closed* interval $[a, b]$, then there exists a c, $a \le c \le b$, such that $f(c)$ is an absolute maximum value, and there exists a d, $a \le d \le b$, such that $f(d)$ is an absolute minimum value. The problem: we don't know where to look for c and d yet; we only know that they exist and are in the interval $[a, b]$.

   Figure 4.13 suggests that absolute maximum and minimum values that occur *between* a and b occur at local maximum and minimum values. Therefore, it seems reasonable to start by searching for *local* extreme values. ■

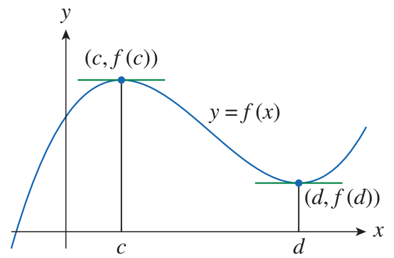


Figure 4.16
For a differentiable function *f*, the tangent lines are horizontal at the local maximum and minimum points.

Figure 4.16 shows the graph of a differentiable function *f* with a local maximum at c and a local minimum at d.

This graph suggests that at the maximum and minimum points, the tangent lines are horizontal, and therefore, each has slope 0. We know that the derivative is the slope of the tangent line, so it appears that $f'(c) = 0$ and $f'(d) = 0$. Fermat's Theorem says that this is always true for differentiable functions.

FERMAT'S THEOREM: Fermat's Theorem is named after Pierre Fermat (1601–1665), a French lawyer who took up mathematics as a hobby. Despite his amateur status, Fermat was one of the two inventors of analytic geometry (Descartes was the other). His methods for finding tangents to curves and maximum and minimum values (before the invention of limits and derivatives) made him a forerunner of Newton in the creation of differential calculus.

Fermat's Theorem

If *f* has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Our intuition suggests that Fermat's Theorem is true. A formal proof, using the definition of a derivative, is given in Appendix E.

Although Fermat's Theorem is very useful, we need to be careful in our interpretation. If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$. But *f* has no maximum or minimum at 0, as illustrated in Figure 4.17. The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at (0, 0). Instead of having a maximum or minimum at (0, 0), the curve crosses its horizontal tangent there.

Therefore, if $f'(c) = 0$, this does not necessarily mean that *f* has a maximum or minimum at c. In other words, the converse of Fermat's Theorem is false in general.

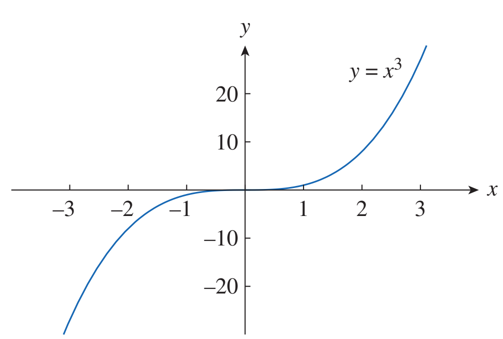


Figure 4.17
If $f(x) = x^3$, then $f'(0) = 0$ but *f* has no maximum or minimum.

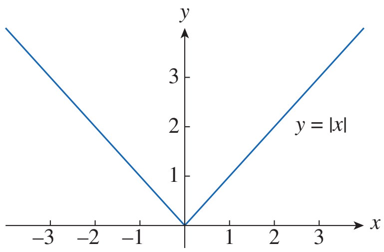


Figure 4.18
If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

We should also consider that there may be an extreme value where $f'(c)$ does not exist. For example, the function $f(x) = |x|$ has its (local and absolute) minimum at 0 (see Figure 4.18), but that value ($x = 0$) cannot be found by setting $f'(x) = 0$ because, as was shown in Example 5 of Section 2.7, $f'(0)$ does not exist.

Fermat's Theorem suggests that we should at least start looking for extreme values of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. These numbers have a special name.

Definition • Critical Number

A **critical number** of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

A critical number is also often called a **critical point** or critical value. Even though the word *number* makes it clear that this is an x-value and not a point in the plane, the term *critical point* is used without ambiguity.

Example 5 Critical Numbers

Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

Solution

Find the derivative.

$$f'(x) = x^{3/5}(-1) + (4 - x)\left(\frac{3}{5}x^{-2/5}\right) \qquad \text{Product Rule.}$$

$$= -x^{3/5} + \frac{3(4 - x)}{5x^{2/5}} \qquad \text{Simplify.}$$

$$= \frac{-5x + 3(4 - x)}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}} \qquad \text{Write as one fraction.}$$

$f'(x) = 0$:

$12 - 8x = 0$ Set the numerator equal to 0.

$-4(2x - 3) = 0$ Factor completely.

$x = \dfrac{3}{2}$ Principle of Zero Products.

$f'(x)$ DNE:

$5x^{2/5} = 0$ Set the denominator equal to 0.

$x = 0$ Solve for x.

The critical numbers are 0 and $\dfrac{3}{2}$.

The graph of f is shown in Figure 4.19. The tangent line to the graph of f is horizontal at $x = \dfrac{3}{2}$ and vertical at $x = 0$, where $f'(x)$ does not exist.

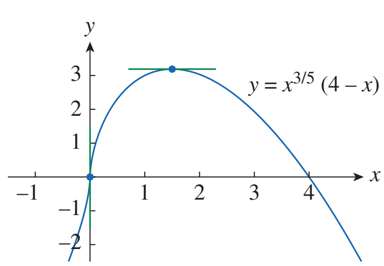


Figure 4.19
The graph of f has a horizontal tangent line at $x = \dfrac{3}{2}$ and a vertical tangent line at $x = 0$.

Graphically, we can see that there is a local (and absolute) maximum value at $x = \frac{3}{2}$, but there is no local extreme value at $x = 0$. ■

The examples we have discussed and the definition of critical numbers suggest a modification of Fermat's Theorem that will help us find maximum and minimum values of a function.

> If f has a local maximum or minimum at c, then c is a critical number of f.

All the preceding examples and theorems suggest that an absolute maximum or minimum value of a continuous function on a closed interval can occur at only one of two places:

(1) At a local extreme value and therefore at a critical number, or

(2) At an endpoint of the interval.

Thus, the following three-step procedure for finding absolute maximum and minimum values always works.

The Closed Interval, or Table of Values, Method, or Candidates Test

To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

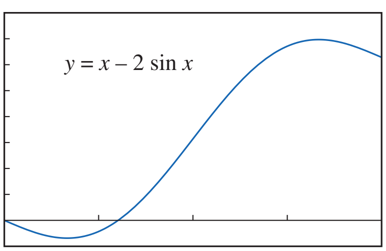


Figure 4.20
$[0, 2\pi] \times [-1, 8]$
Graph of the function
$f(x) = x - 2\sin x, 0 \le x \le 2\pi$.

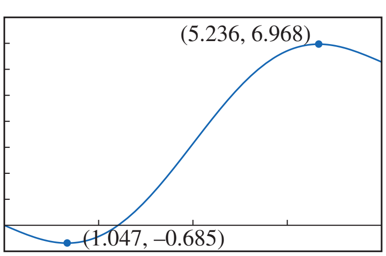


Figure 4.21
$[0, 2\pi] \times [-1, 8]$
Estimates of the absolute minimum and absolute maximum values.

Example 6 Extreme Values on a Closed Interval

Find the absolute maximum and minimum values of the function $f(x) = x - 2\sin x$, $0 \le x \le 2\pi$.

Solution

The function is continuous on a closed interval.

Therefore, we can use the Closed Interval Method.

We will use technology to explore the solution before solving the problem analytically.

Figure 4.20 shows the graph of f, and Figure 4.21 shows estimates of the absolute minimum and maximum values. The absolute maximum value is approximately 6.968, and it occurs at $x \approx 5.236$. The absolute minimum is approximately -0.685, and it occurs at $x \approx 1.047$. We could obtain more accurate estimates by zooming in, but we can find the exact values using calculus.

Find the derivative and all the critical numbers.

$f'(x) = 1 - 2\cos x$

$f'(x) = 0$:

$\cos x = \frac{1}{2}$ Solve for $\cos x$.

$x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ Common angles, $0 < x < 2\pi$.

$f'(x)$ DNE: None

Construct a table of values.

x	$f(x)$	
0	0	
$\dfrac{\pi}{3}$	$\dfrac{\pi}{3} - \sqrt{3}$	← absolute minimum value
$\dfrac{5\pi}{3}$	$\dfrac{5\pi}{3} + \sqrt{3}$	← absolute maximum value
2π	2π	

The absolute minimum value is $\dfrac{\pi}{3} - \sqrt{3} \approx -0.685$, and the absolute maximum value is $\dfrac{5\pi}{3} + \sqrt{3} \approx 6.968$. This confirms our exploratory graphical analysis. ■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Example 7 Extreme Acceleration

The final space Shuttle mission was flown by Atlantis on July 8, 2011. The four-person crew delivered the Multi-Purpose Logistics Module and other spare parts to the International Space Station. A model for the acceleration of the shuttle during this mission, from liftoff at time $t = 0$ until $t = 120$ seconds, the approximate time the rocket boosters were jettisoned, is given by

$$a(t) = 2.01438 + 1.61951t - 0.0544442t^2 + 0.000708145t^3 - 0.00000303802t^4$$

(in m/s^2). Using this model, estimate the absolute maximum and the minimum values of the *acceleration* of the shuttle between liftoff and $t = 120$ seconds.

Solution

The acceleration function is continuous on the closed interval [0, 120].
Use the Closed Interval Method.

Find the derivative:

$a'(t) = 1.61951 - 0.1088884t + 0.002124435 - 0.00001215208t^3$

Find the critical numbers.

$a'(t) = 0$:

Using technology, the solutions to this equation are $t \approx 26.487, 52.507, 95.827$.

$f'(x)$ DNE: None

Construct a table of values.

t	$a(t)$	
0.000	2.014	← absolute minimum value
26.487	18.378	
52.507	16.368	
95.827	24.218	← absolute maximum value
120.000	6.070	

The absolute maximum acceleration is $24.218\ \text{m/s}^2$, and the minimum acceleration is $2.014\ \text{m/s}^2$. ■

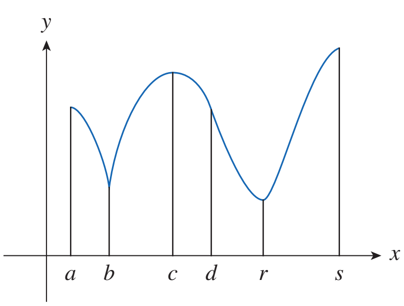
4.2 Exercises

1. Explain the difference between an absolute minimum and a local minimum.

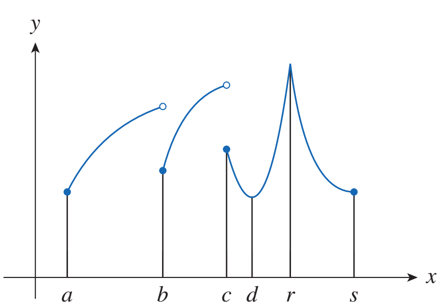
2. Suppose f is a continuous function defined on a closed interval $[a, b]$.
(a) What theorem guarantees the existence of an absolute maximum value and an absolute minimum value?
(b) Explain in your own words the procedure to find the maximum and minimum values.

For each of the numbers a, b, c, d, r, and s, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

3.

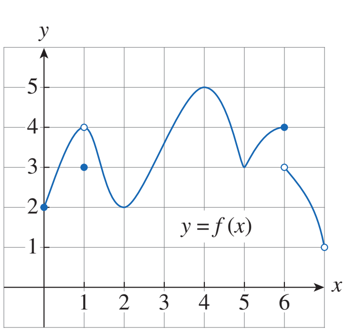


4.

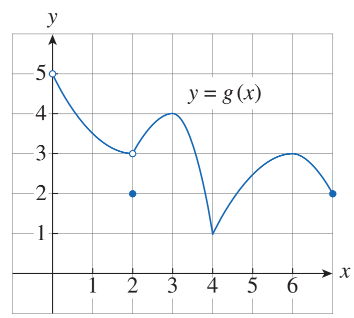


Use the graph to find the absolute and local maximum and minimum values of the function.

5.



6.



Sketch the graph of a function f that is continuous on $[1, 5]$ and has the given properties.

7. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4

8. Absolute maximum at 4, absolute minimum at 5, local maximum at 2, local minimum at 3

9. Absolute minimum at 3, absolute maximum at 4, local maximum at 2

10. Absolute maximum at 2, absolute minimum at 5, 4 is a critical number, but there is no local maximum or minimum there

11. (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.
(b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.
(c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.

12. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no local maximum.
(b) Sketch the graph of a function on $[-1, 2]$, that has a local maximum but no absolute maximum.

13. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no absolute minimum.
(b) Sketch the graph of a function on $[-1, 2]$ that is discontinuous but has both an absolute maximum and an absolute minimum.

14. (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.
(b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

Sketch the graph of f and use your sketch to find the absolute and local maximum and minimum values of f.

15. $f(x) = \frac{1}{2}(3x - 1), \quad x \leq 3$

16. $f(x) = 2 - \frac{1}{3}x, \quad x \geq -2$

17. $f(x) = \frac{1}{x}, \quad x \geq 1$

18. $f(x) = x^2, \quad 0 < x < 2$

19. $f(x) = e^x$

20. $f(x) = \sin x, \quad 0 \leq x < \frac{\pi}{2}$

21. $f(t) = \cos t, \quad -\frac{3\pi}{2} \leq t \leq \frac{3\pi}{2}$

22. $f(x) = \ln x, \quad 0 < x \leq 2$

23. $f(x) = |x|$

24. $f(x) = 1 - \sqrt{x}$

25. $f(x) = \begin{cases} x^2 & \text{if } -1 \leq x \leq 0 \\ 2 - 3x & \text{if } 0 < x \leq 1 \end{cases}$

26. $f(x) = \begin{cases} 2x + 1 & \text{if } 0 \leq x < 1 \\ 4 - 2x & \text{if } 1 \leq x \leq 3 \end{cases}$

Find the critical numbers of the function.

27. $f(x) = 4 + \frac{1}{3}x - \frac{1}{2}x^2$

28. $f(x) = x^3 + 6x^2 - 15x$

29. $f(x) = 2x^3 - 3x^2 - 36x$

30. $f(x) = 2x^3 + x^2 + 2x$

31. $g(t) = t^4 + t^3 + t^2 + 1$

32. $g(t) = |3t - 4|$

33. $g(y) = \dfrac{y - 1}{y^2 - y + 1}$

34. $h(p) = \dfrac{p - 1}{p^2 + 4}$

35. $h(t) = t^{3/4} - 2t^{1/4}$

36. $g(x) = \sqrt[3]{4 - x^2}$

37. $F(x) = x^{4/5}(x - 4)^2$

38. $g(\theta) = 4\theta - \tan \theta$

39. $f(\theta) = 2\cos \theta + \sin^2\theta$

40. $h(t) = 3t - \arcsin t$

41. $f(x) = x^2e^{-3x}$

42. $f(x) = x^{-2} \ln x$

The derivative of a function f is given. How many critical numbers does f have?

43. $f'(x) = 5e^{-0.1|x|} \sin x - 1$

44. $f'(x) = \dfrac{100 \cos^2 x}{10 + x^2} - 1$

Find the absolute maximum and absolute minimum values of f on the given interval.

45. $f(x) = 12 + 4x - x^2, \quad [0, 5]$

46. $f(x) = 5 + 54x - 2x^3, \quad [0, 4]$

47. $f(x) = 2x^3 - 3x^2 - 12x + 1, \quad [-2, 3]$

48. $f(x) = x^3 - 6x^2 + 5, \quad [-3, 5]$

49. $f(x) = 3x^4 - 4x^3 - 12x^2 + 1, \quad [-2, 3]$

50. $f(t) = (t^2 - 4)^3, \quad [-2, 3]$

51. $f(x) = x + \frac{1}{x}, \quad [0.2, 4]$

52. $f(x) = \dfrac{x}{x^2 - x + 1}, \quad [0, 3]$

53. $f(t) = t - \sqrt[3]{t}, \quad [-1, 4]$

54. $f(t) = \dfrac{\sqrt{t}}{1 + t^2}, \quad [0, 2]$

55. $f(t) = 2 \cos t + \sin 2t, \quad \left[0, \frac{\pi}{2}\right]$

56. $f(t) = t + \cos\left(\frac{t}{2}\right), \quad \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$

57. $f(x) = x^{-2} \ln x, \quad \left[\frac{1}{2}, 4\right]$

58. $f(x) = xe^{x/2}, \quad [-3, 1]$

59. $f(x) = xe^{-x/8}, \quad [-1, 4]$

60. $f(x) = \ln(x^2 + x + 1), \quad [-1, 1]$

61. $f(x) = x - 2\tan^{-1} x, \quad [0, 4]$

62. $f(t) = t + \cot\left(\frac{t}{2}\right), \quad \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$

63. Find the maximum value of $f(x) = \dfrac{x^2}{e^x}$ on the interval $[-1, 3]$.

64. Consider the function $g(x) = 3 - 2x - x^2$ on the interval $(-4, 2)$. Determine whether each statement is true or false.
I. g has an absolute maximum value on the interval.
II. g has a relative maximum on the interval.
III. g has no absolute minimum on the interval.

65. A particle moves along a horizontal line so that its position at time t, $t > 0$, is given by $s(t) = t^3 - 2t^2 - 4t + 8$, where t is measured in seconds and s in feet. Find the absolute minimum value of the velocity.

66. Determine whether the function has both a relative maximum and a relative minimum on the given interval.

(a) $y = \dfrac{1}{x}, \quad [-1, 1]$

(b) $y = \dfrac{\sin x}{x}, \quad [-5, 5]$

(c) $y = \dfrac{e^x}{x}, \quad [-2, 2]$

(d) $y = \dfrac{x - 1}{x^2 + 1}, \quad [-4, 4]$

67. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$, $0 \le x \le 1$.

68. Use technology to estimate the critical numbers of $f(x) = |1 + 5x - x^3|$.

Use technology to estimate the absolute maximum and minimum values and then use calculus to find the exact maximum and minimum values.

69. $f(x) = x^5 - x^3 + 2, \quad -1 \le x \le 1$

70. $f(x) = e^x + e^{-2x}, \quad 0 \le x \le 1$

71. $f(x) = x\sqrt{x - x^2}$

72. $f(x) = x - 2\cos x, \quad -2 \le x \le 0$

73. Let $f(x) = x^4 - 4x^3 - 2x^2 + 12x - 5$.
(a) Find all the critical numbers of f.
(b) Find the local minimum value(s) of f.
(c) Find the local maximum value(s) of f.
(d) Find the absolute minimum value of f.

74. The function $R(t) = e^t - 2t^2 + 1$ is used to model the rate, in thousands of cubic feet per minute, that water is released through an irrigation gate over the 3-minute interval $0 \le t \le 3$.
(a) Using correct units, how fast is water being released at time $t = 2$ minutes?
(b) Find the coordinates of all local extrema on the graph of R and indicate whether each is a local maximum or local minimum.
(c) Find the absolute maximum and absolute minimum release rates over the given interval. Show the work that leads to your answer and use correct units.

75. After a person consumes an alcoholic beverage, the concentration of alcohol in the bloodstream (blood alcohol concentration, or BAC) surges as the alcohol is absorbed, then declines gradually as the alcohol is metabolized. The function

$$C(t) = 1.35t\, e^{-2.802t}$$

models the average BAC, measured in mg/mL, of a group of eight male subjects, t hours after rapid consumption of 15 mL of ethanol (corresponding to one alcoholic drink). What is the maximum average BAC during the first 3 hours? When does it occur?

76. After an antibiotic tablet is swallowed, the concentration of antibiotic in the bloodstream is modeled by the function

$$C(t) = 8(e^{-0.4t} - e^{-0.6t})$$

where the time t is measured in hours and C is measured in μg/mL. What is the maximum concentration of antibiotic during the first 12 hours?

77. Between 0°C and 30°C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given approximately by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

Find the temperature at which water has its maximum density.

78. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

where μ is a positive constant called the *coefficient of friction* and where $0 \le \theta \le \pi/2$. Show that F is minimized when $\tan\theta = \mu$.

79. A model for the US average price of a pound of white sugar from 1993 to 2003 is given by the function

$$S(t) = -0.00003237t^5 + 0.0009037t^4 - 0.008956t^3 + 0.03629t^2 - 0.04458t + 0.4074$$

where t is measured in years since August 1993. Estimate the times when sugar was least expensive and most expensive during the period 1993–2003.

80. On May 7, 1992, the space shuttle Endeavour was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

(a) Use technology to find a cubic model for the velocity of the shuttle for the time interval [0, 125]. Graph this polynomial.
(b) Find a model for the acceleration of the shuttle and use it to estimate the maximum and minimum values of the acceleration during the first 125 seconds.

81. When a foreign object lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward causing an increase in pressure in the lungs. This is accompanied by a contraction of the trachea, making a narrower channel for the expelled air to flow through. For a given amount of air to escape in a fixed time, it must move faster through the narrower channel than the wider one. The greater the velocity of the airstream, the greater the force on the foreign object. X-rays show that the radius of the circular tracheal tube contracts to about two-thirds of its normal radius during a cough. According to a mathematical model of coughing, the velocity v of the airstream is related to the radius r of the trachea by the equation

$$v(r) = k(r_0 - r)r^2, \qquad \frac{1}{2}r_0 \le r \le r_0$$

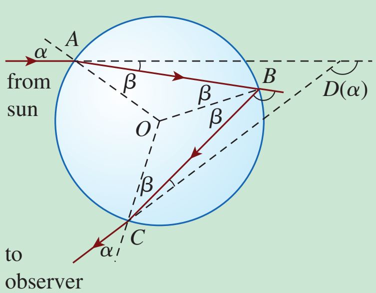
where k is a constant and r_0 is the normal radius of the trachea. The restriction on r is due to the fact that the tracheal wall stiffens under pressure and a contraction greater than $\frac{1}{2}r_0$ is prevented (otherwise the person would suffocate).

(a) Determine the value of r in the interval $\left[\frac{1}{2}r_0,\ r_0\right]$ at which v has an absolute maximum. How does this compare with experimental evidence?
(b) What is the absolute maximum value of v on the interval?
(c) Sketch the graph of v on the interval $[0,\ r_0]$.

82. A cubic function is a polynomial of degree 3, that is, it has the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \ne 0$.
(a) Show that a cubic function can have two, one, or no critical number(s). Give examples and graphs to illustrate three possibilities.
(b) How many local extreme values can a cubic function have? Justify your answer.

Applied Project | The Calculus of Rainbows

Rainbows are created when raindrops scatter sunlight. They have fascinated mankind since ancient times and have inspired attempts at scientific explanation since the time of Aristotle. In this project, we will use the ideas of Descartes and Newton to explain the shape, location, and colors of rainbows.



Formation of the primary rainbow

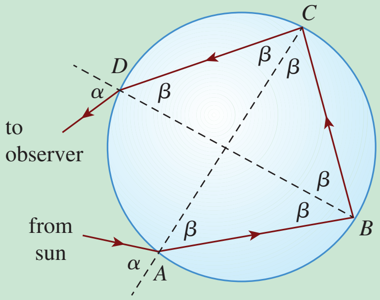
1. The figure shows a ray of sunlight entering a spherical raindrop at A. Some of the light is reflected, but the line AB shows the path of the (light) part that enters the drop. Notice that the light is refracted toward the normal line AO, and in fact, Snell's Law says that $\sin \alpha = k \sin \beta$, where α is the angle of incidence, β is the angle of refraction, and $k \approx \frac{4}{3}$ is the index of refraction for water.

At B some of the light passes through the drop and is refracted into the air, but the line BC shows the part that is reflected. (The angle of incidence equals the angle of reflection.) When the ray reaches C, part of it is reflected, but for the time being, we are more interested in the part that leaves the raindrop at C. (Notice that it is refracted away from the normal line.) The *angle of deviation* is the amount of clockwise rotation that the ray has undergone during this three-stage process. Thus

$$D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta) = \pi + 2\alpha - 4\beta$$

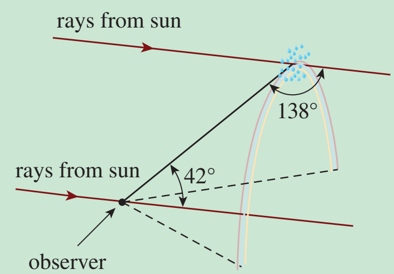
Show that the minimum value of the deviation is $D(\alpha) \approx 138°$ and occurs when $\alpha = 59.4°$.

The significance of the minimum deviation is that when $\alpha \approx 59.4°$, we have $D'(\alpha) \approx 0$, so $\dfrac{\Delta D}{\Delta \alpha} \approx 0$. This means that many rays with $\alpha \approx 59.4°$ become deviated by approximately the same amount.



Formation of the secondary rainbow

It is the *concentration* of rays coming from near the direction of minimum deviation that creates the brightness of the primary rainbow. The following figure shows that the angle of elevation from the observer up to the highest point on the rainbow is $180° - 138° = 42°$. This angle is called the *rainbow angle.*



2. Problem 1 explains the location of the primary rainbow, but how do we explain the colors? Sunlight comprises a range of wavelengths, from the red range through orange, yellow, green, blue, indigo, and violet. As Newton discovered in his prism experiments of 1666, the index of refraction is different for each color. (The effect is called *dispersion.*)

For red light, the refractive index is $k \approx 1.3318$, whereas for violet light, it is $k \approx 1.3435$. By repeating the calculation of Problem 1 for these values of k, show that the rainbow angle is about 42.5° for the red bow and 40.6° for the violet bow. So the rainbow really consists of seven individual bows corresponding to the seven colors.

dominic8/Shutterstock.com

3. Sometimes we can see a fainter secondary rainbow above the primary bow. That is caused by part of a ray that enters a raindrop and is refracted at A, reflected twice (at B and C), and refracted as it leaves the drop at D (see the figure to the left). This time the deviation angle $D(\alpha)$ is the total amount of counterclockwise rotation that the ray undergoes in this four-stage process.

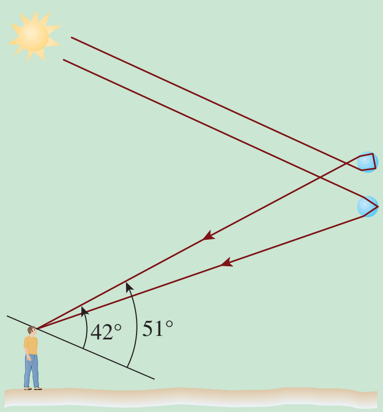
Show that

$$D(\alpha) = 2\alpha - 6\beta + 2\pi$$

and that $D(\alpha)$ has a minimum value when

$$\cos\,\alpha = \sqrt{\frac{k^2 - 1}{8}}$$

If we let $k = \frac{4}{3}$, show that the minimum deviation is about 129° and therefore that the rainbow angle for the secondary rainbow is about 51°, as shown in the following figure.



4. Show that the colors in the secondary rainbow appear in the opposite order from those in the primary rainbow. Did you ever notice this phenomenon?

4.3 Derivatives and the Shapes of Curves

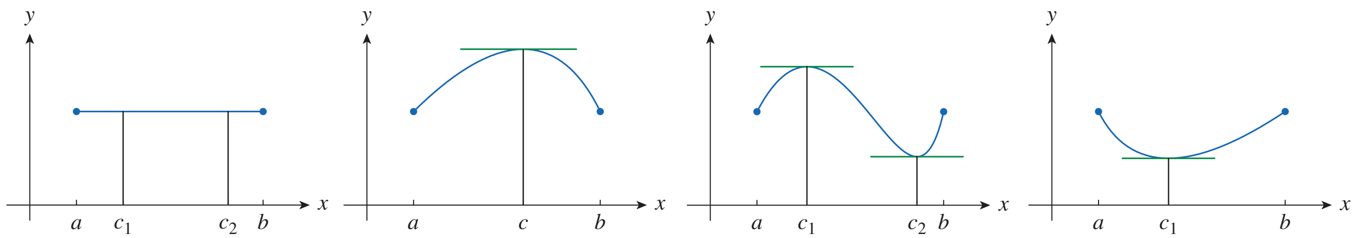
The Mean Value Theorem

There are several very important results in this chapter, and many of these results depend upon one key concept, the Mean Value Theorem. Before we discuss the Mean Value Theorem, consider the following preliminary result.

Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:
(1) f is continuous on the closed interval $[a, b]$.
(2) f is differentiable on the open interval (a, b).
(3) $f(a) = f(b)$
Then there is a number c in (a, b) such that $f'(c) = 0$.

Figure 4.22 shows graphs of typical functions that satisfy the hypotheses of Rolle's Theorem. In each case, it appears that there is at least one point $(c, f(c))$ on the graph where the tangent line is horizontal and therefore $f'(c) = 0$. These graphs suggest Rolle's Theorem is indeed reasonable.



(a) $f'(c_1) = f'(c_2) = 0$ **(b)** $f'(c) = 0$ **(c)** $f'(c_1) = f'(c_2) = 0$ **(d)** $f'(c) = 0$

Figure 4.22
Graphs of functions that satisfy the three hypotheses in Rolle's Theorem.

Lagrange and the Mean Value Theorem

Joseph-Louis Lagrange (1736–1813) was born in Italy of a French father and an Italian mother. He was a child prodigy and became a professor in Turin at the age of 19. Lagrange made great contributions to number theory, theory of functions, theory of equations, and analytical and celestial mechanics. In particular, he applied calculus to the analysis of the stability of the solar system. At the invitation of Frederick the Great, he succeeded Euler at the Berlin Academy and, when Frederick died, Lagrange accepted King Louis XVI's invitation to Paris, where he was given apartments in the Louvre and became a professor at the Ecole Polytechnique.

The main use of Rolle's Theorem is in proving the following important theorem, which was first stated by French mathematician, Joseph-Louis Lagrange.

The Mean Value Theorem

Let f be a function that satisfies the following hypotheses:
(1) f is continuous on the closed interval $[a, b]$.
(2) f is differentiable on the open interval (a, b).
Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

or equivalently,

$$f(b) - f(a) = f'(c)(b - a) \tag{2}$$

A Closer Look

1. The Mean Value Theorem (MVT) is an existence theorem. Similar to the Intermediate Value Theorem, the Extreme Value Theorem, and Rolle's Theorem, it guarantees that there exists a number with a certain property, but it doesn't tell us how to find the number.

2. Rolle's Theorem is a special case of the MVT. If $f(a) = f(b)$ (the additional hypothesis in Rolle's Theorem), then

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0}{b - a} = 0$$

3. We can interpret the MVT geometrically: Figures 4.23 and 4.24 show the points $A(a, f(a))$ and $B(b, f(b))$ on the graphs of two differentiable functions. The slope of the secant line is

$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

which is the same expression as on the right side of Equation 1 in the MVT.
The value $f'(c)$ is the slope of the tangent line to the graph of f at the point $(c, f(c))$.

The MVT says that there is at least one point $P(c, f(c))$ on the graph of f where the slope of the tangent line is the same as the slope of the secant line AB.

Another way to think of this result: there is a point P between A and B on the graph of f where the tangent line is parallel to the secant line AB.

Common Error

The value of c that satisfies the conclusion of the MVT is always the midpoint of the interval $[a, b]$.

Correct Method

A value c that satisfies the conclusion of the MVT *could* be at the midpoint of the interval, but is not necessarily at the midpoint.

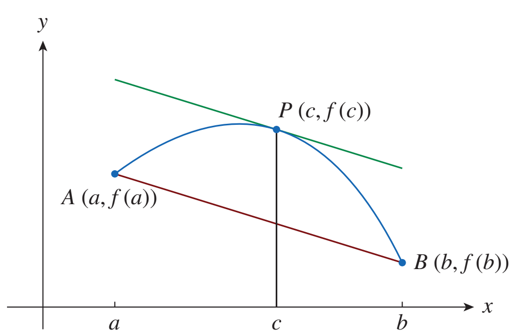


Figure 4.23
The slope of the tangent line to the graph of f at P is the same as the slope of the secant line AB.

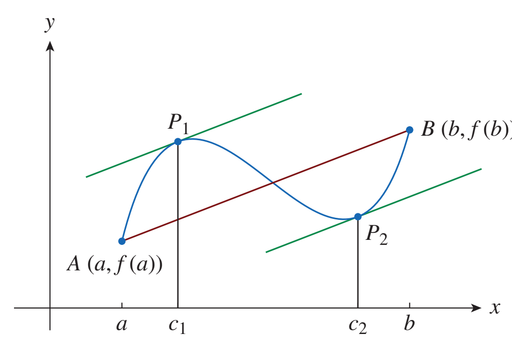


Figure 4.24
The tangent lines to the graph of f at P_1 and P_2 are parallel to the secant line AB.

Because our intuition tells us that the Mean Value Theorem is true, we use this as the starting point for the development of the main facts of calculus. When calculus is developed from first principles, however, the Mean Value Theorem is proved as a consequence of the axioms that define the real number system.

Example 1 Particle Motion

Suppose a particle (in general, an object) moves along a line so that its position at time t is given by $s(t)$.

The average velocity of the particle between $t = a$ and $t = b$ is $\dfrac{s(b) - s(a)}{b - a}$.

The velocity of the particle at $t = c$ is $s'(c)$.

The Mean Value Theorem tells us that at some time $t = c$ between a and b, the instantaneous velocity $s'(c)$ is equal to the average velocity.

For example, if a car has traveled 120 mi in 2 hours, then the speedometer must have read 60 mi/h at least once during those 2 hours.

In general, the Mean Value Theorem can be interpreted to mean that there is a number at which the instantaneous rate of change is equal to the average rate of change over an interval.

The main significance of the Mean Value Theorem is that it enables us to obtain information about a function from its derivative. We'll use this principle to prove the basic facts about increasing and decreasing functions.

Increasing and Decreasing Functions

In Section 1.1, we defined increasing functions and decreasing functions. In several examples and exercises, we observed from graphs that a function is increasing where its derivative is positive and decreasing where its derivative is negative. This fact follows from the Mean Value Theorem.

Let's abbreviate the name of this test to the I/D Test.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval I, then f is increasing on that interval.
(b) If $f'(x) < 0$ on an interval I, then f is decreasing on that interval.

Proof

Part (a): Let x_1 and x_2 be any two numbers in the interval such that $x_1 < x_2$.

Using the definition of an increasing function, we need to show that $f(x_1) < f(x_2)$.

Since $f'(x) > 0$, we know that f is differentiable on $[x_1, x_2]$.

By the Mean Value Theorem, there is a number c between x_1 and x_2 such that

$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$.

$f'(c) > 0$ by assumption and $x_2 - x_1 > 0$ since $x_1 < x_2$.

Therefore, the product $f'(c)(x_2 - x_1)$ is positive.

So, $f(x_2) - f(x_1) > 0 \quad \Rightarrow \quad f(x_1) < f(x_2)$.

This shows that f is increasing.

Part (b) is proved similarly. ■

A Closer Look

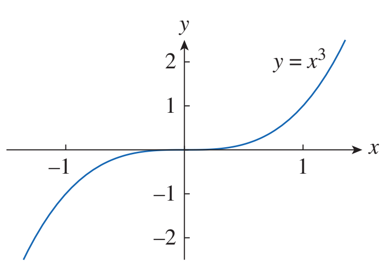


Figure 4.25
Graph of $f(x) = x^3$. The function is increasing on the interval $(-1, 1)$.

1. The converse of the Increasing/Decreasing (I/D) Test is not true. That is, if f is increasing on (a, b), then we cannot conclude that $f'(x) > 0$ for all x in (a, b). Consider the function $f(x) = x^3$ on the interval $(-1, 1)$.

   $f'(x) = 3x^2$ and $f'(x) > 0$ for all x in $(-1, 1)$ except for $x = 0$; $f'(0) = 0$.

   Let x_1 and x_2 be any two values in the interval $(-1, 1)$.

   Using Figure 4.25, $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

   Therefore, f is increasing on $(-1, 1)$, but $f'(0) = 0$.

2. If $f'(x) > 0$ on (a, b), then f is increasing on (a, b), and there are no local extrema in (a, b). Local extrema occur only at critical numbers, where $f'(c) = 0$ or where $f'(c)$ does not exist.

   Similarly, if $f'(x) < 0$ on (a, b), then f is decreasing on (a, b), and there are no local extrema in (a, b).

3. The I/D Test allows us to draw a conclusion about the behavior of f if $f'(x) > 0$ or $f'(x) < 0$. It doesn't say anything about critical numbers c; where $f'(c) = 0$ or $f'(c)$ does not exist.

Remember, the function f must be defined at a and b in order to include these values in an interval on which f is increasing or decreasing.

If $f'(x) > 0$ for all x in (a, b), f is continuous on $[a, b]$, and $f'(a) = 0$ or $f'(a)$ DNE, then we include a in the interval on which f is increasing. Similarly, if $f'(b) = 0$ or $f'(b)$ DNE, we also include b in the interval.

Example 2 Use the I/D Test

For the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$, find the intervals on which it is increasing or decreasing.

Solution

Begin by differentiating f.

$$f'(x) = 12x^3 - 12x^2 - 24x \qquad \text{Differentiate term by term.}$$

$$= 12x(x - 2)(x + 1) \qquad \text{Factor completely.}$$

Find the critical numbers.

$f'(x) = 0$: $x = 0, 2, -1$ — Principle of Zero Products.

$f'(x)$ DNE: None — There are no values where $f'(x)$ DNE.

The critical numbers of f divide the domain into four intervals.

Within each interval, $f'(x)$ must be always positive or always negative.

We can determine the sign of $f'(x)$ for each interval from the signs of the three factors of the derivative $f'(x)$.

Pick a representative number in each interval and use the factored form of $f'(x)$ to determine its sign.

Or, argue in general for numbers in each interval.

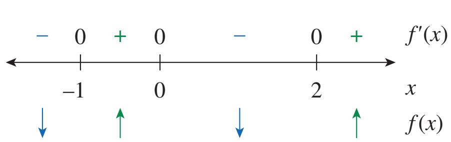
For example, in the interval $(-\infty, -1)$: $12x < 0$, $(x - 2) < 0$, and $(x + 1) < 0$. Therefore, the product $f'(x) = 12x(x - 1)(x + 1)$ is negative.

We can construct a chart to determine and visualize the sign of $f'(x)$ for every interval.

Interval	$12x$	$(x-2)$	$(x+1)$	$f'(x)$
$(-\infty, -1)$	$-$	$-$	$-$	$-$
$(-1, 0)$	$-$	$-$	$+$	$+$
$(0, 2)$	$+$	$-$	$+$	$-$
$(2, \infty)$	$+$	$+$	$+$	$+$

We can summarize this information and the characteristics of f by using a sign chart. Use the sign chart to write the intervals on which f is increasing and on which it is decreasing.

The horizontal axis does not have to be drawn to scale. An additional row is added here with arrows indicating whether f is increasing (↑) or decreasing (↓) on each interval.



f increasing: $[-1, 0]$, $[2, \infty)$

f decreasing: $(-\infty, -1]$, $[0, 2]$

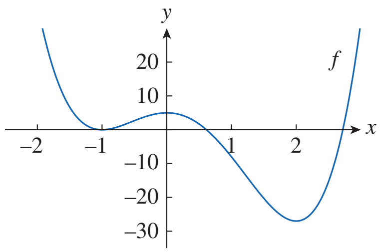


Figure 4.26
Graph of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. The intervals on which the function is increasing and on which it is decreasing match our analytical results.

Figure 4.26 shows a graph of f and confirms our analysis. ■

Local Extreme Values

Recall that if f has a local maximum or minimum at c, then c must be a critical number of f (Fermat's Theorem). But, not every critical number indicates a maximum or a minimum. Therefore, we need a method to determine whether f has a local maximum, or a local minimum, or neither, at a critical number.

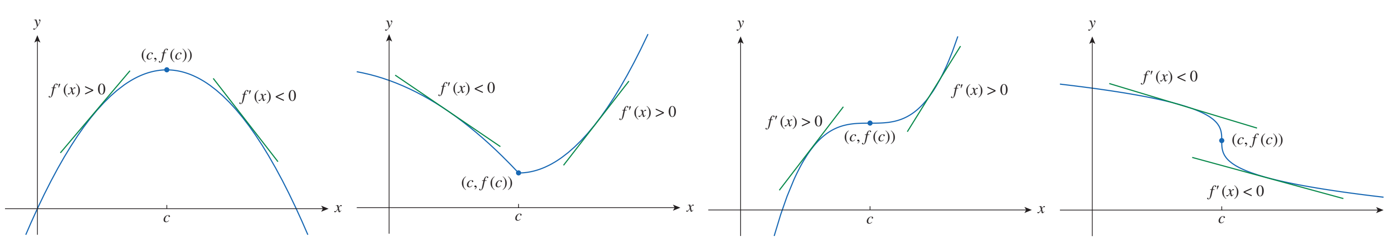
From Figure 4.26, we can see that $f(0) = 5$ is a local maximum value of f because f is increasing on $[-1, 0]$ and decreasing on $[0, 2]$. In terms of the derivative, $f'(x) > 0$ on the interval $(-1, 0)$ (to the left of 0) and $f'(x) < 0$ on the interval $(0, 2)$ (to the right of 0). The sign of f' changes from positive to negative. This observation is the basis of the First Derivative Test.

The First Derivative Test

Suppose that c is a critical number of a continuous function f.
(a) If f' changes from positive to negative at c, then f has a local maximum at c.
(b) If f' changes from negative to positive at c, then f has a local minimum at c.
(c) If f' does not change sign at c (for example, if f' is positive on both sides of c, or negative on both side), then f has no local maximum or minimum at c.

The First Derivative Test follows from the I/D Test. For example, in part (a) since the sign of $f'(x)$ changes from positive to negative at c, f is increasing to the left of c and decreasing to the right of c. It follows that f has a local maximum at c.

Figure 4.27 helps to illustrate the First Derivative Test. Suppose f is a continuous function and c is a critical point.



(a) $f'(x)$ changes from positive to negative at c. Therefore, f has a local maximum at c.

(b) $f'(x)$ changes from negative to positive at c. Therefore, f has a local minimum at c.

(c) $f'(x)$ does not change sign at c. Therefore, f has neither a local maximum nor a local minimum at c.

(d) $f'(x)$ does not change sign at c. Therefore, f has neither a local maximum nor a local minimum at c.

Figure 4.27
Graphs of functions that illustrate the First Derivative Test.

Example 3 Local Extreme Values

Let $f(x) = x^3 + 3x^2 + 2$. Find the interval(s) on which f is increasing or decreasing, and find the local extreme value(s) of f.

Solution

Note that f is continuous on $(-\infty, \infty)$.

Remember that a critical number is also often called a critical point or critical value.

$f'(x) = 3x^2 + 6x = 3x(x + 2)$ Find $f'(x)$; factor completely.

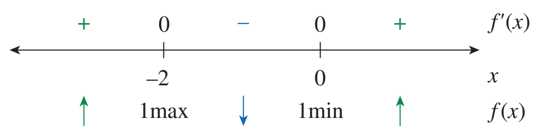
Find the critical numbers.

$f'(x) = 0$: $3x(x+2) = 0 \quad \Rightarrow \quad x = 0,\ -2$ Principle of Zero Products.

$f'(x)$ DNE: None

Construct a sign chart.

We use the abbreviations lmax to mean **l**ocal **max**imum and lmin to mean **l**ocal **min**imum in sign charts.



$f'(x) > 0$ on $(-\infty, -2)$ and $(0, \infty)$.
$f'(x) < 0$ on $(-2, 0)$.

Common Error

If c is a critical point, then f must have a local maximum or a local minimum at c.

Correct Method

If $f'(c) = 0$ or $f'(c)$ DNE, then c is only a *candidate* for a local extreme value. We still need to check whether f' changes from positive to negative at c or from negative to positive at c.

f is increasing on $(-\infty, -2]$ and $[0, \infty)$.
f is decreasing on $[-2, 0]$.

$f'(x)$ changes from positive to negative at -2. Therefore, f has a local maximum at -2, indicated on the sign chart by lmax. The local maximum value is $f(-2) = 6$.

$f'(x)$ changes from negative to positive at 0. Therefore, f has a local minimum at 0, indicated on the sign chart by lmin. The local minimum value is $f(0) = 2$.

Figure 4.28 shows a graph of f and confirms our analysis. ■

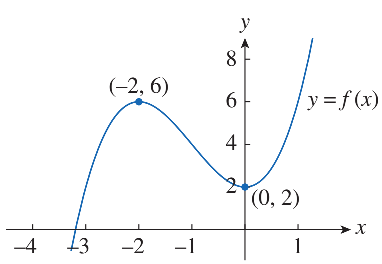
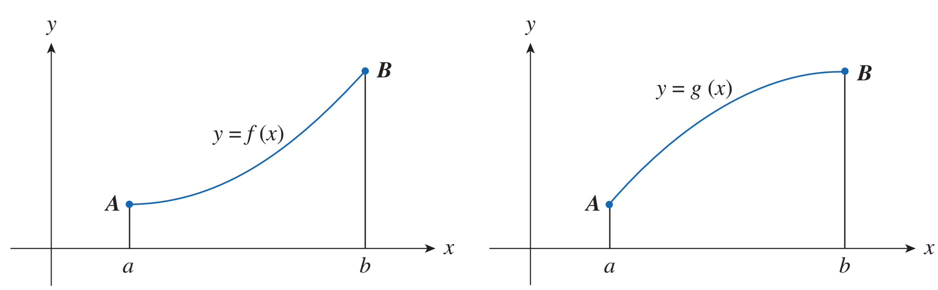


Figure 4.28
$f(-2) = 6$ is a local maximum value, and $f(0) = 2$ is a local minimum value.

Concavity

Figure 4.29 shows the graphs of two increasing functions on the interval $[a, b]$. Even though f and g are both increasing on the same interval, their graphs look different because they bend in different directions; one bends upward, the other downward. Calculus can help us distinguish between these different characteristics.

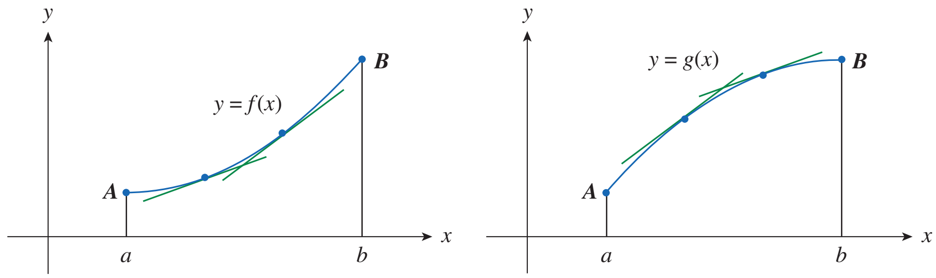


(a) The graph of f looks like it *bows in*.

(b) The graph of g looks like it *bows out*.

Figure 4.29
Both f and g are increasing on $[a, b]$, but their graphs look different.

Figure 4.30 shows the graphs of f and g with tangent lines to these curves drawn at several points. Notice that the graph of f lies *above* each tangent line and that the graph of g lies *below* each tangent line. This property leads to the definition of *concavity.*



(a) The graph of f lies above the tangent lines.

(b) The graph of g lies below the tangent lines.

Figure 4.30
The concavity of the graph of a function is defined using the tangent line to the graph.

Definition • Concavity

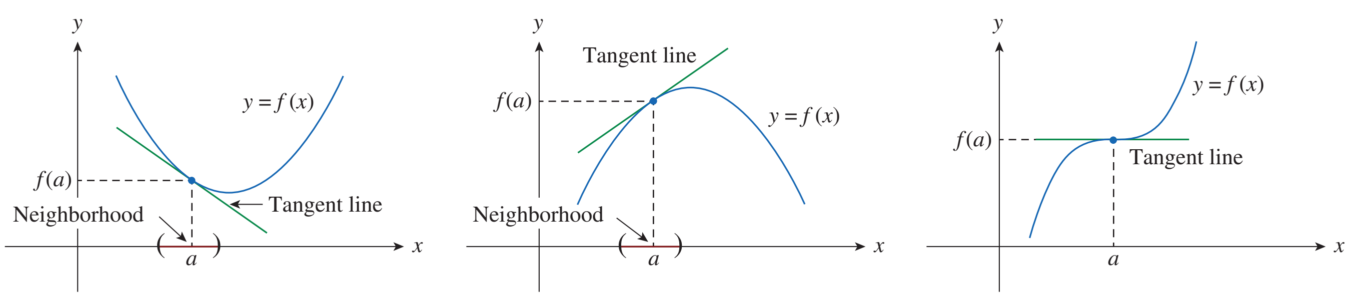
Let f be a differentiable function.

The graph of f is **concave up** at a if it is above the tangent line to the graph of f at a for all x in a neighborhood containing a (but not equal to a).

The graph of f is **concave down** at a if it is below the tangent line to the graph of f at a for all x in a neighborhood containing a (but not equal to a).

A Closer Look

1. Remember, the concept of a neighborhood containing a means an open interval containing a.
2. The concept of concavity is defined in terms of a single number, not an interval. The graph of a function can be concave up, or concave down, or have no concavity at a number. See Figure 4.31.



(a) The graph of f is concave up at a. The graph of f lies above the tangent line in a neighborhood of a.

(b) The graph of f is concave down at a. The graph of f lies below the tangent line in a neighborhood of a.

(c) The graph of f has no concavity at a. There is no neighborhood of a such that the graph of f is always above or always below the tangent line.

Figure 4.31
The graph of a function can be concave up, concave down, or have no concavity at a number a.

3. Figure 4.32 shows a graph that is concave up (CU) on the intervals (b, c), (d, e), and (e, p), and concave down (CD) on the intervals (a, b), (c, d), and (p, q).

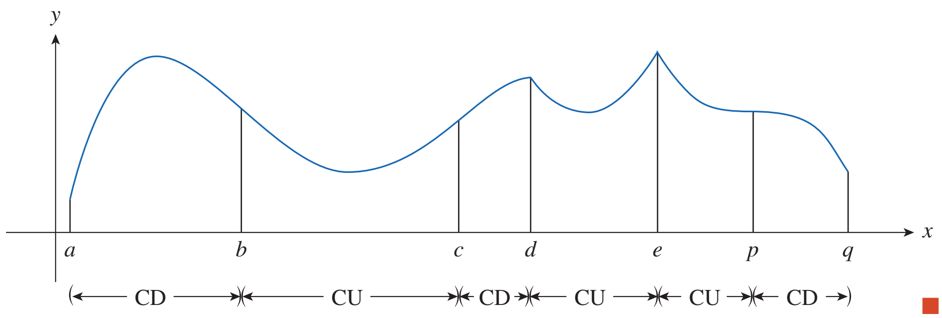


Figure 4.32
The graph of a function and the intervals on which the graph is concave up or concave down.

The second derivative can be used to help determine the intervals of concavity. In Figure 4.30(a), the graph of f is concave up and, as x moves from left to right, the slope of the tangent line is increasing. This means that the derivative f' is an increasing function, and therefore, its derivative, f'', is positive.

Similarly, in Figure 4.30(b), the graph of f is concave down and, as x moves from left to right, the slope of the tangent line is decreasing. This means that the derivative f' is a decreasing function, and therefore, its derivative, f'', is negative.

This reasoning leads to a theorem, the Concavity Test. The Mean Value Theorem is used to prove this result.

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.

A Closer Look

1. The Concavity Test allows us to draw a conclusion about the behavior of the graph of f if $f''(x) > 0$ or $f''(x) < 0$. It doesn't say anything about the numbers a where $f''(a) = 0$ or $f''(a)$ DNE. We need to determine concavity at each of these numbers separately.

2. The graph of a function is either concave up, concave down, or has no concavity at a number. Therefore, we would not include a number in two intervals, one on which the graph is concave up and one on which the graph is concave down.

3. In Figure 4.32 there are several values at which the graph changes concavity, from concave up to concave down, or from concave down to concave up. The point on the graph where this change occurs is called an *inflection point.*

Definition • Inflection Point

A point P on the graph of f is called an **inflection point** if f is continuous there and the graph changes from concave up to concave down or from concave down to concave up at P.

A Closer Look

1. If $f''(a)$ exists and $f''(a) \neq 0$, then the concavity of the graph of f is known and the graph cannot change concavity at $(a, f(a))$.

   $f''(x)$ can change sign only when $f''(x) = 0$ or $f''(x)$ DNE.

2. Using the Concavity Test, we can determine that there is a point of inflection at any value in the domain of f where the second derivative changes sign. Therefore, the graph of f *may* have a point of inflection only where $f''(x) = 0$ or $f''(x)$ DNE.

   We can use a sign chart for the second derivative to help determine concavity and points of inflection.

3. If the graph of f has a tangent line at a point of inflection, then the graph crosses its tangent line at that point. ■

A consequence of the Concavity Test is the following test for maximum and minimum values.

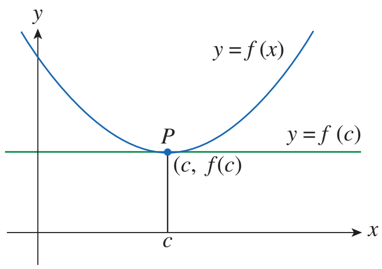


Figure 4.33
$f'(c) = 0$ and $f''(c) > 0$ in a neighborhood of c. The graph of f is concave up near c. Therefore, f has a local minimum at c.

Common Error

If $f''(c) > 0$ (< 0) then f has a local minimum (maximum) at c.

Correct Method

Make sure that $f'(c) = 0$ before checking the sign of $f''(c)$ and using the Second Derivative Test.

The Second Derivative Test

Suppose f'' is continuous in a neighborhood of c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

A Closer Look

1. Here is an explanation and illustration of the Second Derivative Test: suppose $f''(x) > 0$ near c. Then f is concave up near c. Therefore, the graph of f lies *above* its horizontal tangent at c, and f has a local minimum at c. See Figure 4.33.
2. If $f'(c) = 0$ and $f''(c) = 0$, then the Second Derivative Test provides no information. We need to use the First Derivative Test to determine whether $f(c)$ is either a local maximum value, a local minimum value, or neither.
3. The Second Derivative Test cannot be used when $f''(x)$ does not exist. ■

Example 4 Analysis of a Curve

Let $f(x) = x^4 - 4x^3$. Find the interval(s) on which f is increasing or decreasing and find the local extreme values of f. Find the interval(s) on which the graph of f is concave up or concave down and find all inflection points. Use this information to sketch the graph of f.

Solution

Find f' and factor completely.

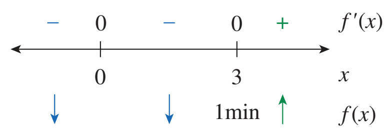
$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

Find the critical points.

$f'(x) = 0$: $\quad 4x^2(x - 3) = 0 \quad \Rightarrow \quad x = 0, 3$

$f'(x)$ DNE: None

Construct a sign chart.



$f'(x) > 0$ on $(3, \ \infty)$.
$f'(x) < 0$ on $(-\infty, \ 0)$ and $(0, 3)$.

f is increasing on $[3, \ \infty)$.
f is decreasing on $(-\infty, \ 3]$.

$f'(x)$ changes sign from negative to positive at 3. Therefore, f has a local minimum at $x = 3$. The local minimum value is $f(3) = -27$.

There is no local maximum value.

Find f'' and factor completely.

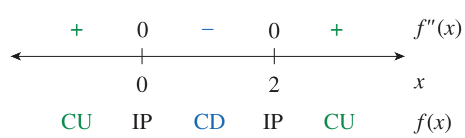
$f''(x) = 12x^2 - 24x = 12x(x - 2)$

Find the inflection point candidates.

$f''(x) = 0$: $\quad 12x(x - 2) = 0 \quad \Rightarrow \quad x = 0, \ 2$

$f''(x)$ DNE: None

Construct a sign chart.



$f''(x) > 0$ on $(-\infty, \ 0)$ and $(2, \ \infty)$.
$f''(x) < 0$ on $(0, 2)$.

f is concave up on $(-\infty, \ 0)$ and $(2, \ \infty)$.
f is concave down on $(0, 2)$.

$f''(x)$ changes sign from positive to negative at 0, and from negative to positive at 2. Therefore, f has inflection points at $(0, f(0)) = (0, \ 0)$ and $(2, f(2)) = (2, \ -16)$.

Figure 4.34 shows a graph of f. ■

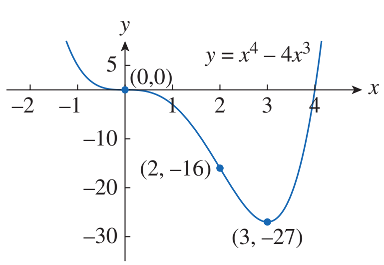


Figure 4.34
The graph shows the local minimum and the inflection points.

Example 5 Analysis of a Curve

Let $f(x) = x^{2/3}(6 - x)^{1/3}$. Find the interval(s) on which f is increasing or decreasing and find the local extreme values of f. Find the interval(s) on which the graph of f is concave up or concave down and find all inflection points. Use this information to sketch the graph of f.

Solution

Find f' and factor completely.

$$f'(x) = x^{2/3} \cdot \frac{1}{3}(6-x)^{-2/3}(-1) + (6-x)^{1/3} \cdot \frac{2}{3}x^{-1/3} \qquad \text{Product Rule.}$$

$$= \frac{-x^{2/3}}{3(6-x)^{2/3}} + \frac{2(6-x)^{1/3}}{3x^{1/3}} \qquad \text{Rewrite as fractions.}$$

$$= \frac{-x^{2/3} \cdot x^{1/3} + 2(6-x)^{1/3} \cdot (6-x)^{2/3}}{3x^{1/3}(6-x)^{2/3}} \qquad \text{Common denominator.}$$

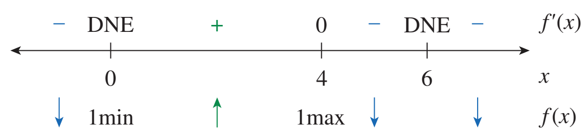
$$= \frac{-(x-4)}{x^{1/3}(6-x)^{2/3}} \qquad \text{Simplify.}$$

Find the critical points.

$f'(x) = 0$: $\quad -(x-4) = 0 \quad \Rightarrow \quad x = 4$

$f'(x)$ DNE: $x^{1/3}(6-x)^{2/3} = 0 \quad \Rightarrow \quad x = 0, 6$

Construct a sign chart.



$f'(x) > 0$ on $(0, 4)$
$f'(x) < 0$ on $(-\infty, 0)$, $(4, 6)$, and $(6, \infty)$

f is increasing on $[0, 4]$.
f is decreasing on $(-\infty, 0]$ and $[4, \infty)$.

$f'(x)$ changes sign from negative to positive at 0. Therefore, f has a local minimum at $x = 0$. The local minimum value is $f(0) = 0$.

$f'(x)$ changes sign from positive to negative at 4. Therefore, f has a local maximum at $x = 4$. The local maximum value is $f(4) = 2^{5/3}$.

Find f'' and factor completely.

$$f''(x) = \frac{x^{1/3}(6-x)^{2/3}(-1) - (-(x-4))\left[x^{1/3}\frac{2}{3}(6-x)^{-1/3}(-1) + (6-x)^{2/3}\frac{1}{3}x^{-2/3}\right]}{\left[x^{1/3}(6-x)^{2/3}\right]^2} \qquad \text{Quotient Rule.}$$

$$= \frac{-x^{1/3}(6-x)^{2/3} - \dfrac{2x^{1/3}(x-4)}{3(6-x)^{1/3}} + \dfrac{(x-4)(6-x)^{2/3}}{3x^{2/3}}}{x^{2/3}(6-x)^{4/3}} \qquad \text{Simplify.}$$

$$= \frac{\dfrac{-3x(6-x) - 2x(x-4) + (6-x)(x-4)}{3x^{2/3}(6-x)^{1/3}}}{x^{2/3}(6-x)^{4/3}} \qquad \text{Common denominator in the numerator.}$$

$$= \frac{-18x + 3x^2 - 2x^2 + 8x + 6x - 24 - x^2 + 4x}{3x^{4/3}(6-x)^{5/3}} \qquad \text{Write as one fraction.}$$

$$= \frac{-8}{x^{4/3}(6-x)^{5/3}} \qquad \text{Simplify.}$$

Find the inflection point candidates.

$f'(x) = 0$: None

$f''(x)$ DNE: $x^{4/3}(6-x)^{5/3} = 0 \quad \Rightarrow \quad x = 0, 6$

Construct a sign chart.

$f''(x) > 0$ on $(6, \infty)$
$f''(x) < 0$ on $(-\infty, 0)$ and $(0, 6)$

f is concave up on $(6, \infty)$
f is concave down on $(-\infty, 0)$ and $(0, 6)$

$f''(x)$ changes sign from negative to positive at 6.
Therefore, f has an inflection point at $(6, f(6)) = (6, 0)$.

Note that $f''(4) = \dfrac{-8}{4^{4/3}(6-4)^{5/3}} = -0.397 < 0$.

By the second derivative test, f has a local maximum at 4. This confirms the result above.

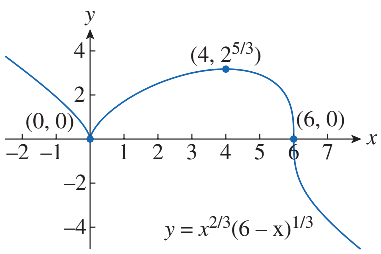


Figure 4.35
The graph shows the local maximum, local minimum, and the inflection point.

Figure 4.35 shows a graph of f. Note that the graph has vertical tangents at (0, 0) and (6, 0) because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$. ■

Example 6 Sketch a Graph

Let $f(x) = e^{1/x}$. Use the first and second derivatives and information about asymptotes to sketch the graph of f.

Solution

The domain of f is $\{x \mid x \neq 0\}$.

Therefore, we need to check for a vertical asymptote on the graph of f at 0.

$$\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty \qquad \text{As } x \to 0^+,\ t = \frac{1}{x} \to \infty.$$

This limit alone shows that $x = 0$ is a vertical asymptote.

Consider the other one-sided limit at 0 to help sketch the graph of f.

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0 \qquad \text{As } x \to 0^-,\ t = \frac{1}{x} \to -\infty.$$

Check for horizontal asymptotes.

$$\lim_{x \to \infty} e^{1/x} = e^0 = 1 \quad \text{and} \quad \lim_{x \to -\infty} e^{1/x} = e^0 = 1$$

These limits show that $y = 1$ is a horizontal asymptote (to both the left and the right).

Find the first derivative.

$$f'(x) = e^{1/x} \cdot (-1x^{-2}) = -\frac{e^{1/x}}{x^2}$$

Find the critical points.

$f'(x) = 0$: $e^{1/x} \neq 0$, therefore, none

$f'(x)$ DNE: None ($x = 0$ is not in the domain of f.)

Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, $f'(x) < 0$ for all $x \neq 0$.

Therefore, f is decreasing on $(-\infty, 0)$ and $(0, \infty)$.

There is no critical point, so the function has no local maximum or minimum.

Find the second derivative.

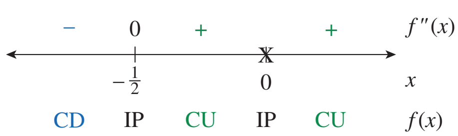
$$f''(x) = -\frac{x^2 \cdot e^{1/x}(-x^{-2}) - e^{1/x} \cdot (2x)}{x^4} = \frac{e^{1/x}(2x+1)}{x^4} \qquad \text{Quotient Rule; simplify.}$$

Find the inflection point candidates.

$f''(x) = 0$: $\quad e^{1/x}(2x+1) = 0 \quad \Rightarrow \quad x = -\dfrac{1}{2}$

$f''(x)$ DNE: None ($x = 0$ is not in the domain of f.)

Construct a sign chart.



$f''(x) > 0$ on $\left(-\dfrac{1}{2}, 0\right)$ and $(0, \infty)$.

$f''(x) < 0$ on $\left(-\infty, -\dfrac{1}{2}\right)$.

f is concave up on $\left(-\dfrac{1}{2}, 0\right)$ and $(0, \infty)$.

f is concave down on $\left(-\infty, -\dfrac{1}{2}\right)$.

$f''(x)$ changes from negative to positive at $-\dfrac{1}{2}$.

Therefore, f has an inflection point at $\left(-\dfrac{1}{2}, f\left(-\dfrac{1}{2}\right)\right) = \left(-\dfrac{1}{2}, e^{-2}\right)$.

To sketch the graph of f, we first draw the horizontal asymptote $y = 1$ (as a dashed line). Use the information about increasing and decreasing intervals, concavity, and the inflection point. Remember that $f(x) \to 0$ as $x \to 0^-$ but $f(0)$ does not exists. And there is a vertical asymptote at $x = 0$.

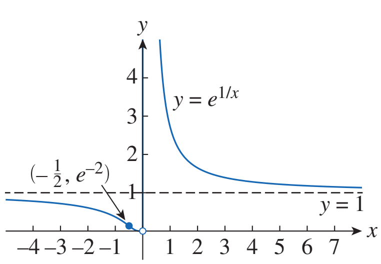


Figure 4.36
This graph of f includes the asymptotes and inflection point.

Figure 4.36 shows a graph of f. ■

Example 7 Bee Population Growth

A population of honeybees raised in an apiary started with 50 bees at time $t = 0$ and was modeled by the function

$$P(t) = \frac{75{,}200}{1 + 1503e^{-0.5932t}}$$

where t is the time in weeks, $0 \leq t \leq 25$. Use a graph to estimate the time at which the bee population was growing fastest. Then use derivatives to provide a more accurate estimate.

Solution

The population grows fastest when the population curve $y = P(t)$ has the steepest tangent line. Use the graph in Figure 4.37 to estimate the time at which the tangent line is steepest. This occurs when $t \approx 12$. So the bee population was growing most rapidly after about 12 weeks.

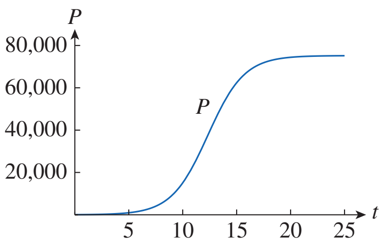


Figure 4.37
Graph of the honeybee population function.

For a better estimate, find the derivative $P'(t)$, the rate of increase of the bee population at time t.

$$P'(t) = -\frac{67{,}046{,}785.92e^{-0.5932t}}{(1 + 1503e^{-0.5932t})^2}$$

The graph of P' is shown in Figure 4.38. This graph suggests that the maximum of P' occurs when $t \approx 12.3$.

To obtain an even better estimate, note that P' obtains a maximum value when P' changes from increasing to decreasing. This occurs when P changes from concave upward to concave downward, that is, when P has an inflection point.

Using a CAS, we find:

$$P''(t) = \frac{119555093144e^{-1.1864t}}{(1 + 1503e^{-0.5932t})^3} - \frac{39772153e^{-0.5932t}}{(1 + 1503e^{-0.5932t})^2}.$$

We could plot this function to see where it changes from positive to negative, but using a CAS to solve the equation $P''(t) = 0$, we find $t \approx 12.3318$. ■

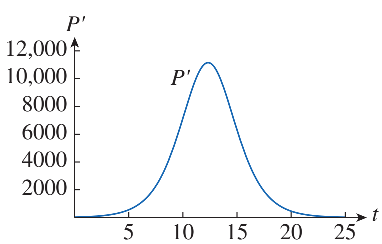


Figure 4.38
Graph of the derivative $P'(t)$.

One final example in this section involves *families* of functions. This means that the functions in the family are related to each other by a formula that contains one or more arbitrary constants. Each value of the constant defines a member of the family, and the idea is to see how the graph of the function changes as the constant changes.

Example 8 A Family of Functions

Investigate the family of functions given by $f(x) = cx + \sin x$. What features do the members of this family have in common? How do they differ?

Solution

The derivative of f is $f'(x) = c + \cos x$.

If $c > 1$, then $f'(x) > 0$ for all x, so f is increasing.

If $c = 1$, then $f'(x) = 0$ when x is an odd multiple of π, but f has horizontal tangents there and is still an increasing function.

If $c \leq -1$, then f is always decreasing.

If $-1 < c < 1$, then the equation $c + \cos x = 0$ has infinitely many solutions; $x = 2n\pi \pm \cos^{-1}(-c)$. Therefore, f has infinitely many minima and maxima.

The second derivative of f is $f''(x) = -\sin x$.

$f''(x) < 0$ when $0 < x < \pi$ and, in general when $2n\pi < x < (2n + 1)\pi$, where n is any integer. Therefore, all members of the family are concave downward on the intervals $(0, \pi), (2\pi, 3\pi), \ldots$ and concave upward on the intervals $(3\pi, 4\pi), \ldots$.

Figure 4.39 shows the graphs of several members of this family of curves.

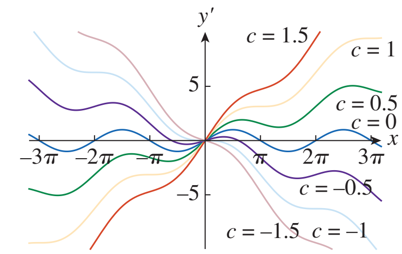
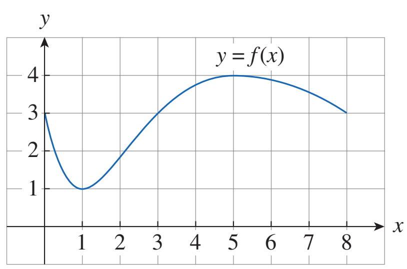


Figure 4.39
The family of curves $f(x) = cx + \sin x$

4.3 Exercises

1. The graph of the function f is shown.



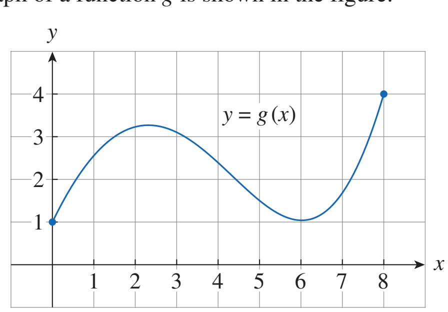
Verify that f satisfies the hypotheses of Rolle's Theorem on the interval [0, 8]. Use the graph to estimate the value(s) of c that satisfies the conclusion of Rolle's Theorem on that interval.

2. Draw the graph of a function defined on [0, 8] such that $f(0) = f(8) = 3$ and the function does not satisfy the conclusion of Rolle's Theorem on this interval.

3. Let $f(x) = \sqrt{4x + 1}$. Find the value of c in the interval [0, 6] guaranteed to exist by the Mean Value Theorem.

4. Let $f(x) = \cos x$. Find the value of c in the interval [1, 3] guaranteed to exist by the Mean Value Theorem.

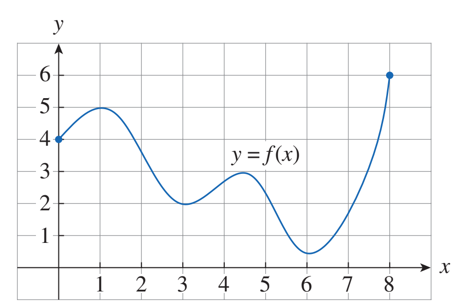
5. The graph of a function g is shown in the figure.



(a) Verify that g satisfies the hypotheses of the Mean Value Theorem on the interval [0, 8].
(b) Estimate the value(s) of c that satisfies the conclusion of the Mean Value Theorem on the interval [0, 8].
(c) Estimate the value(s) of c that satisfies the conclusion of the Mean Value Theorem on the interval [2, 6].

6. Draw the graph of a function that is continuous on [0, 8], where $f(0) = 1$ and $f(8) = 4$, but does not satisfy the conclusion of the Mean Value Theorem on [0, 8].

7. The graph of a function f is shown in the figure.

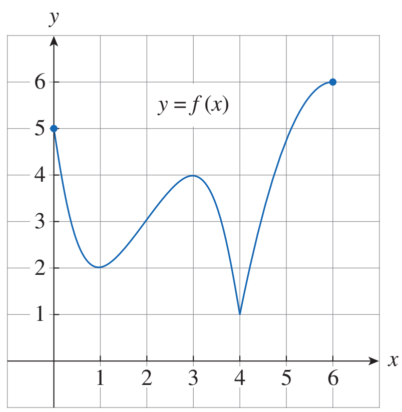


Use the graph to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval [0, 8].

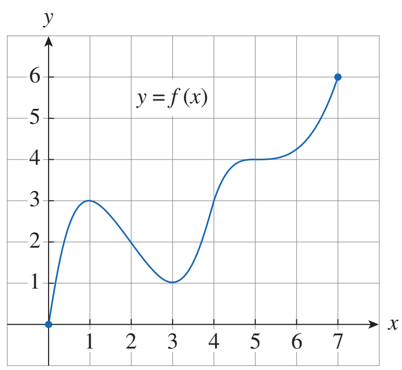
Use the given graph of f to find the following.

(a) The interval(s) on which f is increasing
(b) The interval(s) on which f is decreasing
(c) The interval(s) on which f is concave up
(d) The interval(s) on which f is concave down
(e) The coordinates of the inflection point(s)

8.



9.



10. Suppose a formula for a function f is given.

(a) Explain how to determine where f is increasing or decreasing.

(b) Explain how to determine where the graph of f is concave upward or concave downward.

(c) Explain how to locate inflection points.

11. (a) State the First Derivative Test.

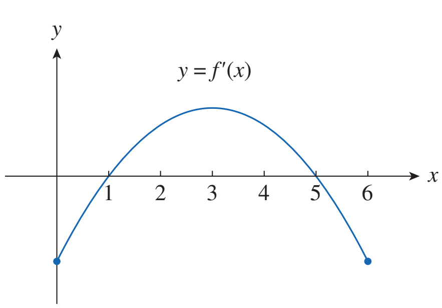
(b) State the Second Derivative Test. Under what circumstances is this test inconclusive? What can you do if the test fails?

The graph of the *derivative* f' of a function f is shown.

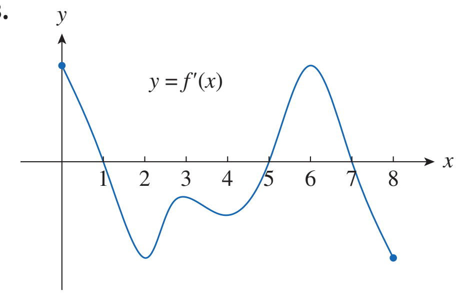
(a) On what intervals is f increasing or decreasing?

(b) At what values of x does f have a local maximum or local minimum?

12.



13.



The graph of the derivative of a continuous function f is shown.

(a) On what intervals is f increasing? Decreasing?

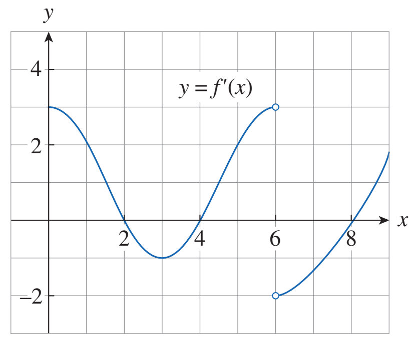
(b) At what values of x does f have a local maximum? Local minimum?

(c) On what intervals is f concave up? Concave down?

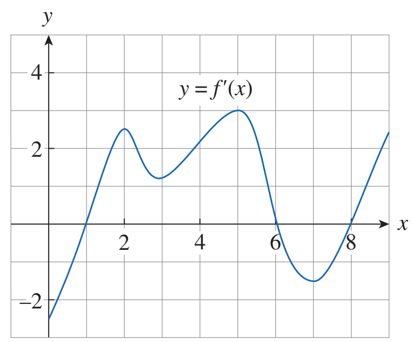
(d) Find the x-coordinate(s) of the inflection point(s).

(e) Assume that $f(0) = 0$ and sketch a graph of f.

14.



15.

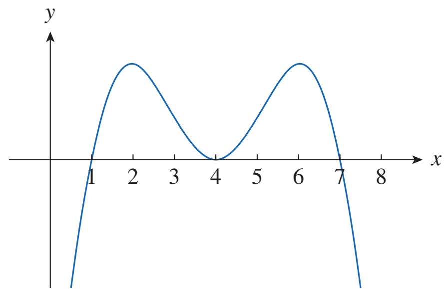


16. In each part state the x-coordinate of the inflection point(s) of f. Give a reason for your answer.

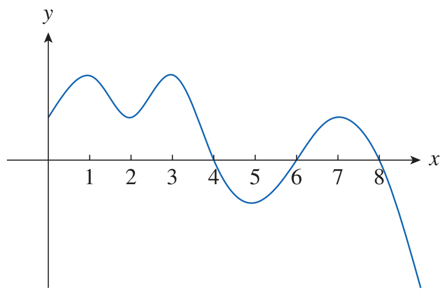
(a) The curve is the graph of f.

(b) The curve is the graph of f'.

(c) The curve is the graph of f''.



17. The graph of the first derivative f' of a function f is shown.



(a) On what interval(s) is f increasing? Explain your reasoning.
(b) At what values of x does f have a local maximum or local minimum? Explain your reasoning.
(c) On what intervals is f concave up or concave down? Explain your reasoning.
(d) What are the x-coordinates of the inflection points? Justify your answer.

18. Sketch the graph of a function f such that $f'(x) < 0$ on $(1, 4)$ and $f''(x) > 0$ on $(1, 4)$.

For the function f,
(a) Find the interval(s) on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the interval(s) of concavity and the inflection point(s).

19. $f(x) = x^3 - 3x^2 - 9x + 4$

20. $f(x) = 2x^3 - 9x^2 + 12x - 3$

21. $f(x) = x^4 - 2x^2 + 3$

22. $f(x) = x^3 + 2x^2 - 4x + 1$

23. $f(x) = \dfrac{x}{x^2 + 1}$

24. $f(x) = \sin x + \cos x, \quad 0 \leq x \leq 2\pi$

25. $f(x) = \cos^2 x - 2 \sin x, \quad 0 \leq x \leq 2\pi$

26. $f(x) = e^{2x} + e^{-x}$

27. $f(x) = x^2 \ln x$

28. $f(x) = \dfrac{\ln x}{\sqrt{x}}$

29. $f(x) = \sqrt{x}e^{-x}$

30. The function f has second derivative given by $f''(x) = x(x - 2)^2(x + 1)^3$. Find the values of x for which f has a point of inflection.

31. Let $f(x) = x\, e^x - e^x$. Find the x-coordinate of each local extrema and each point of inflection.

Find the local maximum and local minimum values of f using both the First and Second Derivative Tests.

32. $f(x) = 1 + 3x^2 - 2x^3$

33. $f(x) = \dfrac{x}{x^2 + 4}$

34. $f(x) = x + \sqrt{1 - x}$

35. $f(x) = \sqrt{x} - \sqrt[4]{x}$

36. Let $f(x) = x^4(x - 1)^3$.
(a) Find the critical numbers of f.
(b) Use the Second Derivative Test to determine the behavior of f at these critical numbers.
(c) Use the First Derivative Test to determine the behavior of f at these critical numbers. Compare your results with part (b).

37. Suppose f' is continuous on $(-\infty, \infty)$.
(a) If $f'(2) = 0$ and $f''(2) = -5$, what can you conclude about f? Explain your reasoning.
(b) If $f'(6) = 0$ and $f''(6) = 0$, what can you conclude about f? Explain your reasoning.

Sketch the graph of a function that satisfies all of the given conditions.

38. (a) $f'(x) < 0$ and $f''(x) < 0$ for all x
(b) $f'(x) > 0$ and $f''(x) > 0$ for all x

39. (a) $f'(x) > 0$ and $f''(x) < 0$ for all x
(b) $f'(x) < 0$ and $f''(x) > 0$ for all x

40. Vertical asymptote $x = 0$, $f'(x) > 0$ if $x < -2$,
$f'(x) < 0$ if $x > -2$ $(x \neq 0)$,
$f''(x) < 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$

41. $f'(0) = f'(2) = f'(4) = 0$,
$f'(x) = 0$ if $x < 0$ or $2 < x < 4$,
$f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
$f''(x) > 0$ if $1 < x < 3$, $f''(x) < 0$ if $x < 1$ or $x > 3$

42. $f'(x) > 0$ for all $x \neq 1$, vertical asymptote $x = 1$,
$f''(x) > 0$ if $x < 1$ or $x > 3$, $f''(x) < 0$ if $1 < x < 3$

43. $f'(5) = 0$, $f'(x) < 0$ if $x < 5$,
$f'(x) > 0$ if $x > 5$, $f''(2) = 0$, $f''(8) = 0$,
$f''(x) < 0$ if $x < 2$ or $x > 8$,
$f''(x) > 0$ if $2 < x < 8$, $\lim_{x \to \infty} f(x) = 3$,
$\lim_{x \to -\infty} f(x) = 3$

44. $f'(0) = f'(4) = 0, \quad f'(x) = 1$ if $x < -1$,
$f'(x) > 0$ if $0 < x < 2$,
$f'(x) < 0$ if $-1 < x < 0$ or $2 < x < 4$ or $x > 4$,
$\lim_{x \to 2^-} f'(x) = \infty, \quad \lim_{x \to 2^+} f'(x) = -\infty$,
$f''(x) > 0$ if $-1 < x < 2$ or $2 < x < 4$,
$f''(x) < 0$ if $x > 4$

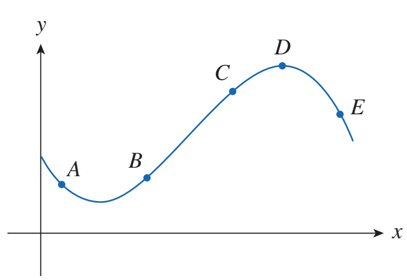
45. $f'(x) > 0$ if $x \neq 2, \quad f''(x) > 0$ if $x < 2$,
$f''(x) < 0$ if $x > 2$, f has inflection point $(2, 5)$,
$\lim_{x \to \infty} f(x) = 8, \quad \lim_{x \to -\infty} f(x) = 0$

46. Suppose $f(3) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all x.

(a) Sketch a possible graph of f.
(b) How many solutions are there to the equation $f(x) = 0$? Explain your reasoning.
(c) Is it possible for $f'(2) = \frac{1}{3}$? Why or why not?

47. Suppose f is a continuous function such that
$f(x) > 0$ for all x, $\quad f(0) = 4$,
$f'(x) > 0$ if $x < 0$ or $x > 2$, $f'(x) < 0$ if $0 < x < 2$,
$f''(-1) = f''(1) = 0$, $f''(x) > 0$ if $x < -1$ or $x > 1$,
$f''(x) < 0$ if $-1 < x < 1$.
(a) Can f have an absolute maximum? If so, sketch a possible graph of f. If not, explain why.
(b) Can f have an absolute minimum? If so, sketch a possible graph of f. If not, explain why.
(c) Sketch a possible graph of f that does *not* achieve an absolute minimum.

48. The graph of a function $y = f(x)$ is shown.



At which point(s) are the following statements true?
(a) $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are both positive.
(b) $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are both negative.
(c) $\frac{dy}{dx}$ is negative but $\frac{d^2y}{dx^2}$ is positive.

49. A function f has derivative given by

$$f'(x) = \frac{\sin\left(\frac{\pi x}{2}\right)}{x}$$

For how many values of x, $0 < x < 4$, does f have a local minimum?

For the given function,
(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values.
(c) Find the intervals on which f is concave up or concave down, and the inflection points.
(d) Use the information from parts (a)–(c) to sketch the graph. Check your work using technology.

50. $f(x) = x^3 - 12x + 2$

51. $f(x) = \frac{1}{2}x^4 - 4x^2 + 3$

52. $f(x) = (x + 1)^5 - 5x - 2$

53. $f(x) = 5x^3 - 3x^5$

54. $f(x) = \frac{x^2}{x^2 - 1}$

55. $f(x) = \frac{x^2}{(x - 2)^2}$

56. $f(x) = \sqrt{x^2 + 1} - x$

57. $f(x) = x\sqrt{6 - x}$

58. $f(x) = 5x^{2/3} - 2x^{5/3}$

59. $f(x) = x^{1/3}(x + 4)$

60. $f(x) = 2\cos x + \cos^2 x, \quad 0 \le x \le 2\pi$

61. $f(x) = x \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

62. $f(x) = x - \sin x, \quad 0 \le x \le 4\pi$

63. $f(x) = \ln(1 - \ln x)$

64. $f(x) = \ln(x^2 + 9)$

65. $f(x) = \frac{e^x}{1 + e^x}$

66. $f(x) = e^{-1/(x+1)}$

67. $f(x) = e^{\arctan x}$

68. Suppose f is a continuous function for all $x \ge 0$. Values of f, f', and f'' are summarized in the table.

x	f	f'	f''
0	0	+	−
(0, 2)		+	−
2	3	0	−
(2, 4)		−	−
4	0	−	−
(4, 5)		−	−
5	−2	DNE	DNE
(5, ∞)		+	+

(a) Write the x- and y-coordinates of all local extreme points of f. Indicate whether each is a local maximum or a local minimum.
(b) Find the coordinates of all points of inflection on the graph of f. Give a reason for your answer.
(c) On what intervals is the function increasing? Give a reason for your answer.

69. Suppose the derivative of a function f is

$$f'(x) = (x + 1)^2(x - 3)^5(x - 6)^4$$

On what interval(s) is f increasing?

70. Use the methods presented in this section to sketch the graph of $f(x) = x^3 - 3a^2x + 2a^3$, where a is a positive constant. What do the members of this family of curves have in common? How do they differ?

For the given function,
(a) Use a graph to estimate the local maximum and minimum values. Find the exact values.
(b) Estimate the value of x at which f increases most rapidly. Find the exact value.

71. $f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$ **72.** $f(x) = x^2 e^{-x}$

For the given function,
(a) Use a graph of f to obtain an estimate of the intervals of concavity and the coordinates of the inflection points.
(b) Use a graph of f'' to obtain similar estimates and compare your results to part (a).

73. $f(x) = \sin 2x + \sin 4x, \quad 0 \le x \le \pi$

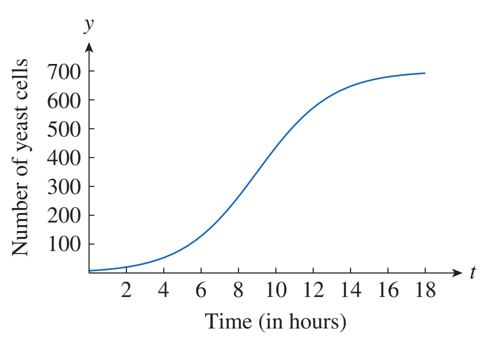
74. $f(x) = (x-1)^2(x+1)^3$

Use technology to graph f'' and estimate the intervals of concavity.

75. $f(x) = \dfrac{x^4 + x^3 + 1}{\sqrt{x^2+x+1}}$ **76.** $f(x) = \dfrac{x^2 \tan^{-1} x}{1 + x^3}$

77. Consider the curve defined by $x^2 - xy + y^2 = 3$.
(a) Show that $x = 1$ is a critical number.
(b) Find $\dfrac{d^2y}{dx^2}$.
(c) Does $x = 1$ correspond to a local minimum, a local maximum, or neither? Justify your answer.

78. The graph shows a population of yeast cells in a new laboratory culture as a function of time.



(a) Describe how the rate of population increase varies.
(b) When is this rate highest?
(c) On what intervals is the population function concave up? Concave down?
(d) Estimate the coordinates of the inflection point.

79. In an episode of *The Simpsons* television show, Homer reads from a newspaper and announces "Here's good news! According to this eye-catching article, SAT scores are declining at a slower rate." Interpret Homer's statement in terms of a function and its first and second derivatives.

80. The president announces that the national debt is increasing, but at a decreasing rate. Interpret this statement in terms of a function and its first and second derivatives.

81. Let $f(t)$ be the temperature at time t where you live and suppose that at time $t = 3$ you feel uncomfortably hot. How do you feel about the given data in each case?
(a) $f'(3) = 2, \quad f''(3) = 4$
(b) $f'(3) = 2, \quad f''(3) = -4$
(c) $f'(3) = -2, \quad f''(3) = 4$
(d) $f'(3) = -2, \quad f''(3) = -4$

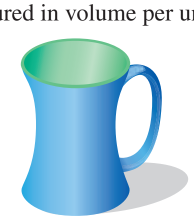
Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. Find the values of t for which the curve is concave upward.

82. $x = t^3 - 12t, \quad y = t^2 - 1$

83. $x = \cos 2t, \quad y = \cos t, \quad 0 < t < \pi$

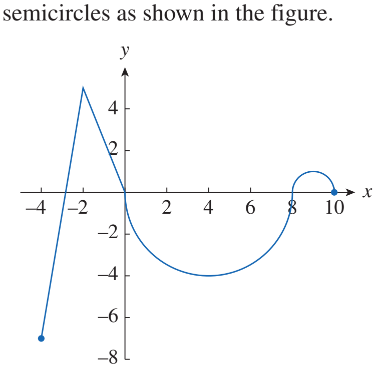
84. Let $K(t)$ be a measure of the knowledge you gain by studying for a test for t hours. Which do you think is larger, $K(8) - K(7)$ or $K(3) - K(2)$? Explain your reasoning. Is the graph of K concave up or down? Why?

85. Coffee is being poured into a mug shown in the figure at a constant rate (measured in volume per unit of time).



Sketch a graph of the depth of the coffee in the mug as a function of time. Account for the shape of the graph in terms of concavity. What is the significance of the inflection point?

86. The continuous function g is defined on the interval $-4 \le x \le 10$. The graph of g consists of two line segments and two semicircles as shown in the figure.



For the function F, $F'(x) = g(x)$.

(a) Find the average rate of change of g for $[-2, 4]$.

(b) Explain why there is no value c, $-3 \le c \le 0$, for which $g'(c)$ equals the average rate of change of g on this interval. Explain why this does not contradict the Mean Value Theorem.

(c) For $-4 \le x \le 10$, find the critical numbers of F. Identify each of these critical numbers as corresponding to a local minimum, a local maximum, or neither. Justify your conclusion.

(d) Find the intervals on which F is concave down.

87. In the theory of relativity, the mass of a particle is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

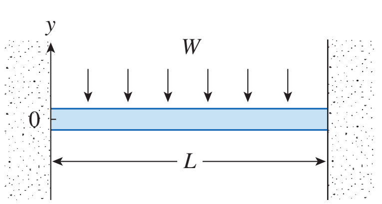
where m_0 is the rest mass of the particle, m is the mass when the particle moves with speed v relative to the observer, and c is the speed of light. Sketch the graph of m as a function of v.

88. In the theory of relativity, the energy of a particle is

$$E = \sqrt{m_0^2 c^4 + h^2 c^2/\lambda^2}$$

where m_0 is the rest mass of the particle, λ is its wave length, and h is Planck's constant. Sketch the graph of E as a function of λ. What does the graph suggest about energy?

89. The figure shows a beam of length L embedded in concrete walls.

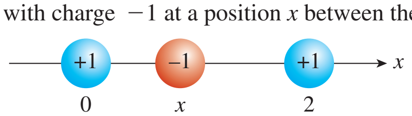


If a constant load W is distributed evenly along its length, the beam takes the shape of the deflection curve

$$y = -\frac{W}{24EI}x^4 + \frac{WL}{12EI}x^3 - \frac{WL^2}{24EI}x^2$$

where E and I are positive constants. (E is Young's modulus of elasticity and I is the moment of inertia of a cross-section of the beam.) Sketch the graph of the deflection curve.

90. Coulomb's Law states that the force of attraction between two charged particles is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The figure shows particles with charge 1 located at positions 0 and 2 on a coordinate line and a particle with charge -1 at a position x between them.



It follows from Coulomb's Law that the net force acting on the middle particle is

$$F(x) = -\frac{k}{x} + \frac{k}{(x-2)^2}, \quad 0 < x < 2$$

where k is a positive constant. Sketch the graph of the net force function. What does this graph suggest about the force?

91. A *drug response curve* describes the level of medication in the bloodstream after a drug is administered. A surge function $S(t) = At^p e^{-kt}$ is often used to model the response curve, reflecting an initial surge in the drug level and then a more gradual decline.

If for a particular drug, $A = 0.01$, $p = 4$, $k = 0.07$, and t is measured in minutes, estimate the times corresponding to the inflection points and explain their significance in the context of this problem. Check your answer using technology.

92. The probability density function for a normal random variable is given by

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

where μ is the *mean* and $\sigma (> 0)$ is the *standard deviation.* This bell-shaped curve occurs often in probability and statistics and is used extensively in inference. Consider the special case where $\mu = 0$ so that

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) Find the asymptote, maximum value, and inflection points of f.

(b) How does σ affect the shape of the graph?

(c) Illustrate your results by graphing several members of this family of functions in the same viewing rectangle.

93. Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at $x = -2$ and a local minimum value of 0 at $x = 1$.

94. For what values of a and b does the function

$$f(x) = axe^{bx^2}$$

have the maximum value $f(2) = 1$?

95. Let $f(x) = x^3 + ax^2 + bx$.

(a) Find values of a and b such that the function has a local minimum value of $-\frac{2}{9}\sqrt{3}$ at $x = \frac{1}{\sqrt{3}}$.

(b) Find an equation of the tangent line to the graph of f that has the smallest slope.

96. Find values of a and b such that (2, 2.5) is an inflection point on the graph of $x^2y + ax + by = 0$. Are there any additional inflection points on this graph? Justify your answer.

97. Show that the graph of $f(x) = \dfrac{1+x}{1+x^2}$ has three points of inflection and that they all lie on one line.

98. Show that the graphs of $y = e^{-x}$ and $y = -e^{-x}$ touch the graph of $y = e^{-x}\sin x$ at its inflection points.

99. Show that the inflection points on the graph of $y = x\sin x$ lie on the graph of $y^2(x^2+4) = 4x^2$.

100. Show that $\tan x > x$ for $0 < x < \frac{\pi}{2}$.

Hint: Show that $f(x) = \tan x - x$ is increasing on $\left[0, \frac{\pi}{2}\right]$.

101. (a) Show that $e^x \geq 1 + x$ for $x \geq 0$.

(b) Deduce that $e^x \geq 1 + x + \frac{1}{2}x^2$ for $x \geq 0$.

(c) Use mathematical induction to show that for $x \geq 0$ and any positive integer n,

$$e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

102. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. The inequality gives a restriction on the rate of growth of f, which then imposes a restriction on the possible values of f. Use the Mean Value Theorem to determine the maximum value of $f(4)$.

103. Suppose that $3 \leq f'(x) \leq 5$ for all values of x. Show that $18 \leq f(8) - f(2) \leq 30$.

104. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed.

Hint: Consider $f(t) = g(t) - h(t)$, where g and h are the position functions of the two runners.

105. At 2:00 PM a car's speedometer reads 30 mi/h. At 2:10 PM it reads 50 mi/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 120 mi/h^2.

106. Show that a cubic function (a third-degree polynomial) always has exactly one point of inflection. If its graph has three x-intercepts x_1, x_2, and x_3, show that the x-coordinate of the inflection point is $\frac{x_1 + x_2 + x_3}{3}$.

107. For what values of c does the polynomial $P(x) = x^4 + cx^3 + x^2$ have two inflection points? One inflection point? None? Illustrate by graphing P for several values of c. How does the graph change as c decreases?

108. For what values of c is the function

$$f(x) = cx + \frac{1}{x^2+3}$$

increasing on $(-\infty, \infty)$?

4.4 Graphing with Calculus and Technology

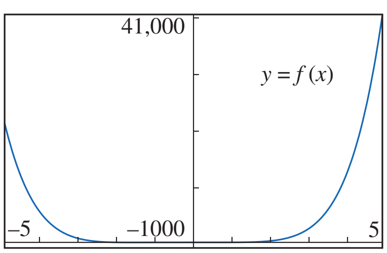


Figure 4.40
Graph of $y = f(x)$.

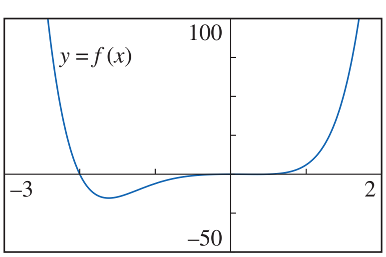


Figure 4.41
Graph of $y = f(x)$, showing more detail.

The method we used to sketch curves in the preceding section was a culmination of our study of differential calculus so far. The graph was the final object that we produced. In this section, our point of view is completely different. Here we start with a graph produced by a graphing calculator or computer and then we try to refine it. We use calculus to make sure that we reveal all the important characteristics of the curve. With the use of graphing devices we can analyze curves that would be far too complicated to consider without technology. The theme is the interaction between calculus and calculators.

Example 1 Discovering Hidden Behavior

Graph the polynomial $f(x) = 2x^6 + 3x^5 + 3x^3 - 2x^2$. Use the graphs of f' and f'' to estimate all maximum and minimum points and intervals of concavity.

Solution

If we use technology to graph this function, we usually need to specify a domain. Many graphing devices will automatically select a reasonable range from the values computed. Figure 4.40 shows a plot using technology with $-5 \leq x \leq 5$. Although this viewing rectangle is useful for showing that the asymptotic behavior (or end behavior) is the same as for $y = 2x^6$, it is hiding some important characteristics of the curve. Consider the viewing rectangle $[-3, 2] \times [-50, 100]$ shown in Figure 4.41.

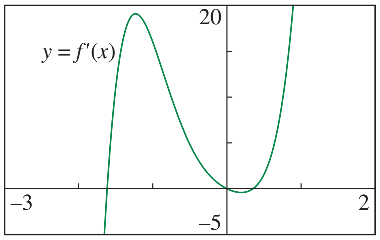


Figure 4.42
Graph of $y = f'(x)$.

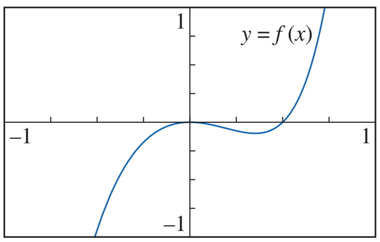


Figure 4.43
Graph of $y = f(x)$, showing more detail.

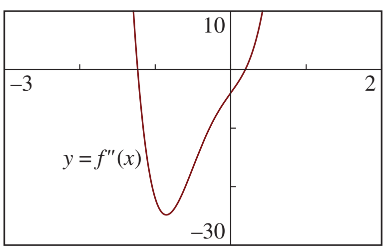


Figure 4.44
Graph of $y = f''(x)$.

From this graph it appears that there is an absolute minimum value of about -15.33 when $x \approx -1.62$ (using the trace feature) and f is decreasing on $(-\infty, -1.62)$ and increasing on $(-1.62, \infty)$. It also appears that there is a horizontal tangent line at the origin and inflection points when $x = 0$ and when x is somewhere in between -2 and -1.

Now let's try to confirm these observations using calculus. Differentiate to find f' and f''.

$$f'(x) = 12x^5 + 15x^4 + 9x^2 - 4x$$

$$f''(x) = 60x^4 + 60x^3 + 18x - 4$$

The graph of f' is shown in Figure 4.42. We can see that $f'(x)$ changes from negative to positive when $x \approx -1.62$; this confirms (by the First Derivative test) the minimum value that we found earlier. But, also notice that $f'(x)$ changes from positive to negative when $x = 0$ and from negative to positive when $x \approx 0.35$. This means that f has a local maximum at 0 and a local minimum when $x \approx 0.35$, but these details where hidden in Figure 4.41. Indeed, if we now zoom in toward the origin as in Figure 4.43, we can see that we missed a local maximum value of 0 when $x = 0$ and a local minimum value of about -0.1 when $x \approx 0.35$.

Now let's consider concavity and inflection points. Figures 4.40 and 4.42 suggest that there are inflection points when x is a little less than -1 and when x is a little greater than 0. But it is difficult to determine inflection points from the graph of f, so consider the graph of f'' in Figure 4.44. We see that f'' changes from positive to negative when $x \approx -1.23$ and from negative to positive when $x \approx 0.19$. Therefore, f is concave upward on $(-\infty, -1.23)$ and $(0.19, \infty)$ and concave downward on $(-1.23, 0.19)$. The inflection points are $(-1.23, -10.18)$ and $(0.19, -0.05)$.

We have discovered that no single graph reveals all the important features of this polynomial. But all of the figures considered collectively provide a more accurate description of the graph of the polynomial. ■

Example 2 Choosing an Appropriate Viewing Rectangle

Draw the graph of the function

$$f(x) = \frac{x^2 + 7x + 3}{x^2}$$

in a viewing rectangle that contains all the important features of the function. Estimate the maximum and minimum values and the intervals of concavity. Then use calculus to find these quantities exactly.

Solution

Figure 4.45 shows a graph of f on the interval $-5 \le x \le 5$ in which the scaling on the y-axis was selected automatically. It doesn't reveal much about the graph of f. Some graphing calculators use $[-10, 10] \times [-10, 10]$ as the default, or standard, viewing rectangle. Figure 4.46 shows the graph of f in this rectangle; it is certainly an improvement.

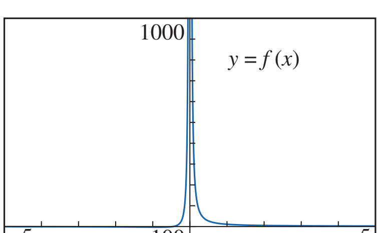


Figure 4.45
y-axis scaling selected automatically.

The y-axis appears to be a vertical asymptote and indeed it is because

$$\lim_{x \to 0} \frac{x^2 + 7x + 3}{x^2} = \infty.$$

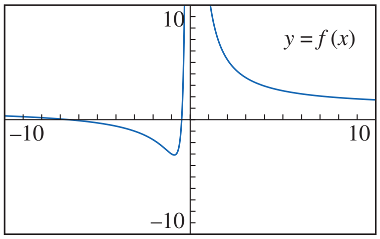


Figure 4.46
Graph of f in a standard viewing rectangle.

Figure 4.46 also allows us to estimate the x-intercepts: about -0.5 and -6.5. The exact values are obtained using the quadratic formula.

$$f(x) = \frac{x^2 + 7x + 3}{x^2} = 0 \quad \Rightarrow \quad x^2 + 7x + 3 = 0 \quad \Rightarrow \quad x = \frac{-7 \pm \sqrt{37}}{2}$$

For a more careful analysis of the horizontal asymptotes, consider the viewing rectangle $[-20, 20] \times [-5, 10]$ in Figure 4.47. It appears that $y = 1$ is the horizontal asymptote. This is confirmed by considering the limits

$$\lim_{x \to \pm\infty} \frac{x^2 + 7x + 3}{x^2} = \lim_{x \to \pm\infty} \left(1 + \frac{7}{x} + \frac{3}{x^2}\right) = 1.$$

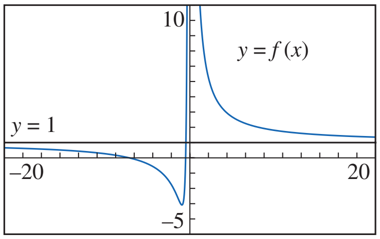


Figure 4.47
The horizontal asymptote $y = 1$ is clearer.

To estimate the minimum value, zoom in and use the viewing rectangle $[-3, 0] \times [-4, 2]$ as in Figure 4.48. The absolute minimum value is about -3.1 when $x \approx -0.9$. We can also see that the function decreases on $(-\infty, -0.9)$ and $(0, \infty)$, and increases on $(-0.9, 0)$.

To find the exact values:

$$f'(x) = -\frac{7}{x^2} - \frac{6}{x^3} = -\frac{7x + 6}{x^3} = 0 \quad \Rightarrow \quad x = -\frac{6}{7}$$

$$f'(x) > 0 \text{ when } -\frac{6}{7} < x < 0 \text{ and } f'(x) < 0 \text{ when } x < -\frac{6}{7} \text{ and when } x > 0$$

The exact minimum value is $f\left(-\frac{6}{7}\right) = -\frac{37}{12} \approx -3.083$.

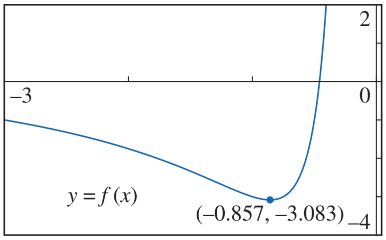


Figure 4.48
Zoom in to estimate the minimum value.

Figure 4.48 also suggests that an inflection point occurs somewhere between $x = -1$ and $x = -2$. We could estimate this point more accurately using the graph of the second derivative. However, in this case it is just as easy to find the exact values.

$$f''(x) = \frac{14}{x^3} + \frac{18}{x^4} = \frac{2(7x + 9)}{x^4} = 0 \quad \Rightarrow \quad x = -\frac{9}{7}$$

$f''(x) > 0$ when $x > -\frac{9}{7}$ $(x \neq 0)$. So f is concave upward on $\left(-\frac{9}{7}, 0\right)$ and $(0, \infty)$ and concave downward on $\left(-\infty, -\frac{9}{7}\right)$. The inflection point is $\left(-\frac{9}{7}, -\frac{71}{27}\right)$.

The analysis using the first two derivatives shows that Figure 4.47 shows all the major characteristics of the curve. ■

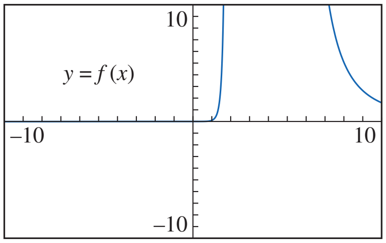


Figure 4.49
Graph of f in a standard viewing rectangle.

Example 3 One Graph Isn't Always Enough

Graph the function $f(x) = \dfrac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4}$.

Solution

Considering the analysis of a rational function in Example 2, start by graphing f in the standard viewing rectangle $[-10, 10] \times [-10, 10]$. Figure 4.49 suggests we need to zoom in to see some finer detail and also zoom out to see the larger picture. Before we zoom in, let's take a closer look at the expression for $f(x)$.

Because of the factors $(x-2)^2$ and $(x-4)^4$ in the denominator, we expect $x=2$ and $x=4$ to be vertical asymptotes on the graph of f. Here is confirmation.

$$\lim_{x\to 2}\frac{x^2(x+1)^3}{(x-2)^2(x-4)^4}=\infty \quad\text{and}\quad \lim_{x\to 4}\frac{x^2(x+1)^3}{(x-2)^2(x-4)^4}=\infty$$

To find the horizontal asymptotes, divide the numerator and denominator by x^6:

$$\frac{x^2(x+1)^3}{(x-2)^2(x-4)^4}=\frac{\dfrac{x^2}{x^3}\cdot\dfrac{(x+1)^3}{x^3}}{\dfrac{(x-2)^2}{x^2}\cdot\dfrac{(x-4)^4}{x^4}}=\frac{\dfrac{1}{x}\left(1+\dfrac{1}{x}\right)^3}{\left(1-\dfrac{2}{x}\right)^2\left(1-\dfrac{4}{x}\right)^4}$$

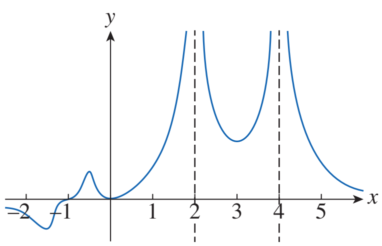


Figure 4.50
A rough sketch of the graph of f.

This shows that as $x\to\pm\infty$, $f(x)\to 0$, so the x-axis is a horizontal asymptote.

It is also very useful to consider the behavior of the graph near the x-intercepts. Since x^2 is positive, $f(x)$ does not change sign at 0 and so its graph does not cross the x-axis at 0. But, because of the factor $(x+1)^3$, the graph does cross the x-axis at -1 and has a horizontal tangent there. Using all of this information, but without considering any derivatives, the curve should look something like the one sketched in Figure 4.50.

Now that we know what characteristics to look for, and where, we can select appropriate viewing rectangles to reveal the important features of the graph of f. See Figures 4.51, 4.52, and 4.53.

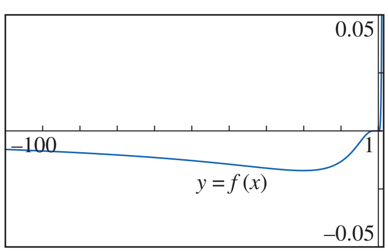


Figure 4.51
This graph shows the absolute minimum value.

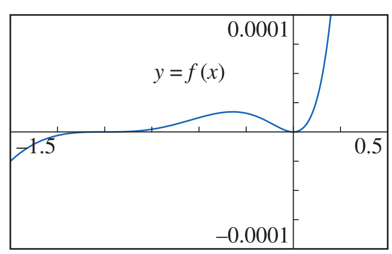


Figure 4.52
This graph shows a local maximum and a local minimum.

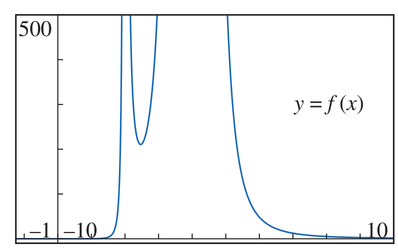


Figure 4.53
This graph shows a local minimum.

We can trace along these three graphs to see the following: an absolute minimum of about -0.02 when $x\approx -20$; a local maximum of about 0.00002 when $x\approx -0.3$; and a local minimum of about 211 when $x\approx 2.5$. These graphs also reveal three inflection points near -35, -5, and -1 and two between -1 and 0. To estimate the inflection points more closely we would need to compute f'', which would be a tedious task, or we could use technology to find f'' analytically or numerically (see Exercise 14).

For this function, at least three (technology) graphs are necessary to convey all of the useful (calculus) information. The only way to convey all of these features of the function on a single graph is to draw it by hand. Despite the distortions and exact scaling, Figure 4.50 does manage to summarize the essential nature of the function. ■

The family of functions $f(x)=\sin(x+\sin\,cx)$ where c is a constant, occurs in applications to frequency modulation (FM) synthesis. A sine wave is modulated by a wave with a different frequency $(\sin\,cx)$. The case where $c=2$ is studied in Example 4. Exercise 22 explores another special case.

Example 4 Estimate Calculus Characteristics

Graph the function $f(x)=\sin(x+\sin\,2x)$. For $0\le x\le\pi$, estimate all maximum and minimum values, intervals on which the function is increasing or decreasing, and inflection points.

Solution

Note that f is periodic with period 2π, f is odd, and $|f(x)| \le 1$ for all x. So, the choice of an initial viewing rectangle is straightforward; consider $[0, \pi] \times [-1.1, 1.1]$. See Figure 4.54.

It appears that there are three local maximum values and two local minimum values in this viewing rectangle. To confirm this and locate them more accurately, first find f'.

$$f'(x) = \cos(x + \sin 2x) \cdot (1 + 2 \cos 2x)$$

Figure 4.55 shows a graph of f and f' in the same viewing rectangle.

Zoom in, use the trace feature and the First Derivative Test to find the following approximate values.

Note that many graphing calculators and CAS have built-in functions to find maximum and minimum values and inflection points on a graph or calculator screen.

Intervals on which the function is increasing:	[0, 0.6], [1.0, 1.6], [2.1, 2.5]
Intervals on which the function is decreasing:	[0.6, 1.0], [1.6, 2.1], [2.5, π]
Local maximum values:	$f(0.6) \approx 1$, $f(1.6) \approx 1$, $f(2.5) \approx 1$
Local minimum values:	$f(1.0) \approx 0.94$, $f(2, 1) \approx 0.94$

Find the second derivative.

$$f''(x) = -(1 + 2 \cos 2x)^2 \sin(x + \sin 2x) - 4 \sin 2x \cos(x + \sin 2x)$$

Figure 4.56 shows a graph of f and f'' in the same viewing rectangle. Zoom in and use the trace feature to find the following approximate values.

Concave upward:	(0.8, 1.3), (1.8, 2.3)
Concave downward:	(0, 0.8), (1.3, 1.8), (2.3, π)
Inflection points:	(0.8, 0.97), (1.3, 0.97), (1.8, 0.97), (2.3, 0.97)

Our analysis confirms that Figure 4.54 does indeed represent f accurately for $0 \le x \le \pi$. Therefore, we can extend the domain so that Figure 4.57 shows an accurate graph of f for $-2\pi \le x \le 2\pi$. ■

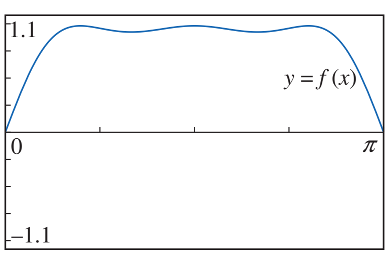


Figure 4.54
Graph of $y = f(x)$ in an initial viewing rectangle.

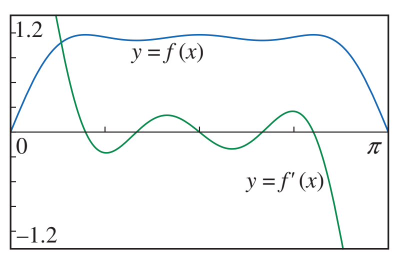


Figure 4.55
Graph of f and f'.

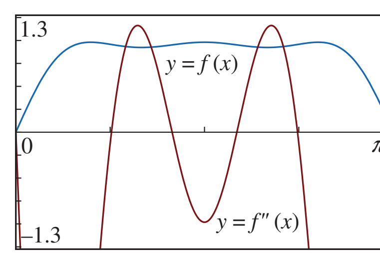


Figure 4.56
Graph of f and f''.

Example 5 Graphing a Family of Functions

Explain how the graph of $f(x) = \dfrac{1}{x^2 + 2x + c}$ changes as c varies.

Solution

Figures 4.58 and 4.59 show graphs for the special cases $c = 2$ and $c = -2$. These curves look very different.

Before drawing any more graphs, let's try to determine what members of this family of curves have in common.

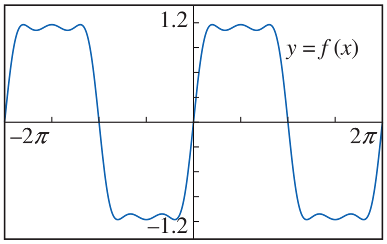


Figure 4.57
Graph of f and f'.

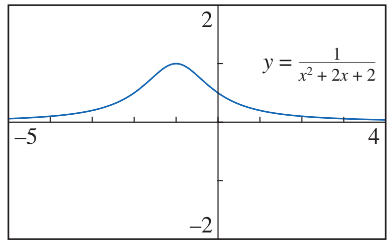


Figure 4.58
Graph of f: $c = 2$.

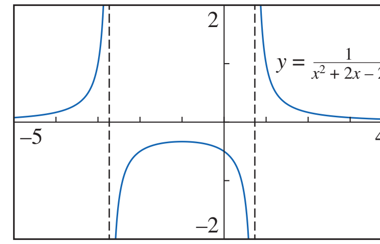


Figure 4.59
Graph of f: $c = -2$.

The limits $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2 + 2x + c} = 0$ are true for any value of c. Therefore, the x-axis is a horizontal asymptote for every member of the family.

A vertical asymptote will occur when $x^2 + 2x + c = 0 \;\Rightarrow\; x = -1 \pm \sqrt{1 - c}$.

When $c > 1$, there is no vertical asymptote, as in Figure 4.58.

When $c = 1$, the graph has a single vertical asymptote $x = -1$ because

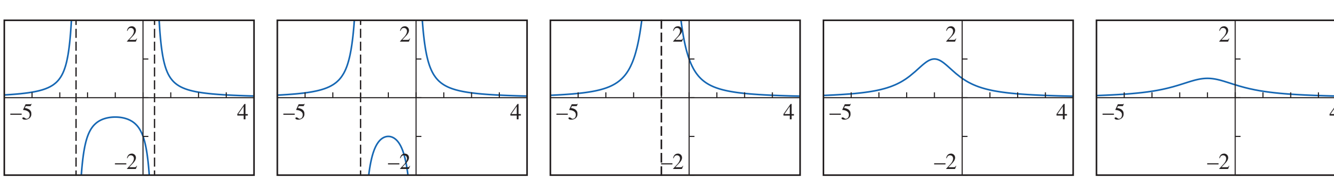
$$\lim_{x\to -1} \frac{1}{x^2 + 2x + 1} = \lim_{x\to -1} \frac{1}{(x+1)^2} = \infty.$$

When $c < 1$, there are two vertical asymptotes: $x = -1 \pm \sqrt{1 - c}$, as in Figure 4.59.

Consider the first derivative: $f'(x) = -\dfrac{2x + 2}{(x^2 + 2x + c)^2}$.

This shows that $f'(x) = 0$ when $x = -1$ (if $c \neq 1$), $f'(x) > 0$ when $x < -1$, and $f'(x) < 0$ when $x > -1$. For $c \geq 1$, this means that f is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$. For $c > 1$, there is an absolute maximum value $f(-1) = \dfrac{1}{c - 1}$. For $c < 1$, $f(-1) = \dfrac{1}{c - 1}$ is a local maximum value and the intervals on which the function is increasing or decreasing are interrupted at the vertical asymptotes.

Figure 4.60 displays five members of the family, all graphed in the viewing rectangle $[-5, 4] \times [-2, 2]$. As our analysis suggested, $c = 1$ is the value at which a transition takes place from two vertical asymptotes to one, and then to none. As c increases from 1, we see that the maximum point becomes lower; this is explained by the fact that $1/(c-1) \to 0$ as $c \to \infty$. As c decreases from 1, the vertical asymptotes become more widely separated because the distance between them is $2\sqrt{1 - c}$, which becomes large as $c \to -\infty$. Again, the maximum point approaches the x-axis because $1/(c-1) \to 0$ as $c \to \infty$.



$c = -1$ $c = 0$ $c = 1$ $c = 2$ $c = 3$

Figure 4.60

The family of functions $f(x) = \dfrac{1}{x^2 + 2x + c}$.

The graphs in Figure 4.60 suggest there is no inflection point when $c \leq 1$. For $c > 1$ consider the second derivative.

$$f''(x) = \frac{2(3x^2 + 6x + 4 - c)}{(x^2 + 2x + c)^3}$$

Analysis of this expression indicates that inflection points occur when $x = -1 \pm \frac{\sqrt{3(c-1)}}{3}$. So the inflection points become more spread out as c increases. This conclusion seems reasonable from the last two graphs in Figure 4.60. ■

In Section 1.7, we used technology to graph parametric curves, and in Section 3.4, we found tangents to parametric curves. But as the next example shows, we can use calculus to ensure that a parameter interval or a viewing rectangle will reveal all the important aspects of a curve.

Example 6 Graph with the Important Features of a Parametric Curve

Graph the curve with parametric equations

$$x(t) = t^2 + t + 1 \qquad y(t) = 3t^4 - 8t^3 - 18t^2 + 25$$

in a viewing rectangle that displays the important features of the curve. Find the coordinates of the significant points on the curve.

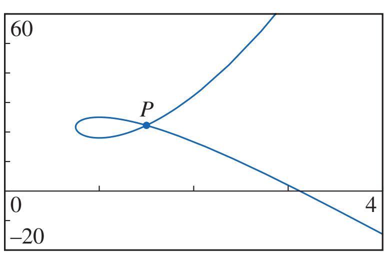


Figure 4.61
The curve intersects itself at the point P.

Solution

Figure 4.61 shows the graph of this curve in the viewing rectangle $[0, 4] \times [-20, 60]$, for $-2 \leq t \leq 1.4$. Zoom in and use the Trace feature to estimate the coordinates of the point where the curve intersects itself, $P(1.50, 22.25)$.

The highest point on the loop is approximately $(1, 25)$, the lowest point is $(1, 18)$, and the leftmost point is about $(0.75, 21.7)$.

Use calculus techniques to make sure that all of the significant points have been identified.

Using Equation 3.4.7: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{12t^3 - 24t^2 - 36t}{2t + 1}$

The vertical tangent line occurs when $\dfrac{dx}{dt} = 2t + 1 = 0 \Rightarrow t = -\dfrac{1}{2}$.

The exact coordinates of the leftmost point of the loop are

$$\left(x\left(-\frac{1}{2}\right), y\left(-\frac{1}{2}\right)\right) = (0.75, 21.6875).$$

Also, $\dfrac{dy}{dt} = 12t(t^2 - 2t - 3) = 12t(t + 1)(t - 3)$.

So horizontal tangents occur when $t = 0, -1$, and 3. The bottom of the loop corresponds to $t = -1$ and its coordinates are $(x(-1), y(-1)) = (1, 18)$. Similarly, the coordinates of the top of the loop are exactly as we estimated: $(x(0), y(0)) = (1, 25)$.

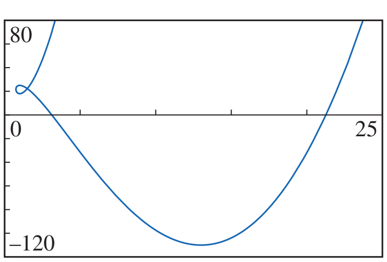


Figure 4.62
Graph that shows the lowest point on the curve.

When $t = 3$, the corresponding point on the curve has coordinates $(x(3), y(3)) = (13, -110)$. Figure 4.62 shows the graph of the parametric curve in the viewing rectangle $[0, 25] \times [-120, 80]$. This shows that the point $(13, -110)$ is the lowest point on the curve, We can now feel confident that there are no hidden maximum or minimum values. ■

4.4 Exercises

Use technology to produce graphs of f that reveal all the important aspects of the curve. In particular, use graphs of f' and f'' to estimate the intervals on which f is increasing or decreasing, the extreme values, intervals of concavity, and inflection points.

1. $f(x) = 4x^4 - 32x^3 + 89x^2 - 95x + 29$

2. $f(x) = x^6 - 15x^5 + 75x^4 - 125x^3 - x$

3. $f(x) = x^6 - 10x^5 - 400x^4 + 2500x^3$

4. $f(x) = \dfrac{x^2 - 1}{40x^3 + x + 1}$

5. $f(x) = \dfrac{x}{x^3 - x^2 - 4x + 1}$

6. $f(x) = \tan x + 5\cos x$

7. $f(x) = x^2 - 4x + 7\cos x, \quad -4 \le x \le 4$

8. $f(x) = \dfrac{e^x}{x^2 - 9}$

9. $f(x) = x - \ln x \cdot \sin x, \quad 0 < x \le 10$

Use technology to produce graphs of f that reveal all the important aspects of the curve. Estimate the intervals on which f is increasing or decreasing and intervals of concavity, and use calculus to find these intervals exactly.

10. $f(x) = 1 + \dfrac{1}{x} + \dfrac{8}{x^2} + \dfrac{1}{x^3}$

11. $f(x) = \dfrac{1}{x^8} - \dfrac{2 \times 10^8}{x^4}$

Sketch the graph of f using information about asymptotes and intercepts, but not derivatives. Then use your sketch as a guide to produce graphs using technology that display the major features of the curve. Use these graphs to estimate the maximum and minimum values.

12. $f(x) = \dfrac{(x + 4)(x - 3)^2}{x^4(x - 1)}$

13. $f(x) = \dfrac{(2x + 3)^2(x - 2)^5}{x^3(x - 5)^2}$

14. Let f be the function presented in Example 3. Use technology to graph f' and use it to confirm the maximum and minimum values found in the example. Use technology to graph f'' and use the graph to estimate the intervals of concavity and inflection point.

15. Let f be the function defined in Exercise 13. Use technology to graph f' and f'' and use these graphs to estimate the intervals on which f is increasing or decreasing and the intervals of concavity.

Use technology to graph f, f', and f''. Use these graphs to estimate the intervals on which f is increasing or decreasing, extreme values, intervals of concavity, and inflection points of f.

16. $f(x) = \dfrac{\sqrt{x}}{x^2 + x + 1}$

17. $f(x) = \dfrac{x^{2/3}}{1 + x + x^4}$

18. $f(x) = \sqrt{x + 5\sin x}, \quad x \le 20$

19. $f(x) = (x^2 - 1)e^{\arctan x}$

20. $f(x) = \dfrac{1 - e^{1/x}}{1 + e^{1/x}}$

21. Graph $f(x) = e^x + \ln|x^2 - 4|$ in as many viewing rectangles as necessary to depict the true nature of the function.

22. In Example 4, we considered a member of the family of functions $f(x) = \sin(x + \sin cx)$ that occur in FM synthesis. In this exercise, we investigate the function with $c = 3$.

Start by graphing f in the viewing rectangle $[0, \pi] \times [-1.2, 1.2]$. How many local maximum points are visible? There are actually more than are visible on this graph. To discover the hidden maximum and minimum points you will need to examine the graph of f' very carefully. In fact, it helps to look at the graph of f'' at the same time. Find all the maximum and minimum values and inflection points. Then graph in the viewing rectangle $[-2\pi, 2\pi] \times [-1.2, 1.2]$ and comment on symmetry.

23. Use a graph to estimate the coordinates of the leftmost point on the curve $x = t^4 - t^2$, $y = t + \ln t$. Then use calculus to find the exact coordinates.

Graph the curve in a viewing rectangle that displays all the important aspects of the curve. Find the points on the curve at which there are vertical or horizontal tangent lines.

24. $x = t^4 - 2t^3 - 2t^2, \quad y = t^3 - t$

25. $x = t^4 + 4t^3 - 8t^2, \quad y = 2t^2 - t$

26. Investigate the family of curves given by the parametric equations $x = t^3 - ct$, $y = t^2$. In particular, determine the values of c for which there is a loop and find the point where the curve intersects itself. What happens to the loop as c increases? Find the coordinates of the leftmost and rightmost points of the loop.

27. The family of functions $f(t) = C(e^{-at} - e^{-bt})$, where a, b, and C are positive numbers and $b > a$, has been used to model the concentration of a drug injected into the bloodstream at time $t = 0$. Graph several members of this family. What do

they have in common? For fixed values of C and a, discover graphically what happens as b increases. Then use calculus to prove what you have discovered.

Describe how the graph of f varies as c varies. Graph several members of the family to illustrate the patterns that you discover. In particular, you should investigate how maximum and minimum points and inflection points move when c changes. You should also identify any transitional values of c at which the basic shape of the curve changes.

28. $f(x) = x^4 + cx^2$

29. $f(x) = x^3 + cx$

30. $f(x) = e^x + ce^{-x}$

31. $f(x) = \ln(x^2 + c)$

32. $f(x) = \dfrac{cx}{1 + c^2x^2}$

33. $f(x) = \dfrac{1}{(1 - x^2)^2 + cx^2}$

34. $f(x) = cx + \sin x$

35. Investigate the family of curves given by the equation $f(x) = x^4 + cx^2 + x$. Start by determining the transitional value of c at which the number of inflection points changes. Then graph several members of the family to see what shapes are possible. There is another transitional value of c at which the number of critical numbers changes. Try to discover it graphically. Then prove what you have discovered.

36. (a) Investigate the family of polynomials given by the equation $f(x) = cx^4 - 2x^2 + 1$. For what values of c does the curve have minimum points?

(b) Show that the minimum and maximum points of every curve in the family lie on the parabola $y = 1 - x^2$. Illustrate by graphing this parabola and several members of the family.

37. (a) Investigate the family of polynomials given by the equation $f(x) = 2x^3 + cx^2 + 2x$. For what values of c does the curve have maximum and minimum points?

(b) Show that the minimum and maximum points of every curve in the family lie on the curve $y = x - x^3$. Illustrate by graphing this curve and several members of the family.

4.5 Indeterminate Forms and L'Hospital's Rule

Suppose we would like to analyze and sketch a graph of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although F is not defined at $x = 1$, we still need to know how F behaves *near* 1. In particular, we need to know the value of the limit

$$\lim_{x \to 1} \frac{\ln x}{x - 1} \tag{1}$$

In order to evaluate this limit, first consider the Limit Laws; the limit of a quotient is the quotient of the limits. As $x \to 1$, $\ln x \to 0$, and as $x \to 1$, $x - 1 \to 0$. Direct substitution leads to $\frac{0}{0}$, an *indeterminate form,* and is an indication we need a different strategy to evaluate this limit.

In general, a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, is called an **indeterminate form of type** $\mathbf{\frac{0}{0}}$ and the limit may or may not exist.

Some limits of this type were presented in Chapter 2, along with some solution techniques. If the quotient is a rational function, then we might be able to cancel common factors. For example,

Direct substitution leads to $\frac{0}{0}$.

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x - 1)}{(x + 1)(x - 1)} = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}$$

A geometric argument was used to show

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Neither of these methods can be used to evaluate the limit in Equation 1. We need a systematic method, known as *l'Hospital's Rule,* to evaluate limits in an indeterminate form.

Consider another situation in which we look for a horizontal asymptote of F; the limit is not obvious and other solution techniques do not help:

$$\lim_{x\to\infty} \frac{\ln x}{x-1} \tag{2}$$

In this case, as $x \to \infty$, both the numerator and the denominator increase without bound. If the numerator increases significantly faster than the denominator, then the limit will be ∞. If the denominator is increasing faster, then the fraction goes to 0. There could also be a compromise, in which case the limit will be some finite positive number.

In general, a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (of $-\infty$), is called an **indeterminate form of type $\frac{\infty}{\infty}$** and the limit may or may not exist.

Some limits of this type were also presented in Chapter 2, along with some solution techniques. If the quotient is a rational function, then we divide the numerator and the denominator by the highest power of x that occurs in the denominator. For example,

$$\lim_{x\to\infty} \frac{x^2-1}{2x^2+1} = \lim_{x\to\infty} \frac{1-\dfrac{1}{x^2}}{2+\dfrac{1}{x^2}} = \frac{1-0}{2+0} = \frac{1}{2}$$

This method does not work for limits like those in Equation 2, but l'Hospital's Rule also applies to this type of indeterminate form.

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x\to a} g(x) = \pm\infty$$

That is, the limit $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is in an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

A Closer Look

1. L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is very important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule. That is, we can apply l'Hospital's Rule only if the limit is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

2. L'Hospital's Rule is also valid for one-sided limits and for limits involving infinity or negative infinity; that is, "$x \to a$" can be replaced by any of these expressions:

$$x \to a^+,\ x \to a^-,\ x \to \infty, \text{ or } x \to -\infty.$$

3. For the special case where $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$, we can show analytically why l'Hospital's Rule is true and also illustrate this situation graphically.

Use the alternate form of the definition of a derivative.

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} \qquad f' \text{ and } g' \text{ are continuous; } g'(a) \neq 0.$$

$$= \frac{\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \qquad \text{Definition of derivative; Limit Laws.}$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)} \qquad \text{Simplify; } f(a) = g(a) = 0.$$

Figure 4.63 shows the graphs of two differentiable functions f and g such that $f(a) = g(a) = 0$.

Since f and g are differentiable, both graphs are locally linear. Figure 4.64 shows the linear approximations to f and g as we zoom in near $x = a$. Therefore, near a, the quotient $f(x)/g(x)$ is the ratio of the linear approximations.

$$\frac{f(x)}{g(x)} \approx \frac{m_1(x-a)}{m_2(x-a)} = \frac{m_1}{m_2}$$

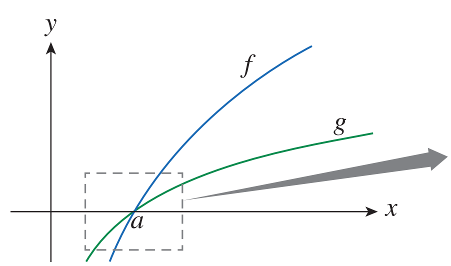


Figure 4.63
f and g are differentiable, and $f(a) = g(a) = 0$.

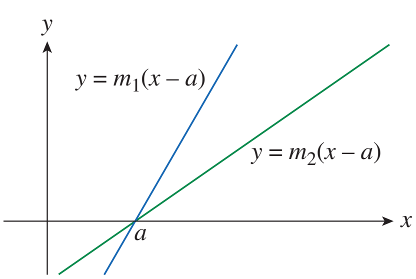


Figure 4.64
Near a, graphs of f and g are both locally linear.

In this case, the quotient simplifies to the ratio of their derivatives. This argument suggests that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Example 1 An Indeterminate Form of Type $\frac{0}{0}$

Find $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$.

Solution

Verify the conditions regarding the limits of the numerator and the denominator.

$$\lim_{x\to 1} \ln x = \ln 1 = 0 \quad \text{and} \quad \lim_{x\to 1}(x-1) = 0$$

Common Error

In applying l'Hospital's Rule, take the derivative of the quotient using the Quotient Rule.

Correct Method

When using l'Hospital's Rule, differentiate the numerator and the denominator *separately*. Do *not* use the Quotient Rule.

The limit is an indeterminate form of type $\frac{0}{0}$. We can apply l'Hospital's Rule.

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1}\frac{\frac{1}{x}}{1} \qquad \text{Differentiate numerator and denominator.}$$

$$= \lim_{x\to 1}\frac{1}{x} = 1 \qquad \text{Simplify; Limit Laws.}$$

Example 2 An Indeterminate Form of Type $\frac{\infty}{\infty}$

Find $\lim\limits_{x\to\infty}\frac{e^x}{x^2}$.

Solution

Verify the conditions regarding the limits of the numerator and the denominator.

$$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \lim_{x\to\infty} x^2 = \infty$$

The limit is an indeterminate form of type $\frac{\infty}{\infty}$. We can apply l'Hospital's Rule.

$$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x\to\infty}\frac{e^x}{2x} \qquad \text{Differentiate numerator and denominator.}$$

Since $\lim\limits_{x\to\infty} e^x = \infty$ and $\lim\limits_{x\to\infty}(2x) = \infty$, the resulting limit is also indeterminate.

Apply l'Hospital's Rule a second time.

$$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2} = \infty \qquad \text{Differentiate numerator and denominator; evaluate the resulting limit.}$$

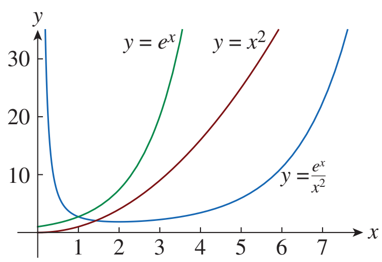


Figure 4.65

As x increases without bound, the numerator increases without bound much faster than the denominator. Therefore, the function $y = \frac{e^x}{x^2}$ is also increasing without bound. Exponential functions grow much faster than power functions.

The graph of $y = \frac{e^x}{x^2}$ in Figure 4.65 shows that this exponential function (the numerator) grows faster than this power function (the denominator).

Since e^x grows faster than x^2 as $x \to \infty$, we could have reasoned that the limit in this example is ∞.

Example 3 Simplify After Applying L'Hospital's Rule

Find $\lim\limits_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}$.

Solution

Verify the conditions regarding the limits of the numerator and the denominator.

$$\lim_{x\to\infty}\ln x = \infty \quad \text{and} \quad \lim_{x\to\infty}\sqrt[3]{x} = \infty$$

The limit is an indeterminate form of type $\frac{\infty}{\infty}$. Apply l'Hospital's Rule.

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}}$$ Differentiate numerator and denominator.

$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{3\sqrt[3]{x^2}}}$$ Rewrite denominator.

$$= \lim_{x\to\infty}\frac{3}{\sqrt[3]{x}} = 0$$ Simplify the expression. Evaluate the limit.

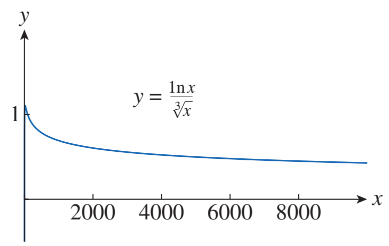


Figure 4.66

Graph of $y = \dfrac{\ln x}{\sqrt[3]{x}}$. As x increases without bound, the function approaches 0. The natural logarithm function grows more slowly than the cube root function.

The graph of $y = \dfrac{\ln x}{\sqrt[3]{x}}$. is shown in Figure 4.66. We have previously seen examples that illustrate the slow growth of logarithms, so it seems reasonable that this ratio approaches 0 as $x \to \infty$.

In Examples 2 and 3, both limits were of indeterminate form $\frac{\infty}{\infty}$, but the final answers were different. In Example 2, the infinite limit indicates that the numerator e^x increases significantly faster than the denominator x^2, which leads to larger and larger ratios. In fact, $y = e^x$ grows more quickly than *all* power functions $y = x^n$. In Example 3, the limit of 0 means that the denominator grows faster than the numerator, and the ratio approaches 0 as x increases without bound.

Example 4 L'Hospital's Rule More Than Once

Find $\displaystyle\lim_{x\to 0}\frac{\tan x - x}{x^3}$.

Solution

Verify the conditions regarding the limits of the numerator and the denominator.

$$\lim_{x\to 0}(\tan x - x) = 0 \quad \text{and} \quad \lim_{x\to 0} x^3 = 0$$

The limit is an indeterminate form of type $\frac{0}{0}$. Apply l'Hospital's Rule.

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}$$ Differentiate numerator and denominator. Resulting limit is still indeterminate of type $\frac{0}{0}$.

$$= \lim_{x\to 0}\frac{2\sec^2 x \tan x}{6x}$$ Apply l'Hospital's Rule again.

Because $\displaystyle\lim_{x\to 0}\sec^2 x = 1$, rewrite the resulting limit.

$$= \frac{1}{3}\lim_{x\to 0}\sec^2 x \cdot \lim_{x\to 0}\frac{\tan x}{x}$$ Write as the product of limits.

This last limit can be evaluated using l'Hospital's Rule again or by writing $\tan x = \dfrac{\sin x}{\cos x}$ and using known trigonometric limits.

$$= \frac{1}{3}\cdot 1 \cdot \lim_{x\to 0}\frac{\tan x}{x} = \frac{1}{3}\lim_{x\to 0}\frac{\sec^2 x}{1} = \frac{1}{3}$$ L'Hospital's Rule again; evaluate resulting limit.

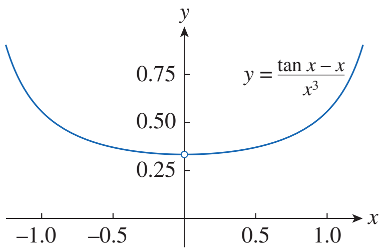


Figure 4.67

Graph of $y = \dfrac{\tan x - x}{x^3}$. As x approaches 0, the function approaches $\dfrac{1}{3}$.

Figure 4.67 confirms this result. Try graphing this function using technology and zoom in near 0. The graph becomes inaccurate because $\tan x$ is close to x when x is small. ■

Example 5 L'Hospital's Rule Without Checking for an Indeterminate Form

Find $\displaystyle\lim_{x\to\pi^-} \frac{\sin x}{1 - \cos x}$.

Solution

Suppose l'Hospital's Rule is used without checking for an indeterminate form.

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \lim_{x\to\pi^-} \frac{\cos x}{\sin x} \qquad \text{Differentiate numerator and denominator.}$$

$$= -\infty \qquad \cos x \to -1 \text{ and } \sin x \to 0 \text{ through small positive values. Therefore, the quotient is decreasing without bound.}$$

This analysis is wrong!

$$\lim_{x\to\pi^-} \sin x = 0 \quad \text{and} \quad \lim_{x\to\pi^-} (1 - \cos x) = 2$$

Therefore, the limit is not an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

L'Hospital's Rule cannot be used here.

Since this function is continuous at π and the denominator is nonzero there, use direct substitution.

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1-(-1)} = 0$$ ■

Common Error

Apply l'Hospital's Rule to any limit involving a quotient.

Correct Method

L'Hospital's Rule can be used only when the limit is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

One word of caution: L'Hospital's Rule is an amazing technique, but many limits may be determined using other, often simpler, methods. So, don't forget all of the other techniques already discussed for evaluating limits, and consider some of these other methods before resorting to l'Hospital's Rule.

Indeterminate Products

Suppose $\displaystyle\lim_{x\to a} f(x) = 0$ and $\displaystyle\lim_{x\to a} g(x) = \infty$ (or $-\infty$). Then the limit

$$\lim_{x\to a} [f(x)g(x)]$$

is not clear. There is a numerical struggle between the functions f and g. If f *wins*, then the overall limit is 0. If g *wins*, then the answer is ∞ (or $-\infty$). Or once again, there may be a compromise in which the answer is a finite nonzero number.

This kind of limit is called an **indeterminate form of type 0 · ∞**. A technique for evaluating a limit of this type is to rewrite the product $f(x)g(x)$ as a quotient in one of two ways:

$$f(x)g(x) = \frac{f(x)}{\left[\dfrac{1}{g(x)}\right]} \qquad \text{This converts the given limit to an indeterminate form of type } \tfrac{0}{0}.$$

$$f(x)g(x) = \frac{g(x)}{\left[\dfrac{1}{f(x)}\right]} \qquad \text{This converts the given limit to an indeterminate form of type } \tfrac{\infty}{\infty}.$$

We can then apply l'Hospital's Rule to this related equivalent expression.

Example 6 Evaluate an Indeterminate Product

Find $\lim\limits_{x \to 0^+} x \ln x$.

Solution

Verify the indeterminate form.

$$\lim_{x \to 0^+} x = 0 \text{ and } \lim_{x \to 0^+} \ln x = -\infty$$

The limit is an indeterminate form of type $0 \cdot \infty$.

Rewrite the product as a quotient and apply l'Hospital's Rule.

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \qquad \text{Rewrite as a quotient; Limit is now of type } \tfrac{\infty}{\infty}.$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \qquad \text{Differentiate numerator and denominator.}$$

$$= \lim_{x \to 0^+} (-x) = 0 \qquad \text{Simplify; evaluate.}$$

Let's consider the graph of $f(x) = x \ln x$.

If $f(x) = x \ln x$, then $f'(x) = x \cdot \dfrac{1}{x} + \ln x = 1 + \ln x$.

$f'(x) = 0 \;\Rightarrow\; \ln x = -1 \;\Rightarrow\; x = e^{-1}$

$f'(x) > 0$ on (e^{-1}, ∞) and $f'(x) < 0$ on $(0, e^{-1})$
f is increasing on $[e^{-1}, \infty)$ and f is decreasing on $(0, e^{-1}]$.

By the First Derivative Test, $f(e^{-1}) = -1/e$ is a local (and absolute) minimum.

$f''(x) = \dfrac{1}{x} > 0$, so f is concave upward on $(0, \infty)$.

The function is undefined at $x = 0$. However, the value of the function approaches 0 as x approaches 0 from the right.

All of this information is used to sketch the graph of $y = x \ln x$, shown in Figure 4.68.

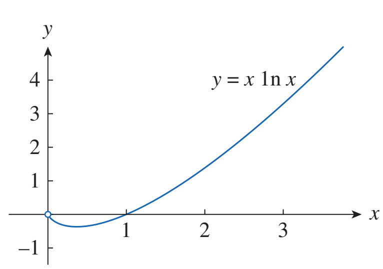


Figure 4.68
Graph of $y = x \ln x$. As $x \to 0^+$, the value of the function approaches 0; the graph approaches the origin.

Note that we could also rewrite this limit as

$$\lim_{x \to 0^+} x \ln x = \frac{x}{\frac{1}{\ln x}}.$$

The resulting limit of this equivalent expression is an indeterminate form of type $\frac{0}{0}$. However, if l'Hospital's Rule is used, a more complicated expression is obtained. In general, when we rewrite an indeterminate product, we try to choose the option that leads to a simpler limit. ■

Indeterminate Differences

If $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} g(x) = \infty$, then

$$\lim_{x \to a} [f(x) - g(x)]$$

We cannot write this limit as the difference of limits. That is

$$\lim_{x\to a}[f(x)-g(x)] \neq \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$$

is called an **indeterminate form of type $\infty - \infty$**. Once again, this is a struggle between f and g. If f is increasing faster, then the limit is ∞. If g is increasing faster, the limit is $-\infty$. Or, there could be a compromise, and the limit could be a finite number.

This indeterminate form of type $\infty - \infty$ really represents two possibilities, $\infty - \infty$ or $(-\infty) - (-\infty)$, a difference of like signs. And we can replace $x \to a$ with a one-sided limit or a limit as x increases or decreases without bound.

To evaluate this limit, we rewrite the difference $f(x) - g(x)$ to obtain either

(1) a quotient that yields the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and then use l'Hospital's Rule or

(2) a product that yields the indeterminate form $0 \cdot \infty$ then rewrite the product as a quotient and use l'Hospital's Rule.

Rewriting often involves a common denominator, rationalizing, or factoring.

Example 7 A Numerical Compromise

Find $\displaystyle\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

Solution

Verify the indeterminate form.

$$\lim_{x\to 1^+}\frac{1}{\ln x} = \infty \quad \text{and} \quad \lim_{x\to 1^+}\frac{1}{x-1} = \infty$$

The limit is an indeterminate from of type $\infty - \infty$.

Rewrite the difference as a quotient and apply l'Hospital's Rule.

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x}$$

Write as one fraction with a common denominator.

Both the numerator and the denominator have limit 0; apply l'Hospital's Rule.

$$= \lim_{x\to 1^+}\frac{1-\dfrac{1}{x}}{(1-x)\cdot\dfrac{1}{x} + \ln x \cdot 1}$$

Differentiate numerator and denominator.

$$= \lim_{x\to 1^+}\frac{x-1}{x-1+x\ln x}$$

Simplify.

Both the numerator and the denominator still have limit 0; apply l'Hospital's Rule again.

$$= \lim_{x\to 1^+}\frac{1}{1+x\cdot\dfrac{1}{x}+1\cdot\ln x}$$

Differentiate numerator and denominator.

$$= \lim_{x\to 1^+}\frac{1}{2+\ln x} = \frac{1}{2}$$

Simplify; evaluate.

Indeterminate Powers

Several indeterminate forms arise from the limit $\lim_{x\to a} [f(x)]^{g(x)}$.

Note that 0^∞ is *not* an indeterminate form. See Exercise 94.

(1) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0

(2) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0

(3) $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = +\infty$ type 1^∞

Each of these indeterminate forms can be evaluated by first taking the natural logarithm of the function $[f(x)]^{g(x)}$.

Let $y = [f(x)]^{g(x)} = e^{g(x)\ln f(x)}$, then $\ln y = g(x)\ln f(x)$.

As $x \to a$ the expression $[g(x)\ln f(x)]$ is an indeterminate form of type $0 \cdot \infty$.

We can rewrite this product as a quotient in an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use l'Hospital's Rule.

Remember, the natural exponential function and the natural logarithmic function are inverse functions. Therefore, $y = e^{\ln y}$.

Finally, to find the original limit, suppose $\lim_{x\to a} \ln y = L$. Then

$$\lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = e^{\lim_{x\to a} \ln y} = e^L \text{ and } \lim_{x\to a} [f(x)]^{g(x)} = e^L.$$

Example 8 An Indeterminate Form of Type 1^∞

Find $\lim_{x\to 0^+} (1 + \sin 4x)^{\cot x}$.

Solution

Verify the indeterminate form.

$\lim_{x\to 0^+} (1 + \sin 4x) = 1$ and $\lim_{x\to 0^+} \cot x = \infty$

The limit is an indeterminate form of type 1^∞.

Let $y = (1 + \sin 4x)^{\cot x}$.

Take the natural logarithm of both sides.

$$\ln y = \ln\left[(1 + \sin 4x)^{\cot x}\right] = \cot x \ln(1 + \sin 4x) = \frac{\ln(1 + \sin 4x)}{\tan x}$$

$\lim_{x\to 0^+} \ln(1 + \sin 4x) = 0$ and $\lim_{x\to 0^+} \tan x = 0$

Use l'Hospital's Rule.

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x}$$

$$= \lim_{x\to 0^+} \frac{\dfrac{4\cos 4x}{1 + \sin 4x}}{\sec^2 x} = \frac{\dfrac{4 \cdot 1}{1 + 0}}{1} = 4$$

Differentiate numerator and denominator; evaluate the limit.

Find the limit of y.

$$\lim_{x\to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^4$$

$y = e^{\ln y}$.

Example 9 An Indeterminate Form of Type 0^0

Find $\lim\limits_{x \to 0^+} x^x$.

Solution

This limit is an indeterminate form of type 0^0.

Note that $0^x = 0$ for any $x > 0$ and $x^0 = 1$ for any $x \neq 0$.

Let $y = x^x$ and take the natural logarithm of both sides.

$\ln y = \ln(x^x) = x \ln x$

$\lim\limits_{x \to 0^+} x = 0 \quad \text{and} \quad \lim\limits_{x \to 0^+} \ln x = -\infty$

The limit $\lim\limits_{x \to 0^+} \ln y$ is an indeterminate form of type $0 \cdot \infty$.

In Example 6, we used l'Hospital's Rule to evaluate the limit $\ln y$ and found that

$\lim\limits_{x \to 0^+} x \ln x = 0.$

Find the limit of y.

$\lim\limits_{x \to 0^+} x^x = \lim\limits_{x \to 0^+} y = \lim\limits_{x \to 0^+} e^{\ln y} = e^0 = 1$

The graph of the function $y = x^x$, for $x > 0$ is shown in Figure 4.69. Notice that 0^0 is undefined, but the values of the function approach 1 as $x \to 0^+$. This graph supports the analytical results obtained using l'Hospital's Rule. ■

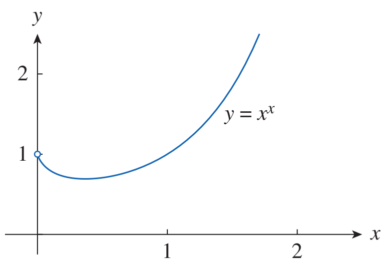


Figure 4.69
Graph of $y = x^x$ for $x > 0$. The values of the function approach 1 as x approaches 0 from the right.

4.5 Exercises

Given that

$\lim\limits_{x \to a} f(x) = 0 \quad \lim\limits_{x \to a} g(x) = 0 \quad \lim\limits_{x \to a} h(a) = 1$

$\lim\limits_{x \to a} p(x) = \infty \quad \lim\limits_{a \to a} q(x) = \infty$

which of the following limits are indeterminate forms? For any limit that is not an indeterminate form, evaluate the limit where possible.

1. (a) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ (b) $\lim\limits_{x \to a} \dfrac{f(x)}{p(x)}$ (c) $\lim\limits_{x \to a} \dfrac{h(x)}{p(x)}$

(d) $\lim\limits_{x \to a} \dfrac{p(x)}{f(x)}$ (e) $\lim\limits_{x \to a} \dfrac{p(x)}{q(x)}$

2. (a) $\lim\limits_{x \to a} [f(x)\, p(x)]$ (b) $\lim\limits_{x \to a} [h(x)\, p(x)]$

(c) $\lim\limits_{x \to a} [p(x)\, q(x)]$

3. (a) $\lim\limits_{x \to a} [f(x) - p(x)]$ (b) $\lim\limits_{x \to a} [p(x) - q(x)]$

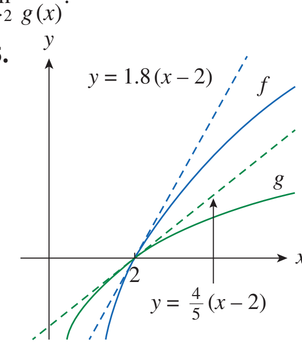
(c) $\lim\limits_{x \to a} [p(x) + q(x)]$

4. (a) $\lim\limits_{x \to a} [f(x)]^{g(x)}$ (b) $\lim\limits_{x \to a} [f(x)]^{p(x)}$ (c) $\lim\limits_{x \to a} [h(x)]^{p(x)}$

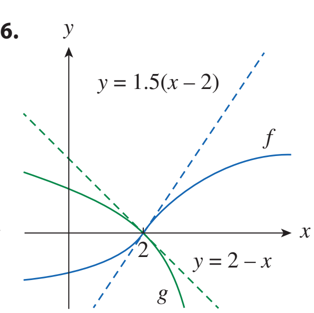
(d) $\lim\limits_{x \to a} [p(x)]^{f(x)}$ (e) $\lim\limits_{x \to a} [p(x)]^{q(x)}$ (f) $\lim\limits_{x \to a} \sqrt[q(x)]{p(x)}$

Use the graphs of f and g and their tangent lines at (2, 0) to find $\lim\limits_{x \to 2} \dfrac{f(x)}{g(x)}$.

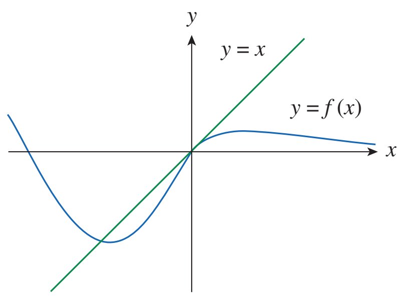
5.



6.



7. The graphs of a function f and its tangent line at the origin are shown in the figure.



Find the value of $\displaystyle\lim_{x\to 0}\frac{f(x)}{e^x-1}$.

8. For which of the following limit expressions does l'Hospital's Rule *not* apply?

(A) $\displaystyle\lim_{x\to 1}\frac{x^3-1}{x-1}$ (B) $\displaystyle\lim_{x\to 0}\frac{\ln(x+1)-x}{\sin x}$

(C) $\displaystyle\lim_{x\to -2}\frac{x^2-4}{x-2}$ (D) $\displaystyle\lim_{x\to 0}\frac{x^2\cos x+2x}{2x}$

Find the limit. Use l'Hospital's Rule where appropriate. If there is an applicable alternate method to l'Hospital's Rule, consider using it instead. If l'Hospital's Rule does not apply, explain why.

9. $\displaystyle\lim_{x\to 3}\frac{x-3}{x^2-9}$

10. $\displaystyle\lim_{x\to 4}\frac{x^2-2x-8}{x-4}$

11. $\displaystyle\lim_{x\to -2}\frac{x^3+8}{x+2}$

12. $\displaystyle\lim_{x\to 1}\frac{x^3-2x^2+1}{x^3-1}$

13. $\displaystyle\lim_{x\to 1/2}\frac{6x^2+5x-4}{4x^2+16x-9}$

14. $\displaystyle\lim_{x\to(\pi/2)^+}\frac{\cos x}{1-\sin x}$

15. $\displaystyle\lim_{x\to 0}\frac{\tan 3x}{\sin 2x}$

16. $\displaystyle\lim_{t\to 0}\frac{e^{2t}-1}{\sin t}$

17. $\displaystyle\lim_{x\to 0}\frac{x^2}{1-\cos x}$

18. $\displaystyle\lim_{\theta\to\pi/2}\frac{1-\sin\theta}{1+\cos 2\theta}$

19. $\displaystyle\lim_{\theta\to\pi}\frac{1+\cos\theta}{1-\cos\theta}$

20. $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$

21. $\displaystyle\lim_{x\to\infty}\frac{x+x^2}{1-2x^2}$

22. $\displaystyle\lim_{x\to 0^+}\frac{\ln x}{x}$

23. $\displaystyle\lim_{x\to\infty}\frac{\ln\sqrt{x}}{x^2}$

24. $\displaystyle\lim_{t\to 1}\frac{t^8-1}{t^5-1}$

25. $\displaystyle\lim_{t\to 0}\frac{8^t-5^t}{t}$

26. $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$

27. $\displaystyle\lim_{u\to\infty}\frac{e^{u/10}}{u^3}$

28. $\displaystyle\lim_{x\to 0}\frac{e^x-1-x}{x^2}$

29. $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}$

30. $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{x}$

31. $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^2}{x}$

32. $\displaystyle\lim_{x\to 0}\frac{x\,3^x}{3^x-1}$

33. $\displaystyle\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}$

34. $\displaystyle\lim_{x\to 0}\frac{\ln(1+\ln x)}{\cos x+e^x-1}$

35. $\displaystyle\lim_{x\to 1}\frac{x\sin(x-1)}{2x^2-x-1}$

36. $\displaystyle\lim_{x\to 0^+}\frac{\arctan(2x)}{\ln x}$

37. $\displaystyle\lim_{x\to 0^+}\frac{x^x-1}{\ln x+x-1}$

38. $\displaystyle\lim_{x\to 1}\frac{x^a-1}{x^b-1},\quad b\neq 0$

39. $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$

40. $\displaystyle\lim_{x\to 0}\frac{\cos x-1+\frac{1}{2}x^2}{x^4}$

41. $\displaystyle\lim_{x\to a^+}\frac{\cos x\ln(x-a)}{\ln(e^x-e^a)}$

42. $\displaystyle\lim_{x\to\infty}x\sin\left(\frac{\pi}{x}\right)$

43. $\displaystyle\lim_{x\to\infty}\sqrt{x}e^{-x/2}$

44. $\displaystyle\lim_{x\to 0}\sin 5x\csc 3x$

45. $\displaystyle\lim_{x\to -\infty}x\ln\left(1-\frac{1}{x}\right)$

46. $\displaystyle\lim_{x\to\infty}x^3e^{-x^2}$

47. $\displaystyle\lim_{x\to\infty}x^{3/2}\sin\left(\frac{1}{x}\right)$

48. $\displaystyle\lim_{x\to 1^+}\ln x\tan\left(\frac{\pi x}{2}\right)$

49. $\displaystyle\lim_{x\to(\pi/2)^-}\cos x\sec 5x$

50. $\displaystyle\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$

51. $\displaystyle\lim_{x\to 0}(\csc x-\cot x)$

52. $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

53. $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\tan^{-1}x}\right)$

54. $\displaystyle\lim_{x\to\infty}(x-\ln x)$

55. $\displaystyle\lim_{x\to 1^+}[\ln(x^7-1)-\ln(x^5-1)]$

56. $\displaystyle\lim_{x\to 0^+}x^{\sqrt{x}}$

57. $\displaystyle\lim_{x\to 0^+}(\tan 2x)^x$

58. $\displaystyle\lim_{x\to 0}(1-2x)^{1/x}$

59. $\displaystyle\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$

60. $\displaystyle\lim_{x\to 1^+}x^{1/(1-x)}$

61. $\displaystyle\lim_{x\to\infty}x^{(\ln 2)/(1+\ln x)}$

62. $\displaystyle\lim_{x\to\infty}x^{1/x}$

63. $\displaystyle\lim_{x\to\infty}x^{e^{-x}}$

64. $\displaystyle\lim_{x\to 0^+}(4x+1)^{\cos x}$

65. $\displaystyle\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$

66. $\displaystyle\lim_{x\to 0^+}(1+\sin 3x)^{1/x}$

67. $\displaystyle\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$

Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

68. $\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

69. $\displaystyle\lim_{x\to 0}\frac{5^x-4^x}{3^x-2^x}$

Illustrate l'Hospital's Rule by graphing both $\dfrac{f(x)}{g(x)}$ and $\dfrac{f'(x)}{g'(x)}$ near $x = 0$ to see that these ratios have the same limit as $x \to 0$. Find the exact value of this limit.

70. $f(x) = e^x - 1, \quad g(x) = x^3 + 4x$

71. $f(x) = 2x \sin x, \quad g(x) = \sec x - 1$

Use l'Hospital's Rule to help find the asymptotes on the graph of f. Use the results, together with information about f' and f'', to sketch the graph of f. Check your work using technology.

72. $f(x) = xe^{-x}$

73. $f(x) = \dfrac{e^x}{x}$

74. $f(x) = \dfrac{\ln x}{x}$

75. $f(x) = xe^{-x^2}$

(a) Graph the function.
(b) Use l'Hospital's Rule to explain the behavior of f as $x \to 0$.
(c) Estimate the minimum value and intervals of concavity. Then use calculus to find the exact values.

76. $f(x) = x^2 \ln x$

77. $f(x) = xe^{1/x}$

(a) Graph the function.
(b) Explain the shape of the graph by computing the limit as $x \to 0^+$ or as $x \to \infty$.
(c) Estimate the maximum and minimum values and then use calculus to find the exact values.
(d) Use a graph of f'' to estimate the x-coordinates of the inflection points.

78. $f(x) = x^{1/x}$

79. $f(x) = (\sin x)^{\sin x}$

80. Investigate the family of curves given by $f(x) = xe^{-cx}$, where c is a real number. Start by computing the limits $\lim_{x \to \pm\infty} f(x)$. Identify any transitional values of c where the basic shape changes. What happens to the maximum or minimum points and inflection points as c changes? Illustrate your results by graphing several members of this family of functions.

81. Investigate the family of curves given by $f(x) = x^n e^{-x}$, where n is a positive integer. What features do these curves have in common? How do they differ from one another? In particular, what happens to the maximum and minimum points and inflection points as n increases? Illustrate your results by graphing several members of this family of functions.

82. Investigate the family of curves $f(x) = e^x - cx$. In particular, find the limits $\lim_{x \to \pm\infty} f(x)$ and determine the values of c for which f has an absolute minimum. What happens to the minimum points as c increases?

83. Explain what happens if l'Hospital's Rule is used to evaluate

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}$$

Evaluate the limit using a different method.

84. Suppose the function f is continuous and twice differentiable for all values of x, $f(x) = 1$, $f'(0) = 1$, and $f''(0) = 2$. Find the value of

$$\lim_{x \to 0} \frac{f(x) - x - 1}{\sin(2x) - x^2 - 2x}$$

85. Show that

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$$

for any positive integer n. This limit demonstrates that the exponential function increases faster, or approaches infinity faster, than *any* power of x.

86. Show that

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$$

for any number $p > 0$. This limit shows that the logarithmic function approaches infinity more slowly than *any* power of x.

87. If an initial amount A_0 of money invested at an interest rate r is compounded n times per year, the value of the investment after t years is

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

The limit of this expression as $n \to \infty$ is called *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after t years is

$$A = A_0 e^{rt}$$

88. If an object with mass m is dropped from rest, one model for its velocity v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

where g is the acceleration due to gravity and c is a positive constant. Note that the air resistance is proportional to the velocity of the object; c is the proportionality constant.
(a) Find $\lim_{t \to \infty} v$. Explain the meaning of this limit in the context of this problem.
(b) For fixed t, use l'Hospital's Rule to calculate $\lim_{c \to 0^+} v$. What can you conclude about the velocity of a falling object in a vacuum?

89. If an electrostatic field E acts on a liquid or a gaseous polar dielectric, the net dipole moment P per unit volume is

$$P(E) = \frac{e^E + e^{-E}}{e^E - e^{-E}} - \frac{1}{E}$$

Show that $\lim_{E \to 0^+} P(E) = 0$.

90. A metal cable has radius r and is covered by insulation so that the distance from the center of the cable to the exterior of the insulation is R. The velocity v of an electrical impulse in the cable is

$$v = -c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$$

where c is a positive constant. Find each of the following limits and interpret your answer in the context of this problem.

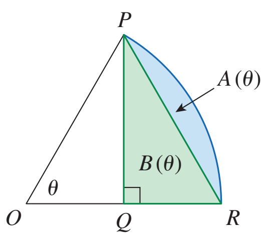
(a) $\lim_{R \to r^+} v$ (b) $\lim_{r \to 0^+} v$

91. The first appearance in print of l'Hospital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hospital in 1696. This was the first calculus textbook ever published and the example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. Note that at the time of publication, it was common to write aa instead of a^2. Find this limit.

92. The figure shows a sector of a circle with central angle θ.



Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}$.

93. Evaluate $\lim_{x \to \infty} \left[x - x^2 \ln\left(\frac{1 + x}{x}\right) \right]$.

94. Suppose f is a positive function. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, show that

$$\lim_{x \to a} [f(x)]^{g(x)} = 0$$

This shows that 0^∞ is not an indeterminate form.

95. If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x \to 0} \frac{f(2 + 3x) + f(2 + 5x)}{x}$$

96. For what values of a and b is the following equation true?

$$\lim_{x \to 0} \left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2} \right) = 0$$

97. If f' is continuous, use l'Hospital's Rule to show that

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x)$$

Explain the meaning of this expression by using a diagram.

98. If f'' is continuous, show that

$$\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x)$$

99. Let

$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(a) Show that f is continuous at 0.

(b) Investigate graphically whether f is differentiable at 0 by zooming in several times near the point (0, 1) on the graph of f.

(c) Show that f is not differentiable at 0. Explain how to reconcile this fact with the appearance of the graphs in part (b).

Writing Project | The Origins of L'Hospital's Rule


ANALYSE
DES
INFINIMENT PETITS,
POUR
L'INTELLIGENCE DES LIGNES COURBES.
Par M. le Marquis De L'Hospital.
SECONDE EDITION.
A PARIS,
Chez ETIENNE PAPILLON, rue S. Jacques, aux Armes d'Angleterre.
MDCCXVI
AVEC APPROBATION ET PRIVILEGE DU ROY.

L'Hospital's Rule was first published in 1696 in the Marquis de l'Hospital's calculus textbook *Analyse des Infiniment Petits*, but the rule was discovered in 1694 by the Swiss mathematician John (Johann) Bernoulli. The explanation is that these two mathematicians had entered into a curious business arrangement in which the Marquis de l'Hospital bought the rights to Bernoulli's mathematical discoveries. The details, including a translation of l'Hospital's letter to Bernoulli proposing the arrangement, can be found in the book by Eves [1].

Write a report on the historical and mathematical origins of l'Hospital's Rule. Start by providing brief biographical details of both men (the dictionary edited by Gillispie [2] is a good source) and outline the business deal between them. Then give l'Hospital's statement of his rule, which is found in Struik's sourcebook [4] and more briefly in the book of Katz [3]. Notice that l'Hospital and Bernoulli formulated the rule geometrically and gave the answer in terms of differentials. Compare their statement with the version of l'Hospital's Rule given in Section 4.5 and show that the two statements are essentially the same.

1. Howard W. Eves, *Mathematical Circles: Volume 1* (Washington, D.C.: Mathematical Association of America, 2003). First published 1969 as *In Mathematical Circles (Volume 2: Quadrants III and IV)* by Prindle Weber and Schmidt.
2. C. C. Gillispie, ed., *Dictionary of Scientific Biography,* 8 vols. (New York: Scribner, 1981). See the article on Johann Bernoulli by E. A. Fellmann and J. O. Fleckstein in Volume II and the article on the Marquis de l'Hospital by Abraham Robinson in Volume III.
3. Victor J. Katz, *A History of Mathematics: An Introduction.* 3rd ed. (New York: Pearson, 2018).
4. Dirk Jan Struik, ed. *A Source Book in Mathematics, 1200 –1800* (1969; repr., Princeton, NJ: Princeton University Press, 2016).

4.6 Optimization Problems

The methods presented in this chapter for finding extreme values have practical applications in many areas of life. For example, a businessperson may want to minimize costs or maximize profits. A traveler usually wants to minimize transportation time. Fermat's Principle in optics states that light follows the path that takes the least time. In this section, we will solve problems involving maximizing area, volumes, and profits, and minimizing distances, times, and costs.

In solving these practical problems, the greatest challenge is to convert the word problem into a mathematical optimization problem, that is, translate the words into mathematics. This involves defining a function to maximize or minimize. Recall the problem-solving principles discussed earlier and adapt them to this situation.

Steps in Solving Optimization Problems

1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?

2. **Draw a diagram** In most cases it is useful to draw a diagram to visualize the problem and to identify the given and required quantities on the diagram.
3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (call it Q for now). In addition, select symbols ($a, b, c, \ldots, x, y$) for other unknown quantities and label the diagram with these symbols. It may help to use initials as reasonable symbols; for example, A for area, h for height, or t for time.
4. **Find an Expression** Express Q in terms of some of the other symbols introduced in Step 3.
5. **Rewrite the Expression** If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Use these *auxiliary* or *helping* equations to eliminate all but one of the variables in the expression for Q. Therefore, Q will be expressed as a function of *one* variable, for example, x, written as $Q = f(x)$. Find the domain of this function in the context of the problem.
6. **Optimize** Use the methods presented in this chapter to find the *absolute* maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 4.2 can be used.

Example 1 Maximizing Area

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. They need no fence along the river. What are the dimensions of the field that has the largest area?

Solution

In order to better understand and visualize this problem, consider a few specific cases. Figure 4.70 (not drawn to scale) shows three possible fields, using all 2400 ft of fencing.

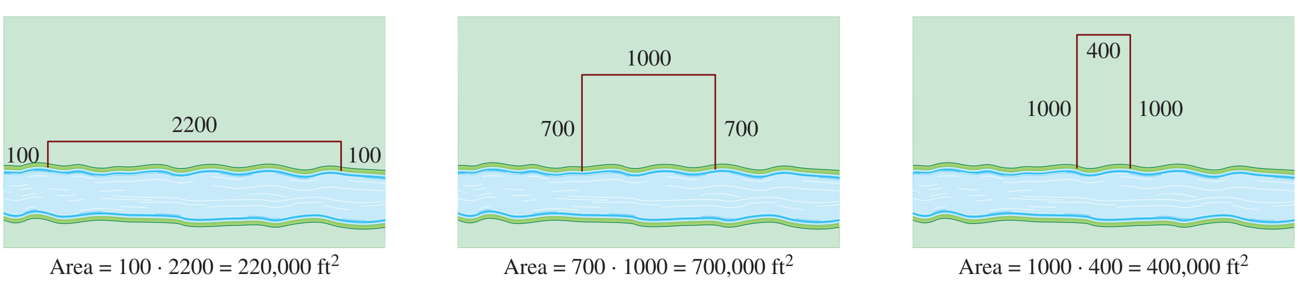


Figure 4.70
Three possible ways to create a rectangular field using all of the fencing available.

This illustration suggests that shallow, wide fields or deep, narrow fields result in relatively small areas. It seems reasonable that there is some intermediate configuration that produces the field with the largest area.

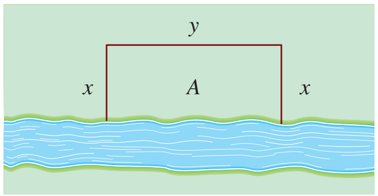


Figure 4.71
The general case: a field of dimensions x by y feet.

Figure 4.71 illustrates the general case.

We need to maximize the area A of the rectangle. Let x and y be the depth and width of the rectangle (in feet). We can express A in terms of x and y.

$A = xy$

Before we can optimize this expression, we need to express A as a function of just one variable. In this case, we can eliminate y by writing y in terms of x.

The total length of the fencing is 2400 ft. Therefore, $2x + y = 2400$.

Solve this equation for y: $y = 2400 - 2x$.

Use this equation for y in the expression for A.

$A = xy = x(2400 - 2x) = 2400x - 2x^2$

Note that the largest possible value for x is 1200 (this uses all the fencing for the depth and none for the width) and x cannot be negative. Therefore, the function we need to maximize is

$A(x) = 2400x - 2x^2, \quad 0 \le x \le 1200.$

NOTE: It may be more realistic to use the domain $0 < x < 1200$ for A. In this case, we can use the First Derivative Test to identify local and absolute extreme values.

Find the derivative of A and the critical numbers.

$A'(x) = 2400 - 4x$

$A'(x) = 0$: $2400 - 4x = 0 \quad \Rightarrow \quad x = 600$

$A'(x)$ DNE: None

Create a table of values for all relevant critical numbers and the endpoints.

x	$A(x)$	
0	0	
600	720,000	← absolute maximum value
1200	0	

Using the Closed Interval Method, or the Candidates Test, the maximum value is $A(600) = 720{,}000$.

Alternatively, we could use the Second Derivative Test to justify the solution.

$A''(4) = -4 < 0$ for all x. Therefore, the graph of A is concave down on the interval $(0, 1200)$, and the local maximum at $x = 600$ must be an absolute maximum.

The corresponding y-value is $y = 2400 - 2(600) = 1200$.

The rectangular field with the largest area has dimensions 600 ft deep by 1200 ft wide. ■

Example 2 Minimizing Cost

A cylindrical can is manufactured to hold 1 L of oil. Find the dimensions of the can that will minimize the cost of the metal to manufacture the can.

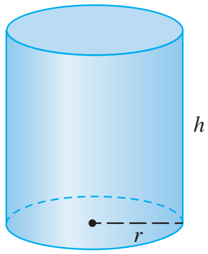


Figure 4.72
The can with radius r and height h.

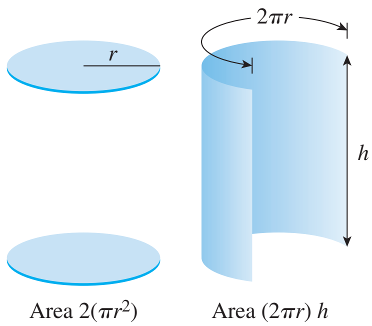


Figure 4.73
A visualization of the surface area.

Solution

Draw a diagram of the can, as in Figure 4.72, where r is the radius and h is the height (both in centimeters).

In order to minimize the cost of the metal, we need to minimize the total surface area of the cylinder, including the top, bottom, and sides. Figure 4.73 shows that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So the surface area is

$$A = 2\pi r^2 + 2\pi rh.$$

We need to express A in terms of one variable, either h or r. It seems easier to eliminate h because it appears only once in the expression for A.

To eliminate h, use the fact that the volume is given as 1 L, which is equivalent to 1000 cm^3. Therefore,

$$\pi r^2 h = 1000 \quad \Rightarrow \quad h = \frac{1000}{\pi r^2}.$$

Substitute this expression for h into the equation for A.

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

We know r must be positive, and there are no (theoretical) limitations on the magnitude of r. Therefore, we need to minimize

$$A = 2\pi r^2 + \frac{2000}{r}, \quad r > 0.$$

Find the derivative of A and the critical numbers.

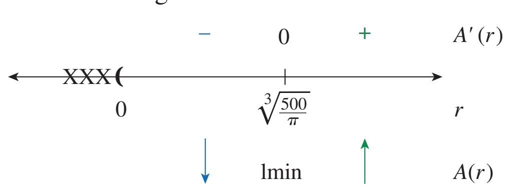
$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

$A'(r) = 0$: $\pi r^3 - 500 = 0 \quad \Rightarrow \quad r = \sqrt[3]{\frac{500}{\pi}}$

$A'(r)$ DNE: None [The domain of A is $(0, \infty)$.]

There is only one critical number and A is continuous, but not on a closed interval. Therefore, we cannot use the Closed Interval Method.

Construct a sign chart.



$A'(r)$ changes sign from negative to positive at $\sqrt[3]{\frac{500}{\pi}}$. Therefore, A has a local minimum at this value. Since this is the only number at which there is an extreme value in the domain, A must also have an *absolute* minimum at this value. Figure 4.74 shows a graph of A, which helps to visualize this result.

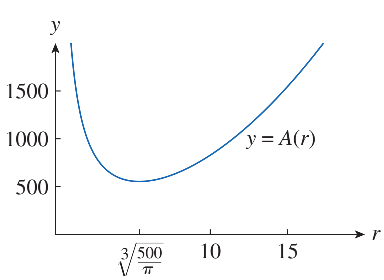


Figure 4.74
The graph of $y = A(r)$. Note that A increases without bound as $r \to 0^+$ and as $r \to \infty$.

The corresponding value of h is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

Therefore, to minimize the cost of the can, the radius should be $\sqrt[3]{\frac{500}{\pi}}$ and the height should be equal to twice the radius, namely, the diameter. ■

A Closer Look

1. The argument used in Example 2 to justify the absolute minimum is a variant of the First Derivative Test (which applies only to local maximum or minimum values) and is given here for future reference.

Common Error

If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ then $f(c)$ is the absolute maximum value of f.

Correct Method

Without the word *all*, that is, for *all* $x < c$ and for *all* $x > c$, the argument justifies only a local extrema.

First Derivative Test for Absolute Extreme Values

Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

2. An alternate method for solving optimization problems is to use implicit differentiation. Here is an illustration of this method applied to Example 2.

Start with the same equations.

$$A = 2\pi r^2 + 2\pi rh \qquad \pi r^2 h = 1000$$

Instead of eliminating h, differentiate both equations implicitly with respect to r. Use the Product Rule where necessary and remember that h is a function of r.

$$A' = 4\pi r + (2\pi rh' + 2\pi h) \qquad \pi r^2 h' + 2\pi rh = 0$$

The minimum occurs at a critical number. Set $A' = 0$ and simplify.

$2r + rh' + h = 0$ Set $A' = 0$; divide by 2π.

$rh' + 2h = 0$ In the second equation, divide by πr.

$2r + rh' + h - (rh' + 2h) = 0 - 0 = 0$ Subtract the second equation from the first.

$2r - h = 0 \Rightarrow h = 2r$ Simplify; solve for h.

This is the same result as in Example 2: the height should be equal to twice the radius. ■

Example 3 Minimizing Distance

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Solution

Figure 4.75 illustrates this situation for an arbitrary point (x, y) on the parabola.

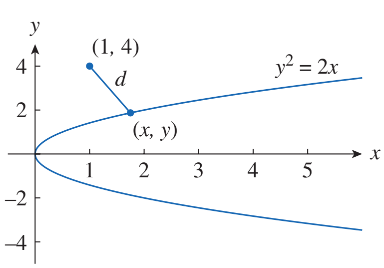


Figure 4.75
We need to minimize the distance d.

The distance between the point $(1, 4)$ and the point (x, y) is

$$d = \sqrt{(x-1)^2 + (y-4)^2}.$$

Alternatively, we could have substituted $y = \sqrt{2x}$ to write d in terms of x alone.

Since (x, y) lies on the parabola $x = \frac{1}{2}y^2$, the expression for d becomes

$d = \sqrt{\left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2}$.

Instead of minimizing d, we will minimize its square.

This technique and is used because d^2 is much easier to work with, and the minimum of d occurs at the same place as the minimum of d^2.

$d^2 = f(y) = \left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2$

Note that there are no restrictions on y. The domain of f is all real numbers.

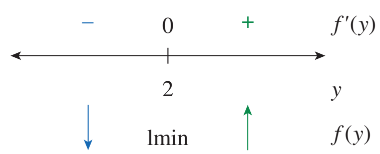
Find the derivative of f and the critical numbers.

$f'(y) = 2\left(\frac{1}{2}y^2 - 1\right)y + 2(y-4) = y^3 - 8$

$f'(y) = 0$: $y^3 - 8 = 0 \;\Rightarrow\; y = 2$

$f'(y)$ DNE: None

Construct a sign chart.



There is only one critical number, 2, and $f'(y) < 0$ for all $y < 2$ and $f'(y) > 0$ for all $y > 2$.

By the First Derivative Test for Absolute Extreme Values, the absolute minimum occurs when $y = 2$.

The corresponding value of x is $x = \frac{1}{2}y^2 = \frac{1}{2} \cdot 2^2 = 2$.

The point on the parabola $y^2 = 2x$ closest to $(1, 4)$ is $(2, 2)$.

The distance between the points is $d = \sqrt{f(2)} = \sqrt{5}$. ■

Example 4 Minimizing Time

A person launches a boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 4.76). The person could row the boat directly across the river to point C and then run to B, or they could row directly to B, or they could row to some point D between C and B and then run to B. If they can row 6 km/h and run 8 km/h, where should they land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)

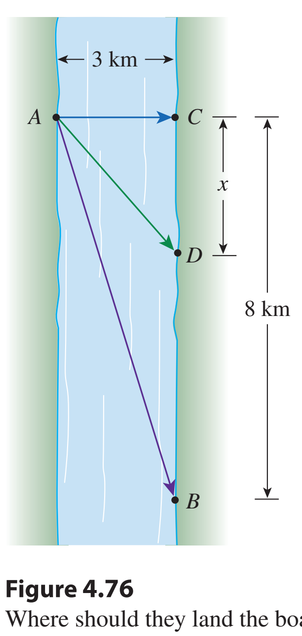


Figure 4.76
Where should they land the boat on the other side of the river in order to reach point B as soon as possible?

Solution

Let x be the distance from C to D.

The running distance is $|DB| = 8 - x$.

The Pythagorean Theorem gives the rowing distance: $|AD| = \sqrt{x^2 + 9}$.

Use the equation time $= \dfrac{\text{distance}}{\text{rate}}$.

The rowing time is $\dfrac{\sqrt{x^2+9}}{6}$ and the running time is $\dfrac{8-x}{8}$.

The total time to point B is T (as a function of x):

$$T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$$

The domain of the function T is $[0, 8]$.

If $x = 0$, then the person rows to the point C, and if $x = 8$, they row directly to B.

Find the derivative of T and the critical numbers.

$$T'(x) = \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8}$$

$T'(x) = 0$:

$$\frac{4x - 3\sqrt{x^2+9}}{24\sqrt{x^2+9}} = 0 \qquad \text{Common denominator.}$$

$$4x - 3\sqrt{x^2+9} = 0 \qquad \text{Set the numerator equal to 0.}$$

$$4x = 3\sqrt{x^2+9} \qquad \text{Rewrite: one term on each side.}$$

$$16x^2 = 9(x^2+9) \qquad \text{Square both sides.}$$

$$7x^2 - 81 = 0 \qquad \text{Simplify.}$$

$$(\sqrt{7}x - 9)(\sqrt{7}x + 9) = 0 \qquad \text{Factor.}$$

$$x = \frac{9}{\sqrt{7}}, \; -\frac{9}{\sqrt{7}} \qquad \text{Principle of Zero Products.}$$

$x = -\dfrac{9}{\sqrt{7}}$ is not in the domain of T.

$T'(x)$ DNE: None

Therefore, $x = \dfrac{9}{\sqrt{7}}$ is the only critical number.

Since T is continuous on a closed interval $[0, 8]$, use the Closed Interval Method.

x	$T(x)$	
0	1.5	
$\dfrac{9}{\sqrt{7}}$	$1 + \dfrac{\sqrt{7}}{8} \approx 1.331$	← absolute minimum value
8	$\dfrac{\sqrt{73}}{6} \approx 1.424$	

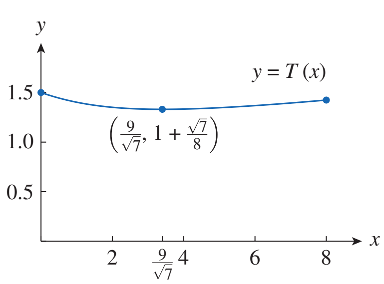


Figure 4.77
Graph of T. It's hard to see the minimum point because the graph of T is relatively flat. Consider graphing T with technology and zooming in near the absolute minimum point.

The absolute minimum value occurs when $x = \dfrac{9}{\sqrt{7}}$.

Therefore, the person should land the boat at a point $\dfrac{9}{\sqrt{7}} = 3.402$ km downstream from their starting point.

Figure 4.77 shows a graph of T to visualize this result. ■

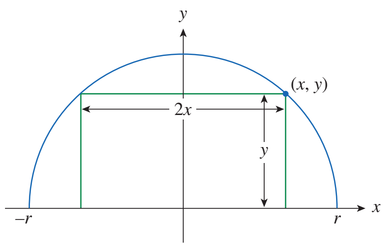


Figure 4.78
A rectangle inscribed in a semicircle of radius r.

Example 5 Finding the Largest Rectangle

Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Solution

Let the semicircle be the upper half of the circle defined by $x^2 + y^2 = r^2$, with center at the origin. The word *inscribed* means that the rectangle is completely inside the semicircle, has two vertices on the semicircle, and two vertices on the x-axis, as shown in Figure 4.78.

Let (x, y) be the vertex of the rectangle that lies in the first quadrant.

The rectangle has sides of lengths $2x$ and y. The area is $A = 2xy$.

To eliminate the variable y, use the fact that (x, y) lies on the circle $x^2 + y^2 = r^2$.

Therefore, $y = \sqrt{r^2 - x^2}$.

The area of the rectangle in terms of x is $A(x) = 2x\sqrt{r^2 - x^2}$.

The domain of the function A is $0 \le x \le r$.

Find the derivative of A and the critical points.

$$A'(x) = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} = \frac{-2(\sqrt{2}x - r)(\sqrt{2}x + r)}{\sqrt{r^2 - x^2}}$$

$A'(x) = 0$: $x = \dfrac{r}{\sqrt{2}}$ $\left(x = -\dfrac{r}{\sqrt{2}} \text{ is not in the domain of } A.\right)$

$A'(x)$ DNE: $x = r$

Use the Closed Interval Method.

x	$A(x)$	
0	0	
$\dfrac{r}{\sqrt{2}}$	r^2	← absolute maximum value
r	0	

The area of the largest inscribed rectangle is

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2.$$

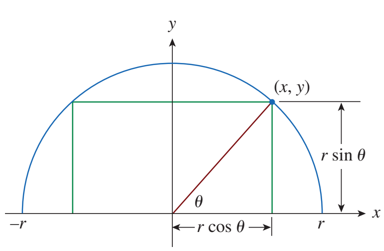


Figure 4.79
Let θ be the angle formed by the x-axis and the line from the origin to the point (x, y) on the semicircle.

Alternate Solution

Consider using an angle as a variable. Let θ be the angle as shown in Figure 4.79.

The area of the rectangle is

$A(\theta) = (2r\cos\theta)(r\sin\theta) = r^2(2\sin\theta\cos\theta) = r^2\sin 2\theta$.

The function $\sin 2\theta$ has a maximum value of 1 and it occurs when $2\theta = \dfrac{\pi}{2}$.

Therefore, $A(\theta)$ has a maximum value of r^2 and it occurs when $\theta = \dfrac{\pi}{4}$. ■

Applications to Business and Economics

In Section 3.8, we introduced the concept of marginal cost. Recall that if $C(x)$, the **cost function**, is the cost of producing x units of a certain product, then the **marginal cost** is the rate of change of C with respect to x; that is, the marginal cost function is the derivative $C'(x)$ of the cost function.

Let's bring the marketing department into the equation. Let $p(x)$ be the price per unit that the company can charge if it sells x units. Then p is called the **demand function** (or **price function**), and we would expect it to be a decreasing function of x. (More units sold corresponds to a lower price.) If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$R(x) = \text{quantity} \times \text{price} = x\,p(x)$$

and R is called the **revenue function**. The derivative R' of the revenue function is called the **marginal revenue function** and is the rate of change of revenue with respect to the number of units sold.

If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

and P is called the **profit function**. The **marginal profit function** is P', the derivative of the profit function. Several exercises at the end of this section require the use of marginal cost, revenue, and profit functions to minimize costs and maximize revenues and profits.

Example 6 Maximizing Revenue

A store has been selling 200 flat-screen television sets a week at \$350 each. A market survey indicates that for each \$10 decrease in price, the number of sets sold will increase by 20 a week. Find the demand function and the revenue function. What price per television set will maximize the store's revenue?

Solution

Let x be the number of sets sold per week.

Since the store sells 200 sets per week, the weekly increase (or decrease) in sales is $x - 200$.

For each increase of 20 sets sold, the price is decreased by \$10.

So, for each additional set sold, the decrease in price is $\dfrac{1}{20} \times 10$.

The demand function is $p(x) = 350 - \frac{10}{20}(x - 200) = 450 - \frac{1}{2}x$.

The revenue function is $R(x) = x\,p(x) = 450x - \frac{1}{2}x^2, \qquad 0 \le x \le 900$.

Find the derivative and the critical points.

$$R'(x) = 450 - \frac{1}{2} \cdot 2 \cdot x = 450 - x$$

$R'(x) = 0$: $\quad 450 - x = 0 \quad \Rightarrow \quad x = 450$

$R'(x)$ DNE: None

The only critical point is $x = 450$. Use the Closed Interval Method.

x	$R(x)$	
0	0	
450	101,250	← absolute maximum value
900	0	

The absolute maximum revenue occurs when $x = 450$ (when the store sells 450 television sets).

The absolute maximum revenue is $101,250, and the corresponding price per television set is $p(x) = 450 - \frac{1}{2}(450) = \$225$.

4.6 Exercises

1. Consider the following problem: find two numbers whose sum is 23 and whose product is a minimum.
(a) Construct a table of values, similar to the one shown, so that the sum of the numbers in the first two columns is always 23.

First number	Second number	Product
1	22	22
2	21	42
3	20	60
⋮	⋮	⋮

On the basis of the evidence in your table, estimate the answer to the problem.
(b) Use calculus to solve this problem and compare with your answer to part (a).

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

5. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

6. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.

7. A rectangle in the coordinate plane has its base along the x-axis and two vertices on the parabola defined by $y = 16 - x^2$. Which of the following equations represents the area of the rectangle $A(x)$?
(A) $A(x) = x^2$ (B) $A(x) = x(16 - x^2)$
(C) $A(x) = 2x(16 - x^2)$ (D) $A(x) = (16 - x^2)^2$

8. Find the point on the line with equation $y = 2x + 1$ that is closest to the point $(3, -1)$.

9. A sporting goods store sells tennis balls for $2.40 per can when a quantity of up to 20 cans is purchased. For each can above 20 purchased, the price per can is reduced by $0.02, with a limit of 60 cans. Find the number of cans of tennis balls sold in one transaction that will maximize the revenue for the store.

10. A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in appropriate units) is

$$Y = \frac{kN}{1 + N^2}$$

where k is a positive constant. Find the nitrogen level that produces the largest yield.

11. The rate (in mg carbon/m^3/h) at which photosynthesis takes place for a species of phytoplankton is modeled by the function

$$P = \frac{100I}{I^2 + I + 4}$$

where I is the light intensity (measured in thousands of foot-candles). Find the light intensity such that P is a maximum.

12. Consider the following problem: a farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
(a) Draw several diagrams to illustrate this situation, some with shallow, wide pens, and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate this area.
(b) Draw a diagram to illustrate the general situation. Introduce notation and label the diagram with your symbols.
(c) Write an expression for the total area.
(d) Use the given information to write an equation that relates the variables.
(e) Use part (d) to write the total area as a function of one variable.
(f) Use calculus to finish solving this problem and compare the answer with your estimate in part (a).

13. Consider the following problem: a box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume of the box.
(a) Draw several diagrams to illustrate this situation, some short boxes with large bases, and some tall boxes with small bases. Find the volume of each configuration. Does it appear that there is a maximum volume? If so, estimate this value.
(b) Draw a diagram to illustrate the general situation. Introduce notation and label the diagram with your symbols.
(c) Write an expression for the volume of the box.
(d) Use the given information to write an equation that relates the variables.

(e) Use part (d) to write the volume of the box as a function of one variable.
(f) Use calculus to finish solving this problem and compare the answer with your estimate in part (a).

14. A cardboard box with a square base and an open top is to be constructed from 108 m^2 of cardboard.
(a) Express the volume of the box as a function of the length x of the side of the square base.
(b) Find the dimensions so that the volume of the box is as large as possible. Justify your answer.

15. A farmer wants to fence in an area of 1.5 million ft^2 in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can this be done in order to minimize the cost of the fence?

16. A packing crate in the shape of a rectangular box has square ends and an open top. Let s represent the length of the square ends and let l represent the length of the crate. If the crate must be built to enclose 1200 ft^3 of space, which of these functions could be used to minimize the amount of material required to build it?

(A) $A(s) = s^2 + \dfrac{4800}{s}$ (B) $A(s) = 2s^2 + \dfrac{3600}{s}$

(C) $A(s) = 4s^2 + \dfrac{1200}{s}$ (D) $A(s) = 2s^2 + \dfrac{4800}{s}$

17. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimizes the amount of material used.

18. Suppose 1200 cm^2 of material is available to make a box with a square base and an open top. Find the largest possible volume of the box.

19. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs $\$10/\mathrm{m}^2$. Material for the sides costs $\$6/\mathrm{m}^2$. Find the cost of materials for the least expensive container.

20. Solve Exercise 19, assuming that the container has a lid that is made from the same material as the sides.

21. A farmer wants to fence in a rectangular plot of land adjacent to the north wall of their barn. No fencing is needed along the barn, and the fencing along the west side of the plot is shared with a neighbor who will split the cost of that portion of the fence. If the fencing costs \$20 per linear foot to install and the farmer is not willing to spend more than \$5000, find the dimensions for the plot that would enclose the greatest area.

22. Suppose the farmer in Exercise 21 wants to enclose 8000 ft^2 of land. Find the dimensions that will minimize the cost of the fence.

23. (a) Show that of all the rectangles with a given area, the one with the smallest perimeter is a square.
(b) Show that of all the rectangles with a given perimeter, the one with the greatest area is a square.

24. Find the point on the line $y = 2x + 3$ that is closest to the origin.

25. Find the point on the graph of $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

26. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

27. Find the coordinates of the point on the graph of $y = \sin x$ that is closest to the point $(4, 2)$.

28. A rectangle with sides parallel to the coordinate axes is inscribed in the region bounded by the graphs of $y = 18 - x^2$ and $y = 2x^2 - 9$. Find the maximum area of such a rectangle.

29. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

30. Find the area of the largest rectangle that can be inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

31. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

32. Find the area of the largest trapezoid that can be inscribed in a circle of radius 1 and whose base is a diameter of the circle.

33. Let $f(x) = 12 - x^2$. Consider the point $P(k, f(k))$ in the first quadrant, so that $0 \le k \le 2\sqrt{3}$.
(a) Find, in terms of k, the equation of the line ℓ tangent to the graph of f at the point P.
(b) Find, in terms of k, the x- and y-intercepts of the line ℓ from part (a).
(c) Find the function $A(k)$ that gives the area of the triangle formed by the tangent line ℓ and the axes.
(d) Find the value of k for which the area of this triangle is a maximum.

34. Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

35. If the two equal sides of an isosceles triangle have length a, find the length of the third side that maximizes the area of the triangle.

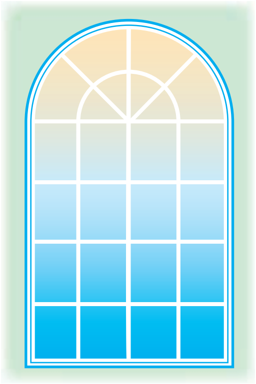
36. A cylindrical can with a closed top is to be constructed so that its volume is 16π in^3. Find the height that will minimize the amount of tin that will be required to construct the can.

37. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

38. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

39. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such a cylinder.

40. A Norman window has the shape of a rectangle surmounted by a semicircle, as shown in the figure.



Let r represent the radius of the circle and h the height of the rectangle. To ensure that the window admits enough light, the area of the window must be 60 ft^2.

(a) Write a formula involving r and h for the area of the window, A, and for the outside perimeter of the window, P.

(b) Write a formula for $P(r)$, the perimeter of the window in terms of r.

(c) Find the values of r and h for which the perimeter, the amount of material required to frame the window, is least. What is the perimeter of the window?

41. A Norman window, as described in Exercises 40, has the shape of a rectangle surmounted by a semicircle. Therefore, the diameter of the semicircle is equal to the width of the rectangle. If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

42. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.

43. A poster is to have an area of 180 in^2 with 1-in margins at the bottom and sides and a 2-in margin at the top. Find the dimensions of the poster that will provide the largest *printed* area.

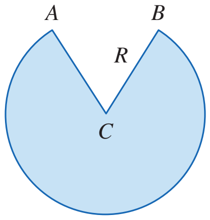
44. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) a minimum?

45. Solve Exercises 44 if one piece is bent into a square and the other into a circle.

46. Suppose you are offered one slice from a round pizza, that is, a sector of a circle, and the slice must have a perimeter of 32 in. What diameter pizza will produce the largest slice?

47. A fence 8 ft tall runs parallel to a building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

48. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB, as shown in the figure.

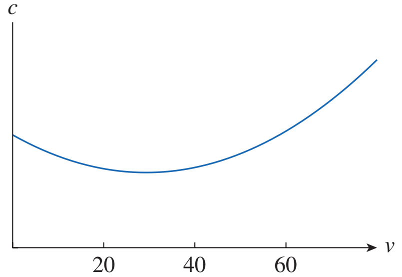


Find the maximum capacity of a cup constructed in this manner.

49. A cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

50. A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.

51. The graph shows the fuel consumption c of a car (measured in gallons per hour) as a function of the speed v of the car.



At very low speeds the engine runs inefficiently, so initially c decreases as the speed increases. But at high speeds the fuel consumption increases. You can see that $c(v)$ is minimized for this car when $v \approx 30$ mi/h. However, for fuel efficiency, what must be minimized is not the consumption in gallons per hour but rather the fuel consumption in gallons *per mile*. Let's call this consumption G. Using the graph, estimate the speed at which G has its minimum value.

52. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with a plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where μ is a constant called the coefficient of friction. Find the value of θ such that F is a minimum.

53. If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2R}{(R + r)^2}$$

If E and r are fixed but R varies, what is the maximum value of the power?

54. For a fish swimming with velocity v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u ($u < v$), then the time required to swim a distance L is $\dfrac{L}{v - u}$ and the total energy required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v - u}$$

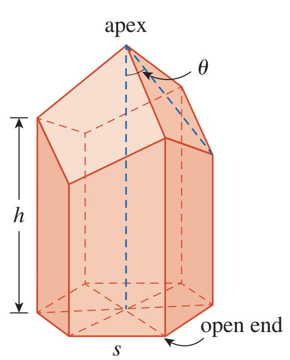
where a is the proportionality constant.

(a) Determine the value of v that minimizes E.

(b) Sketch the graph of E.

Note: This result has been verified experimentally; migrating fish swim against a current at a speed 50% greater than the current speed.

55. In a beehive, each cell is a regular hexagonal prism, open at one end with a trihedral angle at the other end as in the figure.



It is believed that bees form their cells in such a way as to minimize the surface area for a given side length and height, thus using the least amount of wax in cell construction. Examination of these cells has shown that the measure of the apex angle θ is amazingly consistent. Based on the geometry of the cell, it can be shown that the surface area S is given by

$$S = 6sh - \frac{3}{2}s^2 \cot\theta + \left(3s^2\frac{\sqrt{3}}{2}\right)\csc\theta$$

where s, the length of the sides of the hexagon, and h, the height, are constants.

(a) Calculate $\dfrac{dS}{d\theta}$.

(b) What angle should the bees prefer?

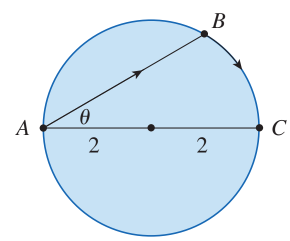
(c) Determine the minimum surface area of the cell (in terms of s and h).

Note: Actual measurements of the angle θ in beehives have been made, and the measures of these angles seldom differ from the calculated value by more than 2°.

56. A boat leaves a dock at 2 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3 PM. At what time were the two boats closest together?

57. Solve the problem as described in Example 4 if the river is 5 km wide and point B is only 5 km downstream from A.

58. A person at a point A on the shore of a circular lake with radius 2 mi wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time, as shown in the figure.



They can walk at the rate of 4 mi/h and row a boat at 2 mi/h. Find the path that takes the minimum time to reach point C.

59. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is \$400,000/km over land to a point P on the north bank and \$800,000/km under the river to the tanks. Find the location of the point P that will minimize the cost of the pipeline.

60. Suppose the refinery in Exercise 59 is located 1 km north of the river. Where should P be located in this case?

61. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 ft apart, where should an object be placed on the line between the sources so as to receive the least illumination?

62. Find an equation of the line through the point (3, 5) that cuts off the least area from the first quadrant.

63. Let a and b be positive numbers. Find the length of the shortest line segment that is cut off by the first quadrant and passes through the point (a, b).

64. Find the points on the curve defined by $y = 1 + 40x^3 - 3x^5$ such that the tangent line has the largest slope.

65. Find the minimum length of the line segment that is cut off by the first quadrant and is tangent to the graph of $y = \dfrac{3}{x}$ at some point.

66. Suppose a triangle is constructed in the first quadrant with sides along the axes and hypotenuse tangent to the parabola defined by $y = 4 - x^2$ at some point. Find the smallest possible area of the triangle.

67. (a) If $C(x)$ is the cost of producing x units of a commodity, the **average cost** per unit is $c(x) = \dfrac{C(x)}{x}$. Show that if the average cost is a minimum, then the marginal cost equals the average cost.

(b) If $C(x) = 16{,}000 + 200x + 4x^{3/2}$, in dollars, find (**i**) the cost, average cost, and marginal cost at a production level of 1000 units; (**ii**) the production level that will minimize the average cost; and (**iii**) the minimum average cost.

68. (a) Show that if the profit $P(x)$ is a maximum, then the marginal revenue equals the marginal cost.

(b) If $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$ is the cost function and $p(x) = 1700 - 7x$ is the demand function, find the production level that will maximize profit.

69. A baseball team plays in a stadium that holds 55,000 fans. With ticket prices at \$50, the average attendance is 40,000. When ticket prices are lowered to \$35 the average attendance is 47,000.

(a) Find the demand function, assuming it is linear.

(b) How should ticket prices be set to maximize revenue?

70. During the summer months a person makes and sells necklaces on the beach. Last summer they sold the necklaces for \$10 each and their sales averaged 20 per day. When they increased the price by \$1, they found that the average decreased by two sales per day.

(a) Find the demand function, assuming it is linear.

(b) If the material for each necklace costs \$6, what should the selling price be to maximize profit?

71. A retailer has been selling 1200 tablet computers a week at \$350 each. The marketing department estimates that an additional 80 tablets will sell each week for every \$10 that the price is lowered.

(a) Find the demand function.

(b) What price will yield the maximum revenue?

(c) If the retailer's weekly cost function is

$$C(x) = 35{,}000 + 12x$$

what price will result in the maximum profit?

72. A company operates 16 oil wells in a designated area. Each pump, on average, extracts 240 barrels of oil daily. The company can add more wells but every added well reduces the average daily output of each of the wells by 8 barrels. How many wells should the company add in order to maximize daily production?

73. Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.

74. Consider the pipeline problem in Exercise 59. Suppose the cost of laying pipe under the river is considerably higher than the cost of laying pipe over land (\$400,000/km). We might suspect that, in some instances, the minimum distance possible under the river should be used and P should be located 6 km from the refinery, directly across from the storage tanks. Show that this is *never* the case, no matter what the *under river* cost is.

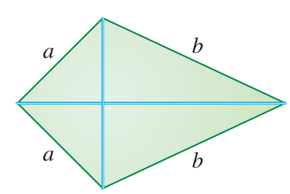
75. Consider the tangent line to the ellipse defined by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at a point (p, q) in the first quadrant.

(a) Show that the tangent line has x intercept $\dfrac{a^2}{p}$ and y-intercept $\dfrac{b^2}{q}$.

(b) Show that the portion of the tangent line cut off by the coordinate axes has minimum length $a + b$.

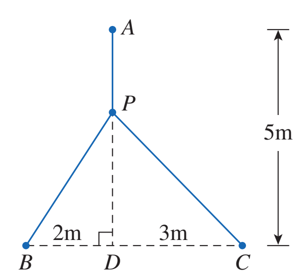
(c) Show that the triangle formed by the tangent line and the coordinate axes has minimum area ab.

76. The frame for a kite is to be made from six pieces of wood. The four exterior pieces have been cut with the lengths indicated in the figure.



Find the length of the diagonal pieces that will maximize the area of the kite.

77. A point P needs to be located somewhere on the line AD so that the total length L of cables linking P to the points A, B, and C is minimized, as shown in the figure.

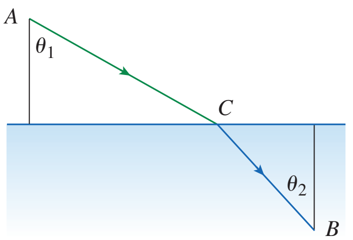


Express L as a function of $x = |AP|$ and use the graphs of L and $\dfrac{dL}{dx}$ to estimate the minimum value of L.

78. Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

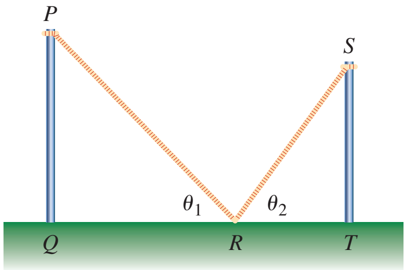
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are defined as shown in the figure.



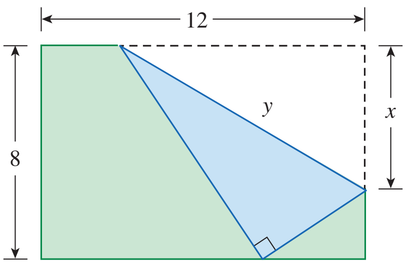
This equation is known as Snell's Law.

79. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the poles and then to the top of the second pole, as shown in the figure.



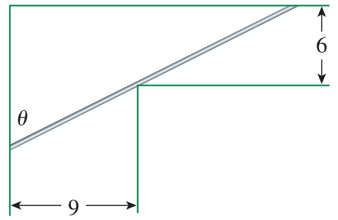
Show that the shortest length of rope occurs when $\theta_1 = \theta_2$.

80. The upper right-hand corner of a piece of paper, 12 in by 8 in, is folded over to the bottom edge as shown in the figure.



How would you fold the paper such that the length of the fold is a minimum? That is, how would you choose x to minimize y?

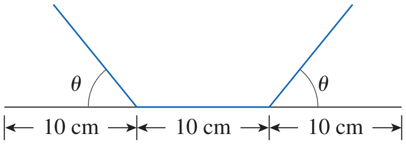
81. A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner?



82. Find the maximum area of a rectangle that can be circumscribed about a given rectangle with length L and width W.

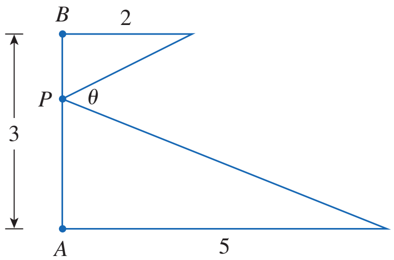
Hint: Express the area as a function of an angle θ.

83. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ, as shown in the figure.

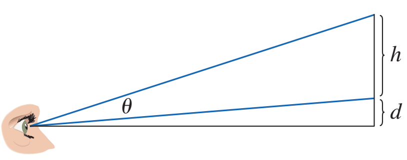


How should θ be chosen so that the gutter will carry the maximum amount of water?

84. Where should the point P be placed on the line segment AB such that the angle θ is a maximum?



85. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer, as shown in the figure.

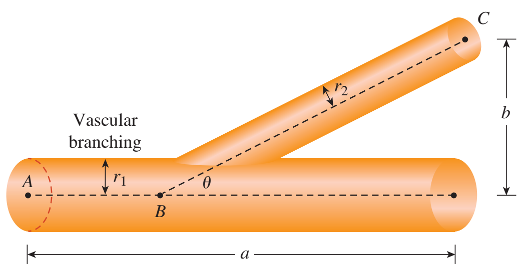


How far from the wall should the observer stand to get the best view? That is, where should the observer stand such that the angle θ subtended at their eye by the painting is a maximum?

86. The blood vascular system consists of blood vessels (arteries, arterioles, capillaries, and veins) that convey blood from the heart to the organs and back to the heart. This system should work so as to minimize the energy expended by the heart in pumping the blood. In particular, this energy is reduced when the resistance of the blood is lowered. One of Poiseuille's Laws gives the resistance R of the blood as

$$R = C\frac{L}{r^4}$$

where L is the length of the blood vessel, r is the radius, and C is a positive constant determined by the viscosity of the blood. (Poiseuille established this law experimentally.) The figure shows a main blood vessel with radius r_1 branching at an angle θ into a smaller vessel with radius r_2.



(a) Use Poiseuille's Laws to show that the total resistance of the blood along the path ABC is

$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

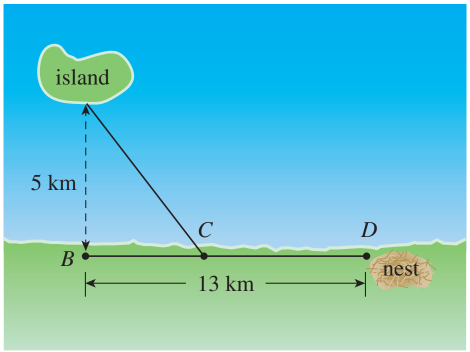
where a and b are the distances shown in the figure.

(b) Show that this resistance is minimized when

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

(c) Find the optimal branching angle when the radius of the smaller blood vessel is two-thirds the radius of the larger vessel.

87. Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than over land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 13 km apart.



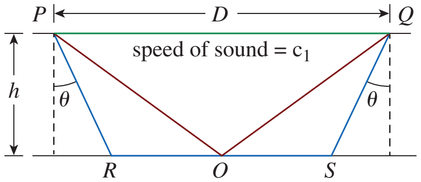
(a) In general, if it takes 1.4 times as much energy to fly over water as it does over land, to what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area?

(b) Let W and L denote the energy (in joules) per kilometer flown over water and land, respectively. What would a large value of the ratio W/L mean in terms of the bird's flight? What would a small value mean? Determine the ratio W/L corresponding to the minimum expenditure of energy.

(c) Determine the value of W/L in order for the bird to fly directly to its nesting area D. Determine the value of W/L for the bird to fly to B and then along the shore to D.

(d) If birds of a certain species reach the shore at a point 4 km from B, how many times more energy does it take a bird to fly over water than over land?

88. The speeds of sound c_1 in an upper layer and c_2 in a lower layer of rock and the thickness h of the upper layer can be determined by seismic exploration if the speed of sound in the lower layer is greater than the speed in the upper layer. Transmitted signals from a controlled sound are recorded at a point Q, which is a distance D from P. The first signal to arrive at Q travels along the surface and takes T_1 seconds. The next signal travels from P to a point R, from R to S in the lower layer, and then to Q, taking T_2 seconds. The third signal is reflected off the lower layer at the midpoint O of RS and takes T_3 seconds to reach Q. See the figure.



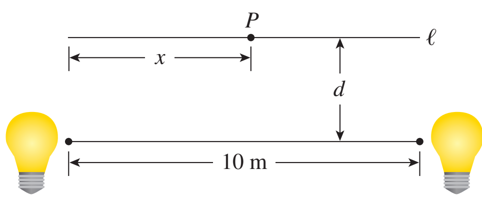
(a) Express T_1, T_2, and T_3 in terms of D, h, c_1, c_2, and θ.

(b) Show that T_2 is a minimum when $\sin\theta = \dfrac{c_1}{c_2}$.

(c) Suppose that $D = 1$ km, $T_1 = 0.26$ s, $T_2 = 0.32$ s, and $T_3 = 0.34$ s. Find c_1, c_2, and h.

Note: Geophysicists use this technique when studying the structure of Earth's crust, in searching for oil or examining fault lines.

89. Two light sources of identical strength are placed 10 m apart. An object is to be placed at a point P on a line ℓ, parallel to the line joining the light sources and at a distance d meters from it, as shown in the figure.



We want to locate P on ℓ so that the intensity of illumination is minimized. We need to use the fact that the intensity of illumination for a single source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source.

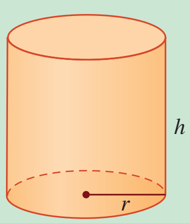
(a) Find an expression for the intensity $I(x)$ at the point P.

(b) If $d = 5$ m, use graphs of $I(x)$ and $I'(x)$ to show that the intensity is minimized when $x = 5$ m, that is, when P is at the midpoint of ℓ.

(c) If $d = 10$ m, show that the intensity (perhaps surprisingly) is *not* minimized at the midpoint.

(d) Somewhere between $d = 5$ m and $d = 10$ m there is a transitional value of d at which the point of minimal illumination abruptly changes. Estimate this value of d using appropriate graphs. Then find the exact value of d.

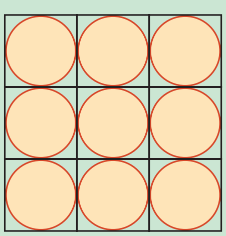
Applied Project | Tin Can Shape



The objective of this project is to investigate the most economical shape for a can. This is interpreted to mean that the volume V of a cylindrical can is given (fixed) and we need to find the height h and radius r that minimize the cost of the metal to make the can (see the figure).

If we disregard any waste metal in the manufacturing process, then the problem is to minimize the surface area of the cylinder. We actually solved this problem in Example 2 in Section 4.6 and found that $h = 2r$, that is, the height should be the same as the diameter.

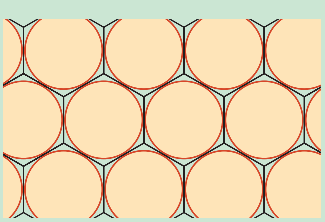
However, if you check a can in your cupboard or your local supermarket with a ruler, you will discover that the height is usually greater than the diameter and the ratio h/r varies from 2 up to about 3.8. Let's see if we can explain this discrepancy.



Discs cut from squares

1. The material for the cans is cut from sheets of metal. The cylindrical sides are formed by bending rectangles; these rectangles are cut from the sheet with little or no waste. But if the top and bottom discs are cut from squares of side $2r$ (as in the figure), this leaves considerable waste metal, which may be recycled but has little or no value to the can makers. If this is the case, show that the amount of metal used is minimized when

$$\frac{h}{r} = \frac{8}{\pi} \approx 2.55$$



Discs cut from hexagons

2. A more efficient packing of the discs is obtained by dividing the metal sheet into hexagons and cutting the circular lids and bases from the hexagons (see the figure). Show that if this strategy is adopted, then

$$\frac{h}{r} = \frac{4\sqrt{3}}{\pi} \approx 2.21$$

3. The values of h/r that we found in Problems 1 and 2 are a little closer to the ones that actually occur on supermarket shelves, but they still don't account for everything. If we look more closely at some real cans, we see that the lid and the base are formed from discs with radius larger than r that are bent over the ends of the can. If we allow for this we would increase h/r. More significantly, in addition to the cost of the metal we need to incorporate the manufacturing of the can into the cost. Let's assume that most of the expense is incurred in joining the sides to the rims of the cans. If we cut the discs from hexagons as in Problem 2, then the total cost is proportional to

$$4\sqrt{3}r^2 + 2\pi rh + k(4\pi r + h)$$

where k is the reciprocal of the length that can be joined for the cost of one unit area of metal. Show that this expression is minimized when

$$\frac{\sqrt[3]{V}}{k} = \sqrt[3]{\frac{\pi h}{r}} \cdot \frac{2\pi - \frac{h}{r}}{\frac{\pi h}{r} - 4\sqrt{3}}$$

4. Plot $\sqrt[3]{V}/k$ as a function of $x = h/r$ and use your graph to argue that when a can is large or joining is inexpensive, we should make h/r approximately 2.21 (as in Problem 2). However, if the can is small or joining is expensive, h/r should be substantially larger.

5. This analysis shows that large cans should be almost square but small cans should be tall and thin. Examine the relative shapes of cans in a supermarket. Is this conclusion usually true in practice? Are there any exceptions? Can you suggest any reasons why small cans are not always tall and thin?

4.7 Newton's Method

Suppose a bank approves a loan of $18,000 for the purchase of a used car. If you are told that the payments will be $375 per month for five years, or 60 months, you might still like to know the interest rate being charged, or applied to the loan. To determine the interest rate using this information we need to solve the equation

The derivation of this equation is part of Exercise 51.

$$48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0 \tag{1}$$

This seems pretty daunting: solve an equation involving a polynomial of degree 61.

However, for a quadratic equation $ax^2 + bx + c = 0$ there is a well-known formula for the solution. For third- and fourth-degree equations there are also formulas for the solutions, but they are more complicated but easily implemented with technology. If f is a polynomial of degree 5 or higher, there is no explicit formula for the solutions to the equation $f(x) = 0$. Similarly, there is no formula that will enable us to find the exact solutions of an equation involving a transcendental function, for example, $\cos x = x$.

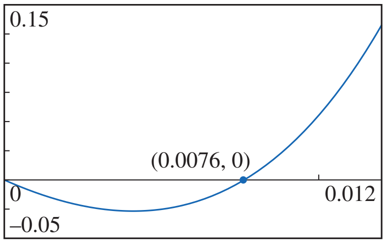


Figure 4.80
A solution to Equation 1 using technology.

We can however find an approximate solution to Equation 1 using technology. Figure 4.80 shows the graph of the left side of the equation in an appropriate viewing rectangle and a solution using a built-in numerical rootfinder. Alternatively, we could zoom in repeatedly, trace the curve, and estimate a solution. Using a computer algebra system, the solution to nine decimal places is 0.07628603.

You may wonder how these numerical rootfinders really work. Well, they use a variety of methods, but most of them make use of **Newton's method**, also called the **Newton–Raphson method**. This method is an application of the concepts of local linearity and linear approximation, and provides some insight into the numerical algorithms used by calculators and computers.

Consider the geometry used to describe Newton's method as shown in Figure 4.81. Suppose we want to solve an equation of the form $f(x) = 0$. Therefore, the roots of the equation correspond to the x-intercepts of the graph of f. The root we want to find is labeled r in the figure.

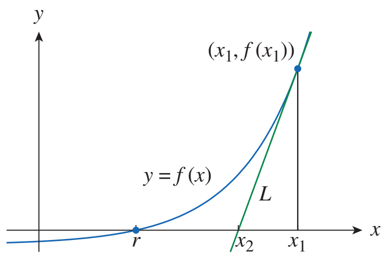


Figure 4.81
The geometry behind Newton's method.

Start with an initial approximation of the root, x_1. We might use a graph of f to obtain this rough first guess of the root. Consider the tangent line L to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$ and find the x-intercept of L, labeled x_2 in the figure. The idea behind Newton's method is that the tangent line is close to the curve, and therefore the x-intercept, x_2, should be close to the x-intercept of the curve, that is, the root r.

To find a formula for x_2 in terms of x_1, use the equation for L. The tangent line has slope $f'(x_1)$ and passes through the point $(x_1, f(x_1))$. An equation for L is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Since the x-intercept of L is x_2, the point $(x_2, 0)$ is on the graph of the tangent line. Therefore, we can write

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

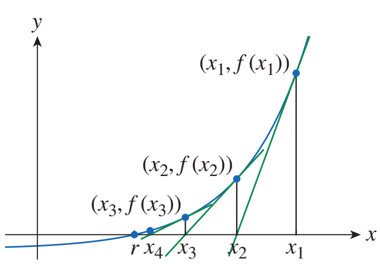


Figure 4.82
A visualization of the continued approximation process.

Now, use x_2 as a second approximation to r. Repeat this procedure with x_1 replaced by the second approximation x_2. Using the tangent line to the graph of $y = f(x)$ at the point $(x_2, f(x_2))$ gives us a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

If we continue this process, we obtain a sequence of approximations $x_1, x_2, x_3, x_4, \ldots$ as shown in Figure 4.82.

In general, the nth approximation is x_n and if $f(x_n) \neq 0$, then the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

If the numbers x_n become closer and closer to r as n increases without bound, then we say that the sequence *converges* to r, and we write

$$\lim_{n \to \infty} x_n = r$$

Although the sequence of successive approximations does indeed converge to the desired root for functions like the one illustrated in Figure 4.82, in some cases, the sequence may not converge. For example, consider the graph in Figure 4.83 and the first few approximations.

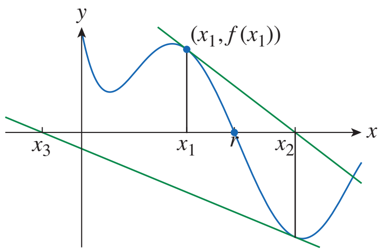


Figure 4.83
Newton's method fails in this case. A better initial guess is necessary.

The figure suggests that x_2 is actually no better at approximating the root; it is at least as far away from the desired root r. This situation arises when $f'(x_1)$ is close to 0. It is also possible for an approximation, like x_3 in Figure 4.83, to fall outside the domain of f. In this case, Newton's method fails, and you simply need to try a better initial approximation. Exercises 41–43 present some specific examples in which Newton's method works very slowly or does not work at all.

Example 1 A Third Approximation

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

Solution

Apply Newton's method.

$$f(x) = x^3 - 2x - 5 \;\Rightarrow\; f'(x) = 3x^2 - 2$$

Use Equation 2 with $n = 1$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2} \qquad \text{Use expressions for } f(x) \text{ and } f'(x).$$

$$= 2 - \frac{-1}{10} = 2 + 0.1 = 2.1 \qquad \text{Simplify.}$$

Use Equation 2 with $n = 2$.

$$x_3 = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \qquad \text{Use expressions for } f(x) \text{ and } f'(x).$$

$$= 2.1 - \frac{0.061}{11.23} = 2.1 - 0.0054 = 2.0946 \qquad \text{Simplify.}$$

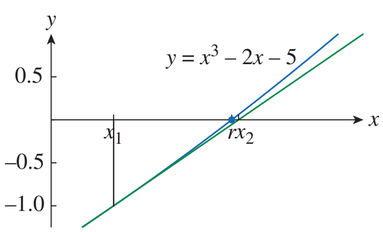


Figure 4.84
The geometry associated with the first step in Newton's method.

Figure 4.84 shows the geometry associated with the first step in Newton's method. The tangent line to the graph of $y = 2x^3 - 2x - 5$ at $(2, -1)$ is $y = 10x - 21$. The x-intercept is $x_2 = 2.1$.

In this case, the third approximation $x_3 = 2.0946$ is accurate to four decimal places. ■

There is a precise statement about the accuracy associated with Newton's method presented in Exercise 39, Section 8.8.

Suppose we want to use Newton's method to estimate a root with a certain degree of accuracy, for example, eight decimal places. We need a way to decide when we can stop the process. In general, we can stop when successive approximations x_n and x_{n+1} agree to eight decimal places.

One more note about Newton's method: Since the procedure for finding the $(n + 1)$st term is the same for all n, this is called an *iterative* process. So, we can readily write code to implement Newton's method for a graphing calculator or computer.

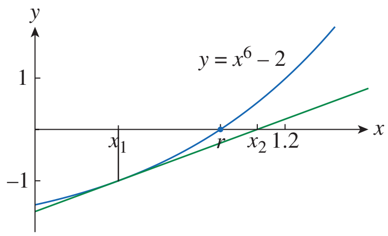


Figure 4.85
Illustration of the first step in Newton's method.

Approximation
$x_1 = 1.$
$x_2 = 1.16666667$
$x_3 = 1.12644368$
$x_4 = 1.12249707$
$x_5 = 1.12246205$
$x_6 = 1.12246205$

Table 4.1
Approximations obtained using Newton's method.

Example 2 Achieve a Desired Accuracy

Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

Solution

Finding $\sqrt[6]{2}$ is equivalent to finding the positive root of the equation $x^6 - 2 = 0$.

Let $f(x) = x^6 - 2 \Rightarrow f'(x) = 6x^5$.

Using Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^6 - 2}{6x_n^5}.$$

Figure 4.85 shows the first step in Newton's method and Table 4.1 shows values of x_n.

Since x_5 and x_6 agree to eight decimal places, we can conclude that $\sqrt[6]{2} = 1.12246205$ to eight decimal places. ■

Example 3 Transcendental Equation

Find, correct to six decimal places, the root of the equation $\cos x = x$.

Solution

Write an equivalent equation in a form for use with Newton's method.

$\cos x - x = 0$

Let $f(x) = \cos x - x \Rightarrow f'(x) = -\sin x - 1$.

Using Newton's method:

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$

Figure 4.86 suggests that a reasonable first approximation is $x_1 = 1$. Table 4.2 shows values of x_n.

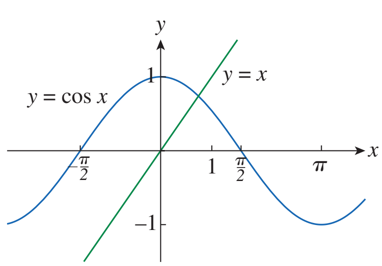


Figure 4.86
This graph suggests that $y = \cos x$ and $y = x$ intersect near the point where $x = 1$.

Approximation
$x_1 = 1.$
$x_2 = 0.75036387$
$x_3 = 0.73911289$
$x_4 = 0.73908513$
$x_5 = 0.73908513$

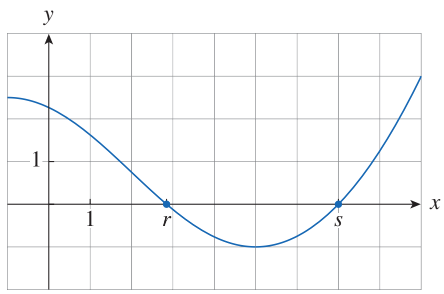
Table 4.2
Approximations obtained using Newton's method.

Since x_4 and x_5 agree to at least six decimal places, we conclude that the root of the equation, correct to six decimal places, is 0.739085.

Note that a different initial guess may produce the same accuracy with fewer steps. For example, if $x_1 = 0.75$ we obtain the same approximation in one less step. ■

4.7 Exercises

1. The figure shows the graph of a function f. Suppose Newton's method is used to approximate the roots of the equation $f(x) = 0$ with initial approximation $x_1 = 6$.



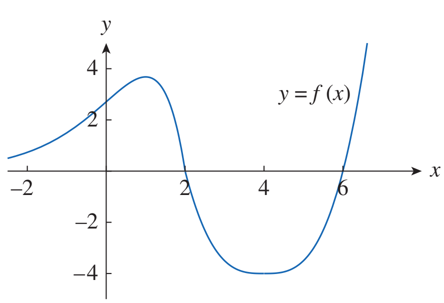
(a) Draw the tangent lines that are used to find x_2 and x_3, and estimate the numerical values of x_2 and x_3.

(b) Would $x_1 = 5$ be a *better* first approximation? Why or why not?

2. Use the information given in Exercise 1. Let $x_1 = 1$ be the initial approximation for finding the root r. Draw the tangent lines that are used to find x_2 and x_3, and estimate the numerical values of x_2 and x_3.

3. Suppose the tangent line to the graph of $y = f(x)$ at the point $(2, 5)$ has equation $y = 9 - 2x$. If Newton's method is used to locate a root of the equation $f(x) = 0$ and the initial approximation is $x_1 = 2$, find the second approximation x_2.

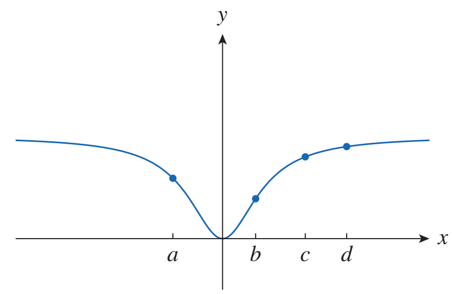
4. The graph of $y = f(x)$ is shown in the figure.



For each initial approximation to a root, describe graphically what happens if Newton's method is used to obtain the next approximation.

(a) $x_1 = 0$ (b) $x_1 = 1$ (c) $x_1 = 3$
(d) $x_1 = 4$ (e) $x_1 = 5$

5. For each initial approximation, $x_1 = a$, b, c, and d, determine whether Newton's method will work and lead to the root of the equation $f(x) = 0$. Justify your answer.



Use Newton's method with the specified initial approximation x_1 to find x_3, the third approximation to the root of the given equation. Give your answer to four decimal places.

6. $x^3 + 2x - 4 = 0, \quad x_1 = 1$

7. $\dfrac{1}{3}x^3 + \dfrac{1}{2}x^2 + 3 = 0, \quad x_1 = -3$

8. $x^5 - x - 1 = 0, \quad x_1 = 1$

9. $x^5 + 2 = 0, \quad x_1 = -2$

10. $\dfrac{2}{x} - x^2 + 1 = 0, \quad x_1 = 2$

11. Use Newton's method with initial approximation $x_1 = -1$ to find x_2, the second approximation to the root of the equation $x^3 + x + 3 = 0$. Draw a graph of the function and the tangent line at $(-1, 1)$ to illustrate this step.

12. Use Newton's method with initial approximation $x_1 = 1$ to find x_2, the second approximation to the root of the equation $x^4 - x - 1 = 0$. Draw a graph of the function and the tangent line at $(1, -1)$ to illustrate this step.

Use Newton's method to approximate the given number correct to eight decimal places.

13. $\sqrt[5]{20}$ **14.** $\sqrt[8]{500}$

15. $\sqrt[4]{75}$ **16.** $\sqrt[100]{100}$

(a) Explain why the given equation must have a root in the given interval. (b) Use Newton's method to approximate the root correct to six decimal places.

17. $3x^4 - 8x^3 + 2 = 0, \quad [2, 3]$

18. $-2x^5 + 9x^4 - 7x^3 - 11x = 0, \quad [3, 4]$

Use Newton's method to approximate the indicated root of the equation correct to six decimal places.

19. The positive root of $\sin x = x^2$

20. The positive root of $3 \sin x = x$

21. The positive root of $\ln x = \tan^{-1} x$

22. The largest root of $x^2 = 2^x$

Use Newton's method to find all solutions of the equation correct to six decimal places.

23. $3 \cos x = x + 1$

24. $\sqrt{x+1} = x^2 - x$

25. $\dfrac{1}{x} = \sqrt[3]{x} - 1$

26. $(x-1)^2 = \sqrt{x}$

27. $x^3 = \cos x$

28. $\sin x = x^2 - 2$

29. $\ln x = 3 \sin x$

30. $e^{-x} = \dfrac{x}{x^2 - 1}$

Use Newton's method to find all the solutions of the equation correct to eight decimal places. Draw a graph and use it to determine your initial approximation.

31. $-2x^7 - 5x^4 + 9x^3 + 5 = 0$

32. $x^5 - 3x^4 + x^3 - x^2 - x + 6 = 0$

33. $\dfrac{x}{x^2+1} = \sqrt{1-x}$

34. $\cos(x^2 - x) = x^4$

35. $x^2\sqrt{2 - x - x^2} = 1$

36. $3 \sin(x^2) = 2x$

37. $4e^{-x^2} \sin x = x^2 - x + 1$

38. $e^{\arctan x} = \sqrt{x^3 + 1}$

39. (a) Apply Newton's method to the equation $x^2 - a = 0$ to derive the following square-root algorithm (used by the ancient Babylonians to compute $\sqrt{a}$):

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) Use part (a) to compute $\sqrt{1000}$ correct to six decimal places.

40. (a) Apply Newton's method to the equation $\dfrac{1}{x} - a = 0$ to derive the following reciprocal algorithm:

$$x_{n+1} = 2x_n - a\,x_n^2$$

(This algorithm enables a computer to find reciprocals without actually dividing.)

(b) Use part (a) to compute $\dfrac{1}{1.6984}$ correct to six decimal places.

41. Explain why Newton's method doesn't work for finding the root of the equation

$$x^3 - 3x + 6 = 0$$

if the initial approximation is $x_1 = 1$.

42. (a) Use Newton's method with $x_1 = 1$ to find the root of the equation $x^3 - x = 1$ correct to six decimal places.

(b) Find the root of the equation in part (a) using $x_1 = 0.6$ as the initial approximation.

(c) Find the root of the equation in part (a) using $x_1 = 0.57$ as the initial approximation. (You may need technology for this part.)

(d) Sketch a graph of $f(x) = x^3 - x - 1$ and the tangent lines to the graph of f at the points were $x = 1$, 0.6, and 0.57. Use your graph to explain why Newton's method is very sensitive to the value of the initial approximation.

43. Explain why Newton's method fails when applied to the equation $\sqrt[3]{x} = 0$ with an initial approximation of $x_1 \neq 0$. Draw a graph to illustrate this result.

44. Let

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$$

The root of the equation $f(x) = 0$ is $x = 0$. Explain why Newton's method fails to produce this root no matter which initial approximation $x_1 \neq 0$ is used. Draw a graph to illustrate this result.

45. (a) Use Newton's method to find the critical numbers of the function

$$f(x) = x^6 - x^4 + 3x^3 - 2x$$

correct to six decimal places.

(b) Find the absolute minimum value of f correct to four decimal places.

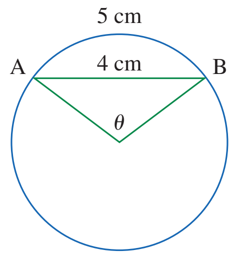
46. Use Newton's method to find the absolute maximum value of the function $f(x) = x \cos x$, $0 \leq x \leq \pi$, correct to six decimal places.

47. Use Newton's method to find the coordinates of the inflection point on the graph of $y = x^2 \sin x$, $0 \leq x \leq \pi$, correct to six decimal places.

48. There are infinitely many lines that are tangent to the graph of $y = -\sin x$ and pass through the origin. However, there is one that has the largest slope. Use Newton's method to find the slope of that line correct to six decimal places.

49. Use Newton's method to find the coordinates, correct to six decimal places, of the point on the parabola $y = (x-1)^2$ that is closest to the origin.

50. In the figure, the length of the chord AB is 4 cm and the length of the arc AB is 5 cm. Find the central angle θ, in radians, correct to four decimal places. Then convert your answer to the nearest degree.



51. Suppose a bank approves a loan of \$18,000 for the purchase of a used car. If you are told that the payments will be \$375 per month for 5 years, or 60 months, find the monthly interest rate that the bank is charging.

To solve this problem, use the formula for the present value A of an annuity consisting of n equal payments of size R with interest rate i per time period:

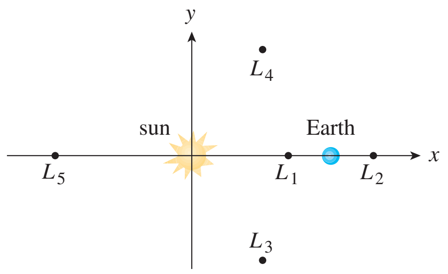
$$A = \frac{R}{i}[1 - (1 + i)^{-n}]$$

Replace i by x and use the other information in this problem to show that

$$48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0$$

Use Newton's method to solve this equation.

52. The figure shows the sun located at the origin and Earth at the point (1, 0). The unit here is the distance between the centers of Earth and the sun, called an *astronomical unit*: 1 AU $\approx 1.496 \times 10^8$ km.



There are five locations L_1, L_2, L_3, L_4, and L_5 in this plane of rotation of Earth about the sun where a satellite remains motionless with respect to Earth. The forces acting on the satellite, including the gravitational attractions of Earth and sun, balance out and allow the satellite to remain in geosynchronous orbit. These locations are called *libration points*. For example, there are several television and radio broadcasting satellites at specific libration points.

If m_1 is the mass of the sun, m_2 is the mass of Earth, and $r = \dfrac{m_2}{m_1 + m_2}$, then the x-coordinate of L_1 is the unique root of the fifth-degree equation

$$p(x) = x^5 - (2 + r)x^4 + (1 + 2r)x^3 - (1 - r)x^2 + 2(1 - r)x + r - 1 = 0$$

and the x-coordinate of L_2 is the root of the equation

$$p(x) - 2rx^2 = 0.$$

Use the value of $r \approx 3.04042 \times 10^{-6}$ to find the locations of the libration points (a) L_1 and (b) L_2.

4.8 Antiderivatives

In previous sections we were given a function f and asked to find the derivative f'. However, often we need to work backward: given the derivative, or rate of change, find the original function. For example, a physicist may use the velocity of a particle to find its position at a certain time. An engineer who can measure the variable rate at which water is leaking from a tank can find the total amount leaked over a certain time period. A biologist who knows the rate at which a bacteria population is increasing can determine the size of the population at some future time. In each case, the problem is to find a function F whose derivative is a known function f. If such a function F exists, it is called an *antiderivative* of f.

Definition • Antiderivative

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

For example, let $f(x) = x^2$. To find an antiderivative of f we just think about the Power Rule.

If $F(x) = \frac{1}{3}x^3$, then $F'(x) = \frac{1}{3} \cdot 3 \cdot x^2 = x^2 = f(x)$.

The function $G(x) = \frac{1}{3}x^3 + 28$ is also an antiderivative of f: $G'(x) = x^2$.

In fact, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant, is an antiderivative of f. This raises an important question: are there any other antiderivatives? The following theorem says that f has no other antiderivative.

Theorem • General Antiderivative

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

The most general antiderivative of $f(x) = x^2$ is the function $F(x) = \frac{1}{3}x^3 + C$. If we let C equal specific values, then we obtain a family of functions whose graphs are vertical translations of one another (see Figure 4.87). This makes sense because each graph must have the same slope (derivative) at any given value of x.

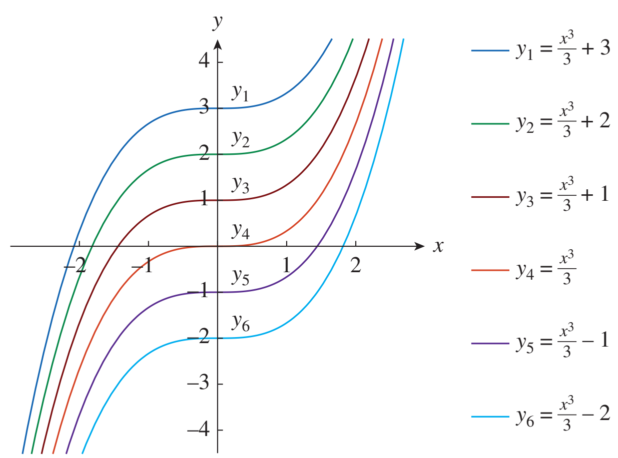


Figure 4.87
Graphs of several members of the family of antiderivatives of $f(x) = x^2$.

Example 1 Find General Antiderivatives

Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$ (b) $f(x) = \dfrac{1}{x}$ (c) $f(x) = x^n, \ n \neq -1$

Solution

(a) If $F(x) = -\cos x$, then $F'(x) = \sin x$.

$F(x) = -\cos x$ is an antiderivative of $\sin x$.

The most general antiderivative is $G(x) = -\cos x + C$.

(b) In Chapter 3, we learned that $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$.

Therefore, on the interval $(0, \infty)$ the general antiderivative of $\dfrac{1}{x}$ is $\ln x + C$.

We also learned that $\dfrac{d}{dx}(\ln |x|) = \dfrac{1}{x}$ for all $x \neq 0$.

Using Theorem 2, the most general antiderivative of $f(x) = \dfrac{1}{x}$ is $\ln |x| + C$ on any interval that does not contain 0.

In particular, this is true on each of the intervals $(-\infty, 0)$ and $(0, \infty)$. So, the general antiderivative of f can be written as

$$F(x) = \begin{cases} \ln x + C_1 & \text{if } x > 0 \\ \ln(-x) + C_2 & \text{if } x < 0 \end{cases}$$

(c) Think about the Power Rule to discover an antiderivative of x^n.

If $n \neq -1$, then $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right) = (n+1) \cdot \dfrac{x^n}{n+1} = x^n$.

Therefore, the most general antiderivative of $f(x) = x^n$ is $F(x) = \dfrac{x^{n+1}}{n+1} + C$.

This is valid for $n \geq 0$ because $f(x) = x^n$ is defined on any interval. If n is negative (but $n \neq -1$), it is valid on any interval that does not contain 0. ■

As Example 1 suggests, every differentiation rule leads to an antidifferentiation rule. Table 4.3 lists some specific antiderivatives. Each rule is true because the derivative of the function in the right column appears in the left column. In particular, the first rule says that the antiderivative of a constant times a function is the constant times the antiderivative of the function. The second rule says that the antiderivative of a sum is the sum of the antiderivatives. The following notation is used: $F' = f$, $G' = g$.

Function	Particular antiderivative	Function	Particular antiderivative
$c\,f(x)$	$c\,F(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\cos x$	$\sin x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
$\dfrac{1}{x}$	$\ln \|x\|$	$\sec x \tan x$	$\sec x$
e^x	e^x	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
b^x	$\dfrac{b^x}{\ln b}$	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$

Table 4.3
Table of Antidifferentiation Formulas. To obtain the most general antiderivative from a particular one, add a constant (or constants) as in Example 1.

Note that there are no rules for the antiderivative of products or quotients. The process of calculating an antiderivative $F(x)$ for a given function $f(x)$ is called **antidifferentiation**.

Example 2 Use Known Antidifferentiation Rules

Find all functions g such that $g'(x) = 4\sin x + \dfrac{2x^5 - \sqrt{x}}{x}$.

Solution

Rewrite the derivative.

$$g'(x) = 4\sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x} = 4\sin x + 2x^4 - \frac{1}{\sqrt{x}} = 4\sin x + 2x^4 - x^{-1/2}$$

Use the rules in Table 4.3 and the general antiderivative theorem.

$$\begin{aligned} g(x) &= 4(-\cos x) + 2\cdot\frac{x^5}{5} - \frac{x^{1/2}}{\frac{1}{2}} + C \\ &= -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + C \end{aligned}$$

■

In applications of calculus, it is very common to encounter a situation as in Example 2, where we need to find a function, given its derivative. An equation involving the derivative of a function is called a **differential equation** (DE).We will study these in greater detail in Chapter 7. For now, we can solve some elementary differential equations. The general solution of a differential equation involves an arbitrary constant (or constants). However, there may be some extra conditions given that will allow us to determine the constants and therefore uniquely specify the particular solution.

Example 3 Solve a Differential Equation

Find f if $f'(x) = e^x + \dfrac{20}{1 + x^2}$ and $f(0) = -2$.

Solution

Find the general antiderivative.

$$f(x) = e^x + 20\tan^{-1}x + C \qquad \text{Table 4.3 rules.}$$

To determine the value of C, use the given information, $f(0) = -2$.

$$f(0) = e^0 + 20\tan^{-1}0 + C = -2 \qquad \text{Let } x = 0.$$

$$C = -2 - 1 = -3 \qquad \text{Solve for } C.$$

The particular solution is $f(x) = e^x + 20\tan^{-1}x - 3$. ■

Example 4 Solve a Differential Equation Involving f''

Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

Solution

Find the general antiderivative of $f''(x)$.

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C \quad \text{Table 4.3 rules.}$$
$$= 4x^3 + 3x^2 - 4x + C \quad \text{Simplify.}$$

Use the antidifferentiation rules again to find the general antiderivative of f'.

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D \quad \text{Table 4.3 rules.}$$
$$= x^4 + x^3 - 2x^2 + Cx + D \quad \text{Simplify.}$$

Use the initial conditions to determine the values C and D: $f(0) = 4$, $f(1) = 1$.

$f(0) = 0 + D = 4 \ \Rightarrow \ D = 4$

$f(1) = 1 + 1 - 2 + C + 4 = 1 \ \Rightarrow \ C = -3$

Therefore, $f(x) = x^4 + x^3 - 2x^2 - 3x + 4$. ■

If we are given the graph of a function f, it seems reasonable that we should be able to sketch the graph of an antiderivative F. Suppose, for example, we are given that $F(0) = 1$. Then we have a place to start the graph of F, at the point $(0, 1)$. The direction in which we draw the graph over each interval is determined by the derivative $F'(x) = f(x)$.

In Example 5, we use the concepts presented in this chapter to draw the graph of F even though we do not have a formula for f. This situation occurs frequently in real-world applications, for example, when $f(x)$ is determined by experimental data.

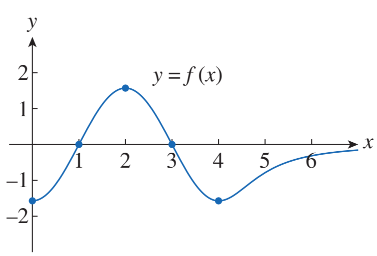


Figure 4.88
Graph of the derivative, f.

Example 5 Sketch an Antiderivative

The graph of a function f is given in Figure 4.88. Draw a rough sketch of an antiderivative F, given that $F(0) = 2$.

Solution

We are guided by the important concept that the slope of $y = F(x)$ is $f(x)$ at any point (x, y).

Start at the point $(0, 2)$.

For $0 < x < 1$, $f(x)$ is negative, so F should be decreasing on this interval.

Notice that $f(1) = f(3) = 0$. F has horizontal tangents when $x = 1$ and $x = 3$.

For $1 < x < 3$, $f(x)$ is positive, so F is increasing.

$f(x)$ changes from negative to positive at $x = 1$, so F has a local minimum there. And $f(x)$ changes from positive to negative at $x = 3$, so F has a local maximum there.

For $x > 3$, $f(x)$ is negative, so F is decreasing on $(3, \infty)$.

Since $f(x) \to 0$ as $x \to \infty$, the graph of F approaches a horizontal line as $x \to \infty$.

Notice that $F''(x) = f'(x)$ changes from positive to negative at $x = 2$ and from negative to positive at $x = 4$, so F has inflection points when $x = 2$ and $x = 4$.

Using all of this information, the graph of the antiderivative is shown in Figure 4.89. ■

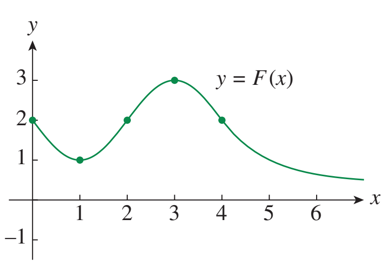


Figure 4.89
Graph of the antiderivative, F.

Rectilinear Motion

Antidifferentiation is useful in analyzing the motion of an object that is moving along a straight line. Recall that if the position of the object at time t is given by $s(t)$, then the velocity function is $v(t) = s'(t)$. This means that the position function is an antiderivative of the velocity function. Similarly, the acceleration function is $a(t) = v'(t)$, so the velocity function is an antiderivative of the acceleration function. Therefore, if the acceleration and the initial values $s(0)$ and $v(0)$ are known, then the position function can be determined by antidifferentiating twice.

Example 6 Find the Position Function, Given Acceleration

A particle moves along a straight line so that its acceleration at time t is given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial position is $s(0) = 9$ cm. Find its position function $s(t)$.

Solution

Find the velocity function, which is the general antiderivative of a.

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$$

Use $v(0) = -6$.

$$v(0) = 0 + 0 + C = -6 \;\Rightarrow\; C = -6 \;\Rightarrow\; v(t) = 3t^2 + 4t - 6$$

Find the position function, the general antiderivative of v.

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

Use $s(0) = 9$ to find D.

$$s(0) = 0 + 0 - 0 + D = 9 \;\Rightarrow\; D = 9$$

The position function is $s(t) = t^3 + 2t^2 - 6t + 9$. ■

Many particle motion problems involve objects moving along a straight line perpendicular to Earth's surface. An object near the surface of Earth is subject to a gravitational force that produces a downward acceleration denoted by g. For motion close to the ground we may assume that g is constant and its value is approximately $9.8\ \text{m/s}^2$ or, equivalently, $32\ \text{ft/s}^2$.

Example 7 Maximum Height

A ball is thrown upward with a velocity of 48 ft/s from the edge of a cliff, 432 ft above the ground. Find the ball's height above the ground t seconds later. When does the ball reach its maximum height? When does the ball hit the ground?

Solution

The motion of the ball is vertical and we generally choose the positive direction to be upward.

Let $s(t)$ be the distance of the ball above the ground at time t seconds.

The only force acting on the ball is gravity, pulling it downward.

Therefore, $a(t) = \dfrac{dv}{dt} = -32$.

Find the general antiderivative of a: $v(t) = -32t + C$.

Use the given information, $v(0) = 48$, to determine C.

$v(0) = 0 + C = 48 \quad \Rightarrow \quad C = 48$

Therefore, the velocity function is $v(t) = -32t + 48$.

The ball's maximum height is reached when $v(t) = 0$.

$$v(t) = 0 \;\Rightarrow\; -32t + 48 = 0 \;\Rightarrow\; t = \frac{48}{32} = 1.5 \text{ seconds}$$

Now find the general antiderivative of s.

$$s(t) = -32\frac{t^2}{2} + 48t + D = -16t^2 + 48t + D$$

Use the given information, $s(0) = 432$, to determine D.

$s(0) = 0 + 0 + D = 432 \quad \Rightarrow \quad D = 432$

Therefore, the position function is $s(t) = -16t^2 + 48t + 432$.

The position function is valid until the time at which the ball hits the ground.

This happens when $s(t) = 0$.

$$s(t) = 0 \quad \Rightarrow \quad -16t^2 + 48t + 432 = 0 \quad \Rightarrow \quad t^2 - 3t - 27 = 0$$

Using the quadratic formula and technology to solve this equation, we find

$$t = \frac{3 + 3\sqrt{13}}{2} = 6.908.$$

Therefore, the ball hits the ground at $t = 6.908$ seconds.

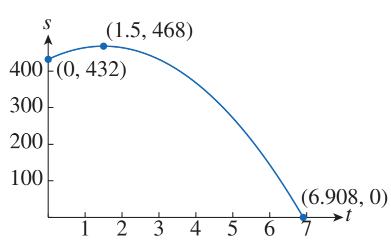


Figure 4.90
Graph of the position function.

Figure 4.90 shows a graph of the position function for the ball and illustrates the solutions in this problem. The ball reaches its maximum height of 468 ft after 1.5 seconds and hits the ground approximately 6.908 seconds after it is thrown. ■

4.8 Exercises

Find the most general antiderivative of the function. Check your answer by differentiation.

1. $f(x) = 4x + 7$

2. $f(x) = x^2 - 3x + 2$

3. $f(x) = 2x^3 - \frac{2}{3}x^2 + 5x$

4. $f(x) = 6x^5 - 8x^4 - 9x^2$

5. $f(x) = x(12x + 8)$

6. $f(x) = (x - 5)^2$

7. $f(x) = 7x^{2/5} + 8x^{-4/5}$

8. $f(x) = x^{3.4} - 2x^{\sqrt{2}-1}$

9. $f(x) = \sqrt{2}$

10. $f(x) = e^2$

11. $f(x) = 3\sqrt{x} - 2\sqrt[3]{x}$

12. $f(x) = \sqrt[3]{x^2} + x\sqrt{x}$

13. $f(x) = \frac{1}{5} - \frac{2}{x}$

14. $f(t) = \frac{3t^4 - t^3 + 6t^2}{t^4}$

15. $g(t) = \frac{1 + t + t^2}{\sqrt{t}}$

16. $f(\theta) = \sec\theta \tan\theta - 2e^\theta$

17. $h(\theta) = 2\sin\theta - \sec^2\theta$

18. $g(v) = 2\cos v - \frac{3}{\sqrt{1 - v^2}}$

19. $f(x) = 2^x + x^2$

20. $f(x) = 1 + 2\sin x + \frac{3}{\sqrt{x}}$

21. $f(x) = \frac{2x^4 + 4x^3 - x}{x^3}, \quad x > 0$

22. $f(x) = \frac{2x^2 + 5}{x^2 + 1}$

Find the antiderivative F of f that satisfies the given condition. Check your answer by comparing the graphs of f and F.

23. $f(x) = 5x^4 - 2x^5, \quad F(0) = 4$

24. $f(x) = 4 - \dfrac{3}{1 + x^2}, \quad F(1) = 0$

Find f.

25. $f''(x) = 20x^3 - 12x^2 + 6x$

26. $f''(x) = x^6 - 4x^4 + x + 1$

27. $f''(x) = 2x + 3e^x$

28. $f''(x) + \dfrac{1}{x^2}$

29. $f'''(t) = 12 + \sin t$

30. $f'''(t) = \sqrt{t} - 2\cos t$

31. $f'(x) = 1 + 3\sqrt{x}, \quad f(4) = 25$

32. $f'(x) = 5x^4 - 3x^2 + 4, \quad f(-1) = 2$

33. $f'(t) = \dfrac{4}{1 + t^2}, \quad f(1) = 0$

34. $f'(t) = t + \dfrac{1}{t^3}, \quad t > 0, \quad f(1) = 6$

35. $f'(x) = 5x^{2/3}, \quad f(8) = 21$

36. $f'(x) = \dfrac{x + 1}{\sqrt{x}}, \quad f(1) = 5$

37. $f'(x) = \dfrac{4}{\sqrt{1 - x^2}}, \quad f\left(\dfrac{1}{2}\right) = 1$

38. $f'(t) = \sec t(\sec t + \tan t), \quad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}, \quad f\left(\dfrac{\pi}{4}\right) = -1$

39. $f'(t) = 3^t - \dfrac{3}{t}, \quad f(1) = 2, \quad f(-1) = 1$

40. $f'(x) = \cos x, \quad f\left(\dfrac{\pi}{2}\right) = 0$

41. $f'(x) = 2x - \dfrac{1}{x}, \quad f(-1) = 1$

42. $f''(x) = -2 + 12x - 12x^2, \quad f(0) = 4, \quad f'(0) = 12$

43. $f''(x) = 8x^3 + 5, \quad f(1) = 0, \quad f'(1) = 8$

44. $f''(\theta) = \sin\theta + \cos\theta, \quad f(0) = 3, \quad f'(0) = 4$

45. $f''(t) = t^2 + \dfrac{1}{t^2}, \quad t > 0, \quad f(2) = 3, \quad f'(1) = 2$

46. $f''(x) = 4 + 6x + 24x^2, \quad f(0) = 3, \quad f(1) = 10$

47. $f''(x) = e^x - 2\sin x, \quad f(0) = 3, \quad f\left(\dfrac{\pi}{2}\right) = 0$

48. $f''(t) = \sqrt[3]{t} - \cos t, \quad f(0) = 2, \quad f(1) = 2$

49. $f''(x) = \dfrac{1}{x^2}, \quad x > 0, \quad f(1) = 0, \quad f(2) = 0$

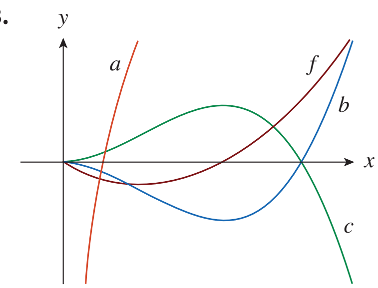
50. $f'''(x) = \cos x, \quad f(0) = 1, \quad f'(0) = 2, \quad f''(0) = 3$

51. Suppose the graph of f passes through the point (2, 5) and that the slope of the tangent line to the graph of f at the point $(x, f(x))$ is $3 - 4x$. Find $f(1)$.

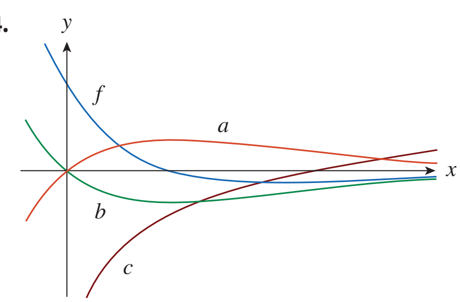
52. Find a function f such that $f'(x) = x^3$ and the graph of the line defined by $x + y = 0$ is tangent to the graph of f.

The graph of a function f is shown in the figure. Which graph is an antiderivative of f? Explain your reasoning.

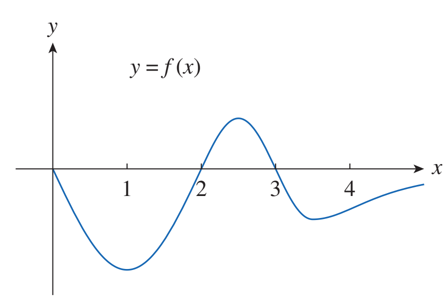
53.



54.

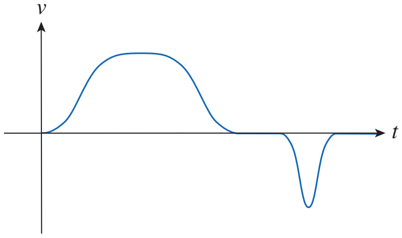


55. The graph of the function f is shown in the figure.



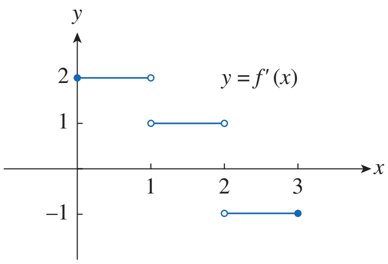
Make a rough sketch of an antiderivative of F, given that $F(0) = 1$.

56. The graph of the velocity function of a particle in motion is shown in the figure.



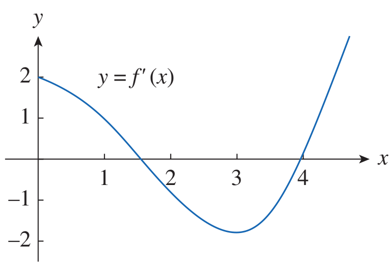
Sketch a graph of a corresponding position function.

57. The graph of f' is shown in the figure.



Sketch the graph of f if f is continuous and $f(0) = -1$.

58. The graph of f' is shown in the figure.



Sketch the graph of f if f is continuous and $f(0) = 0$.

59. Let $f(x) = 2x - 3\sqrt{x}$.
(a) Sketch the graph of f.
(b) Use the graph in part (a) to sketch a rough graph of the antiderivative F such that $F(0) = 1$.
(c) Use the antidifferentiation formulas in this section to find an expression for $F(x)$.
(d) Use technology to graph F using the expression in part (c). Compare this with your sketch in part (b).

60. Let $f(x) = e^x - 2x$.
(a) Use technology to sketch the graph of f.
(b) Use the graph in part (a) to sketch a rough graph of the antiderivative F such that $F(0) = 1$.
(c) Use the antidifferentiation formulas in this section to find an expression for $F(x)$.
(d) Graph F using the expression in part (c). Compare this graph with your sketch in part (b).

Use technology to graph f and use your graph to make a rough sketch of the antiderivative that passes through the origin.

61. $f(x) = \dfrac{\sin x}{1 + x^2} \quad -2\pi \le x \le 2\pi$

62. $f(x) = \sqrt{x^4 - 2x^2 + 2} - 2, \quad -3 \le x \le 3$

A particle is moving along a straight line with the given information. Find the position function of the particle.

63. $v(t) = \sin t - \cos t, \quad s(0) = 0$

64. $v(t) = t^2 - 3\sqrt{t}, \quad s(4) = 8$

65. $a(t) = 2t + 1, \quad s(0) = 3, \quad v(0) = -2$

66. $a(t) = 3\cos t - 2\sin t, \quad s(0) = 0, \quad v(0) = 4$

67. $a(t) = 10\sin t + 3\cos t, \quad s(0) = 0, \quad s(2\pi) = 12$

68. $a(t) = t^2 - 4t + 6, \quad s(0) = 0, \quad s(1) = 20$

69. A stone is dropped from the upper observation deck (the Space Deck) of the CN Tower, 450 m above the ground.
(a) Find the distance of the stone above the ground at time t.
(b) How long does it take the stone to reach the ground?
(c) With what velocity does it strike the ground?
(d) If the stone is thrown downward with a velocity of 5 m/s, how long does it take to reach the ground?

70. Oil is leaking from a tank at a rate of $r(t)$ barrels per hour. Let $A(t)$ be the number of barrels of oil that have leaked out after t hours, and suppose $r'(t)$, the rate of change of $r(t)$, is given by $r'(t) = \dfrac{1}{t^2} + \dfrac{t^2}{4}$. It is also known that $r(1) = 3$ and $A(1) = 4$.
(a) Write an equation for $r(t)$, the rate at which oil is leaking from the tank at time t. Using correct units, find $r(2)$.
(b) Write an equation for $A(t)$, the total amount of oil that has leaked after t hours.
(c) At time $t = 2$, is the rate of leakage increasing or decreasing? Justify your answer.

71. Show that for motion of an object in a straight line with constant acceleration a, initial velocity v_0, and initial position s_0, the position function at time t is

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0$$

72. Suppose an object is projected upward with initial velocity v_0 meters per second from a point s_0 meters above the ground. Show that

$$[v(t)]^2 = v_0^2 - 19.6[s(t) - s_0]$$

73. Two balls are thrown upward from the edge of a cliff as in Example 7. The first is thrown with a velocity of 48 ft/s and the other is thrown a second later with a speed of 24 ft/s. Do the balls ever pass each other?

74. A company estimates that the marginal cost (in dollars per item) of producing x items is $1.92 - 0.002x$. If the cost of producing one item is \$562, find the cost of producing 100 items.

75. The linear density of a rod of length 1 m is given by $\rho(x) = \dfrac{1}{\sqrt{x}}$, in grams per centimeter, where x is measured in centimeters from the end of the rod. Find the mass of the rod.

76. A stone was dropped off a cliff and hit the ground with a speed of 120 ft/s. What is the height of the cliff?

77. Since raindrops grow as they fall, their surface area increases, and therefore, the resistance to their falling increases. A raindrop has an initial downward velocity of 10 m/s and its downward acceleration is

$$a(t) = \begin{cases} -9 - 0.9t & \text{if } 0 \le t \le 10 \\ 0 & \text{if } t > 10 \end{cases}$$

If the raindrop is initially 500 m above the ground, how long does it take to fall?

78. A car is traveling at 50 mi/h when the brakes are fully applied, producing a constant deceleration of 22 ft/s^2. What is the distance traveled before the car comes to a complete stop?

79. What constant acceleration is required to increase the speed of a car from 30 mi/h to 50 mi/h in 5 seconds?

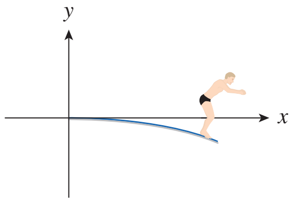
80. A car braked with a constant deceleration of 16 ft/s^2, producing skid marks measuring 200 ft before coming to a stop. How fast was the car traveling when the brakes were first applied?

81. A car is traveling at 100 km/h when the driver sees an accident in the road 80 m ahead and slams on the brakes. What constant deceleration is required to stop the car in time to avoid the accident?

82. If a diver of mass m stands at the end of a diving board with length L and linear density ρ, then the board takes on the shape of the graph of $y = f(x)$, where

$$EIy'' = mg(L - x) + \frac{1}{2}\rho g(L - x)^2$$

E and I are positive constants that depend on the material of the board and $g(< 0)$ is the acceleration due to gravity.



(a) Find an expression for y.
(b) Use $f(L)$ to estimate the distance below the horizontal at the end of the board.

83. A model rocket is fired vertically upward from rest. Its acceleration for the first 3 seconds is $a(t) = 60t$, at which time the fuel is exhausted and it becomes a freely falling body. Fourteen seconds later, the rocket's parachute opens, and the (downward) velocity slows linearly to -18 ft/s in 5 seconds. The rocket then floats to the ground at that rate.
(a) Determine the position function s and the velocity function v (for all times t). Sketch the graphs of s and v.
(b) At what time does the rocket reach its maximum height, and what is that height?
(c) At what time does the rocket land?

84. A high-speed train accelerates and decelerates at the rate of 4 ft/s^2. Its maximum cruising speed is 90 mi/h.
(a) What is the maximum distance the train can travel if it accelerates from rest until it reaches its cruising speed and then runs at that speed for 15 minutes?
(b) Suppose that the train starts from rest and must come to a complete stop in 15 minutes. What is the maximum distance it can travel under these conditions?
(c) Find the minimum time that the train takes to travel between two consecutive stations that are 45 mi apart.
(d) The trip from one station to the next takes 37.5 minutes. How far apart are the stations?

4 Review

Concepts and Vocabulary

1. Explain the difference between an absolute maximum and a local maximum. Illustrate with a sketch.

2. (a) Explain the Extreme Value Theorem in your own words.
(b) Explain how the Closed Interval Method is used.

3. (a) State Fermat's Theorem.
(b) Define a critical number of a function f.

4. (a) State Rolle's Theorem.
(b) State the Mean Value Theorem and provide a geometric interpretation.

5. (a) State the Increasing/Decreasing Test.
(b) What does it mean to say that f is concave up on an interval I?
(c) State the Concavity Test.
(d) What are inflection points? How do you find them?

6. (a) State the First Derivative Test for local extrema.
(b) State the Second Derivative Test.
(c) What are the advantages and disadvantages of these tests?

7. (a) How is l'Hospital's Rule used to find a limit?
(b) How can l'Hospital's Rule be used to evaluate the limit of a product $f(x)g(x)$ where $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$?
(c) How can l'Hospital's Rule be used to evaluate the limit of a difference $f(x) - g(x)$ where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$?
(d) How can l'Hospital's Rule be used to evaluate the limit of an expression in the form $[f(x)]^{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$?

8. State whether each of the following limit forms is indeterminate. Where possible, state the limit.

(a) $\frac{0}{0}$ (b) $\frac{\infty}{\infty}$ (c) $\frac{0}{\infty}$
(d) $\frac{\infty}{0}$ (e) $\infty + \infty$ (f) $\infty - \infty$
(g) $\infty \cdot \infty$ (h) $\infty \cdot 0$ (i) 0^0
(j) 0^∞ (k) ∞^0 (l) 1^∞

9. Even though sophisticated technology is available, explain why we still need calculus to graph a function.

10. (a) Given an initial approximation x_1 to a root of the equation $f(x) = 0$, explain geometrically, with a graph, how the second approximation x_2 in Newton's method is obtained.
(b) Write an expression for x_2 in terms of x_1, $f(x_1)$ and $f'(x_1)$.
(c) Write an expression for x_{n+1} in terms of x_n, $f(x_n)$, and $f'(x_n)$.
(d) Explain the conditions in which Newton's method is likely to fail or to produce a root very slowly.

11. (a) Explain the relationship between a function f and an antiderivative F.
(b) Suppose F_1 and F_2 are both antiderivatives of f on an interval I. How are F_1 and F_2 related?

True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. If $f'(c) = 0$, then f has a local maximum or minimum at c.

2. If f has an absolute minimum value at c, then $f'(c) = 0$.

3. If f is continuous on (a, b), then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in (a, b).

4. If f is differentiable and $f(-1) = f(1)$, then there is a number c such that $|c| < 1$ and $f'(c) = 0$.

5. If $f'(x) < 0$ for $1 < x < 6$, then f is decreasing on $(1, 6)$.

6. If $f''(2) = 0$, then $(2, f(2))$ is an inflection point on the graph of $y = f(x)$.

7. If $f'(x) = g'(x)$ for $0 < x < 1$, then $f(x) = g(x)$ for $0 < x < 1$.

8. There exists a function f such that $f(1) = -2$, $f(3) = 0$, and $f'(x) > 1$ for all x.

9. There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

10. There exists a function f such that $f(x) < 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

11. If f and g are increasing on an interval I, then $f + g$ is increasing on I.

12. If f and g are increasing on an interval I, then $f - g$ is increasing on I.

13. If f and g are increasing on an interval I, then fg is increasing on I.

14. If f and g are positive increasing functions on an interval I, then fg is increasing on I.

15. If f is increasing and $f(x) > 0$ on I, then $g(x) = \dfrac{1}{f(x)}$ is decreasing on I.

16. If f is even, then f' is even.

17. If f is periodic, then f' is periodic.

18. If $f'(x)$ exists and is nonzero for all x, then $f(0) \neq f(1)$.

19. The most general antiderivative of $f(x) = x^{-2}$ is

$$F(x) = -\frac{1}{x} + C$$

20. If $\lim_{x \to \infty} f(x) = 1$ and $\lim_{x \to \infty} g(x) = \infty$, then

$$\lim_{x \to \infty} [f(x)]^{g(x)} = 1$$

21. $\displaystyle \lim_{x \to \infty} \frac{x}{e^x} = 1$

22. If the equation $f(x) = 0$ has three roots, then using Newton's method with any initial approximation will always eventually produce a reasonable estimate of one root.

Exercises

Find the local and absolute extreme values of the function on the given interval.

1. $f(x) = x^3 - 9x^2 + 24x - 2, \quad [0, 5]$

2. $f(x) = x\sqrt{1 - x}, \quad [-1, 1]$

3. $f(x) = \dfrac{3x - 4}{x^2 + 1}, \quad [-2, 2]$

4. $f(x) = \sqrt{x^2 + x + 1}, \quad [-2, 1]$

5. $f(x) = x + 2\cos x, \quad [-\pi, \pi]$

6. $f(x) = x^2 e^{-x}, \quad [-1, 3]$

7. $f(x) = \dfrac{\ln x}{x^2}, \quad [1, 3]$

Sketch the graph of the function that satisfies the given conditions.

8. $f(0) = 0, \quad f'(-2) = f'(1) = f'(9) = 0,$

$\lim_{x \to \infty} f(x) = 0, \quad \lim_{x \to 6} f(x) = -\infty,$

$f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$,

$f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$,

$f''(x) > 0$ on $(-\infty, 0)$ and $(12, \infty)$,

$f''(x) < 0$ on $(0, 6)$ and $(6, 12)$

9. $f(0) = 0, \quad f$ is continuous and even,

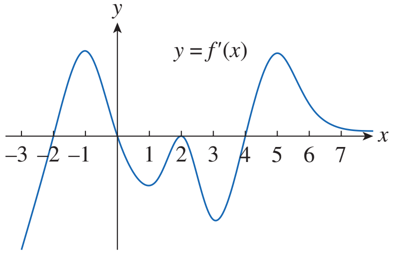
$f'(x) = 2x$ if $0 < x < 1, \quad f'(x) = -1$ if $1 < x < 3,$
$f'(x) = 1$ if $x > 3$

10. f is odd, $\quad f'(x) < 0$ for $0 < x < 2,$

$f'(x) > 0$ for $x > 2, \quad f''(x) > 0$ for $0 < x < 3,$

$f''(x) < 0$ for $x > 3, \quad \lim_{x \to \infty} f(x) = -2$

11. The figure shows the graph of the *derivative* f' of a function f.



(a) On what intervals is f increasing or decreasing?
(b) For what values of x does f have a local maximum or minimum?
(c) Sketch the graph of f''.
(d) Sketch a possible graph of f.

For each function

(a) Find the vertical and horizontal asymptotes, if any.
(b) Find the intervals on which the function is increasing or decreasing.
(c) Find the local maximum and minimum values.
(d) Find the intervals of concavity and the inflection points.
(e) Use the information from parts (a)–(d) to sketch the graph of f. Check your work with technology.

12. $f(x) = 2 - 2x - x^3$

13. $f(x) = x^4 + 4x^3$

14. $f(x) = -2x^3 - 3x^2 + 12x + 5$

15. $f(x) = 3x^4 - 4x^3 + 2$

16. $f(x) = \dfrac{1}{1 - x^2}$

17. $f(x) = \dfrac{x}{1 - x^2}$

18. $f(x) = \dfrac{1}{x(x - 3)^2}$

19. $(x) = \dfrac{(x - 1)^3}{x^2}$

20. $f(x) = x + \sqrt{1 - x}$

21. $f(x) = \sqrt{1 - x} + \sqrt{1 + x}$

22. $f(x) = \dfrac{1}{x^2} - \dfrac{1}{(x - 2)^2}$

23. $f(x) = x\sqrt{2 + x}$

24. $f(x) = x^{2/3}(x - 3)^2$

25. $f(x) = e^x \sin x, \quad -\pi \le x \le \pi$

26. $f(x) = 4x - \tan x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

27. $f(x) = e^{2x - x^2}$

28. $f(x) = (x - 2)e^{-x}$

29. $f(x) = e^x + e^{-3x}$

30. $f(x) = \sin^{-1}\left(\dfrac{1}{x}\right)$

31. $f(x) = \ln(x^2 - 1)$

32. $f(x) = x + \ln(x^2 + 1)$

Sketch a graph of the function that includes all important characteristics. Use the graphs of f' and f'' to estimate the intervals on which the function is increasing or decreasing, extreme values, intervals of concavity, and inflection points. In Exercise 33, use calculus to find these quantities analytically.

33. $f(x) = \dfrac{x^2 - 1}{x^3}$

34. $f(x) = \dfrac{x^3 + 1}{x^6 + 1}$

35. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2$

36. $f(x) = x^2 + 6.5 \sin x, \quad -5 \le x \le 5$

37. Graph $f(x) = e^{-1/x^2}$ in a viewing rectangle that shows all the important characteristics of this function. Estimate the inflection points. Use calculus to find them analytically.

38. Let $f(x) = \dfrac{1}{1 + e^{1/x}}$.

(a) Sketch the graph of f.

(b) Explain the shape of the graph by computing the limits of $f(x)$ as x approaches $\infty, -\infty, 0^+$, and 0^-.

(c) Use the graph to estimate the coordinates of the inflection points.

(d) Find $f'(x)$ and use technology to sketch the graph of $y = f'(x)$.

(e) Use the graph in part (d) to estimate the inflection points more accurately.

Use the graphs of f, f', and f'' to estimate the x-coordinate of the maximum and minimum points and inflection points of f.

39. $f(x) = \dfrac{\cos^2 x}{\sqrt{x^2 + x + 1}}, \quad -\pi \le x \le \pi$

40. $f(x) = e^{-0.1x} \ln(x^2 - 1)$

41. Investigate the family of functions $f(x) = \ln(\sin x + C)$. What features do the members of this family have in common? How do they differ? For which values of C is f continuous on $(-\infty, \infty)$? For which values of C is f undefined for all real numbers? What happens as $C \to \infty$?

42. Investigate the family of functions $f(x) = cxe^{-cx^2}$. What happens to the maximum and minimum points and the inflection points as c changes? Illustrate your conclusions by graphing several members of this family of functions.

43. Show that the equation $3x + 2\cos x + 5 = 0$ has exactly one real root.

44. Suppose that f is continuous on $[0, 4]$, $f(0) = 1$, and $2 \le f'(x) \le 5$ for all x in $(0, 4)$. Show that $9 \le f'(x) \le 21$.

45. Consider the function $f(x) = x^{1/5}$ on the interval $[32, 33]$. Use the Mean Value Theorem to show that

$$2 < \sqrt[5]{33} < 2.0125$$

46. Find the values of a and b such that the point $(1, 3)$ is a point of inflection on the graph of $y = ax^3 + bx^2$.

47. Find the values of a and b such that the point $(1, 6)$ is a point of inflection on the graph of $y = x^3 + ax^2 + bx + 1$.

48. Let $g(x) = f(x^2)$, where f is twice differentiable for all x, $f'(x) > 0$ for all $x \ne 0$, and f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

(a) Find the x-coordinate(s) for which g has an extreme value.

(b) Discuss the concavity of the graph of g.

Evaluate the limit.

49. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{\tan x}$

50. $\displaystyle\lim_{x\to 0} \frac{\tan x}{x + \sin 2x}$

51. $\displaystyle\lim_{x\to 0} \frac{e^{2x} - e^{-2x}}{\ln(x + 1)}$

52. $\displaystyle\lim_{x\to \infty} \frac{e^{2x} - e^{-2x}}{\ln(x + 1)}$

53. $\displaystyle\lim_{x\to -\infty} (x^2 - x^3)e^{2x}$

54. $\displaystyle\lim_{x\to \pi^-} (x - \pi)\csc x$

55. $\displaystyle\lim_{x\to 1^+} \left(\frac{x}{x - 1} - \frac{1}{\ln x}\right)$

56. $\displaystyle\lim_{x\to (\pi/2)^-} (\tan x)^{\cos x}$

57. Find two positive integers such that the sum of the first number and four times the second number is 1000 and the product of the numbers is as large as possible.

58. Show that the shortest distance from the point (x_1, y_1) to the straight line defined by $Ax + By + C = 0$ is

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

59. Find the point on the hyperbola defined by $xy = 8$ that is closest to the point $(3, 0)$.

60. Find the smallest possible area of an isosceles triangle that is circumscribed about a circle of radius r.

61. Find the volume of the largest circular cone that can be inscribed in a sphere of radius r.

62. In ΔABC, D lies on AB, $CD \perp AB$, $|AD| = |BD| = 4$ cm, and $|CD| = 5$ cm. Where should point P be chosen on CD so that the sum $|PA| + |PB| + |PC|$ is a minimum?

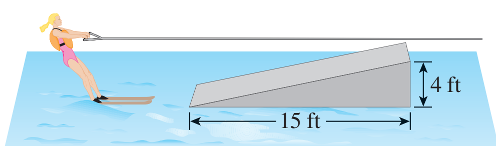
63. Solve Exercise 62 when $|CD| = 2$ cm.

64. The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h. How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun $\pi/6$?

65. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?

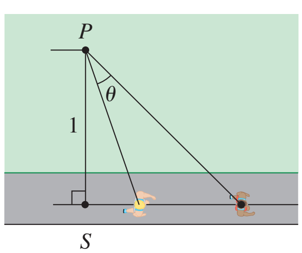
66. A balloon is rising at a constant speed of 30 ft/s. A person is cycling along a straight road at a speed of 15 ft/s. When they pass under the balloon, it is 45 ft above them. How fast is the distance between the person and the balloon increasing 3 s later?

67. A water-skier skis over the ramp shown in the figure at a speed of 30 ft/s. How fast is the person rising as they leave the ramp?



68. An observer stands at a point P, one unit away from a track. Two runners start at the point S in the figure and run along the track. One runner runs three times as fast as the other. Find the maximum value of the observer's angle of sight θ between the runners.

Hint: Maximize $\tan\theta$.



69. The velocity of a wave of length L in deep water is

$$v = K\sqrt{\frac{L}{C} + \frac{C}{L}}$$

where K and C are known positive constants. What is the length of the wave that gives the minimum velocity?

70. A metal storage tank with volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions require the least amount of metal?

71. A hockey team plays in an arena with seating capacity of 15,000 spectators. If the ticket price is \$12, average attendance at a game is 11,000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?

72. A manufacturer determines that the cost of making x units of a commodity is

$$C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$$

and the demand function is $p(x) = 48.2 - 0.03x$.

(a) Graph the cost and revenue functions and use the graphs to estimate the production level for maximum profit.

(b) Use calculus to find the production level for maximum profit.

(c) Estimate the production level that minimizes the average cost.

73. Use Newton's method to find all roots of the equation $\sin x = x^2 - 3x + 1$ correct to six decimal places.

74. Use Newton's method to find the root of the equation

$$x^5 - x^4 + 3x^2 - 3x - 2 = 0$$

in the interval [1, 2] correct to six decimal places.

75. Use Newton's method to find the absolute maximum value of the function $f(x) = \cos x + x - x^2$ correct to eight decimal places.

Find the most general antiderivative of the function.

76. $f(x) = \sqrt{2}x^2 - 7 + x^{-3/5}$ **77.** $f(x) = \sin x - \sec^2 x$

78. $f(x) = e^x - \dfrac{2}{\sqrt{x}}$ **79.** $f(x) = \dfrac{1+x}{\sqrt{x}}$

Find $f(x)$.

80. $f'(x) = 2x - 3\sin x, \quad f(0) = 5$

81. $f'(x) = \dfrac{x^2 + \sqrt{x}}{x}, \quad f(1) = 3$

82. $f''(x) = 1 - 6x + 48x^2, \quad f(0) = 1, \quad f'(0) = 2$

83. $f''(x) = 2x^3 + 3x^2 - 4x + 5, \quad f(0) = 2, \quad f(1) = 0$

Suppose a particle is moving along a horizontal line with the given data. Find the position function $s(t)$ of the particle.

84. $v(t) = 2t - \dfrac{1}{1+t^2}, \quad s(0) = 1$

85. $a(t) = \sin t + 3\cos t, \quad s(0) = 0, \quad v(0) = 2$

---

86. Let $f(x) = 0.1e^x + \sin x, \quad -4 \le x \le 4$.

(a) Use a graph of f to sketch a rough graph of the antiderivative F of f such that $F(0) = 0$.

(b) Find an expression for $F(x)$.

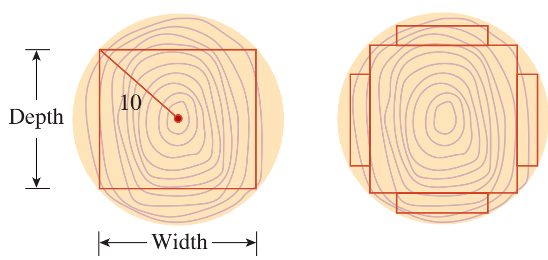
(c) Graph F using the expression in part (b). Compare this with your sketch in part (a).

87. A canister is dropped from a helicopter 500 m above the ground. Its parachute does not open, but the canister has been designed to withstand an impact velocity of 100 m/s. Will the canister land intact or burst upon impact?

88. Investigate the family of functions $f(x) = x^4 + x^3 + cx^2$. Determine the transitional value of c at which the number of critical points changes and the transitional value at which the number of inflection points changes. Illustrate the various possible shapes with graphs.

89. In an automobile race along a straight road, Car A passed Car B twice. Show that at some time during the race their accelerations were equal. State any assumptions that you make in reaching your conclusion.

90. A rectangular beam will be cut from a cylindrical log of radius 10 in as shown in the figure.



(a) Show that the beam of maximal cross-sectional area is a square.

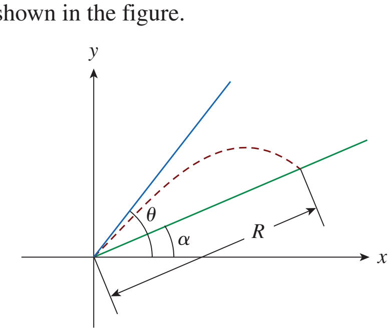
(b) Four rectangular planks will be cut from sections of the log that remain after cutting the square beam. Determine the dimensions of the planks that will have maximal cross-sectional area.

(c) Suppose that the strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from the cylindrical log.

91. If a projectile is fired with an initial velocity v at an angle of inclination θ from the horizontal, then its trajectory, neglecting air resistance, is given by

$$y = (\tan\theta)x - \frac{g}{2v^2\cos^2\theta}x^2, \quad 0 < \theta < \frac{\pi}{2}$$

(a) Suppose the projectile is fired from the base of a plane that is inclined at an angle α, $\alpha > 0$, from the horizontal, as shown in the figure.



Show that the range of the projectile, measured up the slope, is given by

$$R(\theta) = \frac{2v^2\cos\theta\sin(\theta - \alpha)}{g\cos^2\alpha}$$

(b) Determine θ so that R is a maximum.

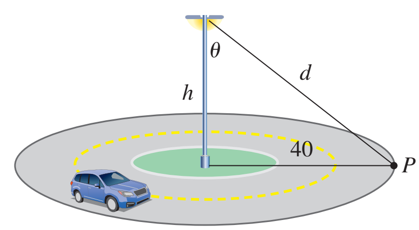
(c) Suppose the plane is at an angle α *below* the horizontal. Determine the range R in this case, and determine the angle at which the projectile should be fired to maximize R.

92. Show that, for $x > 0$,

$$\frac{x}{1 + x^2} < \tan^{-1}x < x$$

93. Sketch the graph of a function f such that $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim_{x\to\pm\infty}[f(x) + x] = 0$.

94. A light is placed atop a pole of height h feet to illuminate a busy traffic circle, which has radius 40 ft. The intensity of illumination I at any point P on the circle is directly proportional to the cosine of the angle θ and inversely proportional to the square of the distance d from the source.



(a) How tall should the light pole be to maximize I?

(b) Suppose that the light pole is h feet tall and that a construction worker is walking away from the base of the pole at the rate of 4 ft/s. At what rate is the intensity of the light at the point on their back 4 ft above the ground decreasing when they reach the outer edge of the traffic circle?

95. Water is flowing at a constant rate into a spherical tank. Let $V(t)$ be the volume of water in the tank and $H(t)$ be the height of the water in the tank at time t.

(a) Interpret $V'(t)$ and $H'(t)$ in the context of this problem. Are these derivatives positive, negative, or zero? Explain your reasoning.

(b) Is $V''(t)$ positive, negative, or zero? Explain your reasoning.

(c) Let t_1, t_2, and t_3 be the times when the tank is one-quarter full, half full, and three-quarters full, respectively. Are the values of $H''(t_1)$, $H''(t_2)$, and $H''(t_3)$ positive, negative, or zero? Explain your reasoning.

Focus on Problem Solving

One of the most important principles of problem solving is *analogy*. In order to get started on a problem, it is often helpful to consider a similar, but simpler, problem. Think about how to relate, or associate, the original problem with one that you have solved before. The following example illustrates this principle.

Example Number Theory

If x, y, and z are positive numbers, prove that

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \geq 8$$

Solution

There are several things we might try first, for example multiple both sides by xyz, or multiply out the numerator. But both of these ideas lead to a more complicated expression.

Consider a similar, simpler problem. When there are several variables involved, it is often helpful to think of an analogous problem with fewer variables.

In this case, we can reduce the number of variables from three to one, and prove the analogous inequality

$$\frac{x^2+1}{x} \geq 2 \quad \text{for } x > 2 \tag{1}$$

If we can prove the inequality in Equation 1, then the original inequality follows because

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} = \left(\frac{x^2+1}{x}\right)\left(\frac{y^2+1}{y}\right)\left(\frac{z^2+1}{z}\right) \geq 2 \cdot 2 \cdot 2 = 8.$$

The key to proving the inequality in Equation 1 is to recognize that this is really just an optimization problem.

Let $f(x) = \dfrac{x^2+1}{x} = x + \dfrac{1}{x}, \qquad x > 0.$

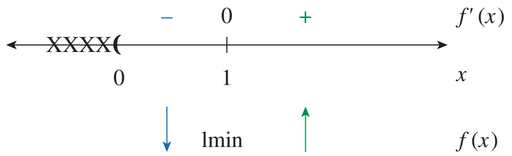
Then $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2-1}{x^2} = \dfrac{(x-1)(x+1)}{x^2}.$

Find the critical numbers.

$f'(x) = 0$: $x = 1$ ($x = -1$ is not in the domain.)

$f'(x)$ DNE: None ($x = 0$ is not in the domain.)

Construct a sign chart.



There is only one critical number, 1, and $f'(x) < 0$ for all $0 < x < 1$ and $f'(x) > 0$ for all $x > 1$.

By the First Derivative Test for Absolute Extreme Values, the absolute minimum occurs when $x = 1$.

The absolute minimum value is $f(1) = \dfrac{1^2+1}{1} = 2.$

Therefore, $f(x) = \dfrac{x^2+1}{x} \geq 2$ for all positive values of x.

And the original inequality follows by multiplication.

Note: The inequality in one variable can also be proved without calculus. If $x > 0$, then

$$\frac{x^2+1}{x} \geq 2 \iff x^2 + 1 \geq 2x \qquad \text{Multiply both sides by } x > 0.$$

$$\iff x^2 - 2x + 1 \geq 0 \qquad \text{Subtract } 2x \text{ from both sides.}$$

$$\iff (x-1)^2 \geq 0 \qquad \text{Factor.}$$

Since the last inequality is true, then the first one is also true (and the original inequality is true). ■

Problems

1. If a rectangle has its base on the x-axis and two vertices on the curve $y = e^{-x^2}$, show that the rectangle has the largest possible area when the two vertices are at the points of inflection of the curve.

2. Show that $|\sin x - \cos x| \leq \sqrt{2}$ for all x.

3. Show that, for all positive values of x and y,

$$\frac{e^{x+y}}{xy} \geq e^2$$

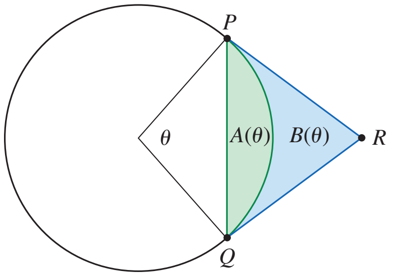
4. Show that $x^2y^2(4 - x^2)(4 - y^2) \leq 16$ for all numbers x and y such that $|x| \leq 2$ and $|y| \leq 2$.

5. Does the function $f(x) = e^{10|x-2|-x^2}$ have an absolute maximum? If so, find it. What about an absolute minimum?

6. Find the point on the parabola $y = 1 - x^2$ at which the tangent line and the coordinate axes form the triangle in the first quadrant with the smallest area.

7. Find the highest and the lowest points on the curve described by the equation $x^2 + xy + y^2 = 12$.

8. An arc PQ of a circle subtends a central angle θ as shown in the figure.



Let $A(\theta)$ be the area between the chord PQ and the arc PQ. Let $B(\theta)$ be the area between the tangent lines PR, QR, and the arc. Find

$$\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}$$

9. If a, b, c, and d are constants such that

$$\lim_{x \to 0} \frac{ax^2 + \sin bx + \sin cx + \sin dx}{3x^2 + 5x^4 + 7x^6} = 8$$

find the value of the sum $a + b + c + d$.

10. Sketch the region in the plane consisting of all points (x, y) such that

$$2xy \leq |x - y| \leq x^2 + y^2$$

11. Determine the values of the number a such that function f has no critical number.

$$f(x) = (a^2 + a - 6)\cos 2x + (a - 2)x + \cos 1$$

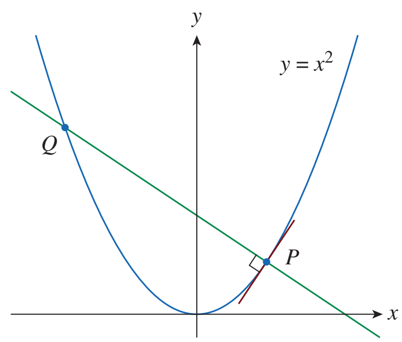
12. For what value of a is the following equation true?

$$\lim_{x \to \infty} \left(\frac{x + a}{x - a} \right)^x = e$$

13. For what values of c does the curve $y = cx^3 + e^x$ have inflection points?

14. Sketch the set of all points (x, y) such that $|x + y| \leq e^x$.

15. Let $P(a, a^2)$ be any point on the parabola $y = x^2$, except for the origin, and let Q be the point where the normal line at P intersects the parabola again, as shown in the figure.



(a) Show that the y-coordinate of Q is smallest when $a = \dfrac{1}{\sqrt{2}}$.

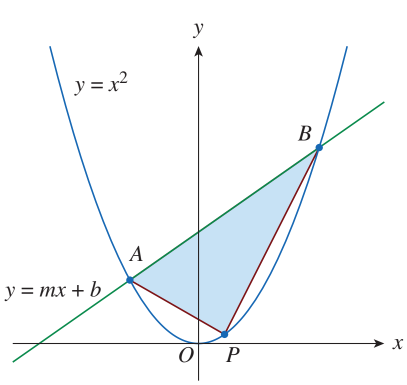
(b) Show that the line segment PQ has the shortest length when $a = \dfrac{1}{\sqrt{2}}$.

16. For what values of c is there a straight line that intersects the graph of

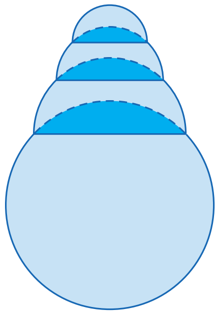
$$y = x^4 + cx^3 + 12x^2 - 5x + 2$$

in four distinct points?

17. The line $y = mx + b$ intersects the parabola $y = x^2$ at the points A and B, as shown in the figure.

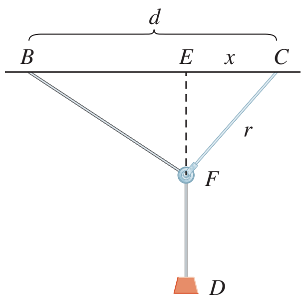


Find the point P on the arc AOB of the parabola that maximizes the area of triangle PAB.



18. $ABCD$ is a square piece of paper with sides of length 1 m. A quarter-circle is drawn from B to D with center A. The piece of paper is folded along EF, with E on AB and F on AD, so that A falls on the quarter-circle. Determine the maximum area of triangle AEF.

19. A hemispherical bubble is placed on a spherical bubble of radius 1. A smaller hemispherical bubble is then placed on the first one. This process is continued until n chambers, including the sphere, are formed. (The figure shows the case $n = 4$.) Use mathematical induction to prove that the maximum height of any bubble tower with n chambers is $1 + \sqrt{n}$.



20. One of the problems posed by the Marquis de l'Hospital in his calculus textbook *Analyse des Infiniment Petits* concerns a pulley that is attached to the ceiling of a room at a point C by a rope of length r. At another point B on the ceiling, at a distance d from C (where $d > r$), a rope of length ℓ is attached and passed through the pulley at F and connected to a weight W. The weight is released and comes to rest at its equilibrium position D. As l'Hospital argued, this happens when the distance $|ED|$ is maximized. Show that when the system reaches equilibrium, the value of x is

$$\frac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$$

Notice that this expression is independent of both W and ℓ.

21. Given a sphere with radius r, find the height of a pyramid of minimum volume whose base is a square and whose base and triangular faces are all tangent to the sphere. What if the base of the pyramid is a regular n-gon? (A regular n-gon is a polygon with equal sides and angles.) (Use the fact that the volume of a pyramid is $\frac{1}{3}Ah$, where A is the area of the base.)

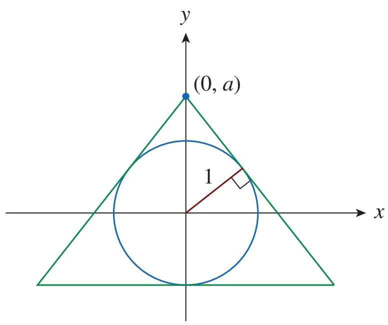
22. A container in the shape of an inverted cone has height 16 cm and radius 5 cm at the top. It is partially filled with a liquid that oozes through the sides at a rate proportional to the area of the container that is in contact with the liquid. (The surface area of a cone is πrl, where r is the radius and l is the slant height.) If we pour the liquid into the container at a rate of $2\ \text{cm}^3/\text{min}$, then the height of the liquid decreases at a rate of 0.3 cm/min when the height is 10 cm. If our goal is to keep the liquid at a constant height of 10 cm, at what rate should we pour the liquid into the container?

23. A cone of radius r centimeters and height h centimeters is lowered point first at a rate of 1 cm/s into a tall cylinder of radius R centimeters that is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?

24. Find the absolute maximum value of the function

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - 2|}$$

25. An isosceles triangle is circumscribed about the unit circle so that the equal sides meet at the point $(0, a)$ on the y-axis, as shown in the figure.



Find the value of a that minimizes the lengths of the equal sides. Hint: The result is not an equilateral triangle.

The Woodheads/Shutterstock.com

The Katse Dam is one of only 30 double-curvature concrete arch dams in the world and is approximately 185 m tall and 60 m thick at the foundation. It is located on the Malibiamatso River in the Kingdom of Lesotho, Africa. The lake produced behind the dam is extremely deep with a relatively small surface area, which results in efficient storage. Calculus can be used to determine a cross-sectional area of the dam, the surface area of the dam, the volume of concrete needed to complete the dam, the amount of water that flows into and out of the dam, the capacity of the dam, and even the amount of electricity produced during a certain time period.

5 Integrals

Contents

The tangent line and instantaneous velocity problems provided motivation and helped to introduce the concept of the derivative, which is the central concept in differential calculus. In a similar manner, the area and distance problems can be used to develop the idea of a definite integral, which is the basic concept of integral calculus.

There is a very important connection between integral calculus and differential calculus: the most important concept in the course, the Fundamental Theorem of Calculus, connects the integral to the derivative. This theorem will also help simplify the solution to many problems.

5.1 Areas and Distances

In this section we will discover that in order to find the area under a curve or the distance traveled by a car, we need to evaluate the same special type of limit.

The Area Problem

One of the most important unifying problems in calculus is the *area problem*. Suppose f is a continuous function and $f(x) \geq 0$. Find the area of the region S bounded by the graph of $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$. See Figure 5.1.

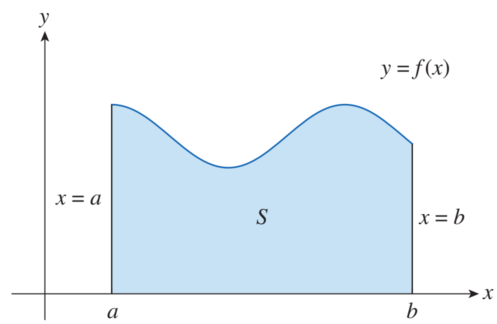
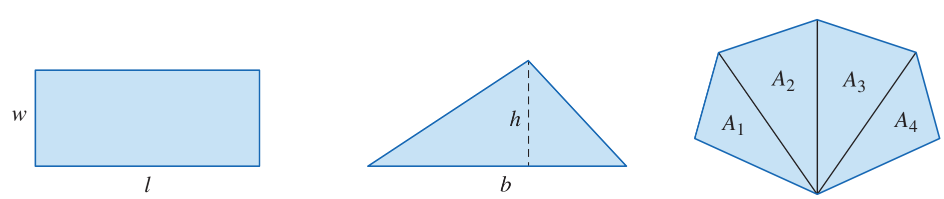


Figure 5.1
The region S is bounded by the graph of $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$:
$S = \{(x, y) \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$.

Before we can begin to solve this problem, we need to think more carefully about the meaning of the word *area*. It seems reasonable that area is positive; that is, the area of any plane region is a positive number. It doesn't really make sense to say that the area of a region is negative. Area is also additive. If we divide a region into nonoverlapping pieces, then the area of the region is the sum of the areas of the individual pieces. And, if region 1 is contained in region 2, then the area of region 1 is less than or equal to the area of region 2.

For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height. The area of a polygon is found by dividing it into triangles (as in Figure 5.2) and adding the areas of the triangles.



(a) $A = lw$.

(b) $A = \frac{1}{2}bh$.

(c) $A = A_1 + A_2 + A_3 + A_4$

Figure 5.2
The area of certain plane regions.

However, it isn't so easy to find the area of a region with curved sides. You probably have an intuitive idea for estimating the area of such a region. But part of the area problem is to find the precise area using an exact mathematical definition.

Recall that in defining a tangent line to a curve at a point we first approximated the slope of the tangent line by slopes of secant lines and then found a limit of these approximations. We will consider an analogous approach for areas. First, we will approximate the area of the region S by rectangles and then consider the limit of the areas of these rectangles as the number of rectangles increases. The following example illustrates this procedure.

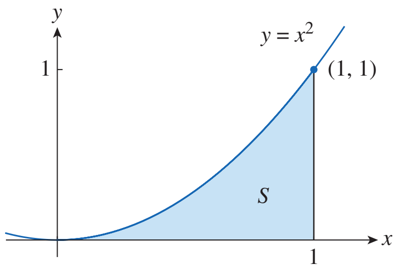


Figure 5.3
Estimate the area of the region S.

Example 1 Estimate an Area

Use rectangles to estimate the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$ as illustrated in Figure 5.3.

Solution

Note that the area of S must be between 0 and 1 because the region S is contained in a square with side of length 1. But we can certainly find a better estimate than this. Divide the region S into four strips, or subregions, S_1, S_2, S_3, S_4, by drawing vertical lines at $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$, as shown in Figure 5.4.

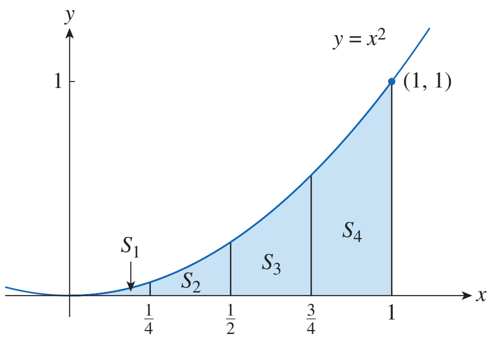


Figure 5.4
Divide the region S into four strips.

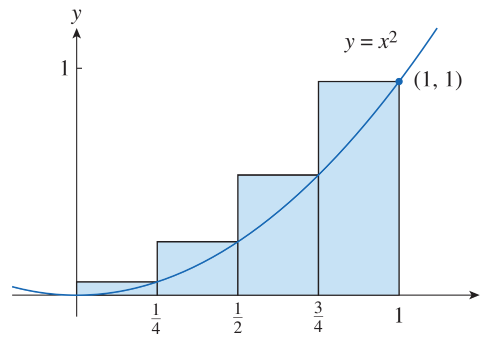


Figure 5.5
Approximate the area of each strip by using a rectangle.

We can approximate the area of each strip by using a rectangle that has the same base as each strip and whose height is the same as the right edge of the strip. See Figure 5.5. Therefore, the height of each rectangle is the value of the function $f(x) = x^2$ at the *right* endpoint of the corresponding subinterval:

$$\left[0, \frac{1}{4}\right], \quad \left[\frac{1}{4}, \frac{1}{2}\right], \quad \left[\frac{1}{2}, \frac{3}{4}\right], \quad \left[\frac{3}{4}, 1\right]$$

Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2, respectively.

Let R_4 be the sum of the areas of these approximating rectangles. Then

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2 = 0.46875$$

Figure 5.5 shows that the area A of the region S is less than R_4, so $A < 0.46875$.

We can also use rectangles constructed in a different way in order to obtain another approximation of the area of S. For example, suppose we use rectangles with heights equal to the values of f at the *left* endpoint of each subinterval. See Figure 5.6.

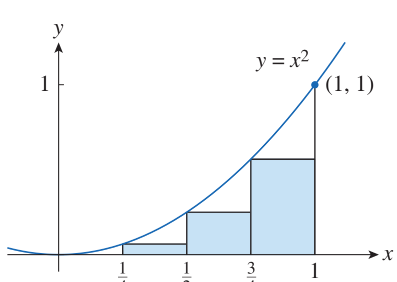


Figure 5.6
An approximation to the area of S using rectangles whose heights are the values of f at the left endpoints of each subinterval. Note that the leftmost rectangle has height 0.

Let L_4 be the sum of the areas of these approximating rectangles. Then

$$L_4 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = 0.21875$$

Figure 5.6 suggests that the area of S is certainly larger than L_4. Therefore, we now have lower and upper estimates for A:

$$0.21875 < A < 0.46875$$

It seems reasonable that in order to obtain a better estimate of the area of S, we should divide the region into smaller pieces and use smaller (thinner) rectangles. Figures 5.7 and 5.8 show better approximations using eight strips (rectangles) of equal width.

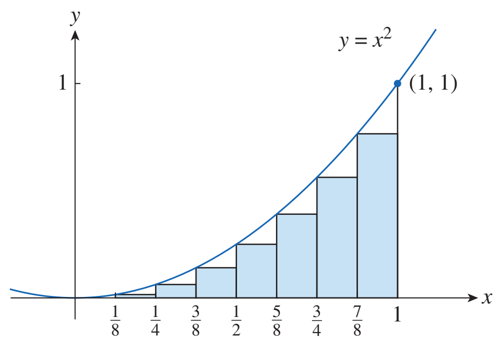


Figure 5.7
An approximation of the area of S with eight rectangles using left endpoints, that is, using the left endpoint of each subinterval to determine the height of each rectangle.

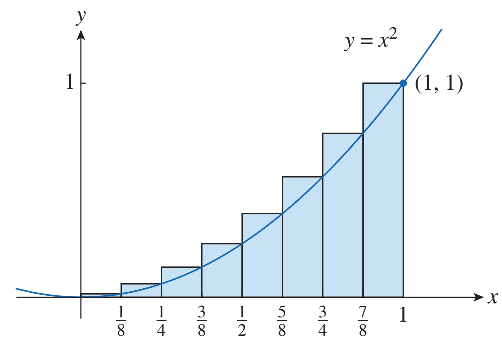


Figure 5.8
An approximation of the area of S with eight rectangles using right endpoints, that is, using the right endpoint of each subinterval to determine the height of each rectangle.

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

Table 5.1
Table of estimates using left endpoints and right endpoints for various values of n, the number of rectangles.

Figures 5.7 and 5.8 suggest that the sum of the areas of the smaller rectangles, L_8, and the sum of the areas of the larger rectangles, R_8, provide better lower and upper estimates for A than L_4 and R_4.

$$L_8 = 0.2734375 < A < 0.3984375 = R_8$$

We can obtain even better estimates by increasing the number of strips, or equivalently, by increasing the number of rectangles. Table 5.1 shows the results of similar calculations using n rectangles whose heights are found using the left endpoints (L_n) or right endpoints (R_n) (to determine the height of each rectangle).

If we use 50 rectangles, then the area A lies between 0.3234 and 0.3434. Using 1000 rectangles, A lies between 0.3328335 and 0.3338335. This process suggests a good estimate for the area of S is the arithmetic mean of these two values, $A \approx 0.3333335$.

The values in this table suggest that R_n and L_n are approaching $\dfrac{1}{3}$. ■

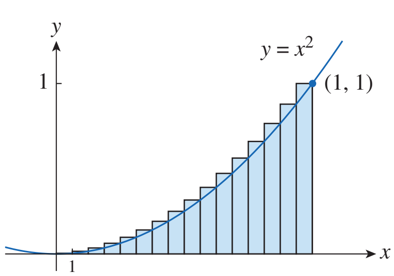


Figure 5.9
R_n is the sum of the areas of n rectangles shown.

Example 2 Find the Exact Area

For the region S in Example 1, show that the sum of the areas of upper approximating rectangles approaches $\dfrac{1}{3}$, that is

$$\lim_{n\to\infty} R_n = \frac{1}{3}$$

Solution

R_n is the sum of the areas of the n rectangles shown in Figure 5.9.

Each rectangle has width $\dfrac{1}{n}$ and the heights are the values of the function $f(x) = x^2$ at the points $\dfrac{1}{n}, \dfrac{2}{n}, \dfrac{3}{n}, \ldots, \dfrac{n}{n}$; that is, the heights are $\left(\dfrac{1}{n}\right)^2, \left(\dfrac{2}{n}\right)^2, \left(\dfrac{3}{n}\right)^2, \ldots, \left(\dfrac{n}{n}\right)^2$. Write the expression for R_n and simplify as much as possible.

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

Sum of the areas of the rectangles.

$$= \frac{1}{n} \cdot \frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

Factor out $\left(\dfrac{1}{n}\right) \times \left(\dfrac{1}{n^2}\right)$.

$$= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

Simplify.

Here we need to use the formula for the sum of the squares of the first n positive integers:

$$1^2 + 2^3 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{1}$$

You may have seen and used this formula before. It is proved in Example 5 in Appendix F.

Use Formula 1 in the expression for R_n.

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Here we are computing the limit of the sequence $\{R_n\}$. Sequences and their limits were discussed in *A Preview of Calculus* and will be studied in detail in Section 8.1. The idea is very similar to a limit at infinity (Section 2.5) except that in writing $\lim_{n \to \infty}$ we restrict n to be a positive integer. In particular, we know that

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

When we write $\lim_{n \to \infty} R_n = \dfrac{1}{3}$, we mean that we can make R_n as close to $\dfrac{1}{3}$ as we like by taking n sufficiently large.

Now consider the limit.

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{(n+1)(2n+1)}{6n^2}$$

Use simplified expression for R_n.

$$= \lim_{n \to \infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

Write as a product.

$$= \lim_{n \to \infty} \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

Rewrite each fraction as a sum of two terms; simplify.

$$= \frac{1}{6} \cdot 1 \cdot 2 = \frac{1}{3}$$

Evaluate the limit. ■

This is a remarkable result. It suggests that the *exact* area of the region S is $\dfrac{1}{3}$.

It can be shown that the lower sums also approach $\dfrac{1}{3}$, that is,

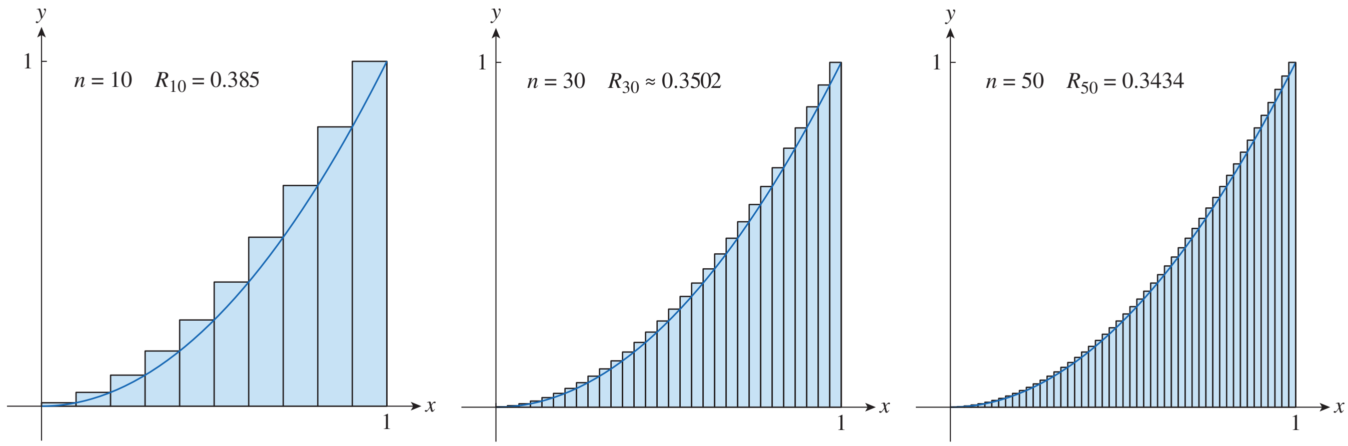
$$\lim_{n \to \infty} L_n = \frac{1}{3}$$

Figures 5.10 and 5.11 illustrate these results.

As n increases, both L_n and R_n become better and better approximations to the area of S. Therefore, we *define* the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

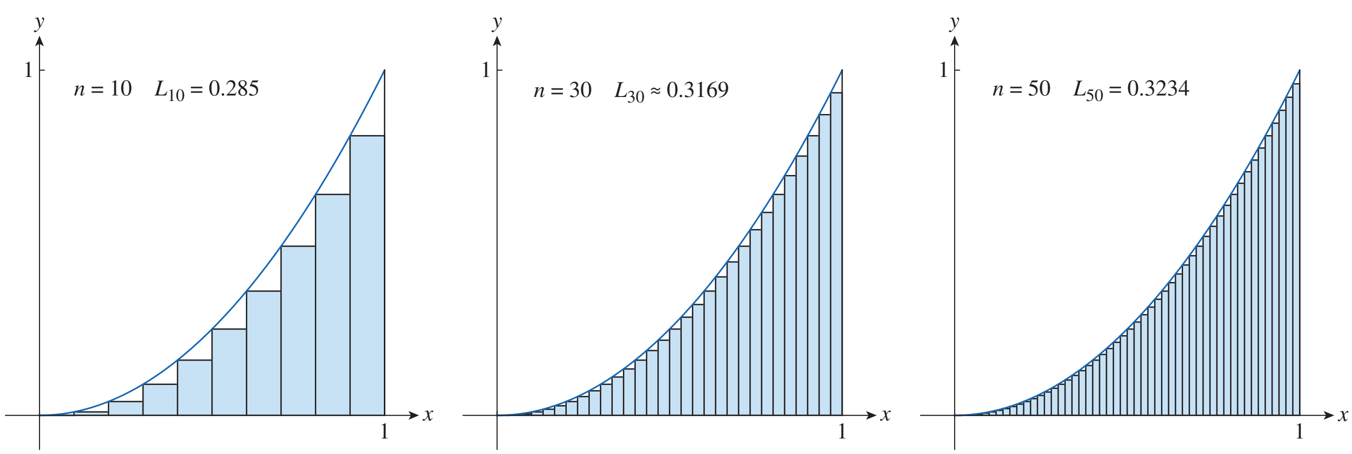
$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \frac{1}{3}$$

■



(a) Right sum, $n = 10$ (b) Right sum, $n = 30$ (c) Right sum, $n = 50$

Figure 5.10
Each sum R_n is an overestimate because $f(x) = x^2$ is increasing.



(a) Left sum, $n = 10$ (b) Left sum, $n = 30$ (c) Left sum, $n = 50$

Figure 5.11
Each sum L_n is an underestimate because $f(x) = x^2$ is increasing.

Let's apply the idea of Examples 1 and 2 to the more general region S illustrated in Figure 5.1. Start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as shown in Figure 5.12.

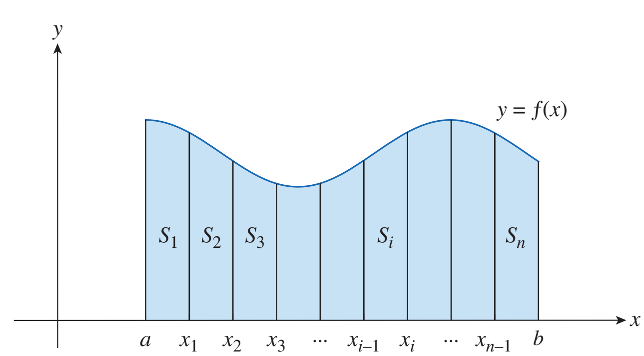


Figure 5.12
Subdivide the region S into n strips.

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \ldots, [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$\begin{aligned} x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ x_3 &= a + 3\Delta x \\ &\vdots \end{aligned}$$

Let's approximate the ith strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint (see Figure 5.13). Then the area of the ith rectangle is $f(x_i)\,\Delta x$. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

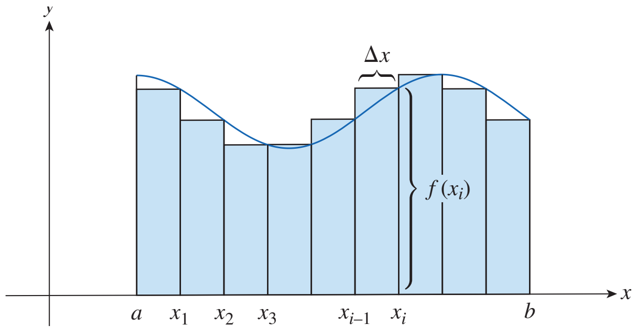
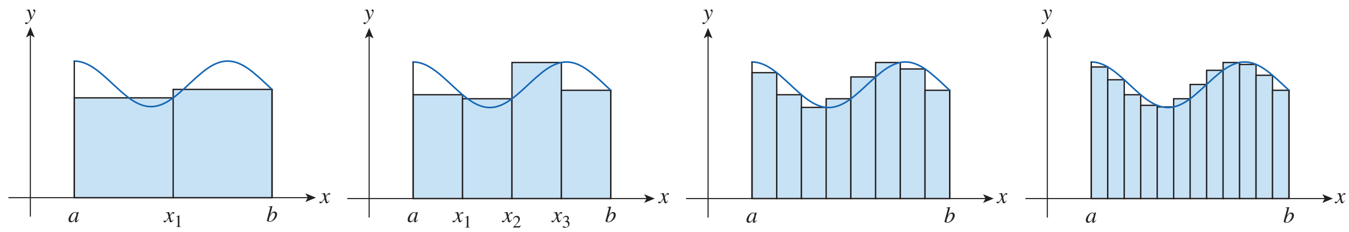


Figure 5.13
The area of the ith rectangle is $f(x_i)\Delta x$.

Figure 5.14 shows this approximation for $n = 2, 4, 8$, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$. Therefore, we define the area A of the region S in the following way.



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

Figure 5.14
The approximation becomes better as n increases.

Definition · Area of the Region

The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of the approximating rectangles:

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

Since we are assuming that f is continuous, then it can be proved that the limit in the definition above always exists. It can also be shown that we get the same value if we use left endpoints:

$$A = \lim_{n \to \infty} L_n = \lim_{n \to \infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x] \tag{2}$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the ith rectangle to be the value of f at *any* number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. The numbers $x_1^*, x_2^*, \ldots, x_n^*$ are called the **sample points**. Figure 5.15 shows approximating rectangles when the sample points are different from endpoints. So a more general expression for the area of S is

$$A = \lim_{n \to \infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] \tag{3}$$

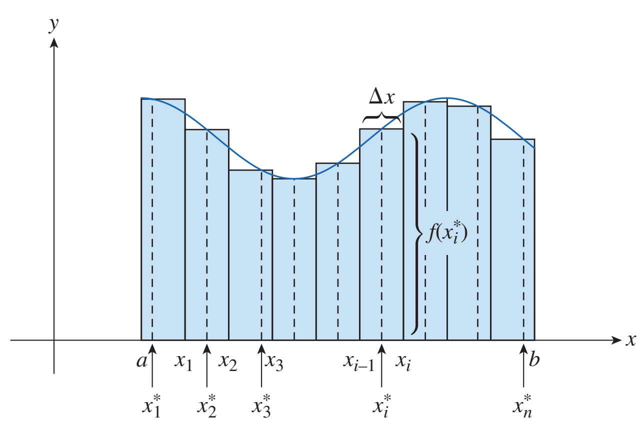


Figure 5.15
The sample points are not endpoints.

We often use **sigma notation** to write long sums in a more compact way. For example,

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

This tells us to end with $i = n$.

This tells us to add. $\sum_{i=m}^{n} f(x_i)\,\Delta x$

This tells us to start with $i = m$.

So the expressions for area in the definition above and Equations 2 and 3 can be written as follows:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

If you need practice with sigma notation, look at the examples and try some of the exercises in Appendix F.

We can also rewrite Equation 1 in the following way:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Example 3 An Area Expressed as a Limit

Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.
(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

Solution

(a) Since $a = 0$ and $b = 2$, the width of a subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}.$$

The endpoints of the subintervals are

$$x_1 = \frac{2}{n}, x_2 = \frac{4}{n}, x_3 = \frac{6}{n}, \ \ldots \ , x_i = \frac{2i}{n}, \ \ldots \ , x_n = \frac{2n}{n}$$

The sum of the areas of the approximating rectangles is

$$\begin{aligned} R_n &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \\ &= e^{-x_1}\Delta x + e^{-x_2}\Delta x + \cdots + e^{-x_n}\Delta x \\ &= e^{-2/n} \cdot \frac{2}{n} + e^{-4/n} \cdot \frac{2}{n} + \cdots + e^{-2n/n} \cdot \frac{2}{n} \end{aligned}$$

Using the definition above, the area of the region S is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2}{n}\left(e^{-2/n} + e^{-4/n} + e^{-6/n} + \cdots + e^{-2n/n}\right).$$

Using sigma notation, we can write the expression for area more compactly:

$$A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}$$

It is difficult to evaluate this limit analytically. Technology can be used to estimate the limit numerically and even find its exact value. This suggests that we need a more efficient, analytical way to find areas.

(b) For $n = 4$, the width of each subinterval is $\Delta x = \dfrac{2-0}{4} = 0.5$.

The subintervals are [0.0.5], [0.5, 1], [1, 1.5], and [1.5, 2].

The midpoints of the subintervals are

$$x_1^* = 0.25, \quad x_2^* = 0.75, \quad x_3^* = 1.25, \quad \text{and} \quad x_4^* = 1.75.$$

The sum of the areas of the four approximating rectangles (shown in Figure 5.16) is

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(x_i^*)\,\Delta x \\ &= f(0.25)\Delta x + f(0.75)\Delta x + f(1.25)\Delta x + f(1.75)\Delta x && \text{Use midpoints.} \\ &= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5) && \text{Evaluate } f\text{; use } \Delta x = 0.5. \\ &= (0.5)(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}) \approx 0.8557 \end{aligned}$$

So an estimate for the area is $A \approx 0.8577$.

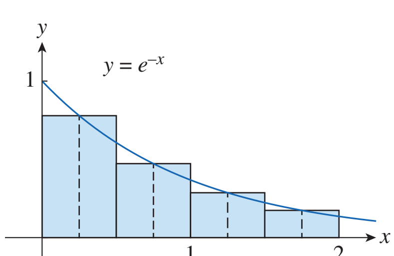


Figure 5.16
Four approximating rectangles.

For $n = 10$, we have $\Delta x = \dfrac{2}{10} = 0.2$.

The subintervals are [0, 0.2], [0.2, 0.4], [0.4, 0.6], . . ., [1.8, 2].

The midpoints of the subintervals are

$x_1^* = 0.1, \quad x_2^* = 0.3, \quad x_3^* = 0.5, \ldots, x_{10}^* = 1.9.$

The midpoint sum is

$$\begin{aligned} M_{10} &= f(0.1)\Delta x + f(0.3)\Delta x + f(0.5)\Delta x + \cdots + f(1.9)\Delta x \\ &= 0.2(e^{-0.1} + e^{-0.3} + e^{-0.5} + \cdots + e^{-1.9}) \approx 0.8632 \approx A \end{aligned}$$

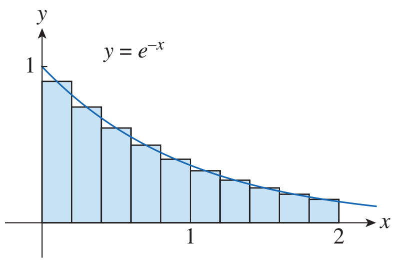


Figure 5.17
Ten approximating rectangles.

Figure 5.17 illustrates this sum and suggests that this estimate is better than the estimate with $n = 4$ rectangles.

The Distance Problem

Now let's consider the *distance problem*: find the distance traveled by an object during a certain time period if the velocity of the object is known at all times. In one sense, this is a *backward* problem: we know the velocity but want the distance traveled.

If the velocity of the object is constant, then the distance problem can be solved using the formula

$$\text{distance} = \text{velocity} \times \text{time}$$

But if the velocity varies, then finding the total distance traveled is more challenging. The good news is that we will be able to visualize the result and relate it back to the area problem. Consider the following example.

Example 4 Estimate a Distance

Suppose the odometer in a car is broken but we still want to estimate the distance driven over a 30-second time interval. The table shows the speedometer readings for selected times, in this case every five seconds.

Time (s)	0	5	10	15	20	25	30
Velocity (m/h)	17	21	24	29	32	31	28

In order to have the time and the velocity in consistent units, let's convert the velocity readings to feet per second (1 mi/h = 5280/3600 ft/s):

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41

During any five-second interval, the velocity doesn't change very much, so we can estimate the distance traveled during a five-second interval by assuming that the velocity is constant.

During the first five seconds, assume the velocity of the car is 25 ft/s (the initial velocity). Then the approximate distance traveled during the first five seconds is

25 ft/s $\times$ 5 s = 125 feet.

Similarly, during the second time interval, assume the velocity is 31 ft/s, the velocity at time $t = 5$ s. The estimate for the distance traveled from $t = 5$ s to $t = 10$ s is

31 ft/s $\times$ 5 s = 155 feet.

Use similar estimates for the remaining time intervals. An estimate for the total distance traveled over the 30 second time interval is

$$(25 \times 5) + (31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (45 \times 5) = 1130 \text{ ft.}$$

We could also have used the velocity at the *end* of each time interval instead of the velocity at the beginning as the assumed constant velocity. Then the estimate for total distance traveled is

$$(31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (45 \times 5) + (41 \times 5) = 1210 \text{ ft.}$$

A more accurate estimate could be obtained if we had velocity readings every two seconds, or even every second. ■

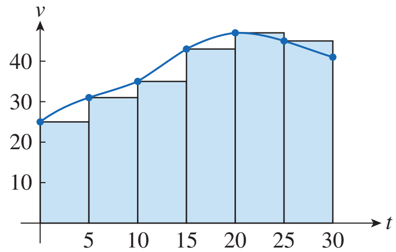


Figure 5.18
A rough sketch of the velocity function and approximating rectangles.

The calculations in Example 4 certainly look like sums we used earlier to estimate areas. The connection is even more apparent if we sketch a graph of the velocity function of the car. Figure 5.18 shows a graph with a smooth curve connecting the points of the velocity function and rectangles with heights equal to the initial velocities in each time interval.

The area of the first rectangle is $25 \times 5 = 125$, which is the estimate for the distance traveled in the first five seconds. The area of each rectangle can be interpreted as a distance because the height represents velocity and the width represents time. The sum of the rectangle areas in Figure 5.18 is $L_6 = 1130$, a left sum, which is the initial estimate for the total distance traveled.

In general, suppose the velocity of an object at time t is given by $v = f(t)$, where $a \le t \le b$ and $f(t) \ge 0$ (so the object is moving in the positive direction). In addition, suppose we know the velocity of the object at times $a = t_0, t_1, t_2, \ldots, t_n = b$. If the times are equally spaced, then $\Delta t = \dfrac{b-a}{n}$.

Assume the velocity is approximately constant over each subinterval. During the first time interval, the velocity of the object is approximately $f(t_0)$ and the distance traveled is $f(t_0)\Delta t$. Similarly, the distance traveled during the second time interval is about $f(t_1)\Delta t$. The total distance traveled during the time interval $[a, b]$ is approximately

$$f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t = \sum_{i=1}^{n} f(t_{i-1})\Delta t$$

If we used the velocity at the right endpoints of each subinterval, the estimate for the total distance traveled is

$$f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t = \sum_{i=1}^{n} f(t_i)\Delta t$$

The more frequently we measure the velocity, the more accurate the estimate becomes. It seems reasonable that the *exact* distance traveled d is the *limit* of these sums as n approaches infinity.

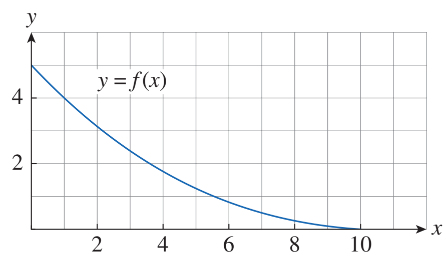
$$d = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_{i-1})\Delta t = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)\Delta t \qquad (4)$$

We will see in Section 5.3 that this is indeed true.

Because Equation 4 has the same form as the expression for area, it follows that the distance traveled is equal to the area under the graph of the velocity function. In Chapter 6, we will see that other quantities in the natural and social sciences, for example, the work done by a variable force or the cardiac output of the heart, can also be interpreted as the area under a curve. So, as we compute areas in this chapter, bear in mind that they can be interpreted in a variety of practical ways.

5.1 Exercises

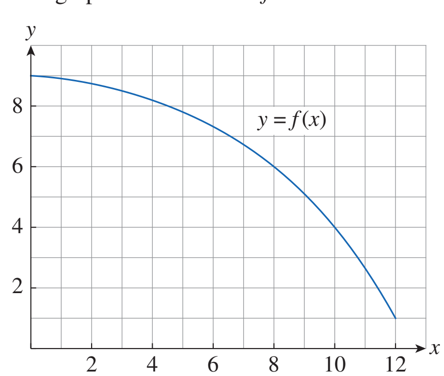
1. The graph of the function f is shown.



(a) Use the graph of f and five rectangles to find a lower estimate and an upper estimate for the area bounded by the graph of f, the x-axis, and the lines $x = 0$ and $x = 10$. In each case, sketch the rectangles to illustrate the estimate.

(b) Find estimates using ten rectangles in each case.

2. The graph of the function f is shown.



(a) Use six rectangles to find estimates of each type for the area bounded by the graph of f, the x-axis, and the lines $x = 0$ and $x = 12$.

(i) L_6 (sample points are left endpoints)

(ii) R_6 (sample points are right endpoints)

(iii) M_6 (sample points are midpoints)

(b) Is L_6 an underestimate or an overestimate of the true area? Explain your reasoning.

(c) Is R_6 an underestimate or an overestimate of the true area? Explain your reasoning.

(d) Which of the numbers L_6, R_6, or M_6 gives the best estimate of the true area? Explain your reasoning.

3. Let $f(x) = \dfrac{1}{x}$.

(a) Estimate the area of the region bounded by the graph of f, the x-axis, and the lines $x = 1$ and $x = 2$ using four rectangles and right endpoints. Sketch the graph of f and the rectangles. Is your estimate an underestimate or an overestimate of the true area? Explain your reasoning.

(b) Repeat part (a) using left endpoints.

4. Let $f(x) = \sqrt{x}$.

(a) Estimate the area under the graph of f from $x = 0$ to $x = 4$ using four approximating rectangles and right endpoints. Sketch the graph of f and the rectangles. Is your estimate an underestimate or an overestimate of the true area? Explain your reasoning.

(b) Repeat part (a) using left endpoints.

5. Let $f(x) = \sin x$.

(a) Estimate the area of the region bounded by the graph of f, the x-axis, and the lines $x = 0$ and $x = \dfrac{\pi}{2}$ using four rectangles and right endpoints. Sketch the graph of f and the rectangles. Is your estimate an underestimate or an overestimate of the true area? Explain your reasoning.

(b) Repeat part (a) using left endpoints.

6. Let $f(x) = 1 + x^2$.

(a) Estimate the area of the region bounded by the graph of f, the x-axis, and the lines $x = -1$ and $x = 2$ using three rectangles and right endpoints. Then use six rectangles to improve your estimate. Sketch the graph of f and the approximating rectangles.

(b) Repeat part (a) using left endpoints.

(c) Repeat part (a) using midpoints.

(d) Using your sketches in parts (a)–(c), which sum appears to give the best estimate? Explain your reasoning.

7. Let $f(x) = x - 2 \ln x, \quad 1 \le x \le 5$.

(a) Use technology to graph the function f.

(b) Estimate the area of the region bounded by the graph of f, the x-axis, and the lines $x = 1$ and $x = 5$ using four rectangles and (i) right endpoints and (ii) midpoints. In each case, sketch the graph and the approximating rectangles.

(c) Improve your estimates in part (b) by using eight rectangles.

Use technology to compute the sum of the areas of approximating rectangles using equal subintervals and right endpoints for $n = 10$, 30, 50, and 100. Use your results to guess the value of the exact area.

8. The region under $y = x^4$ from 0 to 1

9. The region under $y = \cos x$ from 0 to $\pi/2$

10. Let $f(x) = \dfrac{1}{1 + x^2}$ and consider the region bounded by the graph of f, the x-axis, and the lines $x = 0$ and $x = 1$.

(a) Find the left and right sums for $n = 10$, 30, and 50.
(b) Show that the exact area of the region lies between 0.780 and 0.791.

11. Let $f(x) = \ln x$ and consider the region bounded by the graph of f, the x-axis, and the lines $x = 1$ and $x = 4$.
(a) Find the left and right sums for $n = 10$, 30, and 50.
(b) Illustrate the sums in part (a) by graphing the rectangles.
(c) Show that the exact area of the region lies between 2.50 and 2.59.

12. Suppose a function f is positive on the interval [6, 24]. Selected values of $f(x)$ are given in the table.

x	6	12	20	24
$f(x)$	5	7	4	7

Estimate the area under the graph of f from $x = 6$ to $x = 24$ using approximating rectangles as defined in the table and right endpoints. Is your estimate an underestimate or an overestimate? Explain your reasoning.

13. Suppose $\sum_{i=1}^{6} f(x_i^*)\Delta x$ represents an approximation for the area under the graph of $f(x) = x^2 + 2$ between the lines $x = 2$ and $x = 5$ using the midpoint of each subinterval. Find the value of $f(x_3^*)$.

14. The function f is positive and selected values of $f(x)$ over the interval [2, 38] are given in the table.

x	2	8	14	20	26	32	38
$f(x)$	7	9	18	20	16	10	8

Suppose this table is used to compute an estimate of the area under the graph of f from $x = 2$ to $x = 38$ using three rectangles and midpoints. Find the value of this approximation.

15. The speed of a runner increased during the first 3 seconds of a race. Their speed at half-second intervals is given in the table.

t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
v (ft/s)	0	6.2	10.8	14.9	18.1	19.4	20.2

Find lower and upper estimates for the distance that they traveled during these 3 seconds.

16. The table shows the speedometer readings at 10-second intervals during a 1-minute period for a car racing at the Daytona International Speedway in Florida.

Time (s)	Velocity (mi/h)
0	182.9
10	168.0
20	106.6
30	99.8
40	124.5
50	176.1
60	175.6

(a) Estimate the distance the race car traveled during this time period using the velocities at the beginning of each subinterval.
(b) Find another estimate of the distance traveled by using the velocities at the end of each subinterval.
(c) Is your estimate in part (a) an upper or lower estimate, or neither? Is your estimate in part (b) an upper or lower estimate, or neither? Explain your reasoning.

17. Oil is leaking from a tank at a rate of $r(t)$ liters per hour. The rate is decreasing over time and the values of the rate at 2-hour intervals are given in the table.

t (h)	0	2	4	6	8	10
$r(t)$ (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

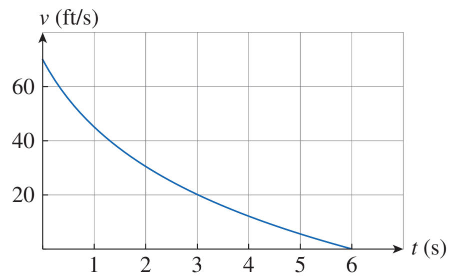
Find lower and upper estimates for the total amount of oil that leaks from the tank over the interval [0, 10].

18. The Hyperloop, designed by Elon Musk, is a type of bullet train with pods carrying passengers through underground tubes. Recently, a team of students designed a pod that reached a top speed of 463 km/h (288 mi/h) in the annual SpaceX Hyperloop Pod Competition. Suppose the table below gives the velocity of the pod at various times.

Time (s)	Velocity (ft/s)
0	0
5	179
10	239
15	277
20	304
25	326
30	343

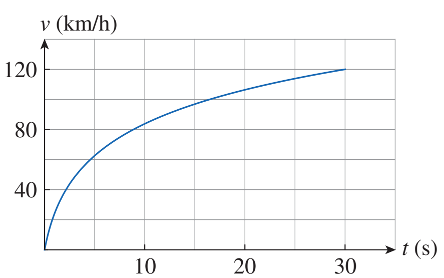
Use these data to estimate the distance the pod traveled in the first 30 seconds.

19. The graph of the velocity of a braking car is shown in the figure.



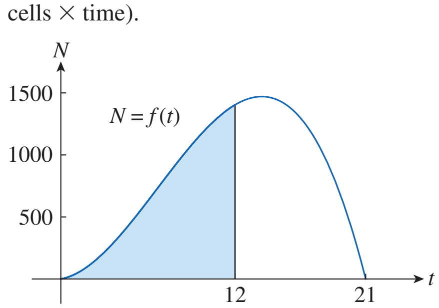
Use this graph with six equal subintervals and midpoints to estimate the distance traveled by the car while the brakes are applied.

20. The graph of the velocity of a car accelerating from rest to a speed of 120 km/h over a period of 30 seconds is shown in the figure.



Estimate the distance traveled during this time period.

21. In someone infected with measles, the virus level N (measured in number of infected cells per mL of blood plasma) reaches a peak density at about $t = 12$ days (when a rash appears) and then decreases fairly rapidly as a result of immune response. The area of the region bounded by the graph of $N(t)$, the t-axis, and the lines $t = 0$ and $t = 12$, as shown in the figure, is equal to the total amount of infection needed to develop symptoms (measured in density of infected cells $\times$ time).



The function $N = f(t)$ has been modeled by

$$f(t) = -t(t - 21)(t + 1)$$

Use this model with six equal subintervals and midpoints to estimate the total amount of infection needed to develop symptoms of measles.

Source: J.M. Heffernan et al., "An In-Host Model of Acute Infection: Measles as a Case Study," *Theoretical Population Biology* 73 (2006): 134–47.

22. The table shows the recorded death rate from the coronavirus in the United States every 3 days beginning on March 10, 2020.

Date	Deaths per day	Date	Deaths per day
March 10	4	March 25	247
March 13	7	March 28	525
March 16	18	March 31	912
March 19	56	April 3	1045
March 22	113	April 6	1255

(a) Estimate the number of people who died from the coronavirus in the United States between March 10 and April 6, 2020, using left endpoints and using right endpoints.

(b) Do you think either of your answers in part (a) is an underestimate or an overestimate? Explain your reasoning.

Use the definition of the area of a region to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

23. $f(x) = \dfrac{2x}{x^2 + 1}, \quad 1 \le x \le 3$

24. $f(x) = x^2 + \sqrt{1 + 2x}, \quad 4 \le x \le 7$

25. $f(x) = x \cos x, \quad 0 \le x \le \dfrac{\pi}{2}$

Describe the region whose area is given by the limit.

26. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1 + \frac{3i}{n}}$

27. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$

28. The expression

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(3 + \frac{6}{n}i\right)^2 + 1\right)\left(\frac{6}{n}\right)$$

represents the exact area of a region S. Estimate the area of the region S using three approximating rectangles with equal width and midpoints.

29. (a) Use the definition of the area of a region to find an expression involving a limit for the area under the curve $y = x^3$ from $x = 0$ to $x = 1$.

(b) The following formula for the sum of cubes of the first n integers is proved in Appendix F.

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Use this formula to evaluate the limit in part (a).

30. (a) Express the area under the curve $y = x^5$ from 0 to 2 as a limit.

(b) Use technology to find the sum of your expression from part (a).

(c) Evaluate the limit in part (a).

31. Find the area of the region under the graph of $y = e^{-x}$ from 0 to 2 by using technology to evaluate the limit in Example 3(a). Compare your answer with the estimate obtained in Example 3(b). Note: This asks for exact area using technology, but that won't generally be possible unless the student is using a computer algebra system.

32. Use technology to find the exact area under the cosine curve $y = \cos x$ from $x = 0$ to $x = b$, where $0 \le b \le \pi/2$. In particular, what is the area if $b = \pi/2$?

33. Consider the area of the region bounded above by the graph of $f(x) = x^2$, below by the x-axis, and between the lines $x = 4$ and $x = 6$. Suppose we estimate the area of the region using 100 approximating rectangles of equal width and left endpoints, and with right endpoints. Find the difference between these two approximations without actually calculating either of the sums.

34. Suppose f is continuous and increasing on the interval $[a, b]$. Let A be the area of a region bounded above by the graph of $y = f(x)$, below by the x-axis, and lying between the lines $x = a$ and $x = b$. Let L_n and R_n be the approximations to A with n subintervals using left and right endpoints, respectively.

(a) How are A, L_n, and R_n related?

(b) Show that

$$R_n - L_n = \frac{b-a}{n}[f(b) - f(a)]$$

Draw a diagram to illustrate this equation by showing that the n rectangles representing $R_n - L_n$ can be reassembled to form a single rectangle whose area is the right side of the equation.

(c) Explain why the following equation is true.

$$R_n - A < \frac{b-a}{n}[f(b) - f(a)]$$

35. Suppose A is the area of the region bounded by the graph of $y = e^x$, the x-axis, and the lines $x = 1$ and $x = 3$. Use Exercise 34 to find a value of n such that $R_n - A < 0.0001$.

36. Let A_n be the area of a polygon with n equal sides inscribed in a circle with radius r.

(a) By dividing the polygon into n congruent triangles with central angle $\dfrac{2\pi}{n}$, show that

$$A_n = \frac{1}{2}nr^2 \sin\left(\frac{2\pi}{n}\right)$$

(b) Show that $\lim_{n\to\infty} A_n = \pi r^2$.

5.2 The Definite Integral

In Section 5.1 we discovered that a limit of the form

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] \tag{1}$$

arises when we try to compute the area of a plane region. We also saw limits of this type when we tried to find the distance traveled by an object. Surprisingly, this same type of limit occurs in a wide variety of applications, even when f is not necessarily a positive function. Later we will see limits of this form occur in finding lengths of curves, volumes of solids, centers of mass, force due to water pressure, and work, as well as many other quantities. This type of limit is so important that it has a special name and notation.

Definition • Definite Integral

Suppose f is a function defined for $a \leq x \leq b$. Divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$ and with endpoints $a = x_0, x_1, x_2, \ldots, x_n = b$.

Let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of *f* from *a* to *b*** is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided the limit exists. If it does exist, we say that f is **integrable** on $[a, b]$.

A Closer Look

1. The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because a definite integral is a limit of sums.

The function $f(x)$ is called the **integrand**.

The values a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**.

For now, the symbol dx has no specific meaning by itself; rather $\int_a^b f(x)\,dx$ is all one **symbol**, representing the definite integral. The dx simply indicates that the independent variable is x.

The procedure of calculating the value of a definite integral is called **integration**.

2. The definite integral, denoted by $\int_a^b f(x)\,dx$, is a number; it does not depend upon x. We could use any letter in place of x without changing the meaning or value of the definite integral, for example,

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

3. A sum of the form

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

that occurs in the definition of a definite integral is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866). So, this definition says that the definite integral is the limit of a Riemann sum and that an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum. ■

Riemann

Bernhard Riemann received his Ph.D. under the direction of the legendary Gauss at the University of Göttingen and remained there to teach. Gauss, who was not in the habit of praising other mathematicians, spoke of Riemann's "creative, active, truly mathematical mind and gloriously fertile originality." The definition of a definite integral of an integral that we use is due to Riemann. He also made major contributions to the theory of functions of a complex variable, mathematical physics, number theory, and the foundations of geometry. Riemann's broad concept of space and geometry turned out to be the right setting, 50 years later, for Einstein's general relativity theory. Riemann's health was poor throughout his life, and he died of tuberculosis at the age of 39.

In Section 5.1 we learned that if f is continuous and positive on the interval $[a, b]$, then a Riemann sum can be interpreted as a sum of areas of approximating rectangles. For example, Figure 5.19 shows a midpoint sum with all subintervals of equal width. By comparing the definition of a definite integral with the definition of area in Section 5.1, we see that the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area of the region bounded above by the graph of $y = f(x)$, below by the x-axis, and between the lines $x = a$ and $x = b$ [or simply, the area under the curve $y = f(x)$ from a to b]. See Figure 5.20.

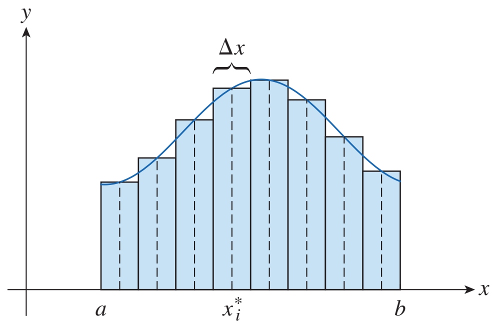


Figure 5.19
If $f(x) \geq 0$, a Riemann sum, in this case a midpoint sum, is the sum of areas of rectangles.

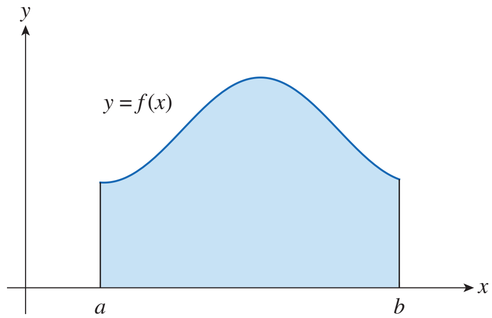


Figure 5.20
If $f(x) \geq 0$, the definite integral $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$, from a to b.

If f takes on both positive and negative values over the interval $[a, b]$, then a Riemann sum is the sum of the area of the rectangles that lie above the x-axis and the *negatives* of the area of the rectangles that lie below the x-axis; that is, $f(x_i^*) < 0$ for some subintervals and therefore the product $f(x_i^*)\,\Delta x_i < 0$. In Figure 5.21, the Riemann sum is the areas of the blue rectangles (above the x-axis) *minus* the areas of the green rectangles (below the x-axis).

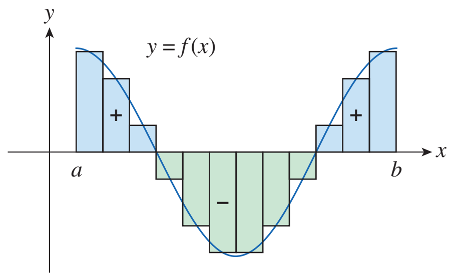


Figure 5.21
The Riemann sum is the areas of the blue rectangles minus the areas of the green rectangles.

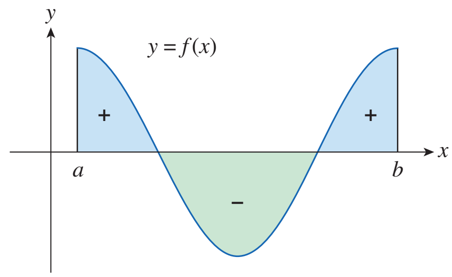


Figure 5.22
$\int_a^b f(x)\,dx$ is the *net* area.

If we take the limit of Riemann sums formed this way, the result is illustrated in Figure 5.22. Therefore, a definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

Although we defined the definite integral $\int_a^b f(x)\,dx$ dividing $[a, b]$ into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width. For example, if we have velocity data at times that are not equally spaced, we can still estimate the distance traveled. And there are methods for numerical integration that take advantage of unequal subintervals.

If the subinterval widths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width, $\max \Delta x_i$, approaches 0. So in this case the definition of a definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

The definite integral is defined for an integrable function, but not all functions are integrable. The following theorem shows that the most commonly occurring functions are, in fact, integrable. The theorem is proved in more advanced mathematics courses.

Theorem • Integrable Functions

If f is continuous on the interval $[a, b]$, or if f has a finite number of jump discontinuities, then f is integrable on $[a, b]$, that is, the definite integral $\int_a^b f(x)\,dx$ exists.

If f is integrable on $[a, b]$, then the limit in the definition of the definite integral exists and gives the same value no matter how we choose the sample points x_i^*. Therefore, to simplify the calculation (for now), we often take the sample points to be the right endpoints. Then $x_i^* = x_i$ and the definition of a definite integral can be written in a simpler form.

Theorem • Definite Integral Using Right Endpoints

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\,\Delta x$$

Example 1 Riemann Sum Interpretation

Write an integral expression for

$$\lim_{n \to \infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x$$

as a definite integral on the interval $[0, \pi]$.

Solution

Compare the given limit with the general limit in the preceding theorem and let $f(x) = x^3 + x \sin x$ in the limit expression. This produces the limit expression in this example.

We are given that $a = 0$ and $b = \pi$.

Therefore, $\displaystyle \lim_{n \to \infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x = \int_0^{\pi} (x^3 + x \sin x)\,dx$. ■

When we apply the definite integral to physical situations, it will be important to recognize limits of (Riemann) sums as definite integrals. When Leibniz chose the notation for a definite integral, he selected the pieces as reminders of the limiting process.

In general, when we write

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

we replace Σ by $\int$, x_i^* by x, and Δx by dx.

Evaluating Integrals

When we use the limit definition to evaluate a definite integral, we will need to use properties of summations and a few summation formulas. Some of the evaluations can be long, but most can be simplified by recognizing familiar patterns. The following three equations are formulas for sums of powers of positive integers. Equation 5 may be familiar to you from a course in algebra. Equations 6 and 7 are discussed in Section 5.1 and are proved in Appendix E.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{3}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{4}$$

The remaining equations are basic rules for working with sigma notation:

Equations 8–11 are proved by writing out each side in expanded form. The left side of Equation 9 is

$ca_1 + ca_2 + \cdots + ca_n$.

The right side is

$c(a_1 + a_2 + \cdots + a_n)$.

These are equal by the distributive property. The other equations are discussed in Appendix F.

$$\sum_{i=1}^{n} nc = nc \tag{5}$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i \tag{6}$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \tag{7}$$

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \tag{8}$$

Example 2 Evaluate an Integral as a Limit of Riemann Sums

Let $f(x) = x^3 - 6x$.

(a) Evaluate the Riemann sum for f taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

Solution

(a) For $n = 6$, the length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

The right endpoints are $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, $x_4 = 2.0$, $x_5 = 2.5$, $x_6 = 3.0$.

The Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x$$

$$= f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x + f(2.0)\,\Delta x + f(2.5)\,\Delta x + f(3.0)\,\Delta x$$

Use the right endpoint of each subinterval.

$$= \frac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9)$$

Evaluate f; use $\Delta x = \frac{1}{2}$.

$$= -3.9375$$

Figure 5.23 illustrates this right Riemann sum.

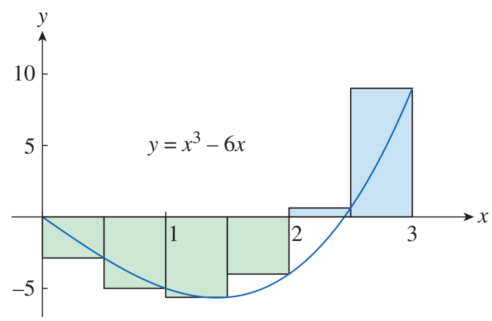


Figure 5.23
An illustration of R_6, a Riemann sum using right endpoints.

Notice that f is not a positive function. Therefore, the Riemann sum does not represent a sum of areas of rectangles. However, it does represent the sum of the areas of the blue rectangles (above the x-axis) minus the sum of the areas of the green rectangles (below the x-axis); the *net* area.

(b) For n subintervals, $\Delta x = \dfrac{b-a}{n} = \dfrac{3}{n}$.

Therefore, $x_0 = 0$, $x_1 = \dfrac{3}{n}$, $x_2 = \dfrac{6}{n}$, $x_3 = \dfrac{9}{n}$, and, in general $x_i = \dfrac{3i}{n}$.

We can use the theorem that gives the definite integral using right endpoints.

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

Evaluate f; constants pass freely through summation symbols.

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]$$

Write as two summations; simplify.

$$= \lim_{n\to\infty} \left(\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right)$$

Summation formulas.

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right)\right]$$

Rearrange terms.

$$= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75$$

Evaluate limit.

This definite integral cannot be interpreted as an area because f takes on both positive and negative values (over the interval [0, 3]). However, it can be

interpreted as the difference of areas $A_1 - A_2$, or net area, where A_1 and A_2 are shown in Figure 5.24.

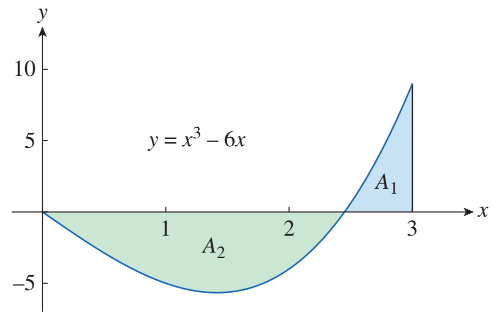


Figure 5.24
A visualization of the definite integral $\int_0^3 (x^3 - 6x)\,dx = A_1 - A_2 = -6.75.$

Figure 5.25 illustrates an approximation to this definite integral by showing the positive and negative terms in the right Riemann sum R_n for $n = 40$. The values in Table 5.2 show the Riemann sums approaching the exact value of the definite integral, -6.75, as $n \to \infty$.

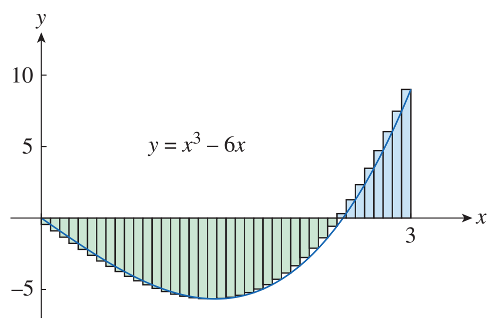


Figure 5.25
An illustration of R_{40}.

n	R_n
40	-6.3998
100	-6.6130
500	-6.7229
1000	-6.7365
5000	-6.7473

Table 5.2
As n increases, $R_n \to -6.75$.

This example reinforces the notion that we need a simpler method to evaluate definite integrals. We'll find one in Section 5.3. ■

Example 3 Use Technology to Evaluate a Definite Integral

Set up, but do not evaluate, an expression for the definite integral $\int_1^3 e^x\,dx$ as a limit of sums. Then use technology to evaluate this expression.

Solution

Let $f(x) = e^x$, $a = 1$, and $b = 3$. Then $\Delta x = \dfrac{b-a}{n} = \dfrac{3-1}{n} = \dfrac{2}{n}$.

The endpoints of the subintervals are $x_0 = 1$, $x_1 = 1 + \dfrac{2}{n}$, $x_2 = 1 + \dfrac{4}{n}$, $x_3 = 1 + \dfrac{6}{n}$,

and the right endpoint of the ith subinterval is $x_i = 1 + \dfrac{2i}{n}$.

Use the theorem for a definite integral using right endpoints to write a limit expression for the definite integral.

n	R_n
50	17.7169
100	17.5415
200	17.4542
500	17.4020
1000	17.3846
2000	17.3759
5000	17.3707
10,000	17.3690
20,000	17.3681
50,000	17.3676

Table 5.3
Sum evaluation using technology.

$$\int_1^3 e^x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x \qquad \text{Limit of a right Riemann sum.}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)\frac{2}{n} \qquad \text{Use values for } x_i \text{ and } \Delta x.$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} e^{1+2i/n} \qquad \text{Evaluate } f\text{; constant passes freely through summation.}$$

We cannot use the summation formulas to evaluate this expression; however, we can use technology to investigate the limit numerically. Table 5.3 shows this sum for various values of n.

If we use a computer algebra system to evaluate the sum and simplify, we obtain

$$\sum_{i=1}^{n} e^{1+2i/n} = \frac{e^{1+2/n}(e^2-1)}{e^{2/n}-1}.$$

We can also use technology to evaluate the limit:

$$\int_1^3 e^x\,dx = \lim_{n\to\infty}\frac{2}{n}\cdot\frac{e^{1+2/n}(e^2-1)}{e^{2/n}-1} \approx e^3-3 \approx 17.3673$$

Figure 5.26 illustrates this definite integral.

A computer algebra system can be used to find a *closed-form expression* for this sum because the sum is actually a geometric series. And, the limit could also be found using l'Hospital's Rule.

A *closed-form expression* can be evaluated in a finite number of operations. It may contain constants, variables, common operations, and common functions but, in general, no limit or definite integral.

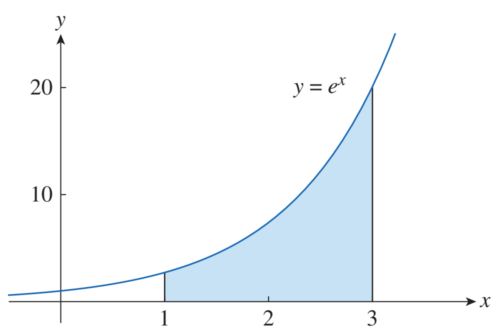


Figure 5.26
Since $f(x) = e^x$ is positive (for all values of x), the definite integral represents the area under the curve.

We still need a quicker way for evaluating this type of definite integral. ■

Example 4 Use Geometry to Evaluate Definite Integrals

Evaluate the following definite integrals by interpreting each in terms of areas.

(a) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$ (b) $\displaystyle\int_0^3 (x-1)\,dx$

Solution

(a) Since $f(x) = \sqrt{1-x^2} \geq 0$, we can interpret this integral as the area under the graph of f and above the x-axis from $x = 0$ to $x = 1$.

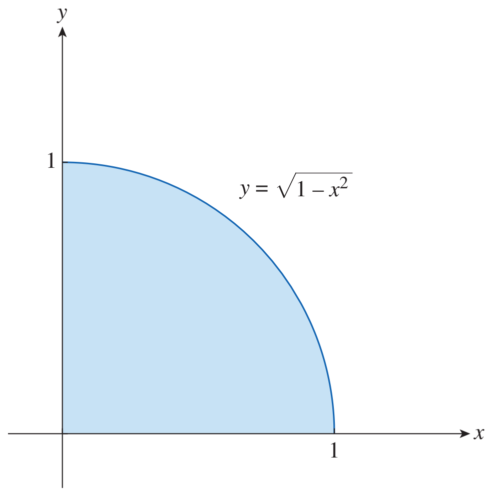


Figure 5.27

The definite integral $\int_0^1 \sqrt{1 - x^2}\,dx$ represents the area of a quarter circle.

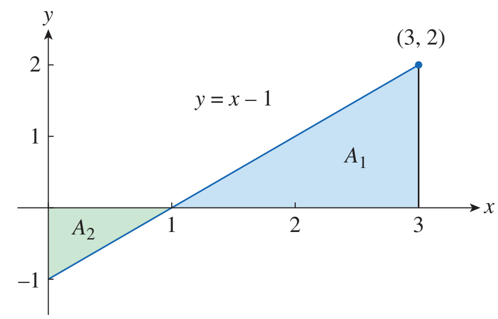


Figure 5.28

The definite integral $\int_0^3 (x - 1)\,dx$ is the net area, $A_1 - A_2$.

The graph of f on [0, 1] is a quarter-circle with radius 1, as shown in Figure 5.27. Therefore, the definite integral is one-quarter of the area of a circle of radius 1.

$$\int_0^1 \sqrt{1 - x^2}\,dx = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$$

(b) The graph of $y = x - 1$ is a line with slope 1 as shown in Figure 5.28.

We can evaluate the definite integral as the difference of the areas of the two triangles.

$$\int_0^3 (x - 1)\,dx = A_1 - A_2 = \frac{1}{2} \cdot 2 \cdot 2 - \frac{1}{2} \cdot 1 \cdot 1 = \frac{3}{2}$$

The Midpoint Rule

In order to evaluate a definite integral, we often choose the sample point x_i^* to be the right endpoint of the ith subinterval because it is convenient for computing the limit. But if the purpose is to find an *approximation* of an integral, it is often better to choose x_i^* to be the midpoint of the interval, which we denote by $\bar{x}_i$. Any Riemann sum is an approximation to a definite integral, but if we use midpoints we get the following approximation.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x = \Delta x[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

where $\quad \Delta x = \dfrac{b - a}{n}$

and $\quad \bar{x}_i = \dfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$

Example 5 Estimate an Integral Using the Midpoint Rule

Use the Midpoint Rule with $n = 5$ subintervals to approximate $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

Solution

The endpoints of the five subintervals are 1, 1.2, 1.4, 1.6, 1.8, and 2.0.

The midpoints are 1.1, 1.3, 1.5, 1.7, and 1.9.

The width of each subinterval is $\Delta x = \dfrac{2-1}{5} = \dfrac{1}{5}$.

Use the Midpoint Rule.

$$\begin{aligned}\int_1^2 \frac{1}{x}\,dx &\approx \Delta x[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]\\ &= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)\\ &\approx 0.691908\end{aligned}$$

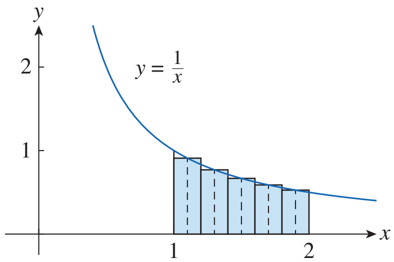


Figure 5.29
The five rectangles as determined by the Midpoint Rule.

Since $f(x) = \dfrac{1}{x} > 0$ for $1 \le x \le 2$, the integral represents an area, and the approximation given by the Midpoint Rule is the sum of the areas of the rectangles shown in Figure 5.29. ■

At the moment we do not know how accurate the approximation in Example 5 is, but in Section 5.9 we will learn a method for estimating the error involved in using the Midpoint Rule. At that time we will also learn other methods for approximating definite integrals.

Figure 5.30 illustrates the Midpoint Rule applied to the integral in Example 2. The approximation $M_{40} \approx -6.7563$ is much closer to the true value -6.75 than the right endpoint approximation, $R_{40} \approx -6.3998$, shown in Figure 5.25.

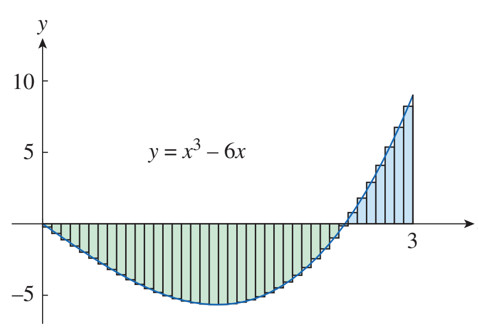


Figure 5.30
An illustration of the Midpoint Rule with $n = 40$: $M_{40} \approx -6.7563$.

Properties of the Definite Integral

In the definition of the definite integral, $\int_a^b f(x)\,dx$, we implicitly assume that $a < b$. However, this definition as a limit of Riemann sums makes sense even if $a > b$. For a regular partition, we can write

$$\Delta x = \frac{a-b}{n} = -\frac{b-a}{n}$$

and therefore,

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

If $a = b$, then $\Delta x = 0$, so

$$\int_a^a f(x)\,dx = 0$$

Here are some basic properties that will helps us to evaluate definite integrals. We assume f and g are continuous functions.

Properties of the Definite Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

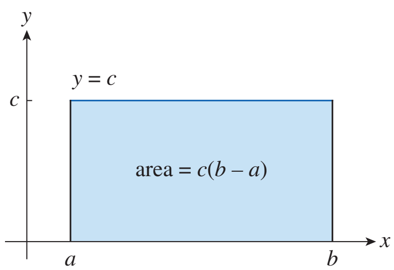


Figure 5.31
If $c > 0$ and $a < b$, then the definite integral in Property 1 is the area of a rectangle: $\int_a^b c\,dx = c(b-a)$.

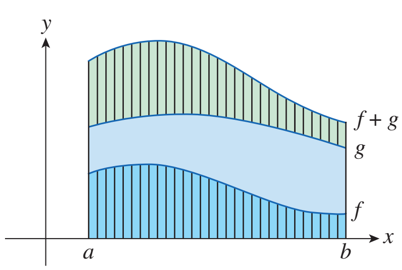


Figure 5.32
If f and g are positive, then the area under f plus the area under g is the area under $f+g$:
$\int_a^b [f(x)+g(x)]\,dx$
$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

A Closer Look

1. In words, Property 1 says that the definite integral of a constant function $f(x) = c$ is the constant times the length of the interval. If $c > 0$ and $a < b$, then $c(b-a)$ is the area of the rectangle bounded above by $f(x) = c$, below by the x-axis, on the left by $x = a$, and on the right by $x = b$. See Figure 5.31.

2. In words, Property 2 says that the definite integral of a sum is the sum of the definite integrals. For positive functions, this can be interpreted as the area under $f+g$ is the area under f plus the area under g. Figure 5.32 illustrates this concept.

Property 2 follows from the definition of a definite integral using right endpoints and the fact that the limit of a sum is the sum of the limits.

$$\begin{aligned}
\int_a^b [f(x)+g(x)]\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i)+g(x_i)]\Delta x && \text{Definition of the definite integral.}\\
&= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\Delta x + \sum_{i=1}^{n} g(x_i)\Delta x\right] && \text{Write as separate summations.}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x + \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\Delta x && \text{Limit of a sum is the sum of the limits.}\\
&= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx && \text{Definition of the definite integral.}
\end{aligned}$$

3. Property 3 can be proved in a similar manner and says that the definite integral of a constant times a function is the constant times the definite integral of the function. In other words, constants pass freely through definite integral signs.

4. Property 4 is proved by writing $f - g = f + (-g)$ and using Properties 2 and 3 with $c = -1$. In words, the definite integral of a difference is the corresponding difference of the definite integrals. ■

Example 6 Use Properties of Integrals

Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\, dx$.

Solution

Use Properties 2 and 3 of integrals.

$$\int_0^1 (4 + 3x^2)\, dx = \int_0^1 4\, dx + 3\int_0^1 x^2\, dx \qquad \text{Properties 2 and 3.}$$

$$= 4(1 - 0) + 3 \cdot \frac{1}{3} \qquad \text{Property 1; Example 2, Section 5.1.}$$

$$= 4 + 1 = 5 \qquad \text{Simplify.}$$

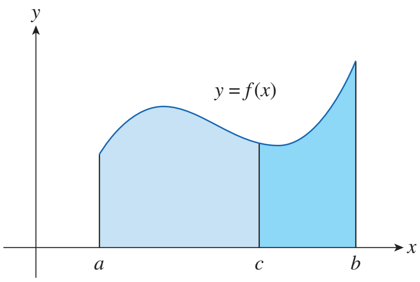


Figure 5.33
Geometric interpretation of Property 5 if $f(x) \geq 0$ and $a < c < b$; area is additive.

Property 5 provides a method to combine definite integrals of the same function over adjacent intervals.

> **Properties of the Definite Integral (continued)**
>
> Suppose that all of the following definite integrals exist.
>
> **5.** $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

The general proof of this property is lengthy; however, there is a very reasonable geometric interpretation if $f(x) \geq 0$ and $a < c < b$. In Figure 5.33, the area under the graph of $y = f(x)$ from a to c plus the area from c to b is equal to the total area from a to b.

Example 7 Property 5 of Integrals

Suppose $\int_0^{10} f(x)\, dx = 17$ and $\int_0^8 f(x)\, dx = 12$, find $\int_8^{10} f(x)\, dx$.

Solution

Use Property 5.

$$\int_0^{10} f(x)\, dx = \int_0^8 f(x)\, dx + \int_8^{10} f(x)\, dx$$

Solve for the unknown definite integral.

$$\int_8^{10} f(x)\, dx = \int_0^{10} f(x)\, dx - \int_0^8 f(x)\, dx = 17 - 12 = 5$$

Properties 1–5 are true whether $a < b$, $a = b$, or $a > b$. The Comparison Properties, in which we compare the magnitude of functions and definite integrals, are true only if $a \leq b$.

> **Comparison Properties of the Definite Integral**
>
> Suppose the following definite integrals exist and $a \leq b$.
>
> **6.** If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\, dx \geq 0$.
>
> **7.** If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$.
>
> **8.** If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$.

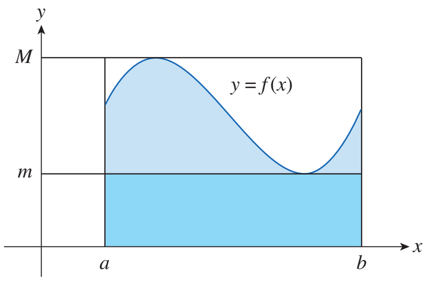


Figure 5.34
Geometric interpretation of Property 8.

A Closer Look

1. Property 6 says that if $f(x) \geq 0$, then $\int_a^b f(x)\,dx$ represents the area of the region bounded above by the graph of $y = f(x)$, below by the x-axis, and lying between the lines $x = a$ and $x = b$. So the geometric interpretation of Property 6 is simply that areas are nonnegative. It also follows directly from the definition of the definite integral because all quantities involved are positive.
2. Property 7 says that a larger function has a larger definite integral. This follows from Properties 6 and 4 and because $f - g \geq 0$.
3. Property 8 is illustrated in Figure 5.34 for the case where $f(x) \geq 0$. If f is continuous, take m and M to be the absolute minimum and maximum values of f on the interval $[a, b]$. In this case, Property 8 says that the area under the graph of f is greater than the area of the rectangle with height m and width $(b - a)$ and less than the area of the rectangle with M and width $(b - a)$. ■

Proof of Property 8

Since $m \leq f(x) \leq M$, use Property 7 to obtain

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx.$$

Use Property 1 to evaluate the definite integrals on the left and right sides.

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$$ ■

Property 8 is useful for finding a very rough estimate of the magnitude of a definite integral. If we can find reasonable values for M and m, then we can easily bound the definite integral.

Example 8 Estimate a Definite Integral

Use Property 8 to estimate $\int_0^1 e^{-x^2}\,dx$.

Solution

Because $f(x) = e^{-x^2}$ is a decreasing function on $[0, 1]$, its absolute maximum value is $M = f(0) = 1$ and its absolute minimum value is $m = f(1) = e^{-1}$.

Using Property 8,

$$e^{-1}(1 - 0) \leq \int_0^1 e^{-x^2}\,dx \leq 1(1 - 0) \quad \text{or} \quad e^{-1} \leq \int_0^1 e^{-x^2}\,dx \leq 1.$$

Since $e^{-1} \approx 0.368$, we can write $0.368 \leq \int_0^1 e^{-x^2}\,dx \leq 1$.

Figure 5.35 illustrates this inequality. ■

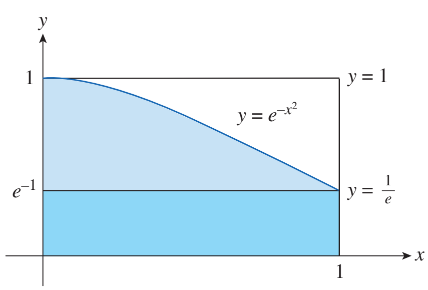


Figure 5.35
The definite integral $\int_0^1 e^{-x^2}\,dx$ is greater than the area of the lower rectangle and less than the area of the upper rectangle with sides of length 1.

5.2 Exercises

1. Evaluate the Riemann sum for $f(x) = 3 - \frac{1}{2}x$, $2 \le x \le 14$, with six subintervals, taking the sample points to be left endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

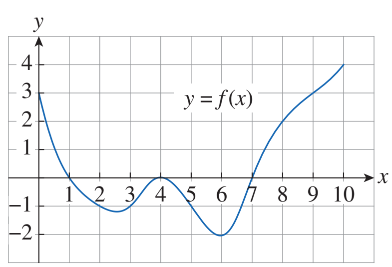
2. If $f(x) = x^2 - 2x$, $0 \le x \le 3$, evaluate the Riemann sum with $n = 6$ subintervals, taking the sample points to be right endpoints. What does the Riemann sum represent? Illustrate with a diagram.

3. If $f(x) = e^x - 2$, $0 \le x \le 2$, find the Riemann sum with $n = 4$ subintervals correct to six decimal places, taking the sample points to be midpoints. What does the Riemann sum represent? Illustrate with a diagram.

4. (a) Find the Riemann sum for $f(x) = \sin x$, $0 \le x \le \frac{3\pi}{2}$, with $n = 6$ subintervals, taking the sample points to be right endpoints. Give your answer correct to six decimal places. Explain what the Riemann sum represents with the aid of a sketch.

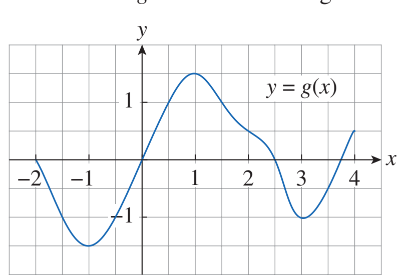
(b) Repeat part (a) with midpoints as sample points.

5. The graph of a function f is shown in the figure.



Estimate $\int_0^{10} f(x)\,dx$ using five equal subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints.

6. The graph of a function g is shown in the figure.



Estimate $\int_{-2}^{4} f(x)\,dx$ using six equal subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints.

7. Selected values of $f(x)$ are given in the table.

x	10	14	18	22	26	30
$f(x)$	-12	-6	-2	1	3	8

Suppose f is an increasing function. Use the table to find a lower estimate and an upper estimate for $\int_{10}^{30} f(x)\,dx$.

8. Selected values of $f(x)$ were obtained from an experiment and are given in the table.

x	3	4	5	6	7	8	9
$f(x)$	-3.4	-2.1	-0.6	0.3	0.9	1.4	1.8

Use the table to estimate $\int_3^9 f(x)\,dx$ using three equal subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints. If the function is increasing, can you say whether your estimates are less than or greater than the exact value of the definite integral? Explain your reasoning.

Use the Midpoint Rule with the given value of n to approximate the definite integral. Round the answer to four decimal places.

9. $\int_0^{10} \sqrt{x^3 + 1}\,dx, \quad n = 4$

10. $\int_0^{8} \sin\sqrt{x}\,dx, \quad n = 4$

11. $\int_0^{\pi/2} \cos^4 x\,dx, \quad n = 4$

12. $\int_0^{\pi} x \sin^2 x\,dx, \quad n = 4$

13. $\int_1^{5} x^2 e^{-x}\,dx, \quad n = 4$

14. $\int_0^{2} \frac{x}{x+1}\,dx, \quad n = 5$

15. Use technology to find the left and right Riemann sums for the function $f(x) = \frac{x}{x+1}$ on the interval $[0, 2]$ with $n = 100$ equal subintervals. Explain why these estimates show that

$$0.895 < \int_0^2 \frac{x}{x+1}\,dx < 0.908$$

16. Use technology to find the left and right Riemann sums for the function $f(x) = \sin(x^2)$ on the interval $[0, 1]$ with $n = 100$ equal subintervals. Explain why these estimates show that

$$0.306 < \int_0^1 \sin(x^2)\,dx < 0.315$$

17. Use technology to construct a table of values of right Riemann sums for the definite integral $\int_0^\pi \sin x \, dx$ with $n = 5$, 10, 50, and 100 equal subintervals. What value do these numbers appear to be approaching?

18. Use technology to construct a table of values of left and right Riemann sums for the definite integral $\int_0^2 e^{-x^2} \, dx$ with $n = 5, 10, 50$, and 100 equal subintervals. Between what two numbers must the value of the definite integral lie? Can you make a similar statement for the definite integral $\int_{-1}^2 e^{-x^2} \, dx$? Explain your reasoning.

Express the limit as a definite integral on the given interval. Assume that the interval $[a, b]$ is divided into n subintervals of equal width $\Delta x = (b - a)/n$. Let $x_0(-a), x_1, x_2, \ldots, x_n(-b)$ be the endpoints of the subintervals and let $x_1^*, x_2^*, \ldots, x_n^*$ be the sample points in these subintervals.

19. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{e^{x_i}}{1 + x_i} \Delta x, \quad [0, 1]$

20. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} x_i \sqrt{1 + x_i} \, \Delta x, \quad [2, 5]$

21. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} x_i \ln(1 + x_i^2) \, \Delta x, \quad [2, 6]$

22. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{\cos x_i}{x_i} \Delta x, \quad [\pi, 2\pi]$

23. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} [5(x_i^*)^3 - 4x_i^*] \, \Delta x, \quad [2, 7]$

24. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{x_i^*}{(x_i^*) + 4} \Delta x, \quad [1, 3]$

Write the definite integral equivalent to the limit.

25. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \left(2 + \dfrac{2i}{n}\right)^2 \cdot \dfrac{2}{n}$

26. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \left[\left(1 + \dfrac{i}{n}\right)^2 + 1\right] \cdot \dfrac{1}{n}$

27. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \left(1 + \dfrac{2i}{n}\right) \ln\left(1 + \dfrac{2i}{n}\right) \cdot \dfrac{2}{n}$

Use the form of the definition of the definite integral given in Theorem 4 to evaluate the integral.

28. $\int_2^5 (4 - 2x) \, dx$

29. $\int_1^4 (x^2 - 4x + 2) \, dx$

30. $\int_{-2}^0 (x^2 + x) \, dx$

31. $\int_0^2 (2x - x^2) \, dx$

32. $\int_0^1 (x^3 - 3x^2) \, dx$

33. $\int_{-2}^2 (x^3 + x) \, dx$

34. Let $f(x) = x^2 - 3x$.

(a) Find an approximation to the definite integral $\int_0^4 f(x) \, dx$ using a right Riemann sum with $n = 8$ equal subintervals.

(b) Draw a diagram to illustrate the approximation in part (a).

(c) Use the theorem about definite integrals using right endpoints to evaluate $\int_0^4 f(x) \, dx$.

(d) Interpret the definite integral in part (c) as a difference of areas and illustrate with a diagram.

35. Show that $\int_a^b x \, dx = \dfrac{1}{2}(b^2 - a^2)$.

36. Show that $\int_a^b x^2 \, dx = \dfrac{1}{3}(b^3 - a^3)$.

Express the definite integral as a limit of sums. Do not evaluate the limit.

37. $\int_1^3 \sqrt{4 + x^2} \, dx$

38. $\int_2^5 \left(x^2 + \dfrac{1}{x}\right) dx$

39. $\int_0^\pi x \sin x \, dx$

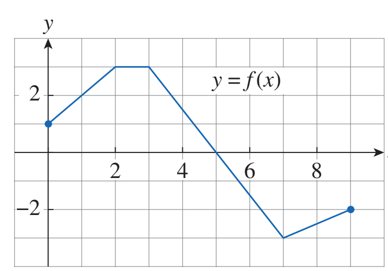
40. $\int_1^2 x^2 \ln x \, dx$

Express the definite integral as a limit of sums. Use technology to find both the sum and the limit.

41. $\int_2^{10} x^6 \, dx$

42. $\int_0^\pi \sin 5x \, dx$

43. The graph of f is shown in the figure.



Evaluate each definite integral by interpreting it in terms of areas.

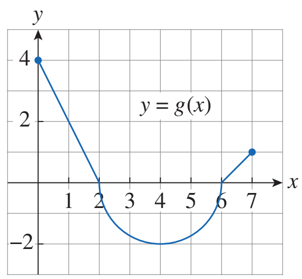
(a) $\int_0^2 f(x) \, dx$

(b) $\int_0^5 f(x) \, dx$

(c) $\int_5^7 f(x) \, dx$

(d) $\int_0^9 f(x) \, dx$

44. The function g is defined on the interval $[0, 7]$. The graph of g consists of two line segments and a semicircle as shown in the figure.



Use the graph to evaluate each definite integral.

(a) $\int_0^2 g(x)\,dx$ (b) $\int_2^6 g(x)\,dx$

(c) $\int_0^7 g(x)\,dx$ (d) $\int_7^2 g(x)\,dx$

Evaluate the definite integral by interpreting it in terms of areas.

45. $\int_{-1}^{2} (1 - x)\,dx$

46. $\int_0^9 \left(\frac{1}{3}x - 2\right) dx$

47. $\int_2^6 (2x - 5)\,dx$

48. $\int_{-3}^{3} \sqrt{9 - x^2}\,dx$

49. $\int_{-3}^{0} \left(1 + \sqrt{9 - x^2}\right) dx$

50. $\int_{-5}^{5} \left(x - \sqrt{25 - x^2}\right) dx$

51. $\int_{-4}^{3} \left|\frac{1}{2}x\right| dx$

52. $\int_1^6 |x - 2|\,dx$

53. $\int_0^1 |2x - 1|\,dx$

54. $\int_{-1}^{4} \sqrt{x^2 - 4x + 4}\,dx$

---

55. Evaluate $\int_1^1 \sqrt{1 + x^4}\,dx$.

56. Given that $\int_0^\pi \sin^4 x\,dx = \frac{3\pi}{8}$, what is $\int_\pi^0 \sin^4 x\,dx$?

57. In Exercise 35 we showed that $\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)$. Use this fact and the properties of definite integrals to evaluate $\int_0^1 (5 - 6x)\,dx$.

58. Use the properties of definite integrals and the result of Example 3 results to evaluate $\int_1^3 (2e^x - 1)\,dx$.

59. Use the result of Example 3 to evaluate $\int_1^3 e^{x+2}\,dx$.

60. Use the properties of definite integrals, the results of Examples 2–8, and the fact that $\int_0^{\pi/2} \cos x\,dx = 1$ to evaluate

$$\int_0^{\pi/2} (2\cos x - 5x)\,dx$$

61. Write the following expression as a single definite integral of the form $\int_a^b f(x)\,dx$.

$$\int_{-2}^{2} f(x)\,dx + \int_2^5 f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

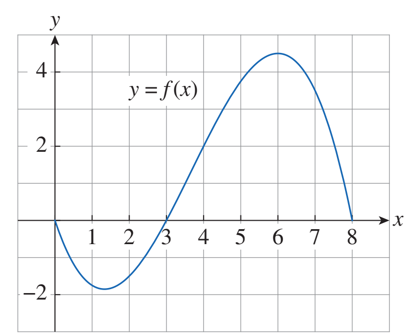
62. If $\int_2^8 f(x)\,dx = 7.3$ and $\int_2^4 f(x)\,dx = 5.9$, find $\int_4^8 f(x)\,dx$.

63. If $\int_0^9 f(x)\,dx = 37$ and $\int_0^9 g(x)\,dx = 16$, find

$$\int_0^9 [2f(x) + 3g(x)]\,dx$$

64. If $f(x) = \begin{cases} 3 & \text{if } x < 3 \\ x & \text{if } x \geq 3 \end{cases}$, find $\int_0^5 f(x)\,dx$.

65. The graph of the function f is given in the figure.

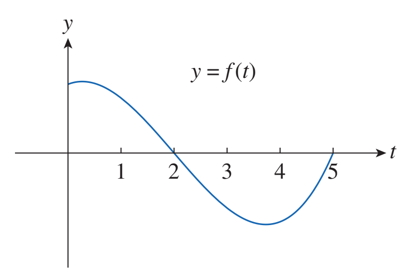


List the following values in increasing order, from smallest to largest, and explain your reasoning.

(i) $\int_0^8 f(x)\,dx$ (ii) $\int_0^3 f(x)\,dx$ (iii) $\int_3^8 f(x)\,dx$

(iv) $\int_4^8 f(x)\,dx$ (v) $f'(1)$

66. The graph of the function f is given in the figure.

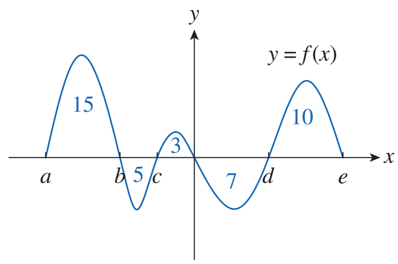


If $F(x) = \int_2^x f(t)\,dt$, which of the following values is largest? Explain your reasoning.

(i) $F(0)$ (ii) $F(1)$ (iii) $F(2)$

(iv) $F(3)$ (v) $F(4)$

67. The figure shows the graph of f. The areas of the regions between the graph of f and the x-axis are labeled.



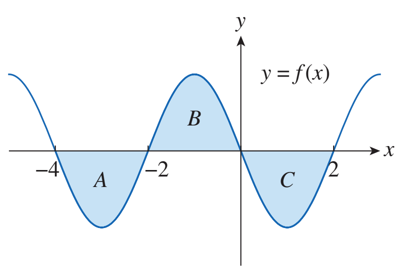
Determine the value of each definite integral or explain why the value cannot be determined.

(a) $\int_0^e f(x)\,dx$ (b) $\int_b^d f(x)\,dx$

(c) $\int_a^0 f(x)\,dx$ (d) $\left|\int_a^0 f(x)\,dx\right|$

(e) $\int_{-e}^{e} f(|x|)\,dx$ (f) $\int_{-c}^{-a} f(-x)\,dx$

68. Each of the shaded regions in the figure bounded by the graph of f and the x-axis has area 3.



Find the value of $\int_{-4}^{2} [f(x) + 2x + 5]\,dx$.

69. If $\int_2^6 f(x)\,dx = 7$, what is the value of $\int_2^6 [f(x) + 2]\,dx$?

70. Suppose the function f has absolute minimum value m and absolute maximum value M. Between what two values must $\int_0^2 f(x)\,dx$ lie? Which property of definite integrals allows you to make this conclusion?

Use the properties of definite integrals to verify the inequality without evaluating the definite integral(s).

71. $\int_0^4 (x^2 - 4x + 4)\,dx \geq 0$

72. $\int_0^1 \sqrt{1 + x}\,dx \leq \int_0^1 \sqrt{1 + x^2}\,dx$

73. $2 \leq \int_{-1}^{1} \sqrt{1 + x^2}\,dx \leq 2\sqrt{2}$

74. $\dfrac{\pi}{12} \leq \int_{\pi/6}^{\pi/3} \sin x\,dx \leq \dfrac{\sqrt{3}\pi}{12}$

Use Property 8 to estimate the value of the definite integral.

75. $\int_0^1 x^3\,dx$

76. $\int_0^5 \dfrac{1}{x + 4}\,dx$

77. $\int_{\pi/4}^{\pi/3} \tan x\,dx$

78. $\int_0^2 (x^3 - 3x + 3)\,dx$

79. $\int_0^2 x e^{-x}\,dx$

80. $\int_{\pi}^{2\pi} (x - 2\sin x)\,dx$

Express the limits as a definite integral.

81. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{i^4}{n^5}$ Hint: Consider $f(x) = x^4$.

82. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{1}{1 + \left(\dfrac{i}{n}\right)^2}$

83. Find $\int_1^2 \dfrac{1}{x^2}\,dx$.

Hint: Choose x_i^* to be the geometric mean of x_{i-1} and x_i, that is, $x_i^* = \sqrt{x_{i-1}x_i}$, and use the identity

$$\frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1}$$

5.3 Evaluating Definite Integrals

In Section 5.2 we learned how to compute the value of a definite integral from the definition, as a limit of Riemann sums. However, this process can be long and tedious. Sir Isaac Newton discovered a much simpler method for evaluating definite integrals and a few years later Leibniz made the same discovery. They realized that they could calculate $\int_a^b f(x)\,dx$ if they happened to know an antiderivative F of f. Their discovery, called the Evaluation Theorem, is part of the Fundamental Theorem of Calculus, which is presented in the next section.

Evaluation Theorem

If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, $F' = f$.

This theorem says that the value of $\int_a^b f(x)\,dx$ can be obtained by finding an antiderivative F of the integrand f, then subtracting, in the proper order, the values of F at the endpoints of $[a, b]$. It is startling that $\int_a^b f(x)\,dx$, defined by a complicated limit procedure involving sums and values of $f(x)$ for $a \le x \le b$, can be found by knowing the values of $F(a)$ and $F(b)$.

For example, we know from Section 4.8 that an antiderivative of the function $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$, so the Evaluation Theorem tells us that

$$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1}{3} \cdot 1^3 - \frac{1}{3} \cdot 0^3 = \frac{1}{3}$$

Compare this method with the calculation in Example 2 in Section 5.1. There we found the area under the parabola $y = x^2$ from 0 to 1 by computing a limit of sums. It's clear that the Evaluation Theorem provides a simple and powerful method for evaluating definite integrals.

Although the Evaluation Theorem may be surprising, it becomes plausible if we interpret the result in physical terms. If $v(t)$ is the velocity of an object and $s(t)$ is its position at time t, then $v(t) = s'(t)$, so s is an antiderivative of v. In Section 5.1 we considered an object that always moves in the positive direction and guessed that the area under the velocity curve is equal to the distance traveled. In symbols,

$$\int_a^b v(t)\,dt = s(b) - s(a)$$

This is the exact interpretation of the Evaluation Theorem in this context.

Proof of the Evaluation Theorem

Divide the interval $[a, b]$ into n equal subintervals with length $\Delta x = \dfrac{b-a}{n}$ and endpoints $a = x_0, x_1, x_2, \ldots, x_n = b$.

Let F be any antiderivative of f.

Write the difference in the F values as the sum of the differences over the subintervals.

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \cdots + F(x_2) - F(x_1) + F(x_1) - F(x_0) \end{aligned}$$

Subtract and add like terms.

$$= \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1}) \right]$$

Use summation notation.

The antiderivative F is continuous because it is differentiable.
Apply the Mean Value Theorem to F on each subinterval $[x_{i-1}, x_i]$.

Therefore, there exists a number x_i^* in (x_{i-1}, x_i) such that

$$F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1}) = f(x_i^*)\,\Delta x.$$

Use this expression in the sum.

$$F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Take the limit of each side of this equation as $n \to \infty$.
The left side is a constant and the right side is a Riemann sum for the function f.

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$ ■

We often use the notation

$$F(x)\Big]_a^b = F(b) - F(a)$$

so the equation in the Evaluation Theorem can be written as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \quad \text{where} \quad F' = f$$

Here is some other common notation for $F(b) - F(a)$: $F(x)\Big|_a^b$ and $[F(x)]_a^b$.

Example 1 Use the Evaluation Theorem

Evaluate $\displaystyle\int_1^3 e^x\,dx$.

Solution

The function $f(x) = e^x$ is continuous everywhere and $F(x) = e^x$ is an antiderivative.

Compare this calculation with the more complicated one in Example 3 in Section 5.2.

If we use $F(x) = e^x + C$, the constant C cancels in $F(3) - F(1)$.

Use the Evaluation Theorem: $\displaystyle\int_1^3 e^x\,dx = e^x\Big]_1^3 = e^3 - e$.

Note that the Evaluation Theorem says we can use *any* antiderivative F of f. So, we may as well use the simplest one, that is, $F(x) = e^x$, instead of $e^x + 7$ or $e^x + C$. ■

Example 2 Area Under a Curve

Find the area under the cosine curve from 0 to b, where $0 \le b \le \dfrac{\pi}{2}$.

Solution

The function $f(x) = \cos x$ is continuous everywhere and $F(x) = \sin x$ is an antiderivative.

$$A = \int_0^b \cos x\,dx = \sin x\Big]_0^b = \sin b - \sin 0 = \sin b$$

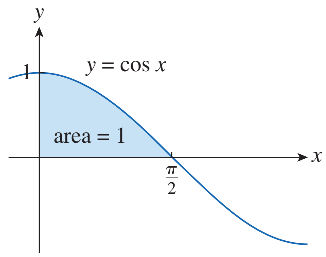


Figure 5.36
The area under the cosine curve from 0 to $\pi/2$.

If we let $b = \frac{\pi}{2}$ then this result shows that the area under the cosine curve from 0 to $\frac{\pi}{2}$ is $\sin\frac{\pi}{2} = 1$. See Figure 5.36. ■

When the French mathematician Gilles de Roberval first considered the area under the sine and cosine curves in 1635, this was a very challenging problem that required a great deal of ingenuity. Without the benefit of the Evaluation Theorem, we would have to compute a difficult limit of sums using obscure trigonometric identities (or use a sophisticated computer algebra system). This problem was even more difficult for Roberval because the concept of limits had not been developed in 1635. But in the 1660s and 1670s, when the Evaluation Theorem was discovered by Newton and Leibniz, such problems were easily solved, as evidenced by Example 2.

Indefinite Integrals

The Evaluation Theorem is a powerful method for evaluating the definite integral of a function, assuming that we can find an antiderivative of the function. However, we need convenient notation for antiderivatives so that we can easily use them in a variety of problems. Because of the relationship between antiderivatives and integrals given by the Evaluation Theorem, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of f and is called an **indefinite integral**. Therefore,

$$\int f(x)\,dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

For example, using this notation we can now write

$$\int x^2\,dx = \frac{x^3}{3} + C \quad \text{because} \quad \frac{d}{dx}\left[\frac{x^3}{3} + C\right] = x^2$$

An indefinite integral is a very general antiderivative, representing an entire *family* of functions (one antiderivative for each value of the constant C).

It is important to distinguish carefully between definite and indefinite integrals. A definite integral, for example, $\int_a^b f(x)\,dx$, represents a *number*. However, an indefinite integral, for example, $\int f(x)\,dx$, represents a *function* (or family of functions). The connection between these two is given by the Evaluation Theorem: if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

We can only use the Evaluation Theorem effectively and efficiently if we have a toolbox, or supply, of antiderivatives of functions. Therefore, the Table of Indefinite Integrals shows a summary of antidifferentiation rules from Section 4.8, together with a few more, using the notation of indefinite integrals. Any rule in this table can be verified by differentiating the function on the right side to obtain the integrand. For example,

$$\int \sec^2 x\,dx = \tan x + C \quad \text{because} \quad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

Table of Indefinite Integrals

$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$	$\int cf(x)\, dx = c\int f(x)\, dx$
$\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \dfrac{1}{x}\, dx = \ln\lvert x\rvert + C$
$\int e^x\, dx = e^x + C$	$\int a^x\, dx = \dfrac{a^x}{\ln a} + C$
$\int \sin x\, dx = -\cos x + C$	$\int \cos x\, dx = \sin x + C$
$\int \sec^2 x\, dx = \tan x + C$	$\int \csc^2 x\, dx = -\cot x + C$
$\int \sec x \tan x\, dx = \sec x + C$	$\int \csc x \cot x\, dx = -\csc x + C$
$\int \dfrac{1}{x^2+1}\, dx = \tan^{-1} x + C$	$\int \dfrac{1}{\sqrt{1+x^2}}\, dx = \sin^{-1} x + C$

Example 3 Evaluate an Indefinite Integral

Find the general indefinite integral $\int (10x^4 - 2\sec^2 x)\, dx$.

Solution

Use the Table of Indefinite Integrals and appropriate properties of indefinite integrals.

A rule in the Table of Indefinite Integrals allows us to split this integral into two separate integrals; in effect, we integrate *term by term*. Then remember that constants pass freely through integral symbols.

$$\begin{aligned}\int (10x^4 - 2\sec^2 x)\, dx &= 10\int x^4\, dx - 2\int \sec^2 x\, dx \\ &= 10\,\frac{x^5}{5} - 2\tan x + C \\ &= 2x^5 - 2\tan x + C\end{aligned}$$

To check the answer, differentiate.

$$\frac{d}{dx}[2x^5 - 2\tan x + C] = 10x^4 - 2\sec^2 x.$$

The graph of the indefinite integral is shown in Figure 5.37 for several values of C. In this example the value of C is the y-intercept. ■

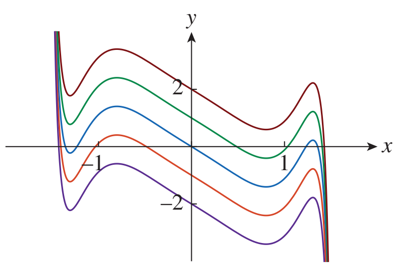


Figure 5.37
Graph of the indefinite integral for several values of C.

Example 4 Evaluate a Definite Integral

Evaluate $\int_0^3 (x^3 - 6x)\, dx$.

Solution

Use the Table of Indefinite Integrals and Evaluation Theorem.

$$\int_0^3 (x^3 - 6x)\, dx = \left[\frac{x^4}{4} - 6\cdot\frac{x^2}{2}\right]_0^3 \qquad \text{Integrate term by term; use the Table of Indefinite Integrals.}$$

$$= \left(\frac{1}{4}\cdot 3^4 - 3\cdot 3^2\right) - \left(\frac{1}{4}\cdot 0^4 - 3\cdot 0^2\right) \qquad \text{Evaluation Theorem.}$$

$$= \frac{81}{4} - 27 - 0 = -6.75 \qquad \text{Simplify.}$$

Compare this calculation with Example 2(b) in Section 5.2. ■

Example 5 Definite Integral Interpreted as Net Area

Find $\int_0^2 \left(2x^3 - 6x + \dfrac{3}{x^2+1}\right) dx$ and interpret the result in terms of areas.

Solution

Use the Table of Indefinite Integrals and the Evaluation Theorem.

$$\begin{aligned}\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx &= 2\left[\frac{x^4}{4} - 6\cdot\frac{x^2}{2} + 3\tan^{-1}x\right]_0^2 \\ &= \left[\frac{1}{2}x^4 - 3x^2 + 3\tan^{-1}x\right]_0^2 \\ &= \left[\frac{1}{2}\cdot 2^4 - 3\cdot 2^2 + 3\tan^{-1}2\right] - 0 \\ &= -4 + 3\tan^{-1}2\end{aligned}$$

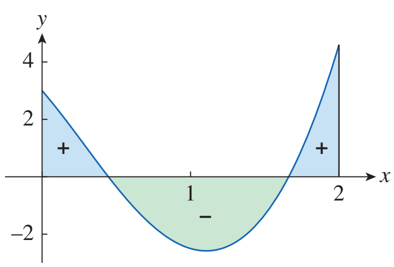


Figure 5.38
A graphical interpretation of the definite integral: net area.

This is the exact value of the definite integral. Using technology, an approximation is -0.67855.

Figure 5.38 shows the graphical interpretation of the definite integral. The value can be interpreted as a net area: the sum of the areas labeled with a plus sign minus the area labeled with a minus sign. ■

Example 6 Simplify Before Integrating

Evaluate $\int_1^9 \dfrac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$.

Solution

Rewrite the integrand as three distinct terms by carrying out the division. Then use the Table of Indefinite Integrals.

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt = \int_1^9 (2 + t^{1/2} - t^{-2})\, dt \qquad \text{Divide.}$$

$$= \left[2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\right]_1^9 = \left[2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\right]_1^9 \qquad \text{Table of Indefinite Integrals; simplify.}$$

$$= \left[2\cdot 9 + \frac{2}{3}\cdot 9^{3/2} + \frac{1}{9}\right] - \left[2\cdot 1 + \frac{2}{3}\cdot 1^{3/2} + \frac{1}{1}\right] \qquad \text{Evaluation Theorem.}$$

$$= \left[18 + 18 + \frac{1}{9}\right] - \left[2 + \frac{2}{3} + 1\right] = \frac{292}{9} \qquad \text{Simplify.}$$

■

■ Applications

The Evaluation Theorem says that if f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Because F is an antiderivative of f, we can write $F' = f$, so the conclusion of the Evaluation Theorem can be rewritten as

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

We have learned that $F'(x)$ represents the rate of change of $y = F(x)$ with respect to x, and $F(b) - F(a)$ is the change in y as x changes from a to b. Note that y could, for example, increase, then decrease, then increase again. Therefore, the value $F(b) - F(a)$ really represents the *net* change in y. Using this interpretation, the Evaluation Theorem can be restated in terms of net change.

Net Change Theorem

The definite integral of a rate of change (F') may be interpreted as the net change in the original function F.

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

A Closer Look

1. The definite integral $\int_a^b F'(x)\,dx$ represents an *accumulation* of the change in F over the interval $[a, b]$.
2. We can rearrange the terms in the Net Change Theorem to provide an alternate interpretation and practical approach for solving many problems.

$$\underbrace{F(b)}_{\text{End amount}} = \underbrace{F(a)}_{\text{Start amount}} + \underbrace{\int_a^b F'(x)\,dx}_{\text{Net change}}$$

This principle has many uses and can be applied to all of the rates of change in the natural and social sciences that we discussed in Section 3.8. Here are a few examples of this idea.

(1) Let $V'(t)$ represent the rate at which water flows into a reservoir at time t. Then $V(t)$ is the volume of water in the reservoir at time t and

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the net change in the amount of water in the reservoir between time t_1 and time t_2.

(2) If $C(t)$ is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $C'(t)$. Therefore,

$$\int_{t_1}^{t_2} C'(t)\,dt = C(t_2) - C(t_1)$$

is the change in the concentration of the chemical from time t_1 to time t_2.

(3) If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. Therefore,

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

(4) If the rate of growth of a population is $P'(t)$, then

$$\int_{t_1}^{t_2} P'(t)\,dt = P(t_2) - P(t_1)$$

is the net change in population during the time period from t_1 to t_2. The population increases when births occur and decreases when deaths happen. The net change takes into account both events: births and deaths.

(5) If $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$. Therefore,

$$\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$$

is the increase in cost when production is increased from x_1 units to x_2 units or the net change in cost.

(6) If a particle moves along a horizontal line so that its position at time t is given by $s(t)$, then its velocity is $v(t) = s'(t)$, and

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1) \tag{1}$$

is the net change in position, or *displacement*, of the particle over the time period from t_1 to t_2. We have already seen examples in which this was true for a particle moving only in a positive direction. However, now we can interpret Equation 1 as the net change in position of the particle over the interval $[t_1, t_2]$.

(7) If we want to find the *total distance* traveled by an object over a time interval, then, theoretically, we have to consider separately the intervals when $v(t) \geq 0$ (the particle is moving to the right) and the intervals where $v(t) \leq 0$ (the particle is moving to the left). However, we can simplify this process if we consider the function $|v(t)|$, the speed of the object. Therefore,

$$\int_{t_1}^{t_2} |v(t)|\,dt = \begin{array}{l}\text{total distance traveled}\\ \text{from time } t_1 \text{ to time } t_2\end{array} \tag{2}$$

Figure 5.39 shows how both displacement and total distance traveled can be interpreted in terms of areas under a velocity curve; A_i represents the area of a shaded region.

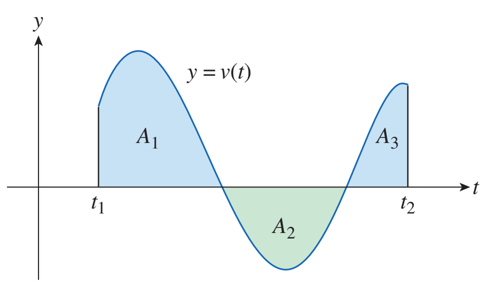


Figure 5.39
The displacement and total distance traveled can be visualized in terms of the area under the graph of a velocity function.

$$\text{Displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{Total distance traveled} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

(8) The acceleration of an object is $a(t) = v'(t)$. Therefore,

$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

is the net change in velocity over the time interval $[t_1, t_2]$.

Example 7 Displacement Versus Total Distance Traveled

A particle moves along a horizontal line so that its velocity at time t, $t \geq 0$, is given by $v(t) = t^2 - t - 6$, where v is measured in meters per second.

(a) Find the displacement of the particle during the time interval [1, 4].

(b) Find the total distance traveled during this same time period.

Solution

(a) Use Equation 1 to find the displacement of the particle.

$$s(4) - s(1) = \int_1^4 v(t)\,dt = \int_1^4 \left(t^2 - t - 6\right) dt \qquad \text{Net Change Theorem.}$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 \qquad \text{Table of Indefinite Integrals.}$$

$$= \left[\frac{4^3}{3} - \frac{4^2}{2} - 6 \cdot 4\right] - \left[\frac{1^3}{3} - \frac{1^2}{2} - 6 \cdot 1\right] = -\frac{9}{2} \qquad \text{Evaluation Theorem.}$$

This means that the particle's net change in position over the time interval [1, 4] is 4.5 m to the left of its position at time $t = 1$ s.

(b) In order to find the total distance traveled, we need to integrate $|v(t)|$.

Therefore, we need to find the intervals on which $v(t) \geq 0$ and those on which $v(t) \leq 0$.

$v(t) = t^2 - t - 6 = (t - 3)(t + 2)$

$v(t) \leq 0$ on the interval [1, 3] and $v(t) \geq 0$ on [3, 4]

Use Equation 2 to find the total distance traveled.

$$\int_1^4 |v(t)|\,dt = \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt \qquad \text{Split the integral into two parts, one on which } v(t) \leq 0 \text{ and one on which } v(t) \geq 0.$$

$$= \int_1^3 [-(t^2 - t - 6)]\,dt + \int_3^4 (t^2 - t - 6)\,dt \qquad \text{Use the expression for } v(t).$$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 \qquad \text{Table of Indefinite Integrals.}$$

$$= \frac{61}{6} \approx 10.167 \qquad \text{Evaluation Theorem; simplify.}$$

The total distance traveled by the particle over the time interval [1, 4] is approximately 10.167 m. ■

Example 8 Power Consumption

On a certain day in September in the city of San Francisco, the rate, in megawatts per hour, at which power is consumed is given by $P(t)$, where t is measured in hours past midnight. The graph of P is shown in Figure 5.40. Estimate the total power used on that day.

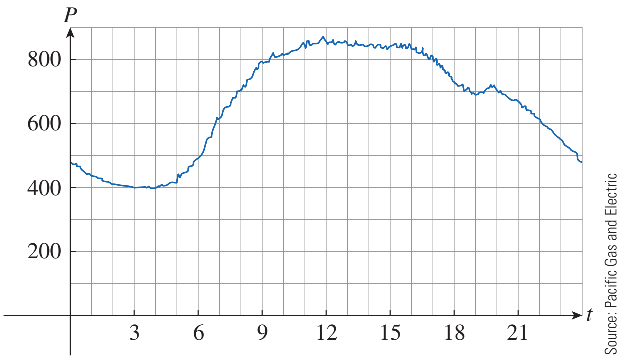


Figure 5.40
A graph of the rate at which power is consumed.

Solution

Since P is the rate of power consumption and $P(t) \geq 0$, we need to find $\int_0^{24} P(t)\, dt$, the total accumulation of power used over the 24-hour period.

There is no formula for $P(t)$, so we cannot find a closed-form expression for the integral.

Therefore, we will approximate the value of the definite integral using the Midpoint Rule with 12 equal subintervals; $\Delta t = 2$.

$$\begin{aligned}\int_0^{24} P(t)\, dt &\approx [P(1) + P(3) + P(5) + \cdots + P(21) + P(23)]\, \Delta t && \text{Midpoint Rule.}\\ &\approx (440 + 400 + 420 + \cdots + 670 + 550)(2) && \text{Read values from the graph.}\\ &= 15{,}840\end{aligned}$$

The total power consumed over the time interval [0, 24] was approximately 15,840 megawatts. ■

A Note on Units

In problems involving net change, it is important to report the correct units. In Example 8, the definite integral $\int_0^{24} P(t)\, dt$ is defined as the limit of the sum of terms of the form $P(t_i^*)\, \Delta t$. The rate $P(t_i^*)$ is measured in megawatts per hour and Δt is measured in hours. Therefore, the product is measured in megawatts:

$$\frac{\text{megawatts}}{\text{hour}} \cdot (\text{hour}) = \text{megawatts}$$

In general, the unit of measurement for the definite integral $\int_a^b f(x)\, dx$ is the product of the unit for $f(x)$ and the unit for x.

5.3 Exercises

Evaluate the integral.

1. $\int_{-2}^{3}(x^2 - 3)\,dx$

2. $\int_{1}^{2}(4x^3 - 3x^2 + 2x)\,dx$

3. $\int_{-2}^{0}\left(\frac{1}{2}t^4 + \frac{1}{4}t^3 - t\right)dt$

4. $\int_{0}^{3}(1 + 6w^2 - 10w^4)\,dw$

5. $\int_{0}^{2}(2x - 3)(4x^2 + 1)\,dx$

6. $\int_{-1}^{1}t(1 - t)^2\,dt$

7. $\int_{0}^{\pi}(5e^x + 3\sin x)\,dx$

8. $\int_{1}^{2}\left(\frac{1}{x^2} - \frac{4}{x^3}\right)dx$

9. $\int_{1}^{4}\left(\frac{4 + 6u}{\sqrt{u}}\right)du$

10. $\int_{0}^{1}\frac{4}{1 + p^2}\,dp$

11. $\int_{0}^{1}x\left(\sqrt[3]{x} + \sqrt[4]{x}\right)dx$

12. $\int_{1}^{4}\frac{\sqrt{y} - y}{y^2}\,dy$

13. $\int_{1}^{2}\left(\frac{x}{2} - \frac{2}{x}\right)dx$

14. $\int_{0}^{1}(5x - 5^x)\,dx$

15. $\int_{0}^{1}(x^{10} + 10^x)\,dx$

16. $\int_{0}^{\pi/4}\sec\theta\tan\theta\,d\theta$

17. $\int_{0}^{\pi/4}\frac{1 + \cos^2\theta}{\cos^2\theta}\,d\theta$

18. $\int_{0}^{\pi/3}\frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta$

19. $\int_{1}^{8}\frac{2 + t}{\sqrt[3]{t^2}}\,dt$

20. $\int_{1}^{5}\left(2e^x + \frac{1}{x}\right)dx$

21. $\int_{0}^{\sqrt{3}/2}\frac{dr}{\sqrt{1 - r^2}}$

22. $\int_{1}^{2}\frac{(x - 1)^3}{x^2}\,dx$

23. $\int_{0}^{1/\sqrt{3}}\frac{t^2 - 1}{t^4 - 1}\,dt$

24. $\int_{0}^{2}|2x - 1|\,dx$

25. $\int_{-1}^{2}(x - 2|x|)\,dx$

26. $\int_{0}^{3\pi/2}|\sin x|\,dx$

27. $\int_{4}^{9}\frac{x + 1}{\sqrt{x}}\,dx$

28. $\int_{0}^{\pi/4}(\cos x - \sin x)\,dx$

Explain why the equation is incorrect.

29. $\int_{-1}^{3}\frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\bigg]_{1}^{3} = -\frac{4}{3}$

30. $\int_{0}^{\pi}\sec^2 x\,dx = \tan x\big]_0^{\pi} = 0$

Use a graph to find a rough estimate of the area of the region that lies beneath the given curve. Then find the exact area.

31. $y = \sin x, \quad 0 \le x \le \pi$

32. $y = \sec^2 x, \quad 0 \le x \le \frac{\pi}{3}$

33. $y = \frac{1}{x}, \quad 1 \le x \le 4$

34. Use technology to estimate the x-intercepts of the graph of $f(x) = 1 - 2x - 5x^4$. Use this information to estimate the area of the region that lies under the graph of f and above the x-axis.

35. Use technology to estimate the x-intercepts of the graph of $f(x) = (x^2 + 1)^{-1} - x^4$. Use this information to estimate the area of the region that lies under the graph of f and above the x-axis.

Evaluate the integral and interpret it as a difference of areas. Illustrate your result with a sketch.

36. $\int_{-1}^{2}x^3\,dx$

37. $\int_{-\pi/2}^{2\pi}\cos x\,dx$

Verify that the formula is correct by differentiation.

38. $\int\cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C$

39. $\int x\cos x\,dx = x\sin x + \cos x + C$

40. $\int\frac{1}{x^2\sqrt{1 + x^2}}\,dx = -\frac{\sqrt{1 + x^2}}{x} + C$

41. $\int x\sqrt{a + bx}\,dx = \frac{2}{15b^2}(3bx - 2a)(a + bx)^{3/2} + C$

Find the general indefinite integral. Graph several members of the family on the same coordinate axes.

42. $\int\left(\cos x + \frac{1}{2}x\right)dx$

43. $\int(e^x - 2x^2)\,dx$

Find the general indefinite integral.

44. $\int(x^{1.3} + 7x^{2.5})\,dx$

45. $\int\sqrt[4]{x^5}\,dx$

46. $\int\left(5 + \frac{2}{3}x^2 + \frac{3}{4}x^3\right)dx$

47. $\int\left(u^6 - 2u^5 - u^3 + \frac{2}{7}\right)du$

48. $\int(u + 4)(2u + 1)\,du$

49. $\int\sqrt{t}(t^2 + 3t + 2)\,dt$

50. $\int\frac{1 + \sqrt{x} + x}{x}\,dx$

51. $\int\left(x^2 + 1 + \frac{1}{x^2 + 1}\right)dx$

52. $\int\frac{1 + r}{r}\,dr$

53. $\int\left(\frac{1 + r}{r}\right)^2 dr$

54. $\int(2 + \tan^2\theta)\,d\theta$

55. $\int\sec t(\sec t + \tan t)\,dt$

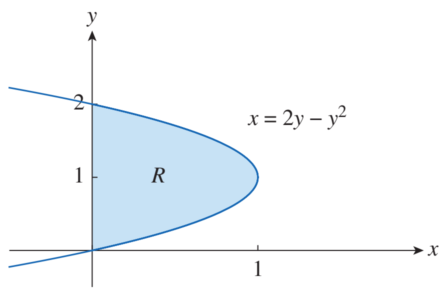
56. $\int 2^t(1 + 5^t)\,dt$

57. $\int\frac{\sin 2x}{\sin x}\,dx$

58. $\int\left(3t^2 + \frac{2}{t^2}\right)dt$

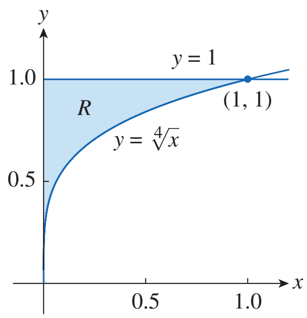
59. $\int(x + 1)^3\,dx$

60. Let R be the region that is bounded on the left by the y-axis and on the right by the graph of $x = 2y - y^2$, as shown in the figure.



The area of R is given by the definite integral $\int_0^2 (2y - y^2)\,dy$. Find the area of the region R.

61. Let R be the region bounded by the graphs of $y = 1$ and $y = \sqrt[4]{x}$ and the y-axis, as shown in the figure.



Find the area of the region R by writing x as a function of y and integrating with respect to y.

62. Suppose $w'(t)$ is the rate of growth of a child in pounds per year. Explain the meaning of $\int_5^{10} w'(t)\,dt$.

63. The current in a wire is defined as the derivative of the charge: $I(t) = Q'(t)$. (See Example 3 in Section 3.8.) Explain the meaning of $\int_a^b I(t)\,dt$.

64. Suppose $r'(t)$ is the rate at which water flows over a dam in cubic feet per second. Explain the meaning of $\int_0^{60} r'(t)\,dt$.

65. Suppose $r(t)$ is the rate at which water flows into a tank and $d(t)$ is the rate at which water flows out of the tank at time t, both measured in cubic feet per hour. Explain the meaning of $\int_3^6 [r(t) - d(t)]\,dt$.

66. Suppose $r'(t)$ is the rate at which oil leaks from a tank in gallons per minute at time t. Explain the meaning of $\int_0^{120} r'(t)\,dt$.

67. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. Explain the meaning of $100 + \int_0^{15} n'(t)\,dt$.

68. Suppose $R'(x)$ is a marginal revenue function (defined in Section 4.6), the derivative of the revenue function $R(x)$, where x is the number of units sold. Explain the meaning of $\int_{1000}^{5000} R'(x)\,dx$.

69. Suppose $f(t)$ is the rate of natural gas production from a well at time t, measured in cubic feet per hour. Explain the meaning of $\int_0^{24} f(t)\,dt$.

70. Suppose $f(x)$ is the slope of a trail at a distance of x miles from the start of the trail. Explain the meaning of $\int_3^5 f(x)\,dx$.

71. If x is measured in meters and $f(x)$ is measured in newtons, what is the unit for $\int_0^{100} f(x)\,dx$?

72. If the units for x is feet and the units for $a(x)$ is pounds per foot, what are the units for $\dfrac{da}{dx}$? What are the units for $\int_2^8 a(x)\,dx$?

The velocity function (in meters per second) is given for a particle moving along a line. Find (a) the displacement and (b) the total distance traveled by the particle during the given time interval.

73. $v(t) = 3t - 5, \quad 0 \le t \le 3$

74. $v(t) = t^2 - 2t - 3, \quad 2 \le t \le 4$

The acceleration function (in m/s^2) and the initial velocity are given for a particle moving along a line. Find (a) the velocity at time t and (b) the distance traveled during the given time interval.

75. $a(t) = t + 4, \quad v(0) = 5, \quad 0 \le t \le 10$

76. $a(t) = 2t + 3, \quad v(0) = -4, \quad 0 \le t \le 3$

77. The linear density of a rod of length 4 meters is given by $\rho(x) = 9 + 2\sqrt{x}$ measured in kilograms per meter, where x is measured in meters from one end of the rod. Find the total mass of the rod.

78. Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \le t \le 50$. Find the amount of water that flows from the tank during the first ten minutes.

79. The velocity of a car was read from its speedometer at 10-second intervals and the values are given in the table.

t (s)	v (mi/h)	t (s)	v (mi/h)
0	0	60	56
10	38	70	53
20	52	80	50
30	58	90	47
40	55	100	45
50	51		

Use the Midpoint Rule to estimate the distance traveled by the car over the interval [0, 100].

80. Suppose that a volcano is erupting and $r(t)$ is the rate at which solid materials are spewed into the atmosphere, where $r(t)$ is measured in tons per second and t is measured in seconds. Selected values for $r(t)$ are given in the table and $r(t)$ is increasing over the interval [0, 6].

t	0	1	2	3	4	5	6
$r(t)$	2	10	24	36	46	54	60

(a) Find upper and lower estimates for the total quantity $Q(6)$ of erupted materials after six seconds.

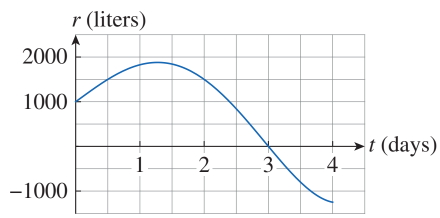
(b) Use the Midpoint Rule to estimate $Q(6)$.

81. The marginal cost of manufacturing x yards of a certain fabric is

$$C'(x) = 3 - 0.01x + 0.000006x^2$$

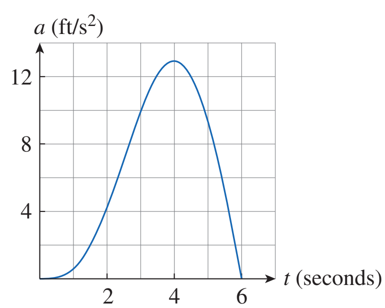
in dollars per yard. Find the increase in cost if the production level is raised from 2000 yards to 4000 yards.

82. Water flows into and out of a storage tank. A graph of the rate of change $r(t)$ of the volume of water in the tank, in liters per day, is shown in the figure.



If the amount of water in the tank at time $t = 0$ is 25,000 L, use the Midpoint Rule with four equal subintervals to estimate the amount of water in the tank four days later.

83. The graph of the acceleration $a(t)$ of a car, measured in ft/s^2, is shown in the figure.



Use the Midpoint Rule with six equal subintervals to estimate the increase in the velocity of the car during the 6-second time interval.

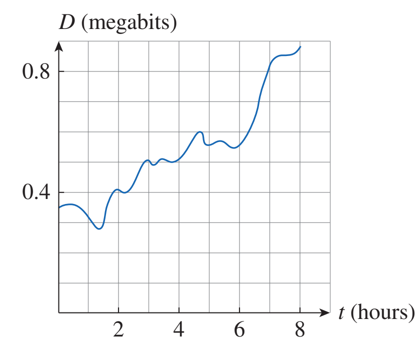
84. The Indian Prairie Canal is located in the Lake Okeechobee basin and the table shows the rate of outflow for selected days in April, in cubic feet per second.

Day	Outflow rate (ft^3/s)	Day	Outflow rate (ft^3/s)
April 01	7.34	April 08	30.70
April 02	5.30	April 09	9.75
April 03	4.75	April 10	15.40
April 04	20.50	April 11	49.50
April 05	30.00	April 12	34.80
April 06	18.50	April 13	16.20
April 07	3.71		

Use the Midpoint Rule to estimate the amount of water that flowed out of Indian Prairie Canal from April 1 to April 13.

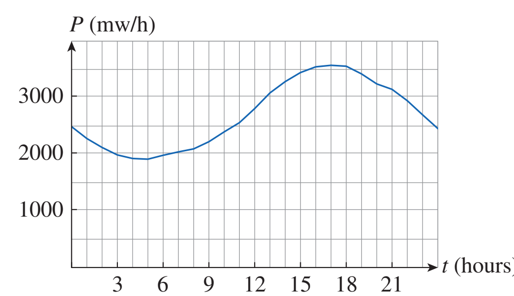
85. A bacteria population is 4000 at time $t = 0$ and its rate of growth is $100 \cdot 2^t$ bacteria per hour at time t hours. What is the population after 1 hour?

86. The function D is the data throughput, the rate at which data flow through an Internet provider's T1 data line, measured in megabits per second. A graph of the function D for midnight to 8:00 AM is shown in the figure.



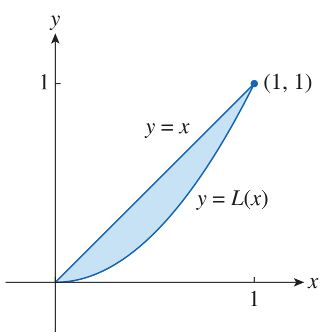
Use the Midpoint Rule to estimate the total amount of data transmitted during that time period.

87. The rate of power consumption, in megawatts per hour, at an electric company on a certain day in April, is given by $P(t)$, where t is measured in hours past midnight. The graph of P is shown in the figure.



Estimate the total power used on that day.

88. Economists use a cumulative distribution called a *Lorenz curve* to describe the distribution of income between households in a given country. Typically, a Lorenz curve is defined on [0, 1] with endpoints (0, 0) and (1, 1) and is continuous, increasing, and concave upward. The points on this curve are determined by ranking all households by income and then computing the percentage of households whose income is less than or equal to a given percentage of the total income of the country. For example, the point $(a/100, b/100)$ is on the Lorenz curve if the bottom $a\%$ of the households receive less than or equal to $b\%$ of the total income. *Absolute equality* of income distribution would occur if the bottom $a\%$ of the households receive $a\%$ of the income, in which case the Lorenz curve would be the line $y = x$. The area between the Lorenz curve and the line $y = x$ measures how much the income distribution differs from absolute equality. The *coefficient of inequality* is the ratio of the area between the Lorenz curve and the line $y = x$ to the area under $y = x$.



(a) Show that the coefficient of inequality is twice the area between the Lorenz curve and the line $y = x$; that is, show that

$$\text{coefficient of inequality} = 2\int_0^1 [x - L(x)]\,dx$$

(b) The income distribution for a certain country is represented by the Lorenz curve defined by the equation

$$L(x) = \frac{5}{12}x^2 + \frac{7}{12}x$$

What is the percentage of total income received by the bottom 50% of the households? Find the coefficient of inequality.

89. (a) Show that $1 \le \sqrt{1 + x^3} \le 1 + x^3$ for $x \ge 0$.

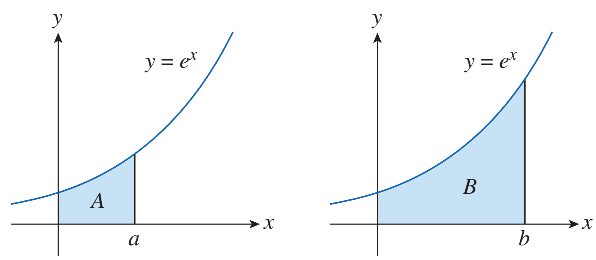
(b) Show that $1 \le \int_0^1 \sqrt{1 + x^3}\,dx \le 1.25$.

90. (a) Show that $\cos(x^2) \ge \cos x$ for $0 \le x \le 1$.

(b) Deduce that $\int_0^{\pi/6} \cos(x^2)\,dx \ge \frac{1}{2}$.

91. Suppose h is a function such that $h(1) = -2$, $h'(1) = 2$, $h''(1) = 3$, $h(2) = 6$, $h'(2) = 5$, $h''(2) = 13$, and h'' is continuous everywhere. Evaluate $\int_1^2 h''(u)\,du$.

92. The area of the region labeled B is three times the area of the region labeled A.



Find an expression for b in terms of a.

Evaluate the limit by recognizing the sum as a Riemann sum for a function defined on [0, 1].

93. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$

94. $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \cdots + \sqrt{\frac{n}{n}}\right)$

Discovery Project | Area Functions

1. (a) Draw the line $y = 2t + 1$ and use geometry to find the area of the region bounded above by this line, below by the t-axis, and between the lines $t = 1$ and $t = 3$.

(b) If $x > 1$, let $A(x)$ be the area of the region bounded above by the line $y = 2t + 1$, below by the t-axis, and between the lines $t = 1$ and $t = x$. Sketch this region and use geometry to find an expression for $A(x)$.

(c) Find the derivative $A'(x)$. How is this derivative related to the equation of the line in part (b)?

2. (a) For $0 \le x \le \pi$, let $A(x) = \int_0^x \sin t\,dt$. The value $A(x)$ represents the area of a region in the plane. Sketch the region.

(b) Use the Evaluation Theorem to find an expression of $A(x)$.

(c) Find the derivative $A'(x)$. How is this derivative related to the integrand in part (a)?

(d) If x is any value between 0 and π, and h is a small positive number, then $A(x + h) - A(x)$ represents the area of a region. Describe and sketch the region.

(e) Draw a rectangle that can be used to approximate the area of the region in part (d). Compare the areas of these two regions and show that

$$\frac{A(x + h) - A(x)}{h} \approx \sin x$$

(f) Use part (e) to present an intuitive explanation for the result in part (c).

3. (a) Use technology to sketch the graph of the function $f(x) = \cos(x^2)$ in the viewing rectangle $[0, 2] \times [-1.25, 1.25]$.

(b) Let the function g be defined by

$$g(x) = \int_0^x \cos(t^2)\, dt$$

The value $g(x)$ is the area of the region bounded above by the graph of f, below by the x-axis, and between 0 and x (until $f(x)$ becomes negative, at which point $g(x)$ becomes a difference of areas, or net area). Use part (a) to determine the first value of $x(>0)$ at which $g(x)$ starts to decrease. Note: Unlike the integral in Problem 2, it is not possible to evaluate the integral defining g to obtain an explicit expression for $g(x)$.

(c) Use technology to estimate $g(0.2), g(0.4), g(0.6), \ldots, g(1.8), g(2)$. Use these values to sketch a graph of g.

(d) Use the graph of g from part (c) to sketch the graph of g' using the interpretation of $g'(x)$ as the slope of a tangent line. How does the graph of g' compare with the graph of f?

4. Suppose f is a continuous function on the interval $[a, b]$ and the function g is defined by the equation

$$g(x) = \int_a^x f(t)\, dt$$

Use your results in Problems 1–3 to make a conjecture for the expression $g'(x)$.

5.4 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is the most important concept in calculus. It establishes a connection between the two branches of calculus: differential calculus and integral calculus. Differential calculus arose from the tangent line problem and integral calculus arose from the seemingly unrelated area problem. Newton's mentor at Cambridge, Isaac Barrow (1630–1677), discovered that these two problems are actually closely related. In fact, he realized that differentiation and integration are inverse processes.

The Fundamental Theorem of Calculus is an amazing result because it provides the precise inverse relationship between the derivative and the integral. Newton and Leibniz recognized this relationship and used it to develop calculus into a systematic mathematical method. In particular, they realized that the Fundamental Theorem of Calculus enabled them to compute areas and integrals without having to find the limit of a Riemann sum.

The first part of the Fundamental Theorem of Calculus deals with functions defined by an equation of the form

$$g(x) = \int_a^x f(t)\, dt \tag{1}$$

where f is a continuous function on $[a, b]$ and x varies between a and b. Note that g depends only on x, which appears as the upper limit in the integral. If x is a fixed number, then the integral in Equation 1 is a finite number. If we let x vary, then the number $\int_a^x f(t)\, dt$ also varies and therefore defines a function of x, denoted by $g(x)$.

If f is a positive function, then $g(x)$ can be interpreted as the area under the graph of f (and above the x-axis) from a to x, where x can vary from a to b. We can interpret g as an *area-so-far* function. See Figure 5.41.

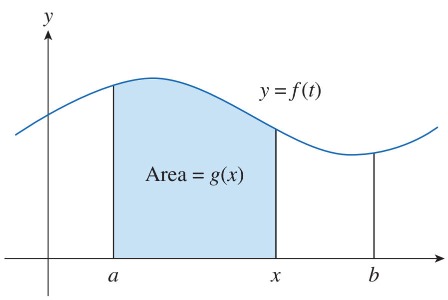


Figure 5.41
A visualization of the *area-so-far* function g.

Example 1 A Function Defined by an Integral

The graph of a function f is given in Figure 5.42.

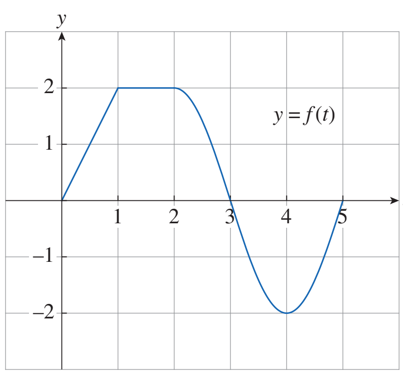


Figure 5.42
Graph of $y = f(t)$.

Let $g(x) = \int_a^x f(t)\, dt$, and find the values $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Use these values to sketch a rough graph of g.

Solution

$$g(0) = \int_0^0 f(t)\,dt = 0$$ By definition, $\Delta x = 0$; no area under one value.

$$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2} \cdot 1 \cdot 2 = 1$$ Area of a triangle; Figure 5.43.

$$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$$ Adjacent intervals; integral property.

$$= 1 + (1 \cdot 2) = 3$$ $g(1)$ plus the area of a rectangle; Figure 5.43.

$$g(3) = g(2) + \int_2^3 f(t)\,dt \approx 3 + 1.3 = 4.3$$ An estimate of the area under the graph of f from 2 to 3 is 1.3; Figure 5.43.

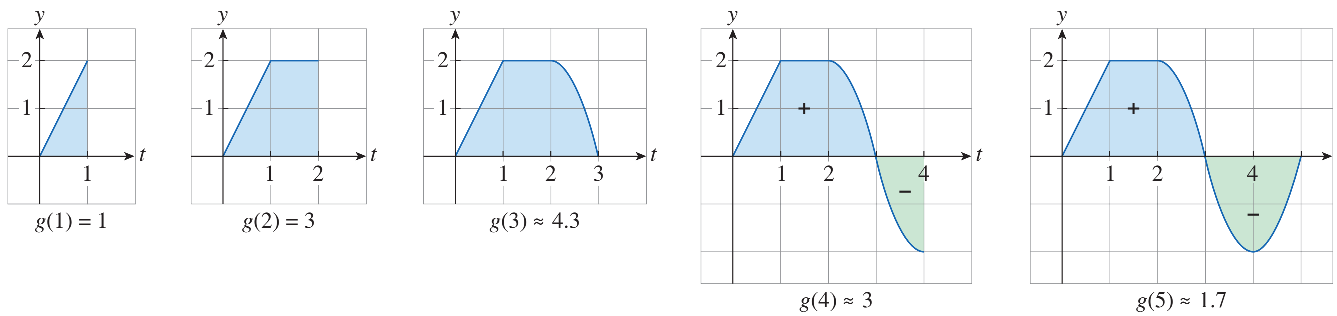


Figure 5.43
A visualization of values of the area-so-far function g.

For $t > 3$, $f(t)$ is negative. Therefore, we need to start subtracting areas.

$$g(4) = g(3) + \int_3^4 f(t)\,dt \approx 4.3 + (-1.3) = 3.0$$ Figure 5.43.

$$g(5) = g(4) + \int_4^5 f(t)\,dt \approx 3 + (-1.3) = 1.7$$ Figure 5.43.

Using these values, a sketch of the graph of g is given in Figure 5.44.

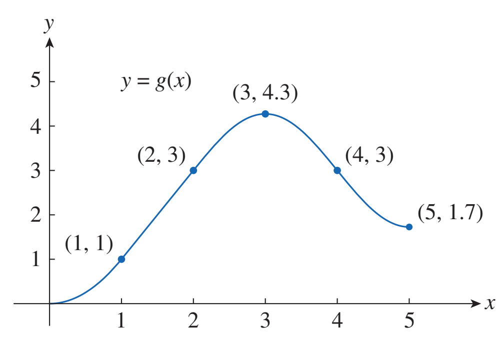


Figure 5.44
A sketch of $y = g(x)$.

Note that because $f(t)$ is positive for $t < 3$, we add areas for $t < 3$, and so g is increasing up to $x = 3$, where it attains a maximum value. For $x > 3$, g is decreasing because $f(t)$ is negative. ■

Example 2 Find a Formula for the Area-So-Far

Let $g(x) = \int_a^x f(t)\,dt$, where $a = 1$ and $f(t) = t^2$. Find a formula for $g(x)$ and differentiate to find $g'(x)$.

Solution

In this example we can compute $g(x)$ explicitly using the Evaluation Theorem.

$$g(x) = \int_1^x t^2\,dt = \frac{t^3}{3}\bigg]_1^x = \frac{x^3 - 1}{3}$$

$$g'(x) = \frac{d}{dx}\left(\frac{1}{3}x^3 - \frac{1}{3}\right) = x^2$$ ■

In Example 2, notice that $g'(x) = x^2$, that is, $g' = f$, or the derivative of the area-so-far function is f. This suggests that if g is defined as the integral of f, as in Equation 1, then g is an antiderivative of f, at least in this case.

This also appears to be the case in Example 1. If we sketch the derivative of the function g shown in Figure 5.44 by estimating slopes of tangents, then we get a graph like the one in Figure 5.42. This also suggests that $g' = f$.

To see why this result might be true in general, consider a continuous function f such that $f(x) \geq 0$. The function $g(x) = \int_a^x f(t)\,dt$ can be interpreted as the area of the region bounded above by the graph of f, below by the x-axis, from a to x: the area-so-far function. See Figure 5.41.

Let's try to compute $g'(x)$ from the definition of a derivative. For $h > 0$, the expression $g(x + h) - g(x)$ is determined by subtracting areas. So, this difference is represented by the area under the graph of f from x to $x + h$, the area of the blue shaded region in Figure 5.45.

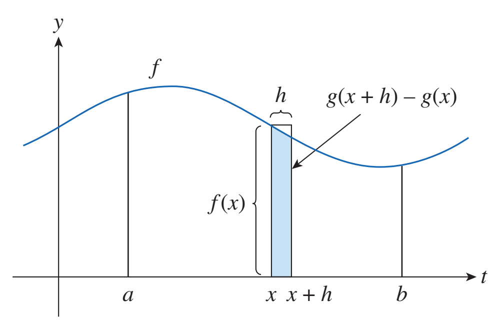


Figure 5.45
A visualization of the difference $g(x + h) - g(x)$.

For small values of h, Figure 5.45 suggests that this area is approximately equal to the area of the rectangle with height $f(x)$ and width h:

$$g(x + h) - g(x) \approx hf(x) \quad \Rightarrow \quad \frac{g(x + h) - g(x)}{h} \approx f(x).$$

Therefore, we expect that

$$g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f(x).$$

This result is in fact true, even when f is not necessarily positive, and is the first part of the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus, Part 1

If f is a continuous function on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt, \quad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

A Closer Look

1. We will abbreviate the name of this theorem as FTC1. In words, it says that the derivative of a definite integral with respect to its upper limit is the integrand (the function to be integrated), evaluated at the upper limit.
2. Here is some other notation for this result involving the derivative of g:

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

   Loosely speaking, this expression says that if we first integrate f and then differentiate the result, we get back the original function f. That is, integration and differentiation are inverse operations; what one does, the other undoes. ■

The proof of the Fundamental Theorem is straightforward if we assume that f has an antiderivative F. This is certainly reasonable; we sketched graphs of antiderivatives in Section 2.8. Then, by the Evaluation Theorem,

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

for any x between a and b. Therefore,

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}[F(x) - F(a)] = F'(x) = f(x)$$

as required. There is a proof presented at the end of this section without the assumption that an antiderivative exists.

Example 3 Differentiate an Integral

Find the derivative of each function.

(a) $g(x) = \int_0^x \sqrt{1 + t^2}\,dt$ (b) $h(x) = \int_1^x \frac{\sin t}{1 + t^2}\,dt$

Solution

(a) $f(t) = \sqrt{1 + t^2}$ is continuous. Use the Fundamental Theorem of Calculus, Part 1.

$$g'(x) = \frac{d}{dx}\left[\int_0^x \sqrt{1 + t^2}\,dt\right] = \sqrt{1 + x^2}$$

(b) $f(t) = \frac{\sin t}{1 + t^2}$ is continuous. Use the FTC1.

$$h'(x) = \frac{d}{dx}\left[\int_1^x \frac{\sin t}{1 + t^2}\,dt\right] = \frac{\sin x}{1 + x^2}$$

■

Example 4 Fresnel Function

Admittedly, the expression $g(x) = \int_a^x f(t)\,dt$ seems like a strange way to define a function. However, functions of this form are common in physics, chemistry, and statistics. For example, the **Fresnel function**

The Fresnel function is an example in which the definition is not in *closed form:* for a given value of x, the Fresnel function cannot be evaluated (exactly) in a finite number of standard operations.

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

named after the French physicist Augustin Fresnel (1788–1827), is used in the study of optics and recently has been applied to the design of highways.

The Fundamental Theorem of Calculus, Part 1, tells us how to differentiate the Fresnel function:

$$S'(x) = \frac{d}{dx}\left[\int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt\right] = \sin\left(\frac{\pi x^2}{2}\right)$$

This means we can use all the methods of differential calculus to analyze the function S.

The graphs of $f(x) = \sin\left(\frac{\pi x^2}{2}\right)$ and the Fresnel function $S(x) = \int_0^x f(t)\,dt$ are shown in Figure 5.46. This figure suggests that $S(x)$ is indeed the area under the graph of f from 0 to x, until $x \approx 1.4$, when $S(x)$ becomes a difference of areas. Figure 5.47 shows a larger part of the graph of S.

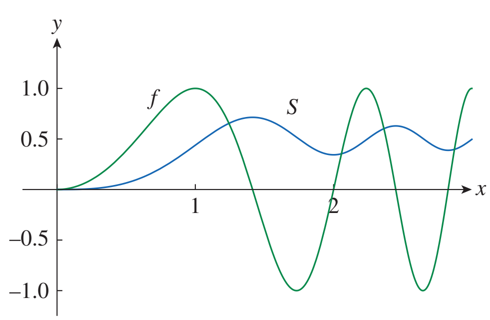


Figure 5.46
Graph of the Fresnel function S and its derivative f.

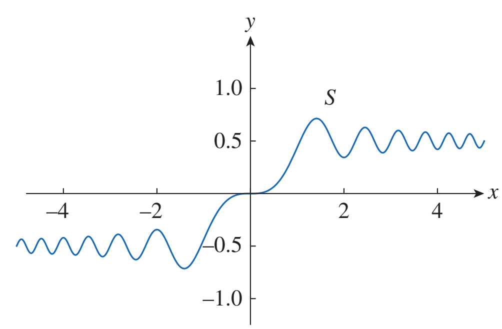


Figure 5.47
A larger part of the graph of the Fresnel function S.

Start with the graph of S in Figure 5.46 and think about what its derivative should look like. It seems reasonable that $S'(x) = f(x)$. For example, S is increasing when $f(x) > 0$ and decreasing when $f(x) < 0$. These graphs suggest a visual confirmation of the Fundamental Theorem of Calculus, Part 1. ■

Example 5 The Chain Rule and the FTC1

Find $\frac{d}{dx}\left[\int_1^{x^4} \sec t\,dt\right]$.

Solution

Because the upper bound is a function of x, we need to use the Chain Rule, together with the FTC1.

$$\frac{d}{dx}\left[\int_1^{x^4} \sec t\, dt\right] = \frac{d}{dx}\left[\int_1^{u} \sec t\, dt\right] \qquad \text{Let } u = x^4.$$

$$= \frac{d}{du}\left[\int_1^{u} \sec t\, dt\right]\frac{du}{dx} \qquad \text{Chain Rule.}$$

$$= \sec u \,\frac{du}{dx} \qquad \text{FTC1.}$$

$$= \sec(x^4) \cdot 4x^3 \qquad \text{Use } u = x^4.$$

Differentiation and Integration as Inverse Processes

Now let's connect the two parts of the Fundamental Theorem of Calculus. Part 1 is *fundamental* because it relates integration and differentiation. But the Evaluation Theorem presented in Section 5.3 also relates integrals and derivatives, so this result is generally considered Part 2 of the Fundamental Theorem.

The Fundamental Theorem of Calculus

Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\, dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\, dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

Earlier in this section, we wrote the result involving the derivative of g as

$$\frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x)$$

Using this notation it may be easier to see that if we integrate f and then differentiate the result, we end up at the original function f.

In Section 5.3, we rewrote Part 2 as the Net Change Theorem. Since $F'(x) = f(x)$, we can write

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

This notation suggests that if we start with a function F, first differentiate and then integrate, we end up with a difference involving the original function F evaluated at the endpoints of the integral, $F(b) - F(a)$. Taken together, these two parts of the FTC say that differentiation and integration are inverse processes. What one does, the other undoes.

The Fundamental Theorem of Calculus is the most important theorem in calculus and, indeed, it ranks as one of the great accomplishments of the human mind. Before it was discovered, from the time of Eudoxus and Archimedes to the time of Galileo and Fermat, problems of finding areas, volumes, and lengths of curves were extremely difficult. But now, armed with the systematic method that Newton and Leibniz developed with the Fundamental Theorem of Calculus, many of these challenging problems are readily accessible.

Proof of FTC1

Here is a more general proof of the Fundamental Theorem of Calculus, Part 1, without assuming the existence of an antiderivative of f.

Let $g(x) = \int_a^x f(t)\,dt$. If x and $x + h$ are in (a, b), then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt && \text{Definition of } g. \\ &= \left(\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt\right) - \int_a^x f(t)\,dt && \text{Property 5 of Definite Integrals.} \\ &= \int_x^{x+h} f(t)\,dt && \text{Simplify.} \end{aligned}$$

For $h \neq 0$,

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt \tag{2}$$

For now, let's assume that $h > 0$. Since f is continuous on $[x, x + h]$, the Extreme Value Theorem says that there are numbers u and v in $[x, x + h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x + h]$. See Figure 5.48.

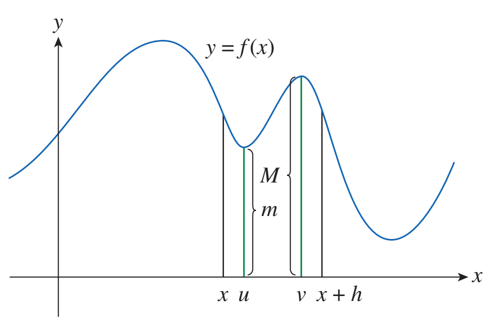


Figure 5.48
Consider the absolute extreme values m and M on the interval $[x, x + h]$.

Use Property 8 of definite integrals: $mh \leq \int_x^{x+h} f(t)\,dt \leq Mh$.

Rewrite this expression using f: $f(u) \cdot h \leq \int_x^{x+h} f(t)\,dt \leq f(v) \cdot h$.

Since $h > 0$, divide this inequality by h: $f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq f(v)$.

Use Equation 2 to replace the middle part of this inequality:

$$f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v) \tag{3}$$

For $h < 0$, we can argue in a similar manner to obtain this same inequality.

As $h \to 0$, both $u \to x$ and $v \to x$ because u and v lie between x and $x + h$.

Therefore, since f is continuous at x:

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x) \quad \text{and} \quad \lim_{h \to 0} f(v) = \lim_{v \to x} f(v) = f(x)$$

Using the inequality in Equation 3 and the Squeeze Theorem,

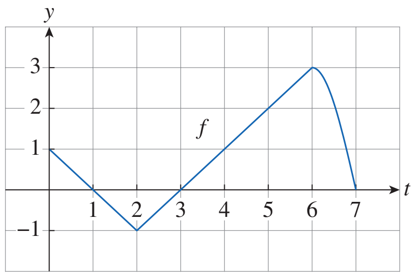
$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x) \tag{4}$$

If $x = a$ or b, then Equation 4 can be interpreted as a one-sided limit. We can then show that g is continuous on $[a, b]$. ■

5.4 Exercises

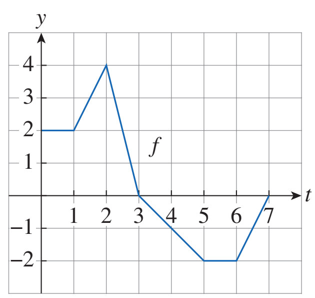
1. Explain in your own words what is meant by the statement that "differentiation and integration are inverse processes."

2. Let $g(x) = \int_a^x f(t)\,dt$, where f is the function whose graph is shown.



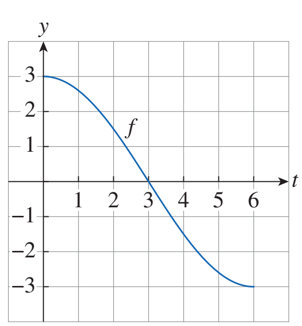
(a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5$, and 6.
(b) Estimate $g(7)$. Where does g have a maximum value? Where does it have a minimum value? Explain your reasoning.
(c) Sketch a rough graph of g.

3. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.



(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.
(b) On what interval is g increasing?
(c) Where does g have a maximum value? Explain your reasoning.
(d) Sketch a rough graph of g.

4. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.



(a) Evaluate $g(0)$ and $g(6)$.
(b) Estimate $g(x)$ for $x = 1, 2, 3, 4$, and 5.
(c) On what interval is g increasing?
(d) Where does g have a maximum value? Explain your reasoning.
(e) Sketch a rough graph of g.
(f) Use the graph in part (e) to sketch the graph of $g'(x)$. Compare your graph with the graph of f.

Sketch the area represented by $g(x)$. Then find $g'(x)$ in two ways: (a) by using Part 1 of the Fundamental Theorem of Calculus and (b) by evaluating the integral using Part 2 and then differentiating.

5. $g(x) = \int_1^x t^2\,dt$

6. $g(x) = \int_1^x (1 + \sqrt{t})\,dt$

7. $g(x) = \int_0^x (2 + \sin t)\,dt$

Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

8. $g(x) = \int_0^x \sqrt{t + t^3}\,dt$

9. $g(x) = \int_1^x \ln(1 + t^2)\,dt$

10. $g(s) = \int_5^s (t - t^2)^8\,dt$

11. $h(u) = \int_0^u \dfrac{\sqrt{t}}{t + 1}\,dt$

12. $f(x) = \int_x^0 \sqrt{1 + \sec t}\,dt$

13. $R(y) = \int_y^2 t^3 \sin t\,dt$

14. $G(x) = \int_x^1 \cos \sqrt{t}\,dt$

15. $h(x) = \int_2^{1/x} \arctan t\,dt$

16. $h(x) = \int_1^{e^x} \ln t\,dt$

17. $h(x) = \int_1^{\sqrt{x}} \dfrac{z^2}{z^4 + 1}\,dz$

18. $g(x) = \int_1^{3x+2} \dfrac{t}{1 + t^3}\,dt$

19. $g(x) = \int_0^{x^4} \cos^2 \theta\,d\theta$

20. $g(x) = \int_0^{\tan x} \sqrt{t + \sqrt{t}}\,dt$

21. $g(x) = \int_{\sqrt{x}}^{\pi/4} \theta \tan \theta\,d\theta$

22. $g(x) = \int_{\sin x}^1 \sqrt{1 + t^2}\,dt$

23. $g(x) = \int_{-x^2}^{x^2} \dfrac{1}{1 + t^2}\,dt$

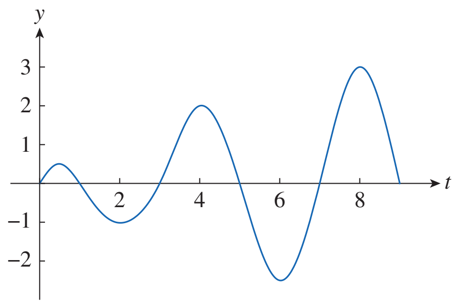
24. $g(x) = \int_{2x}^{x^2} te^t\,dt$

25. $g(x) = \int_{\sin x}^{\cos x} (1 + t^2)^{10}\,dt$

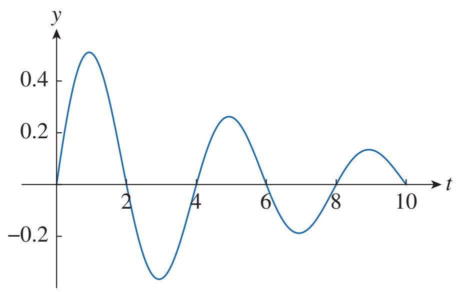
Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

(a) Find the values of x at which the local maximum and minimum values of g occur.
(b) Find the value of x at which g attains its absolute maximum value.
(c) Find the intervals on which g is concave downward.
(d) Sketch the graph of g.

26.



27.



28. Let $f(x) = \int_0^x (1 - t^2)e^{t^2}\, dt$. Find the interval(s) on which the graph of f is increasing.

29. Let $f(x) = \int_0^{\sin x} \sqrt{1 + t^2}\, dt$ and $g(y) = \int_3^y f(x)\, dx$. Find $g''(\pi/6)$.

30. Let $g(x) = \int_0^x \frac{t^2}{t^2 + t + 2}\, dt$. Find the interval(s) on which the graph of g is concave down.

31. Find the slope of the tangent line to the curve with parametric equations

$$x = \int_0^t \sqrt{1 + u^3}\, du \qquad y = 1 + 2t - t^3$$

at the point (0, 1).

32. If $f(1) = 12$, f' is continuous, and $\int_1^4 f'(x)\, dx = 17$, find the value of $f(4)$.

33. The **error function** defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

is used in probability, statistics, and engineering.

(a) Show that $\int_a^b e^{-t^2}\, dt = \frac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b) - \operatorname{erf}(a)]$.

(b) Show that the function $y = e^{x^2} \cdot \operatorname{erf}(x)$ satisfies the differential equation $y' = 2xy + \frac{2}{\sqrt{\pi}}$.

34. The Fresnel function S was defined in Example 4 and graphs of S and its derivative are shown in Figures 5.46 and 5.47.

(a) At what values of x does S have local maximum values? Justify your answer.

(b) Find the intervals on which the graph of S is concave up.

(c) Use technology to solve the equation

$$\int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt = 0.2$$

35. The **sine integral function** is defined by

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$$

and is important in electrical engineering. Note that the integrand $f(t) = \frac{\sin t}{t}$ is not defined when $t = 0$. However, its limit is 1 as $t \to 0$. So we define $f(0) = 1$ and this makes f a continuous function everywhere.

(a) Use technology to graph Si.

(b) At what values of x does this function have local maximum values? Justify your answer.

(c) Find the coordinates of the first inflection point to the right of the origin.

(d) Does the graph of this function have horizontal asymptotes? Explain your reasoning.

(e) Use technology to solve the equation

$$\int_0^x \frac{\sin t}{t}\, dt = 1$$

36. Find a function f such that $f(1) = 0$ and $f'(x) = \frac{2^x}{x}$.

37. Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 < x \le 2 \\ 0 & \text{if } x > 2 \end{cases}$$

and $g(x) = \int_0^x f(t)\, dt$.

(a) Find an expression for $g(x)$ as a piecewise defined function.

(b) Sketch the graphs of f and g.

(c) Where is f differentiable? Where is g differentiable?

38. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\, dt = 2\sqrt{x} \qquad \text{for all } x > 0$$

39. A high-tech company purchases a new computing system whose initial value is V. The system will depreciate at a rate given by the function $f(t)$ and will accumulate maintenance costs at a rate given by the function $g(t)$, where t is the time measured in months. The company wants to determine the optimal time to replace the system.

(a) Let

$$C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]\, ds$$

Show that the critical numbers of C occur at the values t where $C(t) = f(t) + g(t)$.

(b) Suppose that

$$f(t) = \begin{cases} \dfrac{V}{15} - \dfrac{V}{450}t & \text{if } 0 < t \le 30 \\ 0 & \text{if } t > 30 \end{cases}$$

and $$g(t) = \frac{Vt^2}{12{,}900} \quad \text{for } t > 0$$

Determine the length of time T for the total depreciation $D(t) = \int_0^t f(s)\, ds$ to equal the initial value V.

(c) Determine the absolute minimum value of C on $(0, T]$.

(d) Sketch the graphs of C and $f + g$ on the same coordinate axes, and verify the result in part (a) in this case.

40. A manufacturing company owns a major piece of equipment that depreciates at a (continuous) rate given by the function $f(t)$, where t is the time measured in months since its last overhaul. Because a fixed cost A is incurred each time the machine is overhauled, the company wants to determine the optimal time T (in months) between overhauls.

(a) Explain why $\int_0^t f(s)\, ds$ represents the loss in value of the machine over the period of time t since the last overhaul.

(b) Let C be defined by

$$C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\, ds\right]$$

What does the function C represent and why would the company want to minimize the value of C?

(c) Show that C has a minimum value at the numbers $t = T$ where $C(T) = f(T)$.

Writing Project | Newton, Leibniz, and the Invention of Calculus

We sometimes read that the inventors of calculus were Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716). But we know that the basic ideas behind integration were investigated 2500 years ago by ancient Greeks, for example, Eudoxus and Archimedes, and methods for finding tangents were initiated by Pierre Fermat (1601–1665), Isaac Barrow (1630–1677), and others. Barrow, who taught at Cambridge and was a major influence on Newton, was the first to understand the inverse relationship between differentiation and integration. Newton and Leibniz used this relationship, in the form of the Fundamental Theorem of Calculus, in order to develop calculus into a systematic mathematical discipline. This work is the reason Newton and Leibniz are credited with the invention of calculus.

Read about the contributions of these people in one or more of the given references and write a report on one of the following three topics. You can include biographical details, but the focus of your report should be a description, in some detail, of their methods and notations. In particular, you should consult one of the sourcebooks, which give excerpts from the original publications of Newton and Leibniz, translated from Latin to English.

- The Role of Newton in the Development of Calculus
- The Role of Leibniz in the Development of Calculus
- The Controversy Between the Followers of Newton and Leibniz over Priority in the Invention of Calculus

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6. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), Chapter 12.
7. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), Chapter 17.

Sourcebooks

1. John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader* (London: MacMillan Press, 1987), Chapters 12 and 13.
2. D. E. Smith, ed., *A Sourcebook in Mathematics* (New York: Dover, 1959), Chapter V.
3. D. J. Struik, ed., *A Sourcebook in Mathematics*, *1200–1800* (Princeton, NJ: Princeton University Press, 1969), Chapter V.

5.5 The Substitution Rule

Because the Fundamental Theorem of Calculus allows us to evaluate a definite integral very quickly, it is important to be able to find antiderivatives. The basic antidifferentiation formulas are useful but don't help us evaluate an integral such as

$$\int 2x\sqrt{1 + x^2}\, dx \tag{1}$$

To evaluate this indefinite integral, we use the problem-solving strategy of *introducing something extra*. Here the *something extra* is a new variable; we change from the variable x to a new variable u. In other words, we transform, or reduce, the given indefinite integral into an equivalent expression in another variable, preferably into a form that is easier to integrate.

Substitution: Indefinite Integrals

It's not clear yet why this is a *good* choice for u.

Here's how the process works. Define u in terms of the variable x: here, let $u = 1 + x^2$ (the expression under the root sign). In general, given a function $u = g(x)$, recall that du and dx are differentials and the relationship between them is given by $du = g'(x)\, dx$ Therefore, in this case, the differential of u is $du = 2x\, dx$.

Solve for the differential dx: $dx = \dfrac{du}{2x}$.

Use these expressions to transform the integral from the variable x to the variable u.

$$\int 2x\sqrt{1+x^2}\,dx = \int 2x\sqrt{u}\,dx \qquad \text{Use the expression for } u.$$

$$= \int 2x\sqrt{u}\,\frac{du}{2x} = \int \sqrt{u}\,du \qquad \text{Use the expression for } dx\text{; simplify.}$$

$$= \frac{2}{3}u^{3/2} + C \qquad \text{Antidifferentiation rule.}$$

$$= \frac{2}{3}(1+x^2)^{3/2} + C \qquad \text{Use the expression for } u \text{ to rewrite the solution in terms of the original variable, } x.$$

We can check this answer by using the Chain Rule to differentiate.

$$\frac{d}{dx}\left[\frac{2}{3}(1+x^2)^{3/2} + C\right] = \frac{2}{3}\cdot\frac{3}{2}(1+x^2)^{1/2}\cdot 2x = 2x\sqrt{1+x^2}$$

In general, the method of substitution works whenever we have an integral that we can write in the form $\int f(g(x))\,g'(x)\,dx$, If $F' = f$, then

$$\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C \tag{2}$$

because, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))] = F'(g(x))\,g'(x)$$

If we make the *change of variable* or *substitution* $u = g(x)$, then from Equation 2 we have

$$\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C = F(u) + C = \int F'(u)\,du$$

or, using $F' = f$, we get

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

This proves the Substitution Rule.

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

A Closer Look

1. As given above, the Substitution Rule for integration is proved using the Chain Rule for differentiation.
2. If $u = g(x)$, then $du = g'(x)\,dx$. So a way to remember and use the Substitution Rule is to think of dx and du as differentials.
3. In applying the Substitution Rule, it is permissible to work with dx and du after the integral signs (as part of the integrands) as if there were differentials. Practically,

this means that we can treat dx and du as if they are *variables*: we can multiply or divide by these variables or use these variables in any other valid mathematical operation.

4. When using the Substitution Rule, a good choice for $u = g(x)$ is one in which the derivative, $g'(x)$, occurs in the integrand as a factor, except perhaps for a constant multiple.

Example 1 Use the Substitution Rule

Find $\int x^3 \cos(x^4 + 2)\, dx$.

Solution

The integrand cannot be evaluated using the Table of Indefinite Integrals; however, the derivative of the *inner* function, $x^4 + 2$, is $4x^3$, and this expression occurs in the integrand as a factor, except for the constant multiple, 4.

Therefore, a reasonable substitution is $u = x^4 + 2$.

Then $du = 4x^3\, dx \;\Rightarrow\; dx = \dfrac{du}{4x^3}$.

$$\int x^3 \cos(x^4 + 2)\, dx = \int x^3 \cos u \, \frac{du}{4x^3} \qquad \text{Change variables.}$$

$$= \int \frac{1}{4} \cos u \, du = \frac{1}{4} \int \cos u \, du \qquad \text{Simplify.}$$

$$= \frac{1}{4} \sin u + C \qquad \text{Table of Indefinite Integrals.}$$

$$= \frac{1}{4} \sin(x^4 + 2) + C \qquad \text{Final answer in terms of the original variable } x.$$

Remember, we can always check the final answer by differentiation.

The idea behind the Substitution Rule is to replace a relatively complicated integral by a simpler, equivalent integral. This is done by transforming the original integral in x to a reduced integral in u. In Example 1 we replaced a relatively complicated integral, $\int x^3 \cos(x^4 + 2)\, dx$, by a much simpler integral, $\frac{1}{4} \int \cos u \, du$.

The challenge (skill) in using the Substitution Rule is to select an appropriate substitution, that is, a *good* function $u = g(x)$. A good choice means we can transform the integral in x into a simpler one in u, one that we know how to solve.

One helpful strategy is to choose u to be some function in the integrand whose derivative also occurs in the integrand (except for a constant factor). This was the case in Example 1. If that strategy fails, try choosing u to be some complicated part of the integrand, perhaps the inner function in a composite function.

Finding the *right* substitution is an art, perfected through practice and pattern recognition. It is not unusual to start with a bad guess for u. If that happens, then the resulting integral, in u, may be even more complicated, or it may be impossible to transform the entire integrand into the variable u. So, if your first guess doesn't work, try another substitution.

Example 2 Two Possible Substitutions

Evaluate $\int \sqrt{2x+1}\,dx$.

Solution 1

Let $u = 2x + 1 \quad \Rightarrow \quad du = 2dx \quad \Rightarrow \quad dx = \dfrac{1}{2}\,du.$

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int u^{1/2}\,du \qquad \text{Change variables; simplify.}$$

$$= \frac{1}{2}\cdot\frac{u^{3/2}}{\frac{3}{2}} + C = \frac{1}{3}u^{3/2} + C \qquad \text{Table of Indefinite Integrals.}$$

$$= \frac{1}{3}(2x+1)^{3/2} + C \qquad \text{Final answer in terms of } x.$$

Solution 2

Common Error

When using the Substitution Rule, leave the antiderivative in terms of the variable u.

Correct Method

When using the Substitution Rule for an indefinite integral, the final answer must be given in terms of the original variable.

Note that other substitutions may also produce a simpler integral that we can solve. For example,

let $u = \sqrt{2x+1} \quad \Rightarrow \quad du = \dfrac{dx}{\sqrt{2x+1}} \quad \Rightarrow \quad dx = \sqrt{2x+1}\,du = u\,du.$

$$\int \sqrt{2x+1}\,dx = \int u\cdot u\,du = \int u^2\,du \qquad \text{Change variables; simplify.}$$

$$= \frac{u^3}{3} + C = \frac{1}{3}(2x+1)^{3/2} + C \qquad \text{Final (same) answer in terms of } x.$$

■

Example 3 Use the Substitution Rule

Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Solution

This looks like a good choice for u because its derivative also occurs in the integrand as a factor, except for a constant multiple.

Let $u = 1 - 4x^2 \quad \Rightarrow \quad du = -8x\,dx \quad \Rightarrow \quad dx = -\dfrac{du}{8x}.$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{x}{\sqrt{u}}\cdot -\frac{du}{8x} = \int -\frac{1}{8}u^{-1/2}\,du \qquad \text{Change variables; simplify.}$$

$$= -\frac{1}{8}\left(2\sqrt{u}\right) + C \qquad \text{Table of Indefinite Integrals.}$$

$$= -\frac{1}{4}\left(\sqrt{1-4x^2}\right) + C \qquad \text{Final answer in terms of } x.$$

We can check this answer by differentiation. However, in this case, let's verify the solution graphically. Figure 5.49 shows the graph of both the integrand, $f(x) = \dfrac{x}{\sqrt{1-4x^2}}$, and the indefinite integral, $g(x) = -\dfrac{1}{4}\sqrt{1-4x^2}$ (let $C = 0$).

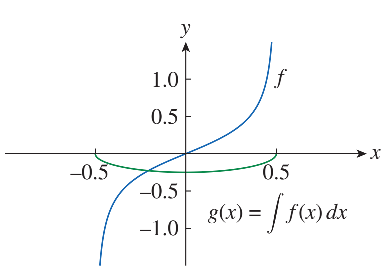


Figure 5.49
Graphs of the integrand and the indefinite integral.

Notice that $g(x)$ decreases when $f(x)$ is negative, increases when $f(x)$ is positive, and has its minimum value when $f(x) = 0$. So the graphical evidence suggests that g is an antiderivative of f. ■

Example 4 Substitution and an Exponential Function

Find $\int e^{5x}\,dx$.

Solution

Let $u = 5x \quad \Rightarrow \quad du = 5\,dx \quad \Rightarrow \quad dx = \dfrac{du}{5}$.

$$\int e^{5x}\,dx = \int e^u \cdot \frac{du}{5} = \frac{1}{5}\int e^u\,du \qquad \text{Change variables; simplify.}$$

$$= \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C \qquad \text{Table of Indefinite Integrals; final answer in terms of } x.$$

■

Example 5 Substitution and a Trigonometric Function

Find $\int \tan x\,dx$.

Solution

There doesn't seem to be any possible substitution here.

However, we can write tangent in terms of sine and cosine.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

This suggests the substitution $u = \cos x \quad \Rightarrow \quad du = -\sin x\,dx \quad \Rightarrow \quad dx = -\dfrac{du}{\sin x}$.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{\sin x}{u}\left(-\frac{du}{\sin x}\right) \qquad \text{Change variables.}$$

$$= -\int \frac{1}{u}\,du = -\ln|u| + C \qquad \text{Simplify; Table of Indefinite Integrals.}$$

$$= -\ln|\cos x| + C \qquad \text{Final answer in terms of } x.$$

■

Example 5 presents an important result that can be added to our Table of Indefinite Integrals.

$$\begin{aligned}\int \tan x\,dx &= -\ln|\cos x| + C \\ &= \ln(|\cos x|^{-1}) = \ln\left(\frac{1}{|\cos x|}\right) \\ &= \ln|\sec x| + C \qquad (3)\end{aligned}$$

Definite Integrals

There are two possible methods for evaluating a *definite* integral when the method of substitution is used. One method is to evaluate the *indefinite* integral first and then use the Evaluation Theorem; that is, temporarily ignore the limits of integration, evaluate the indefinite integral in terms of the original variable, and then use the Evaluation Theorem to find the final answer.

Here's how this method works using the result from Example 2.

$$\int_0^4 \sqrt{2x+1}\,dx = \int \sqrt{2x+1}\,dx\Big]_0^4$$ Find the indefinite integral first.

$$= \frac{1}{3}(2x+1)^{3/2}\Big]_0^4$$ Change variables; integrate; return to the variable x.

$$= \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2} = \frac{1}{3}(27-1) = \frac{26}{3}$$ Use the Evaluation Theorem in terms of x.

Another method, which is usually preferable, is to also transform the limits of integration; write, and use, the appropriate bounds in terms of the new variable.

This rule says that when using the method of substitution on a definite integral, we need to transform *everything* in terms of the new variable u: not only x and dx but also the limits of integration. The new limits of integration are the values of u that correspond to $x = a$ and $x = b$.

The Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof

Let F be an antiderivative of f.

Then $F(g(x))$ is an antiderivative of $f(g(x))\,g'(x)$.

Use the Evaluation Theorem.

$$\int_a^b f(g(x))\,g'(x)\,dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

Apply the Evaluation Theorem a second time to obtain the same result.

$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$ ■

Example 6 Substitution in a Definite Integral

Evaluate $\displaystyle\int_0^4 \sqrt{2x+1}\,dx$.

Solution

Use the substitution from Solution 1 of Example 2.
Here is the information we need to change variables from x to u.

$$u = 2x + 1 \qquad x = 0 : u = 2(0) + 1 = 1$$

$$du = 2dx \qquad x = 4 : u = 2(4) + 1 = 9$$

$$dx = \frac{du}{2}$$

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \sqrt{u} \cdot \frac{du}{2}$$ Change variables.

$$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big]_1^9$$ Table of Indefinite Integrals.

$$= \frac{1}{3}(9^{3/2} - 1^{3/2}) = \frac{26}{3}$$ FTC2; simplify.

The geometric interpretation of this result is shown in Figure 5.50. The substitution $u = 2x + 1$ stretches the interval [0, 4] by a factor of 2 and translates it to the right by 1 unit. The Substitution Rule shows that the two areas are equal.

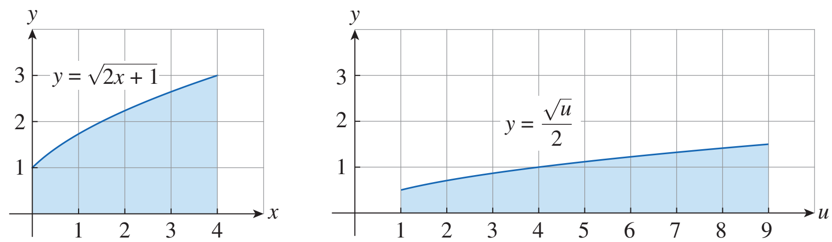


Figure 5.50
A geometric interpretation of the substitution rule for definite integrals: the areas of the shaded blue regions are the same.

Note that when using the Substitution Rule for Definite Integrals, we do *not* return to the original variable x after integrating. We simply evaluate the expression in u using the appropriate values of u. ■

Example 7 Substitution in a Definite Integral

Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$.

Solution

The inner function in the denominator is a good choice for u.

$u = 3 - 5x \qquad x = 1: u = 3 - 5 \cdot 1 = -2$

$du = -5dx \qquad x = 2: u = 3 - 5 \cdot 2 = -7$

$dx = -\frac{1}{5}du$

$$\int_1^2 \frac{dx}{(3-5x)^2} = \int_{-2}^{-7} \frac{1}{u^2}\left(-\frac{1}{5}du\right) = -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2}$$ Change variables; notice that the bounds become *reversed* in terms of u.

$$= -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = \frac{1}{5u}\Big]_{-2}^{-7}$$ Table of Indefinite Integrals; simplify.

$$= \frac{1}{5}\left[-\frac{1}{7} + \frac{1}{2}\right] = \frac{1}{14}$$ FTC2. ■

Example 8 Substitution in a Definite Integral

Find $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.

Solution

The choice for u isn't clear here because there doesn't appear to be an inner function. However, both $\ln x$ and its derivative occur in the integrand.

$$u = \ln x \qquad x = 1 : u = \ln 1 = 0$$

$$du = \frac{1}{x}\,dx \qquad x = e : u = \ln e = 1$$

$$dx = x\,du$$

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 \frac{u}{x}\cdot x\,du = \int_0^1 u\,du \qquad \text{Change variables.}$$

$$= \frac{u^2}{2}\bigg]_0^1 \qquad \text{Table of Indefinite Integrals.}$$

$$= \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2} \qquad \text{FTC2; simplify.}$$

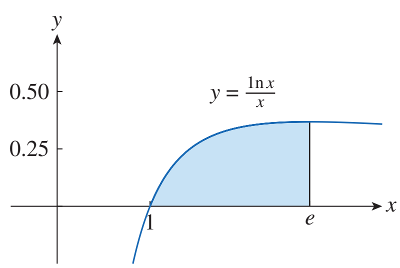


Figure 5.51
The integral in Example 8 represents the area of the shaded region.

Since the function $f(x) = \dfrac{\ln x}{x}$ is positive for $x > 1$, the definite integral represents the area of the shaded region in Figure 5.51. ■

Symmetry

We can now consider a theorem that uses the Substitution Rule for Definite Integrals to simplify the calculation of integrals of functions which exhibit certain symmetry properties.

Integrals of Symmetric Functions

Suppose f is continuous on $[-a, a]$.

(a) If f is even, $f(-x) = f(x)$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd, $f(-x) = -f(x)$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Proof

Split the integral into two parts.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx \tag{4}$$

Use the substitution $u = -x$ in the first integral of the last sum on the right side of Equation 4.

$$u = -x \qquad x = 0 : u = -(0) = 0$$

$$du = -dx \qquad x = -a : u = -(-a) = a$$

$$dx = -du$$

$$-\int_0^{-a} f(x)\,dx = -\int_0^a f(-u)(-du) = \int_0^a f(-u)\,du$$

Use this result to rewrite Equation 4.

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx \tag{5}$$

(a) If f is even, then $f(-u) = f(u)$. Equation 5 becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

(b) If f is odd, then $f(-u) = -f(u)$. Equation 5 becomes

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0.$$ ■

Figures 5.52 and 5.53 illustrate the results in the theorem for Integrals of Symmetric Functions in terms of area under a curve.

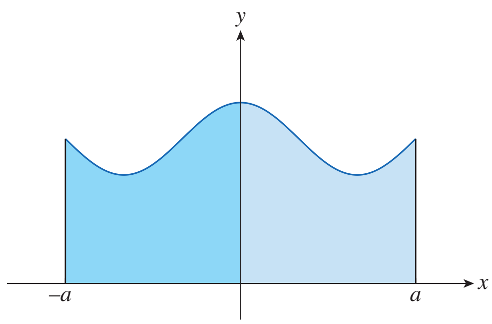


Figure 5.52
If f is an even function, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

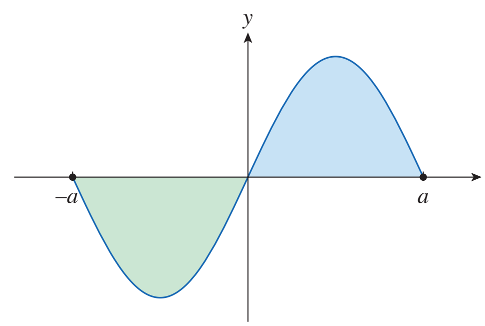


Figure 5.53
If f is an odd function, $\int_{-a}^{a} f(x)\,dx = 0$.

For the case where f is positive and even, part (a) says that the area under $y = f(x)$ from $-a$ to a is twice the area from 0 to a because of symmetry. Remember that an integral $\int_{a}^{b} f(x)\,dx$ can represent the area above the x-axis and below $y = f(x)$ minus the area below the x-axis and above the curve. Therefore, part (b) says the integral is 0 because the areas cancel.

Example 9 Integrals Involving Symmetry

Evaluate each definite integral.

(a) $\int_{-2}^{2} (x^6 + 1)\,dx$

(b) $\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx$

Solution

(a) Let $f(x) = x^6 + 1$.

$$f(-x) = (-x)^6 + 1 = x^6 + 1 = f(x)$$

Therefore, f is an even function.

$$\begin{aligned}
\int_{-2}^{2} (x^6 + 1)\,dx &= 2\int_{0}^{2} (x^6 + 1)\,dx && \text{Integral of a symmetric (even) function.} \\
&= 2\left[\frac{1}{7}x^7 + x\right]_0^2 && \text{Integrate term by term; Table of Indefinite Integrals.} \\
&= 2\left[\left(\frac{128}{7} + 2\right) - \left(\frac{0^7}{7} + 0\right)\right] = \frac{284}{7} && \text{FTC2; simplify.}
\end{aligned}$$

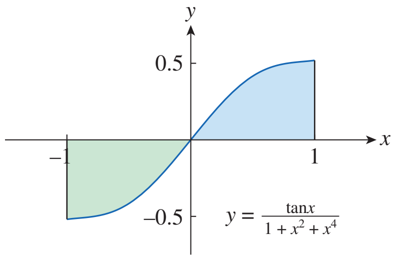


Figure 5.54
The function f is odd,
$\int_{-1}^{1} f(x)\,dx = 0$.

(b) Let $f(x) = \dfrac{\tan x}{1 + x^2 + x^4}$.

$$f(-x) = \frac{\tan(-x)}{1 + (-x)^2 + (-x)^4} = \frac{-\tan x}{1 + x^2 + x^4} = -f(x)$$

Therefore, f is an odd function.

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0$$

Figure 5.54 illustrates this result. ■

5.5 Exercises

Evaluate the indefinite integral by using the indicated substitution.

1. $\int \cos 2x\,dx, \quad u = 2x$

2. $\int x^3(2 + x^4)^5\,dx, \quad u = 2 + x^4$

3. $\int x\,e^{-x^2}\,dx, \quad u = -x^2$

4. $\int x^2\sqrt{x^3 + 1}\,dx, \quad u = x^3 + 1$

5. $\int \dfrac{dt}{(1 - 6t)^4}, \quad u = 1 - 6t$

6. $\int \sin^2\theta \cos\theta\,d\theta, \quad u = \sin\theta$

7. $\int \dfrac{\sec^2(1/x)}{x^2}\,dx, \quad u = \dfrac{1}{x}$

8. $\int \dfrac{\cos\sqrt{t}}{\sqrt{t}}\,dt, \quad u = \sqrt{t}$

9. $\int \sqrt{z - 1}\,dz, \quad u = z - 1$

Evaluate the indefinite integral.

10. $\int x\sqrt{1 - x^2}\,dx$

11. $\int x^2 e^{x^3}\,dx$

12. $\int (1 - 2x)^9\,dx$

13. $\int \sin t\sqrt{1 + \cos t}\,dt$

14. $\int \cos\left(\dfrac{\pi t}{2}\right) dt$

15. $\int \sec^2 2\theta\,d\theta$

16. $\int \dfrac{dx}{5 - 3x}$

17. $\int y^2(4 - y^3)^{2/3}\,dy$

18. $\int \cos^3\theta \sin\theta\,d\theta$

19. $\int e^{-5r}\,dr$

20. $\int \dfrac{e^u}{(1 - e^u)^2}\,du$

21. $\int \dfrac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

22. $\int \dfrac{a + bx^2}{\sqrt{3ax + bx^3}}\,dx$

23. $\int \dfrac{z^2}{z^3 + 1}\,dz$

24. $\int \dfrac{(\ln x)^2}{x}\,dx$

25. $\int \sin x \sin(\cos x)\,dx$

26. $\int \sec^2\theta \tan^3\theta\,d\theta$

27. $\int x\sqrt{x + 2}\,dx$

28. $\int e^x\sqrt{1 + e^x}\,dx$

29. $\int \dfrac{dx}{ax + b}\ (a \neq 0)$

30. $\int \dfrac{\cos x}{\sin^2 x}\,dx$

31. $\int \dfrac{\tan^{-1} x}{1 + x^2}\,dx$

32. $\int (x^2 + 1)(x^3 + 3x)^4\,dx$

33. $\int e^{\cos t}\sin t\,dt$

34. $\int 5^t \sin(5^t)\,dt$

35. $\int \dfrac{\sec^2 x}{\tan x}\,dx$

36. $\int \dfrac{(\arctan x)^2}{x^2 + 1}\,dx$

37. $\int \dfrac{x}{x^2 + 4}\,dx$

38. $\int \cos(1 + 5t)\,dt$

39. $\int \dfrac{\cos\left(\dfrac{\pi}{x}\right)}{x^2}\,dx$

40. $\int \sqrt{\cot x}\csc^2 x\,dx$

41. $\int \dfrac{2^t}{2^t + 3}\,dt$

42. $\int \sqrt{\sin x}\cos x\,dx$

43. $\int \dfrac{dt}{\cos^2 t\sqrt{1 + \tan t}}$

44. $\int \dfrac{\sin 2x}{1 + \cos^2 x}\,dx$

45. $\int \dfrac{\sin x}{1 + \cos^2 x}\,dx$

46. $\int \cot x\,dx$

47. $\int \dfrac{\cos(\ln t)}{t}\,dt$

48. $\int \dfrac{dx}{\sqrt{1 - x^2}\sin^{-1} x}$

49. $\int \dfrac{x}{1 + x^4}\,dx$

50. $\int \dfrac{1 + x}{1 + x^2}\,dx$

51. $\int x^2\sqrt{2 + x}\,dx$

52. $\int x(2x + 5)^8\,dx$

53. $\int x^3\sqrt{x^2 + 1}\,dx$

Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the integrand and its antiderivative on the same coordinate axes (use $C = 0$).

54. $\int x(x^2 - 1)^3\,dx$

55. $\int \tan^2\theta \sec^2\theta\,d\theta$

56. $\int e^{\cos x}\sin x\,dx$

57. $\int \sin x \cos^4 x\,dx$

Evaluate the definite integral.

58. $\int_0^1 \cos\left(\frac{\pi t}{2}\right)dt$

59. $\int_0^1 (3t - 1)^{50}\,dt$

60. $\int_0^1 \sqrt[3]{1 + 7x}\,dx$

61. $\int_0^3 \frac{dx}{5x + 1}$

62. $\int_0^1 x^2(1 + 2x^3)^5\,dx$

63. $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$

64. $\int_0^{\pi/6} \frac{\sin t}{\cos^2 t}\,dt$

65. $\int_{\pi/3}^{2\pi/3} \csc^2\left(\frac{t}{2}\right)dt$

66. $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

67. $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

68. $\int_{-\pi/4}^{\pi/4} (x^3 + x^4\tan x)\,dx$

69. $\int_0^{\pi/2} \cos x \sin(\sin x)\,dx$

70. $\int_0^{13} \frac{dx}{\sqrt[3]{(1 + 2x)^2}}$

71. $\int_0^a x\sqrt{a^2 - x^2}\,dx$

72. $\int_0^a x\sqrt{x^2 + a^2}\,dx \quad (a > 0)$

73. $\int_{-\pi/3}^{\pi/3} x^4 \sin x\,dx$

74. $\int_1^2 x\sqrt{x - 1}\,dx$

75. $\int_0^4 \frac{x}{\sqrt{1 + 2x}}\,dx$

76. $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

77. $\int_0^2 (x - 1)e^{(x-1)^2}\,dx$

78. $\int_0^1 \frac{e^z + 1}{e^z + z}\,dz$

79. $\int_0^{T/2} \sin\left(\frac{2\pi t}{T} - \alpha\right)dt$

80. $\int_0^1 \frac{dx}{(1 + \sqrt{x})^4}$

81. Verify that $f(x) = \sin\sqrt[3]{x}$ is an odd function and use that fact to show that

$$0 \le \int_{-2}^{3} \sin\sqrt[3]{x}\,dx \le 1$$

Use a graph to provide a rough estimate of the area of the region that lies under the curve. Then find the exact area.

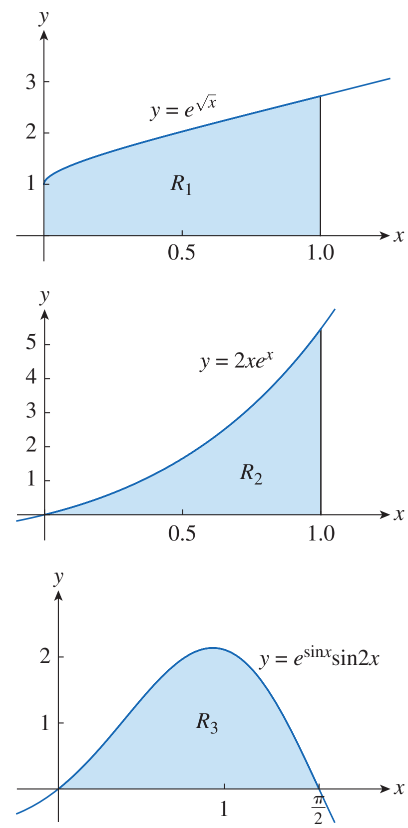
82. $y = \sqrt{2x + 1}, \quad 0 \le x \le 1$

83. $y = 2\sin x - \sin 2x, \quad 0 \le x \le \pi$

84. Evaluate $\int_{-2}^{2} (x + 3)\sqrt{4 - x^2}\,dx$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

85. Evaluate $\int_0^1 x\sqrt{1 - x^4}\,dx$ by making a substitution and interpreting the resulting integral in terms of area.

86. Which of the following shaded regions have equal area? Why?



87. A model for the basal metabolism rate, in kcal/h, of a young man is $R(t) = 85 - 0.18\cos\left(\frac{\pi t}{12}\right)$, where t is the time in hours measured from 5:00 AM. What is the total basal metabolism of this man, $\int_0^{24} R(t)\,dt$, over a 24-hour period?

88. Suppose an oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the first hour?

89. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How many bacteria will there be after 3 hours?

90. Breathing is cyclic, and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about five seconds. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin\left(\frac{2\pi t}{5}\right)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs from time $t = 0$ to time $t = T$.

91. The rate of growth of a fish population was modeled by the equation

$$G(t) = \frac{60{,}000\, e^{-0.6t}}{(1 + 5e^{-0.6t})^2}$$

where t is measured in years since 2000 and G in kilograms per year. If the biomass was 25,000 kg in year 2000, what is the predicted biomass for the year 2030?

92. Dialysis treatment removes urea and other waste products from a patient's blood by diverting some of the blood flow externally through a machine called a dialyzer. The rate at which urea is removed from the blood (in mg/min) is often well described by the equation

$$u(t) = \frac{r}{v} C_0 e^{-rt/V}$$

where r is the rate of flow of blood through the dialyzer (in mL/min), V is the volume of the patient's blood (in mg), and C_0 is the amount of urea in the blood (in mg) at time $t = 0$, and t is measured in minutes. Evaluate the integral $\int_0^{30} u(t)\, dt$ and interpret your answer in the context of this problem.

93. Alabama Instruments Company has set up a production line to manufacture a new calculator. The rate of production of these calculators after t weeks is

$$\frac{dx}{dt} = 5000\left(1 - \frac{100}{(t + 10)^2}\right) \quad \text{calculators/week}$$

(Notice that the production approaches 5000 per week as time goes on, but the initial production is lower because of the workers' unfamiliarity with the new assembly techniques.) Find the number of calculators produced from the beginning of the third week to the end of the fourth week.

94. Herbst Glacier is located in Glacier National Park in Montana and has been steadily losing area since the mid-19th century. Suppose the rate at which the glacier is losing area can be modeled by the function $A(t) = -16647\, e^{-0.0485t}$, where t is measure in years since 1966 and A is measure in m^2/year. Find the total glacier area lost between the years 2005 and 2015.

95. If f is continuous and $\int_0^4 f(x)\, dx = 10$, find $\int_0^2 f(2x)\, dx$.

96. If f is continuous and $\int_0^9 f(x)\, dx = 4$, find $\int_0^3 x f(x^2)\, dx$.

---

97. If f is continuous for all real numbers, show that

$$\int_a^b f(-x)\, dx = \int_{-b}^{-a} f(x)\, dx$$

For the case where $f(x) \geq 0$ and $0 < a < b$, draw a diagram to interpret this equation geometrically as an equality of areas.

98. If f is continuous for all real numbers, show that

$$\int_a^b f(x + c)\, dx = \int_{a+c}^{b+c} f(x)\, dx$$

For the case where $f(x) \geq 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

99. If a and b are positive numbers, show that

$$\int_0^1 x^a(1 - x)^b\, dx = \int_0^1 x^b(1 - x)^a\, dx$$

100. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi x f(\sin x)\, dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\, dx$$

101. Use Exercise 100 to evaluate the integral

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\, dx$$

102. (a) If f is continuous, show that

$$\int_0^{\pi/2} f(\cos x)\, dx = \int_0^{\pi/2} f(\sin x)\, dx$$

(b) Use part (a) to evaluate $\int_0^{\pi/2} \cos^2 x\, dx$ and $\int_0^{\pi/2} \sin^2 x\, dx$.

5.6 Integration by Parts

Every differentiation rule has a corresponding integration rule. For example, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called *integration by parts*.

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Using the notation for indefinite integrals, this equation can be written as

$$\int [f(x)g'(x) + g(x)f'(x)]dx = f(x)\,g(x)$$

or

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)\,g(x)$$

We can rearrange this equation as

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \qquad (1)$$

This expression is called the **formula for integration by parts**. It is probably easier to remember, and use, with the following notation.

Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the formula for integration by parts can be written as

Read this as "the integral of $u\,dv$ is equal to u (times) v minus the integral of $v\,du$."

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

A Closer Look

1. The idea, and the reason for using integration by parts, is to express a complicated integral in terms of another, hopefully simpler, integral. The choices for u and dv are important, but often the process of trial and error is necessary in order to appropriately identify each expression.
2. Choose dv so that it can be integrated easily.
3. In general, choose u so that $\int v\,du$ is simpler than $\int u\,dv$. ■

Here is an example that illustrates the notation in both Equations 1 and 2.

Example 1 Integration by Parts, Notational Fluency

Find $\int x \sin x\,dx$.

Solution

This is not a formula in the Table of Indefinite Integrals.

We will learn through practice and pattern recognition that this integral is a good candidate for integration by parts. If we let $f(x) = x$, then the integral on the right of Equation 1 will indeed be simpler, with x raised to a smaller power.

Let $f(x) = x$ and $g'(x) = \sin x$. Then

$$f'(x) = 1 \quad \text{and} \quad \int g'(x)\,dx = \int \sin x\,dx = -\cos x$$

Remember, we can choose *any* antiderivative for g; it's easiest to let $C = 0$.

Use the formula for integration by parts.

$$\int x \sin x\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$ Integration by parts, Equation 1.

$$= x(-\cos x) - \int (-\cos x)(1)\,dx$$ Use expressions for $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$.

$$= -x\cos x + \int \cos x\,dx$$ Simplify the remaining integrand.

$$= -x\cos x + \sin x + C$$ Antiderivative of $\cos x$.

Remember that you can always check the answer by differentiation.

Here is the solution using the notation in Equation 2.

It is helpful to use this pattern:

$u = \square \qquad dv = \square$

$du = \square \qquad v = \square$

Choose u and dv, and find du and v.

$u = x \qquad dv = \sin x\,dx$

$du = dx \qquad v = \int \sin x\,dx = -\cos x$

$$\int x\sin x\,dx = \int \underbrace{x}_{u}\ \underbrace{\sin x\,dx}_{dv}$$ Identify u and dv.

$$= \underbrace{x}_{u}\ \underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v}\ \underbrace{dx}_{du}$$ Integration by parts, Equation 2.

$$= -x\cos x + \int \cos x\,dx$$ Simplify the remaining integrand.

$$= -x\cos x + \sin x + C$$ Antiderivative of $\cos x$.

■

The goal in using integration by parts is to *reduce* the given integral to obtain a simpler integral than the original. In Example 1, we started with $\int x \sin x\,dx$ and were able to express this in terms of the simpler integral $\int \cos x\,dx$.

If we had instead chosen $u = \sin x$ and $dv = x\,dx$, then $du = \cos x\,dx$ and $v = \dfrac{x^2}{2}$.

Using integration by parts, we would obtain

$$\int x \sin x\,dx = (\sin x)\frac{x^2}{2} - \frac{1}{2}\int x^2 \cos x\,dx.$$

Although this expression is valid, the integral on the right, $\int x^2 \cos x\,dx$, is a more complicated integral than the original; it involves x raised to a higher power.

In general, when choosing u and dv, try to select $u = f(x)$ such that the derivative is simpler, or at least no more complicated, than $f(x)$ and $dv = g'(x)\,dx$ so that we can readily integrate to find v.

Example 2 Integration by Parts; One Choice

Evaluate $\int \ln x\, dx$.

Solution

The integrand is indeed a product of functions:

$$\int \ln x\, dx = \int \ln x \cdot 1\, dx$$

The only other choice is $u = 1$ and $dv = \ln x\, dx$. Finding v in this case is the original integral.

We don't have much choice for u and dv.

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x}\, dx \qquad v = \int dx = x$$

Integrate by parts.

Remember that we usually write $\int 1\, dx = \int dx$ and that you can always check your final answer by differentiation.

$$\int \ln x\, dx = x \ln x - \int x \frac{1}{x}\, dx \qquad \text{Integration by parts, Equation 2.}$$
$$= x \ln x - \int dx \qquad \text{Simplify the integrand.}$$
$$= x \ln x - x + C \qquad \text{Antiderivative of 1.}$$

du is really the *differential*. However, in the context of integration by parts, we will often say *the derivative du*.

Note that the choice for u is consistent with our integration by parts strategy; the derivative, du, is *simpler* than u. ■

Example 3 Integration by Parts Twice

Find $\int t^2 e^t\, dt$.

Solution

A good choice is $u = t^2$ because the derivative, du, is simpler. If $u = e^t$, then $du = e^t\, dt$ remains the same; it isn't any simpler.

$$u = t^2 \qquad dv = e^t\, dt$$
$$du = 2t\, dt \qquad v = \int e^t\, dt = e^t$$

Integrate by parts.

$$\int t^2 e^t\, dt = t^2 e^t - 2\int t e^t\, dt \qquad \text{Integration by parts, Equation 2.}$$

The remaining integral, $\int te^t\, dt$, is simpler but still not in the Table of Indefinite Integrals. Therefore, we need to use integration by parts a second time.

In the remaining integral,

$$u = t \qquad dv = e^t\, dt$$
$$du = dt \qquad v = \int e^t\, dt = e^t$$

Use integration by parts on the remaining integral.

$$\int t^2 e^t dt = t^2 e^t - 2\left(te^t - \int e^t dt\right) \qquad \text{Integration by parts.}$$
$$= t^2 e^t - 2(te^t - e^t + C) \qquad \text{Antiderivative of } e^t.$$
$$= t^2 e^t - 2te^t + 2e^t + C_1 \qquad \text{Distribute; } C_1 = -2C.$$

■

It is precise to write $C_1 = -2C$. However, in this context, we often simply use the same arbitrary constant, C.

Example 4 Integration by Parts, Integrand Reappears

Evaluate $\int e^x \sin x\, dx$.

Solution

Neither e^x nor $\sin x$ becomes simpler when differentiated.

However, let's try the following.

$$u = e^x \qquad dv = \sin x\, dx$$
$$du = e^x dx \qquad v = \int \sin x\, dx = -\cos x$$

Integrate by parts.

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx \qquad \text{Integration by parts; simplify.}$$

The remaining integral is no simpler, but at least it's no more difficult. Because using integration by parts twice worked in Example 3, it seems reasonable to try integration by parts again.

Make *consistent* choices for u and dv.

$$u = e^x \qquad dv = \cos x\, dx$$
$$du = e^x dx \qquad v = \int \cos x\, dx = \sin x$$

Use integration by parts on the remaining integral.

$$\int e^x \sin x\, dx = -e^x \cos x + \left(e^x \sin x - \int e^x \sin x\, dx\right) \qquad \text{Integration by parts.}$$

It doesn't seem as though we have accomplished much. However, the remaining integral is exactly the same as the original. Think of the original integral as an unknown quantity in an algebraic equation and solve for this unknown quantity.

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx \qquad \text{Equation after two applications of integration by parts.}$$

$$2\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x \qquad \text{Add } \int e^x \sin x\, dx \text{ to both sides.}$$

$$\int e^x \sin x\, dx = \frac{1}{2} e^x(\sin x - \cos x) + C \qquad \text{Divide both sides by 2; add the constant of integration.}$$

Figure 5.55 provides some graphical confirmation of this solution.

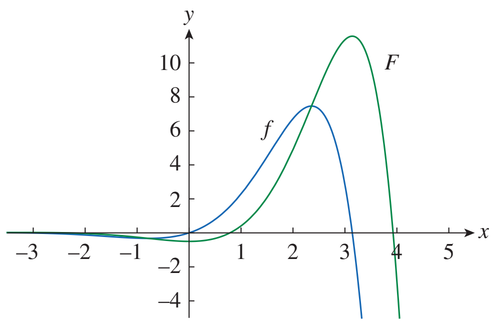


Figure 5.55
The graph of $f(x) = e^x \sin x$ and an antiderivative

$F(x) = \frac{1}{2} e^x(\sin x - \cos x)$. Notice that $f(x) = 0$ where F attains a relative extreme value.

We can combine the formula for integration by parts with Part 2 of the Fundamental Theorem of Calculus to evaluate definite integrals by parts. Assume that f' and g' are continuous, evaluate Equation 1 between a and b, and use the Fundamental Theorem.

$$\int_a^b f(x)g'(x)\, dx = \left[f(x)g(x)\right]_a^b - \int_a^b g(x) f'(x)\, dx \qquad (3)$$

Example 5 Integration by Parts with a Definite Integral

Evaluate $\displaystyle\int_0^1 \tan^{-1} x\,dx$.

Solution

The only real choice here is $u = \tan^{-1} x$.

$$u = \tan^{-1} x \qquad dv = dx$$

$$du = \frac{1}{1+x^2}\,dx \qquad v = \int dx = x$$

Even though du is simpler, a rational function versus a transcendental function, v, is more complicated. Let's see what happens.

$$\int_0^1 \tan^{-1} x\,dx = \Big[x\tan^{-1} x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx \qquad \text{Integration by parts.}$$

$$= (1\cdot\tan^{-1} 1 - 0\cdot\tan^{-1} 0) - \int_0^1 \frac{x}{1+x^2}\,dx \qquad \text{Evaluate the first expression.}$$

$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx \qquad \text{Simplify.}$$

Use substitution to evaluate the remaining integral.

$$t = 1 + x^2 \qquad x = 0:\ t = 1 + 0^2 = 1$$

$$dt = 2x\,dx \qquad x = 1:\ t = 1 + 1^2 = 2$$

$$dx = \frac{dt}{2x}$$

$$\int_0^1 \frac{x}{1+x^2}\,dx = \int_1^2 \frac{x}{t}\,\frac{dt}{2x} \qquad \text{Change variables.}$$

$$= \frac{1}{2}\int_1^2 \frac{dt}{t} = \left[\frac{1}{2}\ln|t|\right]_1^2 \qquad \text{Simplify; Table of Indefinite Integrals.}$$

$$= \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2 \qquad \text{FTC2; simplify.}$$

Use this result to find the final answer.

$$\int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx = \frac{\pi}{4} - \frac{\ln 2}{2}$$

Notice that since $\tan^{-1} x \ge 0$ for $x \ge 0$, the original integral in this example can be interpreted as the area under the graph of $y = \tan^{-1} x$ between 0 and 1, as illustrated in Figure 5.56. ■

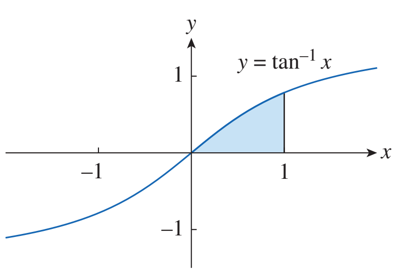


Figure 5.56
The definite integral $\int_0^1 \tan^{-1} x\,dx$ can be interpreted as the area under the graph of $y = \tan^{-1} x$ between $x = 0$ and $x = 1$.

Reduction Formulas

The integration by parts formula can often be used to repeatedly reduce the complexity of certain integrals. Consider the next example.

Example 6 Reduction

Prove the reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \quad \text{where } n \ge 2 \text{ is an integer.}$$

This equation is called a *reduction formula* because the exponent, n, in the original integral has been *reduced* to $n - 2$ in the remaining integral.

Solution

The choice for u here is a little different.

Write the integrand as $\sin^n x = \sin^{n-1} x \cdot \sin x$.

$$u = \sin^{n-1} x \qquad dv = \sin x\, dx$$

$$du = (n-1)\sin^{n-2} x \cos x\, dx \qquad v = \int \sin x\, dx = -\cos x$$

Integrate by parts.

$$\int \sin^n x\, dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx \qquad \text{Integration by parts.}$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x (1 - \sin^2 x)\, dx \qquad \text{Trigonometric identity.}$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

Property of integrals.

$$n\int \sin^n x\, dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\, dx \qquad \text{Solve for the original integral.}$$

$$\int \sin^n x\, dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\, dx \qquad \text{Divide both sides by } n.$$

Reduction formulas are useful because repeated application eventually results in a remaining, straightforward integral and therefore an antiderivative for the original integral.

If we use the reduction formula in this example repeatedly, then eventually the remaining integral will be either $\int \sin x\, dx$ (if n is odd) or $\int (\sin x)^0\, dx$ (if n is even). ■

5.6 Exercises

Evaluate the integral using integration by parts with the indicated choices for u and dv.

1. $\int xe^{2x}\, dx$; $\quad u = x, \quad dv = e^{2x}\, dx$
2. $\int \sqrt{x} \ln x\, dx$; $\quad u = \ln x, \quad dv = \sqrt{x}\, dx$

Evaluate the integral.

3. $\int x \cos 5x\, dx$
4. $\int ye^{0.2y}\, dy$
5. $\int te^{-3t}\, dt$
6. $\int (x-1)\sin \pi x\, dx$
7. $\int (x^2 + 2x)\cos x\, dx$
8. $\int t^2 \sin \beta t\, dt$
9. $\int \cos^{-1} x\, dx$
10. $\int \ln \sqrt{x}\, dx$
11. $\int t^4 \ln t\, dt$
12. $\int \tan^{-1} 2y\, dy$
13. $\int t \csc^2 t\, dt$
14. $\int (\ln x)^2\, dx$
15. $\int \frac{z}{10^z}\, dz$
16. $\int e^{2\theta} \sin 3\theta\, d\theta$
17. $\int e^{-\theta} \cos 2\theta\, d\theta$
18. $\int z^3 e^z\, dz$
19. $\int x \tan^2 x\, dx$
20. $\int \frac{xe^{2x}}{(1+2x)^2}\, dx$
21. $\int (\arcsin x)^2\, dx$
22. $\int_0^{1/2} x \cos \pi x\, dx$
23. $\int_0^1 (x^2 + 1)e^{-x}\, dx$
24. $\int_1^2 w^2 \ln w\, dw$

25. $\displaystyle\int_1^5 \frac{\ln R}{R^2}\,dR$

26. $\displaystyle\int_0^{2\pi} t^2 \sin 2t\,dt$

27. $\displaystyle\int_0^{\pi} x \sin x \cos x\,dx$

28. $\displaystyle\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx$

29. $\displaystyle\int_1^5 \frac{M}{e^M}\,dM$

30. $\displaystyle\int_1^2 \frac{(\ln x)^2}{x^3}\,dx$

31. $\displaystyle\int_0^{\pi/3} \sin x \ln(\cos x)\,dx$

32. $\displaystyle\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr$

33. $\displaystyle\int_1^2 x^4(\ln x)^2\,dx$

34. $\displaystyle\int_0^t e^s \sin(t-s)\,ds$

35. $\displaystyle\int_0^1 x^2 e^x\,dx$

36. $\displaystyle\int_{\pi/2}^{\pi} x \sin x\,dx$

37. $\displaystyle\int_0^t \frac{x}{e^x}\,dx, \quad t \neq 0$

38. $\displaystyle\int_1^2 x \ln x\,dx$

39. The functions f and g are each twice differentiable for all real numbers. The function h has the property $h'(x) = g(x) \cdot f'(x)$. The table gives values of $f(x)$, $g(x)$, and $h(x)$ for selected values of x.

x	1	3
$f(x)$	2	5
$g(x)$	4	6
$h(x)$	7	8

Find the value of $\displaystyle\int_1^3 f(x) \cdot g'(x)\,dx$.

40. The functions f and g are each twice differentiable for all real numbers. The table gives the values of $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ for selected values of x.

x	1	2
$f(x)$	3	2
$f'(x)$	-2	1
$g(x)$	-1	3
$g'(x)$	1	-3

If $\displaystyle\int_1^2 f'(x)g(x)\,dx = 5$, then find the value of $\displaystyle\int_1^2 f(x)g'(x)\,dx$.

First make a substitution (change variables) and then use integration by parts to evaluate each integral.

41. $\displaystyle\int e^{\sqrt{x}}\,dx$

42. $\displaystyle\int \cos(\ln x)\,dx$

43. $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\,d\theta$

44. $\displaystyle\int_0^{\pi} e^{\cos t} \sin 2t\,dt$

45. $\displaystyle\int x \ln(1+x)\,dx$

46. $\displaystyle\int \frac{\arcsin(\ln x)}{x}\,dx$

Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$) on the same coordinate axes.

47. $\displaystyle\int xe^{-2x}\,dx$

48. $\displaystyle\int x^{3/2} \ln x\,dx$

49. $\displaystyle\int x^3 \sqrt{1+x^2}\,dx$

50. $\displaystyle\int x^2 \sin 2x\,dx$

51. (a) Use the reduction formula in Example 6 to show that

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) Use part (a) and the reduction formula to evaluate $\displaystyle\int \sin^4 x\,dx$.

52. (a) Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

(b) Use part (a) to evaluate $\displaystyle\int \cos^2 x\,dx$.

(c) Use parts (a) and (b) to evaluate $\displaystyle\int \cos^4 x\,dx$.

53. (a) Use the reduction formula in Example 6 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

where $n \geq 2$ is an integer.

(b) Use part (a) to evaluate $\displaystyle\int_0^{\pi/2} \sin^3 x\,dx$ and $\displaystyle\int_0^{\pi/2} \sin^5 x\,dx$.

(c) Use part (a) to show that, for odd powers of sine,

$$\int_0^{\pi/2} \sin^{2n+1}x\,dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}$$

54. Show that, for even powers of sine,

$$\int_0^{\pi/2} \sin^{2n}x\,dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} \cdot \frac{\pi}{2}$$

Use integration by parts to prove the reduction formula.

55. $\displaystyle\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

56. $\displaystyle\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

57. $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx \quad (n \neq 1)$

58. $\displaystyle\int \sec^n x\,dx = \frac{\tan x \sec^{n-2}x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$

59. Use Exercise 55 to find $\int (\ln x)^3\, dx$.

60. Use Exercise 56 to find $\int x^4 e^x\, dx$.

61. A particle moves along a horizontal line so that its velocity at time t is given by $v(t) = t^2 e^{-t}$, where t is measured in seconds and v is in meters per second. Find the total distance traveled by the particle during the first t seconds.

62. A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is m, the fuel is consumed at rate r, and the exhaust gases are ejected with constant velocity v_e (relative to the rocket). A model for the velocity of the rocket at time t is given by the equation

$$v(t) = -gt - v_e \ln \frac{m - rt}{m}$$

where g is the acceleration due to gravity and t is not too large. If $g = 9.8\text{ m/s}^2$, $m = 30{,}000$ kg, $r = 160$ kg/s, and $v_e = 3000$ m/s, find the height of the rocket 1 minute after liftoff.

63. Suppose $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and f'' is continuous. Find the value of $\int_1^4 x f''(x)\, dx$.

64. If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that
$\int_0^a f(x)g''(x)\, dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)dx.$

65. (a) Use integration by parts to show that

$$\int f(x)\, dx = x f(x) - \int x f'(x)\, dx$$

(b) If f and g are inverse functions and f' is continuous, show that

$$\int_a^b f(x)\, dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\, dy$$

Hint: Use part (a) and make the substitution $y = f(x)$.

(c) In the case in which f and g are positive functions and $b > a > 0$, draw a diagram to illustrate a geometric interpretation of part (b).

(d) Use part (b) to evaluate $\int_1^e \ln x\, dx$.

66. Let $I_n = \int_0^{\pi/2} \sin^n x\, dx$.

(a) Show that $I_{2n+2} \le I_{2n+1} \le I_{2n}$.

(b) Use Exercise 54 to show that

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}$$

(c) Use parts (a) and (b) to show that

$$\frac{2n+1}{2n+2} \le \frac{I_{2n+1}}{I_{2n}} \le 1$$

and deduce that $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.

(d) Use part (c) and Exercises 53 and 54 to show that

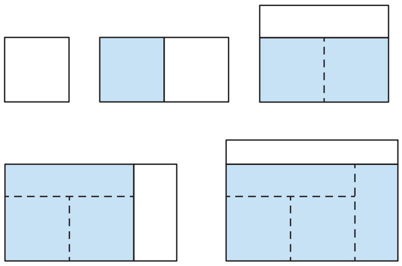
$$\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot \cdots \cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}$$

This formula is usually written as

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$$

and is called the *Wallis product*.

(e) Suppose rectangles are constructed in the following manner. Start with a square of area 1 and attach rectangles of area 1 alternately beside or on top of the previous rectangle as in the figure below.



Find the limit of the ratios of width to height of these rectangles.

5.7 Additional Techniques of Integration

We have learned the two basic techniques of integration, substitution and parts, in Sections 5.5 and 5.6. Here we briefly discuss other methods that are useful for particular classes of functions, for example, trigonometric functions and rational functions.

Trigonometric Integrals

We can use trigonometric identities to integrate certain combinations of trigonometric functions.

Example 1 Integrand an Odd Power of $\cos x$

Evaluate $\int \cos^3 x \, dx$.

Solution

A simple substitution of $u = \cos x$ isn't helpful because $du = \sin x \, dx$ and there is no $\sin x$ factor in the integrand.

To integrate powers of cosine, we need a $\sin x$ factor. Similarly, a power of sine would require a $\cos x$ factor.

In this example, we can factor $\cos^3 x$ as $(\cos^2 x)(\cos x)$ and rewrite the squared factor as an expression involving sine using a trigonometric identity.

$$\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x) \cos x$$

We can then use the substitution:

$$u = \sin x \quad \Rightarrow \quad du = \cos x \, dx \quad \Rightarrow \quad dx = \frac{du}{\cos x}$$

$$\int \cos^3 x \, dx = \int \cos^2 x \cdot \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$$ Factor $\cos^3 x$; use a trigonometric identity.

$$= \int (1 - u^2) \cos x \frac{du}{\cos x} = \int (1 - u^2) \, du$$ Change variables; simplify.

$$= u - \frac{1}{3} u^3 + C$$ Table of Indefinite Integrals.

$$= \sin x - \frac{1}{3} \sin^3 x + C$$ Final answer in terms of x.

In general, to integrate $\cos^n x$ where n is odd, factor $\cos^n x$ as $(\cos^{n-1} x)(\cos x)$ and rewrite the first term as an expression involving $\sin x$. Similarly, to integrate $\sin^m x$ where m is odd, factor $\sin^m x$ as $(\sin^{m-1} x)(\sin x)$ and rewrite the first term as an expression involving $\cos x$. The identity $\sin^2 x + \cos^2 x = 1$ enables us to convert back and forth between even powers of sine and cosine.

To integrate $\sin^m x \cos^n x$ where at least one power is odd, save one sine (or cosine) factor and use $\sin^2 x + \cos^2 x = 1$ to express the remaining factors in terms of cosine (or sine).

If the integrand contains only even powers of both sine and cosine, then this strategy fails. In this case, use the half-angle identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example 2 Integrand an Even Power of sin *x*

Evaluate $\int_0^{\pi} \sin^2 x\, dx$.

Solution

If we write $\sin^2 x = 1 - \cos^2 x$, the resulting integral is no simpler than the original.

Use the half-angle formula for $\sin^2 x$.

The integral $\int \cos nx\, dx$ occurs often in this type of problem. The solution involves a routine substitution, $u = nx$.

$$\int_0^{\pi} \sin^2 x\, dx = \frac{1}{2}\int_0^{\pi} (1 - \cos 2x)\, dx \qquad \text{Half-angle formula.}$$

$$= \left[\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^{\pi} \qquad \text{Table of Indefinite Integrals.}$$

$$= \frac{1}{2}\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right] = \frac{\pi}{2} \qquad \text{FTC2; simplify.}$$

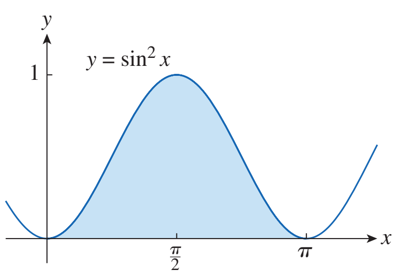


Figure 5.57
$\int_0^{\pi} \sin^2 x\, dx$ can be interpreted as the area under the graph of $y = \sin^2 x$ between 0 and π.

Since $\sin^2 x \geq 0$ for all x, the original integral in this example can be interpreted as the area under the graph of $y = \sin^2 x$ between $x = 0$ and $x = \pi$, as illustrated in Figure 5.57.

We can use a similar strategy to integrate powers of $\tan x$ and $\sec x$ using the identity $\sec^2 x = 1 + \tan^2 x$.

Trigonometric Substitution

There are many practical problems that involve the integral of an algebraic function that contains an expression of the form

$$\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2}, \qquad \text{or} \qquad \sqrt{x^2 - a^2}$$

where $a > 0$. Often, the best way to solve this problem is to use a trigonometric substitution that eliminates the root sign.

Example 3 Area of a Circle

Prove that the area of a circle with radius r is πr^2.

Solution

This is a well-known formula; someone probably told you that it's true a long time ago, but we can actually prove it using integration.

For simplicity, let's consider the circle with its center at the origin, so its equation is $x^2 + y^2 = r^2$.

Solve this equation for y: $y = \pm\sqrt{r^2 - x^2}$.

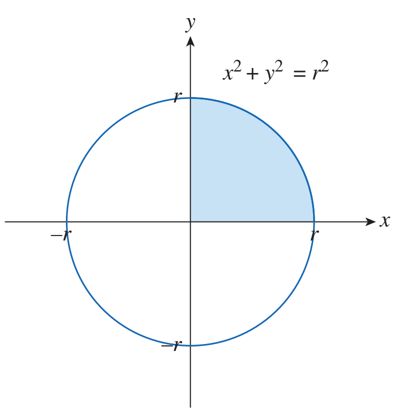


Figure 5.58
Total area A is four times the area of the shaded region.

Because the circle is symmetric with respect to both axes, the total area A is four times the area in the first quadrant, as illustrated in Figure 5.58.

The part of the circle in the first quadrant is given by the function

$y = \sqrt{r^2 - x^2}, \quad 0 \leq x \leq r$.

An expression involving the total area is $\frac{1}{4}A = \int_0^r \sqrt{r^2 - x^2}\, dx$.

To simplify the integral, we would like to make a substitution that transforms $r^2 - x^2$ into a square and, therefore, eliminates the root sign. Use the trigonometric identity $1 - \sin^2\theta = \cos^2\theta$.

If we use the substitution $x = r\sin\theta$, then

$$r^2 - x^2 = r^2 - r^2\sin^2\theta = r^2(1 - \sin^2\theta) = r^2\cos^2\theta.$$

This kind of substitution is called *inverse substitution*. Here the old variable x is a function of the new variable θ instead of the other way around. But the substitution $x = r\sin\theta$ is equivalent to saying that $\theta = \sin^{-1}(x/r)$.

Since $0 \le x \le r$, we restrict θ so that $0 \le \theta \le \dfrac{\pi}{2}$.

$x = r\sin\theta \quad \Rightarrow \quad dx = r\cos\theta\, d\theta$ and

$$\sqrt{r^2 - x^2} = \sqrt{r^2 - r^2\sin^2\theta} = \sqrt{r^2\cos^2\theta} = r\cos\theta \qquad \cos\theta \ge 0 \text{ when } 0 \le \theta \le \frac{\pi}{2}.$$

Use the Substitution Rule:

$$x = 0 = r\sin\theta \quad \Rightarrow \quad \sin\theta = 0 \quad \Rightarrow \quad \theta = 0$$

$$x = r = r\sin\theta \quad \Rightarrow \quad \sin\theta = 1 \quad \Rightarrow \quad \theta = \pi/2$$

$$\int_0^r \sqrt{r^2 - x^2}\, dx = \int_0^{\pi/2} (r\cos\theta) r\cos\theta\, d\theta = r^2 \int_0^{\pi/2} \cos^2\theta\, d\theta$$

This trigonometric integral is similar to the one in Example 2; use a half-angle formula.

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\begin{aligned}
\frac{1}{4}A &= r^2 \int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{1}{2} r^2 \int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta && \text{Half-angle formula.} \\
&= \frac{1}{2} r^2 \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} && \text{Table of Indefinite Integrals; substitution } u = 2\theta. \\
&= \frac{1}{2} r^2 \left[\left(\frac{\pi}{2} + 0\right) - (0)\right] && \text{FTC2.} \\
&= \frac{1}{4}\pi r^2 && \text{Simplify.}
\end{aligned}$$

Therefore, $A = \pi r^2$. ■

Example 3 suggests that if an integrand contains a factor of the form $\sqrt{a^2 - r^2}$, then a trigonometric substitution $x = a\sin\theta$ may be effective. However that does not mean that this kind of substitution is *always* the best method. For example, to evaluate $\int x\sqrt{a^2 - x^2}\, dx$, a simpler substitution is $u = a^2 - x^2$ because $du = -2x\, dx$.

If an integral contains an expression of the form $\sqrt{a^2 + x^2}$, the substitution $x = a\tan\theta$ should be considered because the identity $1 + \tan^2\theta = \sec^2\theta$ eliminates the root sign. Similarly, if the factor $\sqrt{x^2 - a^2}$ occurs, the substitution $x = a\sec\theta$ is often effective.

Partial Fractions

One method for integrating a rational function (the ratio of polynomials) is to rewrite the function as a sum of simpler fractions, called *partial fractions*, that we already know how to integrate. The next example illustrates the simplest case.

Example 4 Distinct Linear Factors

Find $\displaystyle\int \frac{5x-4}{2x^2+x-1}\,dx$.

Solution

Factor the denominator completely, in this case, into a product of linear factors.

$$2x^2 + x - 1 = (x+1)(2x-1)$$

Since the degree of the numerator is less than the degree of the denominator, and the denominator is the product of distinct linear factors, we can write the given rational function as a sum of partial fractions.

$$\frac{5x-4}{2x^2+x-1} = \frac{A}{x+1} + \frac{B}{2x-1}, \text{ where } A \text{ and } B \text{ are constants}$$

To determine the values of A and B, first find the least common denominator on the right side, which is the product of the two linear terms, and write as one fraction.

$$\frac{5x-4}{2x^2+x-1} = \frac{A(2x-1)+B(x+1)}{2x^2+x-1}$$

If two rational functions are equal and their denominators are the same, then their numerators must be the same.

Equate the numerators.

$$\begin{aligned} 5x-4 &= A(2x-1)+B(x+1) \\ &= (2A+B)x + (-A+B) && \text{Expand; simplify.} \end{aligned}$$

Since these two polynomials are equal, their coefficients must be the same. Equating the coefficients leads to a system of equations, in this case, two equations in two unknowns.

$$\begin{cases} 2A + B = 5 \\ -A + B = -4 \end{cases}$$

Solving this system, we get $A = 3$ and $B = -1$.

We can now rewrite the original integral using this *partial fraction decomposition* (PFD) and integrate term by term.

$$\begin{aligned} \int \frac{5x-4}{2x^2+x-1}\,dx &= \int \left(\frac{3}{x+1} - \frac{1}{2x-1}\right)dx && \text{PFD.} \\ &= 3\ln|x+1| - \frac{1}{2}\ln|2x-1| + C && u = x+1;\ u = 2x-1. \end{aligned}$$

■

Partial Fraction Details

Here are a few more guidelines for integrating a rational expression $\int f(x)\,dx$, where $f(x) = \dfrac{P(x)}{Q(x)}$.

1. If f is *improper*, that is, $\deg(P) \geq \deg(Q)$, then we first divide Q into P using long division until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$. This division process is written as

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are polynomials. For example,

$$\frac{2x^3 - 11x^2 - 2x + 2}{2x^2 + x - 1} = x - 6 + \frac{5x - 4}{(x + 1)(2x - 1)}$$

2. If the denominator $Q(x)$ is a product of distinct linear factors, then we can write $Q(x)$ as

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

where no linear factor is repeated and no factor is a constant multiple of another. In this case we need to include a term corresponding to each factor. There exist constants $A_1, A_2, \ldots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

For example

$$\frac{x + 6}{x(x - 3)(4x + 5)} = \frac{A}{x} + \frac{B}{x - 3} + \frac{C}{4x + 5}$$

where A, B, and C are constants determined by solving a system of three equations in three unknowns.

3. If $Q(x)$ is a product of linear factors, some of which are repeated, then we need to include extra terms in the PFD. Suppose the linear factor $(ax + b)$ is repeated r times; that is, $(ax + b)^r$ is a factor of $Q(x)$. Then this factor contributes r terms to the partial fraction decomposition.

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_r}{(ax + b)^r}$$

For example, here is a partial fraction decomposition in which there are repeated linear factors.

$$\frac{x^3 - x + 1}{x^2(x - 1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2} + \frac{E}{(x - 1)^3}$$

4. If $Q(x)$ has the irreducible quadratic factor $ax^2 + bx + c$, that is, the discriminant $b^2 - 4ac < 0$, then the partial fraction decomposition will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined. This term can be integrated by completing the square and using the formula

The constant a in Equation 1 represents a different arbitrary constant than the constant a in the quadratic denominator of the partial fraction.

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad (1)$$

5. If $Q(x)$ contains an irreducible quadratic factor, a term of the form $(ax^2 + bx + c)^r$ where $b^2 - 4ac < 0$, then this term contributes r terms to the partial fraction decomposition.

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Each of these terms can be integrated by using a substitution or by first completing the square if necessary. ■

Example 5 Linear and Quadratic Factor

Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

Solution

Factor the denominator $x^3 + 4x = x(x^2 + 4)$.

Each factor contributes one term to the partial fraction decomposition.

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Use the least common denominator on the right-hand side and equate numerators.

$$\begin{aligned} 2x^2 - x + 4 &= A(x^2 + 4) + (Bx + C)x \\ &= (A + B)x^2 + Cx + 4A \end{aligned}$$

Equate coefficients. The system of equations is

$$\begin{cases} A + B \quad\;\; = 2 \\ \qquad\quad C = -1 \\ 4A \qquad\;\;\, = 4 \end{cases}$$

Therefore, $A = 1$, $B = 1$, and $C = -1$.

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x - 1}{x^2 + 4}\right) dx \qquad \text{PFD.}$$

$$= \int \frac{1}{x}\,dx + \int \frac{x}{x^2 + 4}\,dx - \int \frac{1}{x^2 + 4}\,dx \qquad \text{Split second term into two separate integrals.}$$

$$= \ln|x| + \frac{1}{2}\ln(x^2 + 4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + K \qquad \text{Use } u = x^2 + 4 \text{ in the middle integral.}$$

5.7 Exercises

Evaluate the integral.

1. $\displaystyle\int \sin^3 x \cos^2 x\,dx$

2. $\displaystyle\int_0^{\pi/2} \cos^5 x\,dx$

3. $\displaystyle\int_{\pi/2}^{\pi/4} \sin^5 x \cos^3 x\,dx$

4. $\displaystyle\int \sin^3(mx)\,dx$

5. $\displaystyle\int_0^{2\pi} \cos^2(6\theta)\,d\theta$

6. $\displaystyle\int_0^{\pi/2} \sin^2 x \cos^2 x\,dx$

7. $\displaystyle\int \sqrt{\cos\theta}\,\sin^3\theta\,d\theta$

8. $\displaystyle\int \frac{\sin^2(\frac{1}{t})}{t^2}\,dt$

Use the substitution $u = \sec x$ to evaluate the integral.

9. $\displaystyle\int \tan^3 x \sec x\,dx$

10. $\displaystyle\int \tan^5 x \sec^3 x\,dx$

Use the substitution $u = \tan x$ to evaluate the integral.

11. $\displaystyle\int_0^{\pi/4} \tan^2 x \sec^4 x\,dx$

12. $\displaystyle\int \tan^4 x \sec^6 x\,dx$

13. Use the substitution $x = 3\sin\theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, and the identity $\cot^2\theta = \csc^2\theta - 1$ to evaluate

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx$$

14. Use the substitution $x = \sec\theta$, where $0 \le \theta < \dfrac{\pi}{2}$ or $\pi \le \theta < \dfrac{3\pi}{2}$, to evaluate

$$\int \frac{\sqrt{x^2 - 1}}{x^4}\,dx$$

15. Use the substitution $x = 2\tan\theta$, $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, to evaluate

$$\int \frac{1}{x^2\sqrt{x^2 + 4}}\,dx$$

16. (a) Verify, by differentiation, that

$$\int \sec^3\theta \, d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C$$

(b) Evaluate $\int_0^1 \sqrt{x^2+1}\, dx$.

Evaluate the integral.

17. $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\, dt$

18. $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\, dx$

19. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

20. $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\, dx$

21. $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\, dx$

22. $\displaystyle\int_0^a x^2\sqrt{a^2-x^2}\, dx$

Write out the form of the partial fraction decomposition of the function. Do not determine the numerical values of the coefficients.

23. (a) $\dfrac{2x}{(x+3)(3x+1)}$ (b) $\dfrac{1}{x^3+2x^2+x}$

24. (a) $\dfrac{x}{x^2+x-2}$ (b) $\dfrac{x^2}{x^2+x+2}$

25. (a) $\dfrac{x^6}{x^2-4}$ (b) $\dfrac{1}{x^2+x^4}$

Evaluate the integral.

26. $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\, dx$

27. $\displaystyle\int_0^1 \frac{x-4}{x^2-5x+6}\, dx$

28. $\displaystyle\int_2^3 \frac{1}{x^2-1}\, dx$

29. $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\, dx$

30. $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\, dx$

31. $\displaystyle\int \frac{2x^2+5}{(x^2+1)(x^2+4)}\, dx$

32. $\displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\, dx$

33. $\displaystyle\int \frac{x^2-x+6}{x^3+3x}\, dx$

34. $\displaystyle\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\, dx$

35. $\displaystyle\int \frac{x^2+x+1}{(x^2+1)^2}\, dx$

Use long division to evaluate the integral.

36. $\displaystyle\int \frac{x}{x-6}\, dx$

37. $\displaystyle\int \frac{x^2}{x+4}\, dx$

38. $\displaystyle\int \frac{x^3+4}{x^2+4}\, dx$

39. $\displaystyle\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\, dx$

Use a substitution to express the integrand as a rational function and then evaluate the integral.

40. $\displaystyle\int_9^{16} \frac{\sqrt{x}}{x-4}\, dx$

41. $\displaystyle\int \frac{dx}{2\sqrt{x+3}+x}$

42. A particle moves along a horizontal line so that its velocity at time t is given by $v(t) = \sin\omega t\cos^2\omega t$. Find its position function if $s(0) = 0$.

43. Evaluate the integral

$$\int \frac{dx}{x^4\sqrt{x^2-2}}$$

Graph the integrand and its indefinite integral on the same coordinate axes to check that your answer is reasonable.

44. Use a trigonometric substitution to verify

$$\int_0^x \sqrt{a^2-t^2}\, dt = \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2-x^2}$$

45. Complete the square in the quadratic x^2+x+1 and make a substitution to evaluate

$$\int \frac{dx}{x^2+x+1}$$

46. Complete the square in the quadratic $3-2x-x^2$ and use a trigonometric substitution to evaluate

$$\int \frac{x}{\sqrt{3-2x-x^2}}\, dx$$

47. The rational number $\frac{22}{7}$ has been used as an approximation to the number π since the time of Archimedes. Show that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx = \frac{22}{7} - \pi$$

5.8 Integration Using Tables and Computer Algebra Systems

In this section we describe how to evaluate integrals using tables and computer algebra systems.

Tables of Integrals

Tables of indefinite integrals are very useful when we consider an integral that is difficult to evaluate and we don't have access to a computer algebra system. A relatively brief table of 120 integrals, categorized by form, is provided on the Reference Pages at the back of the book. More extensive tables are available in other publications, some containing hundreds or even thousands of integrals. Note that integrals do not often occur in exactly the form listed in a table. Usually we need to use the Substitution Rule or algebraic manipulation to transform a given integral into one of the forms in the table.

Example 1 Table of Integrals, Long Division

Use the Table of Integrals to evaluate $\displaystyle\int_0^2 \frac{x^2+12}{x^2+4}\,dx$.

Solution

The only formula in the table that resembles the given integral is entry 17:

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

Since the integrand is a rational function and improper, perform long division on the integrand.

$$\frac{x^2+12}{x^2+4} = 1 + \frac{8}{x^2+4}$$

Now use Formula 17 with $a = 2$.

$$\int_0^2 \frac{x^2+12}{x^2+4}\,dx = \int_0^2 \left(1 + \frac{8}{x^2+4}\right)dx \qquad \text{Long division.}$$

$$= \left[x + 8\cdot\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^3 \qquad \text{Table of Indefinite Integrals; Formula 17, } a = 2.$$

$$= 2 + 4\tan^{-1}1 = 2 + \pi \qquad \text{FTC2; simplify.}$$

■

Example 2 Table of Integrals, Root Form

Use the Table of Integrals to evaluate $\displaystyle\int \frac{x^2}{\sqrt{5-4x^2}}\,dx$.

Solution

Consider the section of the table entitled *Forms involving* $\sqrt{a^2-u^2}$. The closest entry is number 34.

$$\int \frac{u^2}{\sqrt{a^2 - u^2}}\,du = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$

This isn't exactly our integral, but first use the substitution $u = 2x$.

$$u = 2x \quad \Rightarrow \quad du = 2dx \quad \Rightarrow \quad dx = \frac{du}{2}$$

$$\int \frac{x^2}{\sqrt{5 - 4x^2}}\,dx = \int \frac{(u/2)^2}{\sqrt{5 - u^2}}\,\frac{du}{2} = \frac{1}{8}\int \frac{u^2}{\sqrt{5 - u^2}}\,du$$

We can now use Formula 34 with $a^2 = 5 \quad \Rightarrow \quad a = \sqrt{5}$.

$$\begin{aligned} \int \frac{x^2}{\sqrt{5 - 4x^2}}\,dx &= \frac{1}{8}\int \frac{u^2}{\sqrt{5 - u^2}}\,du && \text{Substitution } u = 2x. \\ &= \frac{1}{8}\left(-\frac{u}{2}\sqrt{5 - u^2} + \frac{5}{2}\sin^{-1}\frac{u}{\sqrt{5}}\right) + C && \text{Formula 34.} \\ &= -\frac{x}{8}\sqrt{5 - 4x^2} + \frac{5}{16}\sin^{-1}\left(\frac{2x}{\sqrt{5}}\right) + C && \text{Final answer in terms of } x. \end{aligned}$$

■

Example 3 Table of Integrals, Trigonometric Form

Use the Table of Integrals to evaluate $\int x^3 \sin x\,dx$.

Solution

In the *Trigonometric Forms* section, none of the entries explicitly includes a u^3 factor. However, we can start by using the reduction formula in entry 84 with $n = 3$.

$$\int x^3 \sin x\,dx = -x^3 \cos x + 3\int x^2 \cos x\,dx$$

85. $\int u^n \cos u\,du$

$= u^n \sin u - n\int u^{n-1} \sin u\,du$

We now need to evaluate $\int x^2 \cos x\,dx$. We can use the reduction formula in entry 85 with $n = 2$, followed by entry 82.

$$\begin{aligned} \int x^2 \cos x\,dx &= x^2 \sin x - 2\int x \sin x\,dx \\ &= x^2 \sin x - 2(\sin x - x\cos x) + K \end{aligned}$$

Combine these results to obtain the final answer.

$$\int x^3 \sin x\,dx = -x^3 \cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + C$$

where $C = 3K$. ■

Example 4 Table of Integrals, Complete the Square

Use the Table of Integrals to evaluate $\int x\sqrt{x^2 + 2x + 4}\,dx$.

Solution

Since the table provides formulas for integrals involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$ but not $\sqrt{ax^2 + bx + c}$, it seems reasonable to first complete the square.

$$x^2 + 2x + 4 = (x^2 + 2x + 1) + 3 = (x + 1)^2 + 3$$

Use the substitution $u = x + 1$.

$$u = x + 1 \quad \Rightarrow \quad du = dx \text{ and } x = u - 1$$

$$\int x\sqrt{x^2+2x+4}\,dx = \int (u-1)\sqrt{u^2+3}\,du \qquad \text{Substitution } u = x+1.$$

$$= \int u\sqrt{u^2+3}\,du - \int \sqrt{u^2+3}\,du \qquad \text{Property of integrals.}$$

The first integral is evaluated using the substitution $t = u^2 + 3$.

$$\int u\sqrt{u^2+3}\,du = \frac{1}{2}\int \sqrt{t}\,dt = \frac{1}{2}\cdot\frac{2}{3}t^{3/2} = \frac{1}{3}(u^2+3)^{3/2}$$

21. $\int \sqrt{a^2+u^2}\,du$
$= \frac{u}{2}\sqrt{a^2+u^2}$
$+ \frac{a^2}{2}\ln\left(u + \sqrt{a^2+u^2}\right) + C$

For the second integral, use Formula 21 with $a = \sqrt{3}$.

$$\int \sqrt{u^2+3}\,du = \frac{u}{2}\sqrt{u^2+3} + \frac{3}{2}\ln(u + \sqrt{u^2+3})$$

Therefore,

$$\int x\sqrt{x^2+2x+4}\,dx$$

$$= \frac{1}{3}(x^2+2x+4)^{3/2} - \frac{x+1}{2}\sqrt{x^2+2x+4} - \frac{3}{2}\ln(x+1+\sqrt{x^2+2x+4}) + C. \blacksquare$$

Computer Algebra Systems

We have seen that the use of tables involves matching the form of the given integrand with the forms of the integrands in the tables. Computers are particularly good at matching patterns. And just as we used substitutions together with tables, a CAS can perform substitutions that transform a given integral into one that occurs in a stored formula. So it isn't surprising that computer algebra systems excel at integration. That doesn't mean that integration is an obsolete or unnecessary skill to learn. We will see that an analytic computation, sometimes produces an indefinite integral in a form that is more convenient than an answer produced by technology.

Let's see what happens when we use a CAS to integrate the relatively simple function $y = \dfrac{1}{3x-2}$. Using the substitution $u = 3x - 2$, this is a straightforward calculation:

$$\int \frac{1}{3x-2}\,dx = \frac{1}{3}\ln|3x-2| + C$$

However, some computer algebra systems return the answer

$$\frac{1}{3}\ln(3x-2)$$

The first thing to notice is that computer algebra systems omit the constant of integration. That is, they return a *particular* antiderivative, not the most general one. Therefore, when using a CAS for integration, we may have to add a constant.

Notice also that the absolute value symbol is omitted in the machine answer. That's fine if our problem involves only values of x greater than $\frac{2}{3}$. But if we are interested in other values of x, then we need to insert the absolute value symbol.

In the next example, we reconsider the integral of Example 4, but this time we will use a CAS.

Example 5 Computer Algebra System for Integration

Use a computer algebra system to evaluate $\int x\sqrt{x^2+2x+4}\,dx$.

Solution

One computer algebra system gives the answer

$$\frac{1}{3}(x^2+2x+4)^{3/2} - \frac{1}{4}(2x+2)\sqrt{x^2+2x+4} - \frac{3}{2}\operatorname{arcsinh}\frac{\sqrt{3}}{3}(1+x).$$

This certainly looks different from the answer we found in Example 4, but it is indeed equivalent. The third term can be rewritten using the identity

This is the formula in Problem 8(c) in the Discovery Project on page 260.

$$\operatorname{arcsinh} x = \ln\left(x+\sqrt{x^2+1}\right).$$

Therefore,

$$\begin{aligned}\operatorname{arcsinh}\frac{\sqrt{3}}{3}(1+x) &= \ln\left[\frac{\sqrt{3}}{3}(1+x) + \sqrt{\frac{1}{3}(1+x)^2+1}\right]\\ &= \ln\left[\frac{1}{\sqrt{3}}\left(1+x+\sqrt{(1+x)^2+3}\right)\right]\\ &= \ln\frac{1}{\sqrt{3}} + \ln\left(x+1+\sqrt{x^2+2x+4}\right)\end{aligned}$$

The resulting extra term $-\frac{3}{2}\ln\frac{1}{\sqrt{3}}$ can be included in the constant of integration.

Another computer algebra system provides the answer

$$\frac{1}{6}\left(\sqrt{x^2+2x+4}(2x^2+x+5) - 9\operatorname{arcsinh}\left(\frac{x+1}{\sqrt{3}}\right)\right).$$

Here the first two terms of Example 4 (and the first computer algebra system result) are combined into a single term by factoring. ■

Example 6 Use a CAS, Polynomial Integrand

Use a CAS to evaluate $\int x(x^2+5)^8\,dx$.

Solution

Some computer algebra systems provide the answer

$$\tfrac{1}{18}x^{18} + \tfrac{5}{2}x^{16} + 50x^{14} + \tfrac{1750}{3}x^{12} + 4375x^{10} + 21{,}875x^8 + \tfrac{218{,}750}{3}x^6 + 156{,}250x^4 + \tfrac{390{,}625}{2}x^2 + \tfrac{1{,}953{,}125}{18}.$$

This answer is obtained by first expanding $(x^2+5)^8$ by the Binomial Theorem and then integrating term by term.

If we integrate without using a CAS, this is a simple substitution with $u = x^2 + 5$.

Using this we get

$$\int x(x^2 + 5)^8\, dx = \frac{1}{18}(x^2 + 5)^9 + C.$$

For most purposes, this is a more convenient form of the answer. ■

Example 7 Use a CAS, Trigonometric Integrand

Use a CAS to evaluate $\int \sin^5 x \cos^2 x\, dx$.

Solution

One CAS returns the answer

$$-\frac{1}{7}\sin^4 x \cos^3 x - \frac{4}{35}\sin^2 x \cos^3 x - \frac{8}{105}\cos^3 x,$$

whereas another CAS returns

$$-\frac{5}{64}\cos x - \frac{1}{192}\cos 3x + \frac{3}{320}\cos 5x - \frac{1}{448}\cos 7x.$$

It seems reasonable that we could use trigonometric identities to show these answers are equivalent. Indeed, if we ask Derive, Maple, and Mathematica to simplify their expression using trigonometric identities, they ultimately produce the same form of the answer.

$$\int \sin^5 x \cos^2 x\, dx = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x$$ ■

Can We Integrate All Continuous Functions?

It seems natural to ask: Will our basic integration formulas, together with the Substitution Rule, integration by parts, tables of integrals, and computer algebra systems, enable us to find the integral of every continuous function? In particular, can we use these techniques to evaluate $\int e^{x^2}\, dx$? The answer is No, at least not in terms of the functions that we are familiar with.

Most of the functions that we have been working with in this book are called **elementary functions**. These are the polynomials, rational functions, power functions (x^a), exponential functions (a^x), logarithmic functions, trigonometric and inverse trigonometric functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition; for example, the function

$$f(x) = \sqrt{\frac{x^2 - 1}{x^3 + 2x - 1}} + \ln(\cos x) - xe^{\sin 2x}$$

is an elementary function.

If f is an elementary function, then f' is also an elementary function, but $\int f(x)\, dx$ need not be an elementary function. Consider $f(x) = e^{x^2}$. Since f is continuous, its integral exists, and if we define the function F by

$$F(x) = \int_0^x e^{t^2}\, dt$$

then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^{x^2}$$

In Chapter 8 we will see how to express $\int e^{x^2}\,dx$ as an infinite series.

Thus $f(x) = e^{x^2}$ has an antiderivative F, but it has been proved that F is not an elementary function. This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2}\,dx$ in terms of the common functions we know. The same is true of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3+1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions do not have elementary antiderivatives.

5.8 Exercises

Use the Table of Integrals to evaluate the integral.

1. $\int \tan^3(\pi x)\,dx$

2. $\int e^{2\theta}\sin 3\theta\,d\theta$

3. $\int \frac{dx}{x^2\sqrt{4x^2+9}}$

4. $\int_2^3 \frac{1}{x^2\sqrt{4x^2-7}}\,dx$

5. $\int e^{2x}\arctan(e^x)\,dx$

6. $\int \frac{\sqrt{2y^2-3}}{y^2}\,dy$

7. $\int_0^{\pi} x^3\sin x\,dx$

8. $\int \frac{dx}{2x^3-3x^2}$

9. $\int \frac{\tan^3(1/z)}{z^2}\,dz$

10. $\int \sin^{-1}\sqrt{x}\,dx$

11. $\int y\sqrt{6+4y-4y^2}\,dy$

12. $\int x\sin(x^2)\cos(3x^2)\,dx$

13. $\int \sin^2 x\cos x\ln(\sin x)\,dx$

14. $\int \frac{\sin 2\theta}{\sqrt{5-\sin\theta}}\,d\theta$

15. $\int \frac{e^x}{3-e^{2x}}\,dx$

16. $\int_0^2 x^3\sqrt{4x^2-x^4}\,dx$

17. $\int \frac{x^4}{\sqrt{x^{10}-2}}\,dx$

18. $\int_0^1 x^4e^{-x}\,dx$

19. $\int \frac{\sqrt{4+(\ln x)^2}}{x}\,dx$

20. $\int \frac{\sec^2\theta\tan^2\theta}{\sqrt{9-\tan^2\theta}}\,d\theta$

21. $\int \sqrt{e^{2x}-1}\,dx$

22. $\int e^t\sin(\alpha t-3)\,dt$

23. Verify Formula 53 in the Table of Integrals (a) by differentiation and (b) by using the substitution $t = a + bu$.

24. Verify Formula 31 in the Table of Integrals (a) by differentiation and (b) by substituting $u = a\sin\theta$.

Use a computer algebra system to evaluate the integral. Compare the answer with the result of using tables. If the answer forms are different, show that they are equivalent.

25. $\int \sec^4 x\,dx$

26. $\int x^2(1+x^3)^4\,dx$

27. $\int x^2\sqrt{x^2+4}\,dx$

28. $\int \frac{dx}{e^x(3e^x+2)}$

29. $\int x\sqrt{1+2x}\,dx$

30. $\int \sin^4 x\,dx$

31. $\int \tan^5 x\,dx$

32. $\int \frac{1}{\sqrt{1+\sqrt[3]{x}}}\,dx$

33. (a) Use the Table of Integrals to evaluate $F(x) = \int f(x)\,dx$, where

$$f(x) = \frac{1}{x\sqrt{1-x^2}}$$

What is the domain of f and F?

(b) Use a CAS to evaluate $F(x)$. What is the domain of the function F that the CAS produces? Explain any discrepancy between this domain and the domain of the function F that you found in part (a).

34. Computer algebra systems often need a little help from us. Try to evaluate

$$\int (1+\ln x)\sqrt{1+(x\ln x)^2}\,dx$$

with a computer algebra system. If it does not return an answer, use a substitution that transforms the integral into one that the CAS can evaluate (and then evaluate the integral).

Discovery Project | Patterns in Integrals

In this project, a computer algebra system is used to investigate indefinite integrals of families of functions. By observing the patterns that occur in the integrals of several members of the family, you will be asked to first guess, and then prove, a general formula for the integral of any member of the family.

1. (a) Use a computer algebra system to evaluate the following integrals.

(i) $\displaystyle\int \frac{1}{(x+2)(x+3)}\,dx$ (ii) $\displaystyle\int \frac{1}{(x+1)(x+5)}\,dx$

(iii) $\displaystyle\int \frac{1}{(x+2)(x-5)}\,dx$ (iv) $\displaystyle\int \frac{1}{(x+2)^2}\,dx$

(b) Based on the pattern of your responses in part (a), guess the value of the integral

$$\int \frac{1}{(x+a)(x+b)}\,dx$$

if $a \neq b$. What if $a = b$?

(c) Check your answer using a CAS to evaluate the integral found in part (b). Then prove this result using partial fractions or by differentiation.

2. (a) Use a computer algebra system to evaluate the following integrals.

(i) $\displaystyle\int \sin x \cos 2x\,dx$ (ii) $\displaystyle\int \sin 3x \cos 7x\,dx$ (iii) $\displaystyle\int \sin 8x \cos 3x\,dx$

(b) Based on the pattern of your responses in part (a), guess the value of the integral

$$\int \sin ax \cos bx\,dx$$

(c) Check your guess using a CAS. Then prove the result by differentiation. For what values of a and b is this result valid?

3. (a) Use a computer algebra system to evaluate the following integrals.

(i) $\displaystyle\int \ln x\,dx$ (ii) $\displaystyle\int x \ln x\,dx$ (iii) $\displaystyle\int x^2 \ln x\,dx$

(iv) $\displaystyle\int x^3 \ln x\,dx$ (v) $\displaystyle\int x^7 \ln x\,dx$

(b) Based on the pattern of responses in part (a), guess the value of

$$\int x^n \ln x\,dx$$

(c) Use integration by parts to prove your conjecture in part (b). For what values of n is this result valid?

4. (a) Use a computer algebra system to evaluate the following integrals.

(i) $\displaystyle\int xe^x\,dx$ (ii) $\displaystyle\int x^2e^x\,dx$ (iii) $\displaystyle\int x^3e^x\,dx$

(iv) $\displaystyle\int x^4e^x\,dx$ (v) $\displaystyle\int x^5e^x\,dx$

(b) Based on the pattern of responses in part (a), guess the value of $\int x^6e^x\,dx$. Then use a CAS to check your guess.

(c) Based on the patterns in parts (a) and (b), make a conjecture as to the value of the integral

$$\int x^n e^x \, dx$$

when n is a positive integer.

(d) Use mathematical induction to prove your conjecture in part (c).

5.9 Approximate Integration

Approximation Techniques

There are two situations in which it is impossible to find the exact value of a definite integral.

The first situation arises from the fact that in order to evaluate the definite integral $\int_a^b f(x)\, dx$ using the Evaluation Theorem we need to know an antiderivative of f. Sometimes, however, it is difficult, or even impossible, to find an antiderivative (see Section 5.8). For example, it is impossible to evaluate the following integrals exactly:

$$\int_0^1 e^{x^2}\, dx \qquad\qquad \int_{-1}^1 \sqrt{1 + x^3}\, dx$$

The second situation arises when the function is determined from a scientific experiment through instrument readings or collected data. In this case, there is no formula for the function (see Example 5).

In both cases we need to find approximate values of definite integrals, and we already know one such method. Recall that the definite integral is defined as a limit of Riemann sums. So any Riemann sum could be used as an approximation to the integral. If we divide the interval $[a, b]$ into n subintervals of equal length $\Delta x = \dfrac{b - a}{n}$, then

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

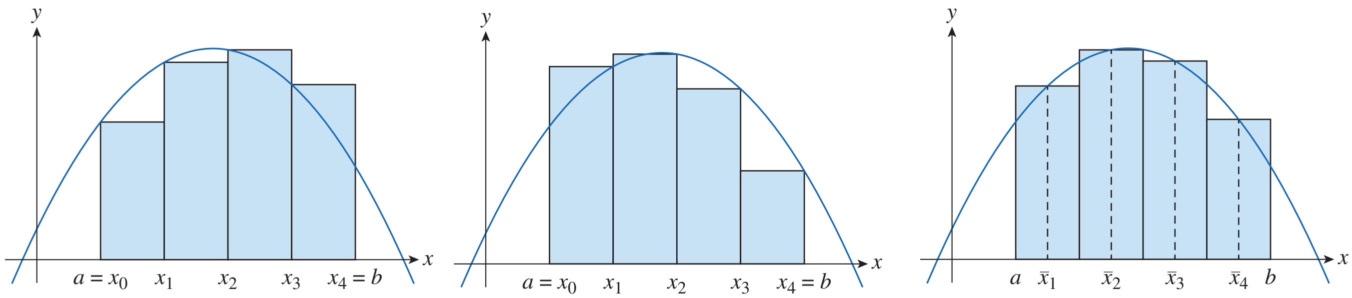
where x_i^* is any point in the ith subinterval $[x_{i-1}, x_i]$. If x_i^* is the left endpoint of each subinterval, then $x_i^* = x_{i-1}$ and

$$\int_a^b f(x)\, dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\, \Delta x \tag{1}$$

If $f(x) \geq 0$, then the integral in Equation 1 represents an approximation to the area under the curve by rectangles, as shown in Figure 5.59(a). If x_i^* is the right endpoint of each subinterval, then $x_i^* = x_i$, and

$$\int_a^b f(x)\, dx \approx R_n = \sum_{i=1}^{n} f(x_i)\, \Delta x \tag{2}$$

See Figure 5.59(b). The approximations L_n and R_n defined in Equations 1 and 2 are called the **left endpoint approximation** and **right endpoint approximation**, respectively.



(a) Left endpoint approximation (b) Right endpoint approximation (c) Midpoint approximation

Figure 5.59
Visualization of various approximation methods.

In Section 5.2 we also considered the case where x_i^* is the midpoint $\bar{x}_i$ of each subinterval $[x_{i-1}, x_i]$. Figure 5.59(c) illustrates a midpoint approximation M_n, which appears to be a better approximation than either L_n or R_n.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x\left[f(\bar{x}_1) + \cdots + f(\bar{x}_n)\right]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

Another approximation, the Trapezoidal Rule, is found by summing the areas of trapezoids. This is also the result of the arithmetic mean of the left endpoint and right endpoint approximations.

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\,\Delta x + \sum_{i=1}^{n} f(x_i)\,\Delta x\right] = \frac{\Delta x}{2}\left[\sum_{i=1}^{n}\left(f(x_{i-1}) + f(x_i)\right)\right]$$

$$= \frac{\Delta x}{2}\left[(f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n))\right]$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$.

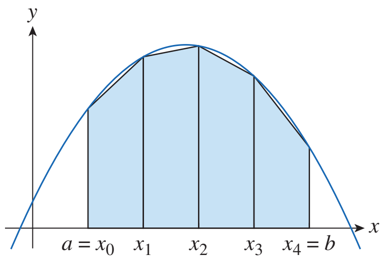


Figure 5.60
Trapezoidal approximation.

Figure 5.60 illustrates the Trapezoidal Rule for the case where $f(x) \geq 0$ and $n = 4$. The area of the trapezoid that lies above the ith subinterval is

$$\Delta x\left(\frac{f(x_{i-1}) + f(x_i)}{2}\right) = \frac{\Delta x}{2}[f(x_{i-1}) + f(x_i)]$$

The sum of the areas of all of these trapezoids is the right side of the Trapezoidal Rule.

Example 1 Integral Approximations

Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 \frac{1}{x}\,dx$.

Solution

(a) For $n = 5$, $a = 1$, and $b = 2$ $\Rightarrow$ $\Delta x = \frac{2-1}{5} = 0.2.$

Use the Trapezoidal Rule.

$$\begin{aligned}\int_1^2 \frac{1}{x}\,dx \approx T_5 &= \frac{0.2}{2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]\\ &= 0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)\\ &\approx 0.695635\end{aligned}$$

This approximation is illustrated in Figure 5.61.

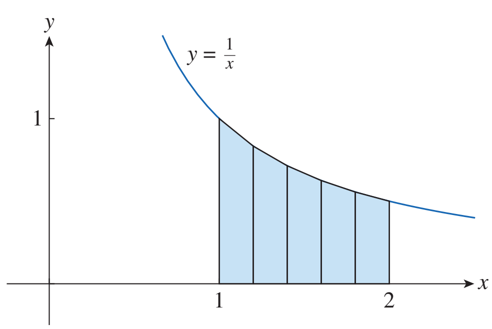


Figure 5.61
Trapezoidal Rule approximation.

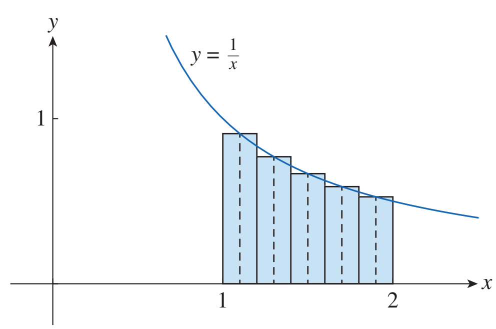


Figure 5.62
Midpoint Rule approximation.

(b) The midpoints of the five subintervals are 1.1, 1.3, 1.5, 1.7, and 1.9. Use the Midpoint Rule.

$$\begin{aligned}\int_1^2 \frac{1}{x}\,dx &\approx \Delta x[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]\\ &= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)\\ &\approx 0.691908\end{aligned}$$

This approximation is illustrated in Figure 5.62. ■

Error Bounds

In Example 1 we can actually find an antiderivative and compute the exact answer. This will allow us to examine the accuracy of the Trapezoidal and Midpoint Rules in this case. Using the Fundamental Theorem of Calculus,

$$\int_1^2 \frac{1}{x}\,dx = \ln x\Big]_1^2 = \ln 2 = 0.693147\ldots$$

error $= \int_a^b f(x)\,dx -$ approximation

The **error** in using an approximation is defined to be the difference between the actual value and the approximation. Using the values obtained in Example 1, the errors in the Trapezoidal and Midpoint Rule approximations for $n = 5$ are

$$E_T \approx -0.002488 \quad \text{and} \quad E_M \approx 0.001239$$

In general, we write

$$E_T = \int_a^b f(x)\,dx - T_n \qquad \text{and} \qquad E_M = \int_a^b f(x)\,dx - M_n$$

The following tables show the results of calculations similar to those in Example 1, For $n = 5$, 10, and 20, Table 5.4 shows the left and right endpoint approximations as well as the Trapezoidal and Midpoint Rule approximations.

Table 5.5 shows the errors corresponding to each of these approximations.

Table 5.4
Approximations to $\int_1^2 \frac{1}{x}\,dx$.

n	L_n	R_n	T_n	M_n
5	0.745635	0.645635	0.695635	0.691908
10	0.718771	0.668771	0.693771	0.692835
20	0.705803	0.680803	0.693303	0.693069

Table 5.5
Corresponding errors.

n	E_L	E_R	E_T	E_M
5	−0.052488	0.047512	−0.002488	0.001239
10	−0.025624	0.024376	−0.000624	0.000312
20	−0.012656	0.012344	−0.000156	0.000078

These observations are actually true in most cases.

Here are several observations from examining these tables.

1. In every method, the approximation becomes more accurate as the value of n increases. But very large values of n result in so many arithmetic operations that there could be accumulated round-off error.
2. The errors in the left and right endpoint approximations are opposite in sign and appear to decrease by a factor of about 2 when we double the value of n.
3. The Trapezoidal and Midpoint Rules are much more accurate than the endpoint approximations.
4. The errors in the Trapezoidal and Midpoint Rules are opposite in sign and appear to decrease by a factor of about 4 when we double the value of n.
5. The size of the error in the Midpoint Rule is about half the size of the error in the Trapezoidal Rule.

Figure 5.63 illustrates the reason we can usually expect the Midpoint Rule to be more accurate than the Trapezoidal Rule. The area of a typical rectangle in the Midpoint Rule is the same as the area of the trapezoid $ABCD$ whose side BC is tangent to the graph at P. The area of this trapezoid is closer to the area under the graph than is the area of the trapezoid $AQRD$ used in the Trapezoidal Rule. The midpoint error (the area of two regions labeled E_M) is smaller than the trapezoidal error (the area of the region labeled E_T).

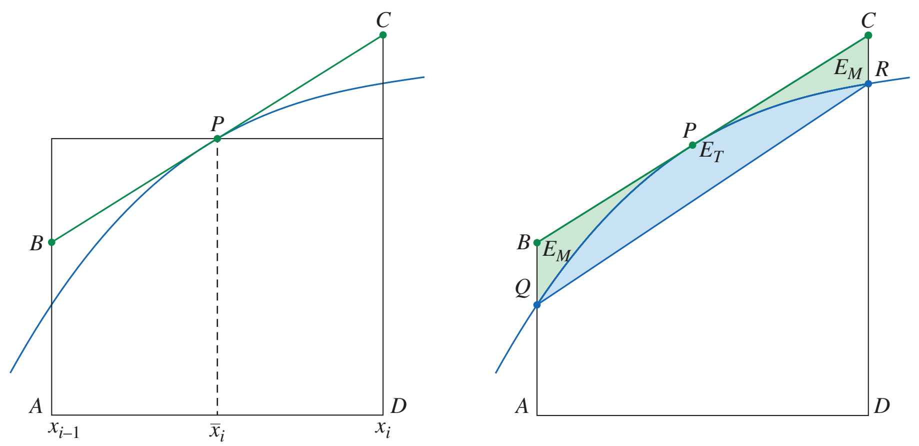


Figure 5.63
Comparison of the Midpoint and Trapezoidal errors.

These observations are supported by the following error estimates, which are proved in numerical analysis texts. Notice that Observation 4 corresponds to the n^2 in each denominator because $(2n)^2 = 4n^2$. The fact that the estimates depend on the size of the second derivative seems reasonable if we consider Figure 5.63, because $f''(x)$ measures how much the graph is curved. Recall that $f''(x)$ measures how fast the slope of $y = f(x)$ is changing.

Error Bounds

Suppose $|f''(x)| \le K$ for $a \le x \le b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2} \tag{3}$$

Let's apply this error estimate to the Trapezoidal Rule approximation in Example 1. If $f(x) = \dfrac{1}{x}$, then $f'(x) = -\dfrac{1}{x^2}$ and $f''(x) = \dfrac{2}{x^3}$. Since $1 \le x \le 2$, then $\dfrac{1}{x} \le 1$, so

$$|f''(x)| = \left|\frac{2}{x^3}\right| \le \frac{2}{1^3} = 2$$

K can be any number larger than $|f''(x)|$, but smaller values of K provide more accurate error bounds.

Therefore, let $K = 2$ and using $a = 1$, $b = 2$, and $n = 5$ in the error bounds estimate, we obtain

$$|E_T| \le \frac{2(2-1)^3}{12(5)^2} = \frac{1}{150} \approx 0.006667$$

Compare this error estimate of 0.006667 with the actual error of about 0.002488. This example shows that the actual error can be substantially less than the upper bound for the error given in Equation 3.

Example 2 Approximation Within a Specified Accuracy

How large should n be in order to guarantee that the Trapezoidal and Midpoint Rule approximations for $\int_1^2 \frac{1}{x}\,dx$ are accurate to within 0.0001?

Solution

In the error estimate to the Trapezoidal Rule approximation in Example 1, we showed that $|f''(x)| \leq 2$ for $1 \leq x \leq 2$.

Let $K = 2$, $a = 1$, and $b = 2$ in Equation 3.

Accuracy to within 0.0001 means that the magnitude of the error should be less than 0.0001.

Therefore, choose n so that $\dfrac{2(1)^3}{12n^2} < 0.0001$.

Solve the inequality for n.

It is possible that a smaller value of n would suffice, but 41 is the smallest value for which the error bound formula can *guarantee* the accuracy will be within 0.0001.

$$n^2 > \frac{2}{12(0.0001)} \quad \Rightarrow \quad n > \frac{1}{\sqrt{0.0006}} \approx 40.8$$

Therefore, $n = 41$ will ensure the desired accuracy.

To achieve the same accuracy using the Midpoint Rule, we choose n so that

$$\frac{2(1)^3}{24n^2} < 0.0001 \quad \Rightarrow \quad n > \frac{1}{\sqrt{0.0012}} \approx 29.$$ ■

Example 3 Estimate the Error in Using the Midpoint Rule

(a) Use the Midpoint Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.

(b) Find an upper bound for the error involved in this approximation.

Solution

(a) Use $a = 0$, $b = 1$, and $n = 10$ in the Midpoint Rule.

$$\begin{aligned}\int_0^1 e^{x^2}dx &\approx \Delta x[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)] \\ &= 0.1[e^{0.0025} + e^{0.022} + \cdots + e^{0.7225} + e^{0.9025}] \\ &\approx 1.460393\end{aligned}$$

Figure 5.64 illustrates this approximation.

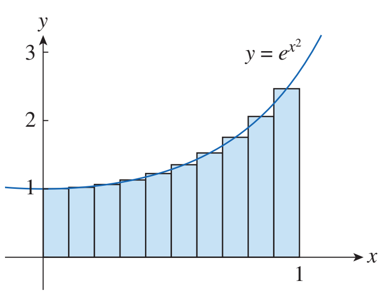


Figure 5.64
Graphical illustration of the Midpoint Rule.

(b) $f(x) = e^{x^2} \quad \Rightarrow \quad f'(x) = 2xe^{x^2} \quad \Rightarrow \quad f''(x) = (2 + 4x^2)e^{x^2}$

And, $0 \leq x \leq 1 \quad \Rightarrow \quad x^2 \leq 1 \quad \Rightarrow \quad 0 \leq f''(x) = (2 + 4x^2)e^{x^2} \leq 6e$

Let $K = 6e$, $a = 0$, $b = 1$, and $n = 10$ in the error estimate (Equation 3). An upper bound for the error is

Error estimates provide upper bounds for the error. They are theoretical, worst-case scenarios. The actual error in this case is approximately 0.0023.

$$\frac{6e(1)^3}{24(10)^2} = \frac{e}{400} \approx 0.007.$$ ■

Simpson's Rule

Instead of using straight line segments to approximate a curve, it seems reasonable to consider a curve, in particular a parabola. As before, divide $[a, b]$ into n subintervals of equal length $h = \Delta x = \dfrac{b - a}{n}$, but this time assume that n is an *even* number. Then on each consecutive pair of intervals, we approximate the curve $y = f(x) \geq 0$ by a parabola as shown in Figure 5.65. If $y_i = f(x_i)$, then $P_i(x_i, y_i)$ is the point on the curve lying above x_i. A typical parabola passes through three consecutive points P_i, P_{i+1}, and P_{i+2}.

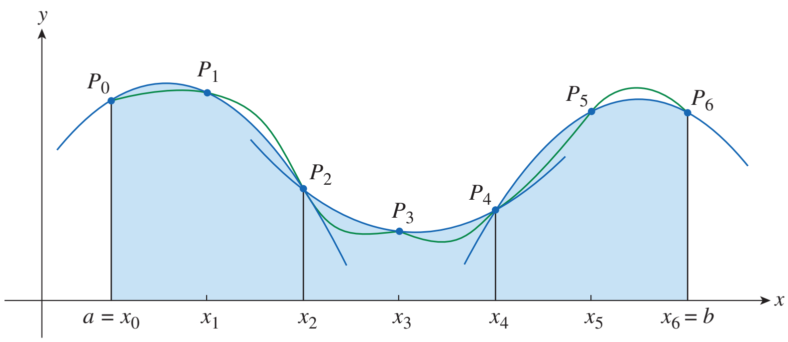


Figure 5.65
Approximating parabolas on each pair of intervals.

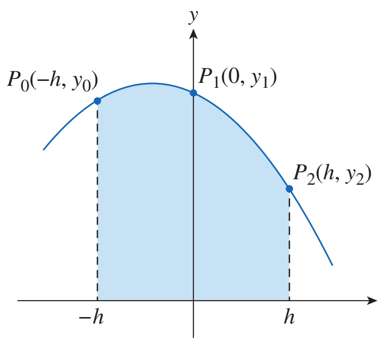


Figure 5.66
Special case for finding the approximating parabola.

To simplify the calculations, consider the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. (See Figure 5.66.) We know that the equation of the parabola through P_0, P_1, and P_2 is of the form $y = Ax^2 + Bx + C$ and the area under the parabola from $x = -h$ to $x = h$ is

$$\int_{-h}^{h} (Ax^2 + Bx + C)\,dx = 2\int_0^h (Ax^2 + C)\,dx \qquad Ax^2 + C \text{ is even; } Bx \text{ is odd.}$$

$$= 2\left[A\frac{x^3}{3} + Cx\right]_0^h \qquad \text{Table of Indefinite Integrals.}$$

$$= 2\left(A\frac{h^3}{3} + Ch\right) = \frac{h}{3}(2Ah^2 + 6C) \qquad \text{FTC2; simplify.}$$

Since the parabola passes through $P_0(-h, y_0)$, $P_1(0, y_1)$, and $P_2(h, y_2)$, we have

$$y_0 = A(-h)^2 + B(-h) + C = Ah^2 - Bh + C$$

$$y_1 = C$$

$$y_2 = Ah^2 + Bh + C$$

and therefore, $$y_0 + 4y_1 + y_2 = 2Ah^2 + 6C$$

So the area under the parabola can be written as

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

If we shift the parabola horizontally, the area under the curve remains the same. Therefore, the area under the parabola through P_0, P_1, and P_2 from $x = x_0$ to $x = x_2$ in Figure 5.65 is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly, the area under the parabola through P_2, P_3, and P_4 from $x = x_2$ to $x = x_4$ is

$$\frac{h}{3}(y_2 + 4y_3 + y_4)$$

If we compute the areas under all the parabolas in this manner and add the results, we get

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)\\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)\end{aligned}$$

Although this approximation applies to the case in which $f(x) \geq 0$, it is a reasonable approximation for any continuous function f and is called Simpson's Rule after the English mathematician Thomas Simpson (1710–1761). Note the pattern of coefficients: 1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 4, 1.

Simpson

Thomas Simpson was a weaver who taught himself mathematics and went on to become one of the best English mathematicians of the 18th century. What we call Simpson's Rule was actually known to Cavalieri and Gregory in the 17th century, but Simpson popularized it in his calculus textbook, *Mathematical Dissertations* (1743).

Simpson's Rule

$$\begin{aligned}\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[&f(x_0) + 4f(x_1) + 2f(x_2) + f(x_3) + \cdots\\ &+ 2f(x_{n-2} + 4f(x_{n-1}) + f(x_n)]\end{aligned}$$

where n is even and $\Delta x = \dfrac{b-a}{n}$

Example 4 Simpson's Rule Approximation

Use Simpson's Rule with $n = 10$ to approximate $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

Solution

Use Simpson's Rule with $f(x) = \dfrac{1}{x}$, $n = 10$, and $\Delta x = \dfrac{2-1}{10} = 0.1$.

$$\begin{aligned}\int_1^2 \frac{1}{x}\,dx &\approx S_{10}\\ &= \frac{\Delta x}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \cdots + 2f(1.8) + 4f(1.9) + f(2)]\\ &= \frac{0.1}{3}\left(\frac{1}{1} + \frac{4}{1.1} + \frac{2}{1.2} + \frac{4}{1.3} + \frac{2}{1.4} + \frac{4}{1.5} + \frac{2}{1.6} + \frac{4}{1.7} + \frac{2}{1.8} + \frac{4}{1.9} + \frac{1}{2}\right)\\ &\approx 0.693150\end{aligned}$$

Notice that, in Example 4, Simpson's Rule produces a much better approximation ($S_{10} \approx 0.693150$) to the true value of the integral ($\ln 2 \approx 0.693147\ldots$) than does the Trapezoidal Rule ($T_{10} \approx 0.693771$) or the Midpoint Rule ($M_{10} \approx 0.692835$). It can be shown (see Exercises 42) that the approximation in Simpson's Rule is a weighted average of corresponding approximations in the Trapezoidal and Midpoint Rules:

$$S_{2n} = \frac{1}{3} T_n + \frac{2}{3} M_n$$

Recall that E_T and E_M usually have opposite signs and $|E_M|$ is about half the size of $|E_T|$.

In many applications of calculus, we need to evaluate an integral even if no explicit formula is known for y as a function of x. A function may be given graphically or as a table of values of collected data. If there is evidence that the values are not changing rapidly, then the Trapezoidal Rule or Simpson's Rule can still be used to find an approximate value for $\int_a^b y\,dx$, the integral of y with respect to x.

Example 5 Estimate the Amount of Transmitted Data

Figure 5.67 shows data traffic on the link from the United States to SWITCH, the Swiss academic and research network, on a day in February. $D(t)$ is the data throughput, measured in megabits per second (Mb/s). Use Simpson's Rule to estimate the total amount of data transmitted on the link from midnight to noon on that day.

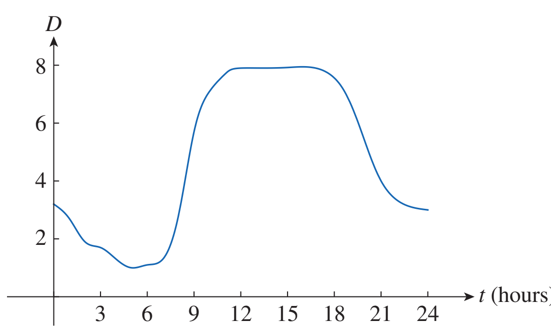


Figure 5.67
Graph of data traffic.

Solution

To make the units consistent, $D(t)$ is measured in megabits per second, so convert the units for t from hours to seconds. If we let $A(t)$ be the amount of data (in megabits) transmitted by time t, where t is measured in seconds, then $A'(t) = D(t)$.

By the Net Change Theorem (Section 5.3), the total amount of data transmitted by noon (when $t = 12 \times 60^2 = 43{,}200$) is

$$A(43{,}200) = \int_0^{43{,}200} D(t)\,dt.$$

Estimate the values of $D(t)$ at hourly intervals from the graph and compile them in a table.

t (hours)	t (seconds)	$D(t)$	t (hours)	t (seconds)	$D(t)$
0	0	3.2	7	25,200	1.3
1	3600	2.7	8	28,800	2.8
2	7200	1.9	9	32,400	5.7
3	10,800	1.7	10	36,000	7.1
4	14,400	1.3	11	39,600	7.7
5	18,000	1.0	12	43,200	7.9
6	21,600	1.1			

Use Simpson's Rule with $n = 12$ and $\Delta t = 3600$ to estimate the integral.

$$\begin{aligned}\int_0^{43,200} A(t)\,dt &\approx \frac{\Delta t}{3}[D(0) + 4D(3600) + 2D(7200) + \cdots + 4D(39,600) + D(43,200)] \\ &\approx \frac{3600}{3}[3.2 + 4(2.7) + 2(1.9) + 4(1.7) + 2(1.3) + 4(1.0) \\ &\qquad + 2(1.1) + 4(1.3) + 2(2.8) + 4(5.7) + 2(7.1) + 4(7.7) + 7.9] \\ &= 143,880\end{aligned}$$

The total amount of data transmitted from midnight to noon is approximately 144,000 megabits, or 144 gigabits. ■

Table 5.6 shows how Simpson's Rule compares with the Midpoint Rule for the integral $\int_1^2 \frac{1}{x}\,dx$, whose true value is about 0.69314718. Table 5.7 shows how the error E_S in Simpson's Rule decreases by a factor of about 16 when n is doubled. (Exercises 25 and 26 involve this concept for two additional integrals.)

n	M_n	S_n
4	0.69121989	0.69315453
8	0.69266055	0.69314765
16	0.69302521	0.69314721

Table 5.6
Simpson's Rule compared with the Midpoint Rule.

n	E_M	E_S
4	0.00192729	−0.00000735
8	0.00048663	−0.00000047
16	0.00012197	−0.00000003

Table 5.7
Simpson's Rule and Midpoint Rule error comparison.

This is consistent with the appearance of n^4 in the denominator of the following error estimate for Simpson's Rule. It is similar to the estimates given in Equation 3 for the Trapezoidal and Midpoint Rules, but it involves the fourth derivative of f.

Error Bound for Simpson's Rule

Suppose that $|f^{(4)}(x)| \le K$ for $a \le x \le b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \le \frac{K(b-a)^5}{180n^4} \tag{4}$$

Example 6 Simpson's Rule with Specified Accuracy

How large should n be in order to guarantee that the Simpson's Rule approximation for $\int_1^2 \frac{1}{x}\,dx$ is accurate to within 0.0001?

Solution

If $f(x) = \dfrac{1}{x}$ then $f^{(4)}(x) = \dfrac{24}{x^5}$.

Since $x \ge 1$, then $\dfrac{1}{x} \le 1 \quad \Rightarrow \quad \left|f^{(4)}(x)\right| = \left|\dfrac{24}{x^5}\right| \le 24.$

Let $K = 24$ in Equation 4. For an error of less than 0.0001, choose n so that

$$\frac{24(1)^5}{180n^4} < 0.0001 \quad \Rightarrow \quad n^4 > \frac{24}{180(0.0001)} \quad \Rightarrow \quad n > \frac{1}{\sqrt[4]{0.00075}} \approx 6.04.$$

Therefore, $n = 8$ (n must be even) will ensure the desired accuracy. Compare this with Example 2, where we found $n = 41$ for the Trapezoidal Rule and $n = 29$ for the Midpoint Rule. ∎

Example 7 Estimate the Error in Using Simpson's Rule

(a) Use Simpson's Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.

(b) Estimate the error involved in this approximation.

Solution

(a) Use Simpson's Rule with $n = 10$ and $\Delta x = 0.1$.

$$\begin{aligned}\int_0^1 e^{x^2}dx &\approx \frac{\Delta x}{3}[f(0) + 4f(0.1) + 2f(0.2) + \cdots + 2f(0.8) + 4f(0.9) + f(1)] \\ &= \frac{0.1}{3}[e^0 + 4e^{0.01} + 2e^{0.04} + 4e^{0.09} + 2e^{0.16} + 4e^{0.25} + 2e^{0.36} \\ &\qquad + 4e^{0.49} + 2e^{0.64} + 4e^{0.81} + e^1] \\ &\approx 1.462681\end{aligned}$$

Figure 5.68 illustrates this calculation. Notice that the parabolic arcs are so close to the graph of $y = e^{x^2}$ that they are difficult to distinguish.

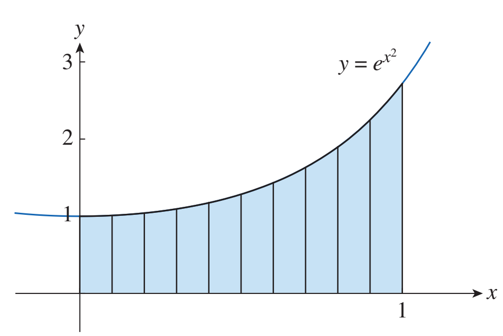


Figure 5.68
Visualization of Simpson's Rule.

(b) $f(x) = e^{x^2} \;\Rightarrow\; f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2}$

Since $0 \le x \le 1$, $0 \le f^{(4)}(x) \le (12 + 48 + 16)e^1 = 76e$.

Let $K = 76e$, $a = 0$, $b = 1$, and $n = 10$ in Equation 4. The error is at most

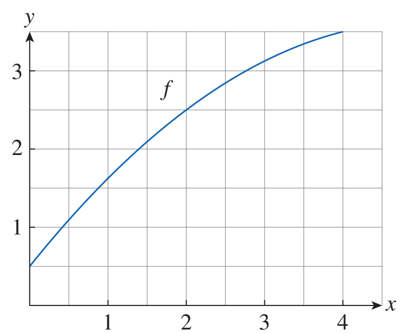
$$\frac{76e(1)^5}{180(10)^4} \approx 0.000115.$$

Compare this with Example 3. Thus, correct to three decimal places, we have

$$\int_0^1 e^{x^2} dx \approx 1.463.$$

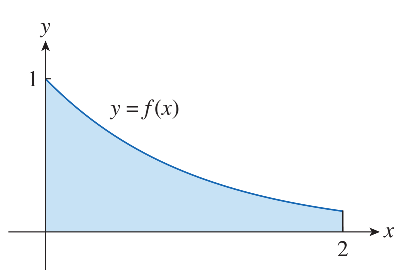
5.9 Exercises

1. Let $I = \int_0^4 f(x)\, dx$, where f is the function whose graph is shown.



(a) Use the graph to find L_2, R_2, and M_2.
(b) Determine whether each value in part (a) is an overestimate or underestimate of I.
(c) Use the graph to find T_2. Is this value an overestimate or underestimate of I ?
(d) For any value of n list the numbers L_n, R_n, M_n, T_n, and I in increasing order.

2. The left, right, Trapezoidal, and Midpoint Rule approximations were used to estimate $\int_0^2 f(x)\, dx$, where f is the function whose graph is shown.



The estimates were 0.7811, 0.8675, 0.8632, and 0.9540, and the same number of subintervals were used in each case.
(a) Determine which rule produced each estimate.
(b) Between which two approximations does the true value of $\int_0^2 f(x)\, dx$ lie?

3. Let $I = \int_0^1 \cos(x^2)\, dx$. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule, each with $n = 4$ to estimate I. Use a graph of the integrand to determine whether each value is an underestimate or an overestimate of I. What can you conclude about the true value of the integral?

4. Draw the graph of $f(x) = \sin\left(\frac{1}{2}x^2\right)$ in the viewing rectangle $[0, 1] \times [0, 0.5]$ and let $I = \int_0^1 f(x)\, dx$.
(a) Use the graph to determine whether each value, L_2, R_2, M_2, and T_2, is an underestimate or overestimate of I.
(b) For any value of n, list the numbers L_n, R_n, M_n, T_n, and I in increasing order.
(c) Compute L_5, R_5, M_5, and T_5. Use the graph to decide which of these values in the best estimate of I.

Use (a) the Midpoint Rule and (b) Simpson's Rule to approximate the given integral with the specified value of n. Round your answers to six decimal places. Compare your results to the actual value to determine the error in each approximation.

5. $\displaystyle\int_0^2 \frac{x}{1 + x^2}\, dx, \quad n = 10$

6. $\displaystyle\int_0^\pi x \cos x\, dx, \quad n = 4$

Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified value of n. Round your answers to six decimal places.

7. $\displaystyle\int_0^2 \sqrt[4]{1 + x^2}\, dx, \quad n = 8$

8. $\displaystyle\int_0^{1/2} \sin(x^2)\, dx, \quad n = 4$

9. $\displaystyle\int_0^2 \frac{\ln x}{1 + x}\, dx, \quad n = 10$

10. $\displaystyle\int_0^3 \frac{dt}{1 + t^2 + t^4}, \quad n = 6$

11. $\int_0^{1/2} \sin(e^{t/2})\,dt, \quad n = 8$

12. $\int_0^4 \sqrt{1 + \sqrt{x}}\,dx, \quad n = 8$

13. $\int_0^4 e^{\sqrt{t}} \sin t\,dt, \quad n = 8$

14. $\int_0^4 \cos\sqrt{x}\,dx, \quad n = 10$

15. $\int_1^5 \frac{\cos x}{x}\,dx, \quad n = 8$

16. $\int_4^6 \ln(x^3 + 2)\,dx, \quad n = 10$

17. (a) Find the approximations T_8 and M_8 for the integral $\int_0^1 \cos(x^2)\,dx$.
(b) Estimate the errors in the approximations of part (a).
(c) How large should n be so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.0001?

18. (a) Find the approximations T_{10} and M_{10} for the integral $\int_1^2 e^{1/x}\,dx$.
(b) Estimate the errors in the approximations of part (a).
(c) How large should n be so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.0001?

19. (a) Find the approximations T_{10}, M_{10}, and S_{10} for the integral $\int_0^{\pi} \sin x\,dx$ and the corresponding errors E_T, E_M, and E_S.
(b) Compare the actual errors in part (a) with the error estimates given by Equations 3 and 4.
(c) How large should n be so that the approximations T_n, M_n, and S_n to the integral in part (a) are accurate to within 0.00001?

20. How large should n be to guarantee that the Simpson's Rule approximation to $\int_0^1 e^{x^2}\,dx$ is accurate to within 0.00001?

21. It is often difficult to determine the fourth derivative of a function and find a reasonable upper bound K for $|f^{(4)}(x)|$. However, we can usually use technology to find and sketch a graph of $f^{(4)}$. We can then use the graph to find a value for K. This exercise involves approximations to the integral

$$I = \int_0^{2\pi} f(x)\,dx, \text{ where } f(x) = e^{\cos x}.$$

(a) Use a graph to obtain a reasonable upper bound for $|f''(x)|$.
(b) Use M_{10} to approximate I.
(c) Use part (a) to estimate the error in part (b).
(d) Use technology to evaluate I.
(e) How does the actual error compare with the error estimate in part (c)?
(f) Use a graph to obtain a reasonable upper bound for $|f^{(4)}(x)|$.
(g) Use S_{10} to approximate I.
(h) Use part (f) to estimate the error in part (g).
(i) How does the actual error compare with the error estimate in part (h)?
(j) How large should n be to guarantee that the magnitude of the error in using S_n is less than 0.0001?

22. Repeat Exercises 21 for the integral $\int_{-1}^1 \sqrt{4 - x^3}\,dx$.

Find the approximations L_n, R_n, T_n, and M_n for $n = 5$, 10, and 20. Then compute the corresponding errors E_L, E_R, E_T, and E_M. Round your answers to six decimal places. What observations can you make? In particular, what happens to the errors when n is doubled?

23. $\int_0^1 xe^x\,dx$

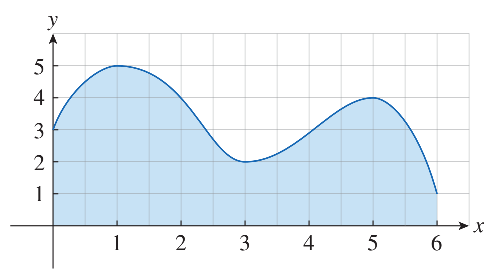
24. $\int_1^2 \frac{1}{x^2}\,dx$

Find the approximations T_n, M_n, and S_n for $n = 6$ and 12. Then compute the corresponding errors E_T, E_M, and E_S. Round your answers to six decimal places. What observations can you make? In particular, what happens to the errors when n is doubled?

25. $\int_0^2 x^4\,dx$

26. $\int_1^4 \frac{1}{\sqrt{x}}\,dx$

27. The graph of $y = f(x)$ is shown in the figure.



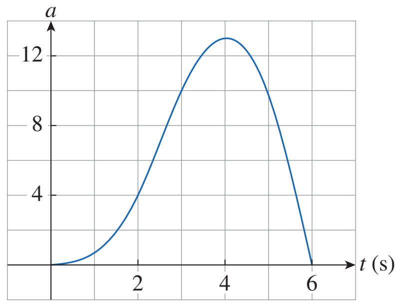
Estimate the area of the shaded region by using (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule, each with $n = 6$.

28. A radar gun was used to record the speed v of a runner during the first 5 seconds of a race. Values of v for selected times t are given in the table.

t (s)	v (m/s)	t (s)	v (m/s)
0	0	3.0	10.51
0.5	4.67	3.5	10.67
1.0	7.34	4.0	10.76
1.5	8.86	4.5	10.81
2.0	9.73	5.0	10.81
2.5	10.22		

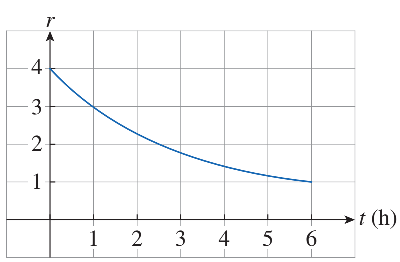
Use Simpson's Rule to estimate the distance the runner covered during the first 5 seconds.

29. The graph of the acceleration a of a car measured in ft/s^2 is shown in the figure.



Use Simpson's Rule to estimate the increase in the velocity of the car during the 6-second time interval.

30. Water is leaking from a tank at a rate of $r(t)$ liters per hour. The graph of r is shown in the figure.



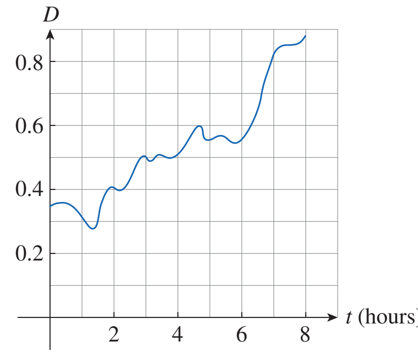
Use Simpson's Rule to estimate the total amount of water that leaked from the tank during the first 6 hours.

31. The table (supplied by San Diego Gas and Electric) gives the power consumption P in megawatts in San Diego County from midnight to 6:00 AM on a day in December.

t	P	t	P
0:00	1814	3:30	1611
0:30	1735	4:00	1621
1:00	1686	4:30	1666
1:30	1646	5:00	1745
2:00	1637	5:30	1886
2:30	1609	6:00	2052
3:00	1604		

Use Simpson's Rule to estimate the energy used during that time period. Use the fact that power is the derivative of energy.

32. The graph shows the traffic on an Internet service provider's T1 data line from midnight to 8:00 AM.



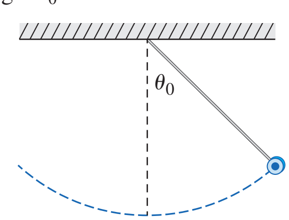
D is the data throughput, measured in megabits per second. Use Simpson's Rule to estimate the total amount of data transmitted during that time period.

33. (a) Use the Midpoint Rule and the data in the table to estimate the value of the integral $\int_0^{3.2} f(x)\,dx$.

x	$f(x)$	x	$f(x)$
0.0	6.8	2.0	7.6
0.4	6.5	2.4	8.4
0.8	6.3	2.8	8.8
1.2	6.4	3.2	9.0
1.6	6.9		

(b) Suppose it is known that $-4 \le f''(x) \le 1$ for all x. Estimate the error involved in the approximation in part (a).

34. The figure shows a pendulum with length L that makes a maximum angle θ_0 with the vertical.



Using Newton's Second Law, it can be shown that the period T (the time for one complete swing) is given by

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity. If $L = 1$ m and $\theta_0 = 42°$, use Simpson's Rule with $n = 10$ to find the period.

35. The intensity of light with wavelength λ traveling through a diffraction grating with N slits at an angle θ is given by

$$I(\theta) = \frac{N^2 \sin^2 k}{k^2}$$

where $k = \dfrac{\pi N d \sin\theta}{\lambda}$ and d is the distance between adjacent slits.

A helium-neon laser with wavelength $\lambda = 632.8 \times 10^{-9}$ m is emitting a narrow band of light, given by $-10^{-6} < \theta < 10^{-6}$, through a grating with 10,000 slits spaced 10^{-4} m apart. Use the Midpoint Rule with $n = 10$ to estimate the total light intensity $\displaystyle\int_{-10^{-6}}^{10^{-6}} I(\theta)\, d\theta$ emerging from the grating.

36. Sketch the graph of a continuous function on the interval [0, 2] for which the right endpoint approximation with $n = 2$ is more accurate than Simpson's Rule.

37. Sketch the graph of a continuous function on the interval [0, 2] for which the Trapezoidal Rule with $n = 2$ is more accurate than the Midpoint Rule.

38. Use the Trapezoidal Rule with $n = 10$ to approximate $\displaystyle\int_0^{20} \cos(\pi x)\, dx$. Compare your result to the actual value. Explain the discrepancy.

39. If f is a positive function and $f''(x) < 0$ for $a \le x \le b$, show that

$$T_n < \int_a^b f(x)\, dx < M_n$$

40. Show that if f is a polynomial of degree 3 or less, then Simpson's Rule gives the exact value of $\displaystyle\int_a^b f(x)\, dx$.

41. Show that $\frac{1}{2}(T_n + M_n) = T_{2n}$.

42. Show that $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

5.10 Improper Integrals

In the definition of the definite integral $\displaystyle\int_a^b f(x)\, dx$, we assumed that the function f was defined on a finite interval $[a, b]$ and that f did not have an infinite discontinuity. In this section, we extend the concept of a definite integral to the case in which the interval is infinite and also to the case in which f has an infinite discontinuity in $[a, b]$. In either case, the integral is called an *improper* integral.

Type 1: Infinite Intervals

Consider the infinite region S that lies under the curve $y = \dfrac{1}{x^2}$, above the x-axis, and to the right of the line $x = 1$. It seems reasonable that since S is infinite in extent, its area must be infinite, but let's take a closer look. The area of the part of S that lies to the left of the line $x = t$, as shown in Figure 5.69, is

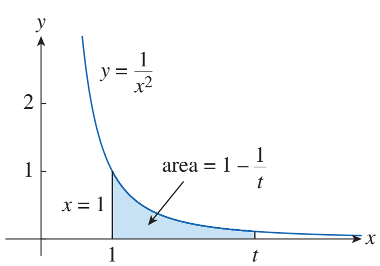


Figure 5.69
The area of the part of the region R to the left of the line $x = t$.

$$A(t) = \int_1^t \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t}$$

Notice that $A(t) < 1$ no matter how large a value of t is selected. However, consider the limit as t increases without bound.

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

This limit indicates that the area of the shaded region approaches 1 as $t \to \infty$. It also suggests the *area* of the infinite region S is 1. We express this result mathematically using the following notation.

$$\int_1^\infty \frac{1}{x^2}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\, dx = 1$$

A graphical interpretation of this expression and limit is shown in Figure 5.70.

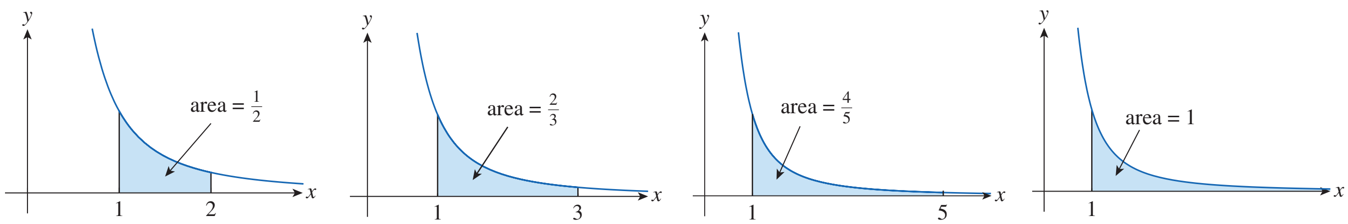


Figure 5.70
The area of the shaded region approaches 1 as $t \to \infty$.

Using this example as a guide, we define the integral of f (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

Definition • Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$

provided that this limit exists as a finite number.

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t \to -\infty} \int_t^b f(x)\,dx$$

provided that this limit exists as a finite number.

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c), any real number a can be used.

A Closer Look

1. Remember that ∞ is a symbol, not a number. You cannot evaluate an antiderivative of f at ∞ or $-\infty$.
2. Evaluating a Type 1 improper integral is a two-step process. First, evaluate a definite integral (and simplify), and then evaluate the limit.
3. Any of the improper integrals in the definition above can be interpreted as an area, provided that f is a positive function. For example, in case (a), if $f(x) \geq 0$ and the integral $\int_a^\infty f(x)\,dx$ is convergent, then we define the area of the region

$S = \{(x, y) \mid x \geq a, 0 \leq y \leq f(x)\}$ in Figure 5.71 to be

$$A(S) = \int_a^\infty f(x)\,dx$$

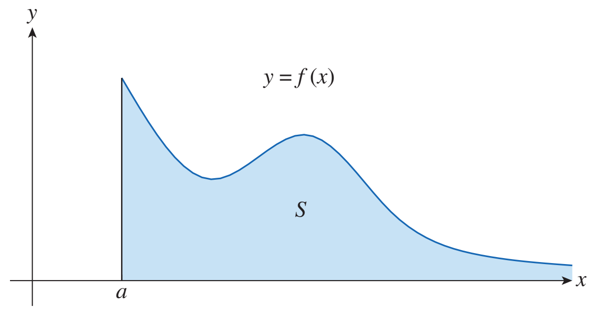


Figure 5.71

$\int_a^\infty f(x)\,dx$ is the limit as $t \to \infty$ of the area under the graph of f from a to t.

Example 1 Infinite Interval

Determine whether the integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent.

Solution

The interval is infinite, so this is a Type 1 improper integral.

$$\begin{aligned}
\int_1^\infty \frac{1}{x}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx && \text{Improper integral definition.} \\
&= \lim_{t\to\infty} \big[\ln|x|\big]_1^t && \text{Table of Indefinite Integrals.} \\
&= \lim_{t\to\infty} [\ln t - \ln 1] && \text{FTC2.} \\
&= \lim_{t\to\infty} \ln t = \infty && \ln t \text{ increases without bound as } t \to \infty.
\end{aligned}$$

Remember this notation means that the limit does not exist as a finite number; it describes the behavior of the function $\ln t$ as $t \to \infty$.

Therefore, the improper integral $\int_1^\infty \frac{1}{x}\,dx$ is divergent.

Compare these first two examples of improper integrals:

$$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges} \qquad \int_1^\infty \frac{1}{x}\,dx \text{ diverges}$$

The graphs of $y = \frac{1}{x^2}$ and $y = \frac{1}{x}$ look very similar for $x > 0$. However, the area under the graph of $y = \frac{1}{x^2}$ to the right of $x = 1$ is finite (see Figure 5.72), but the area under the graph of $y = \frac{1}{x}$ is infinite (see Figure 5.73).

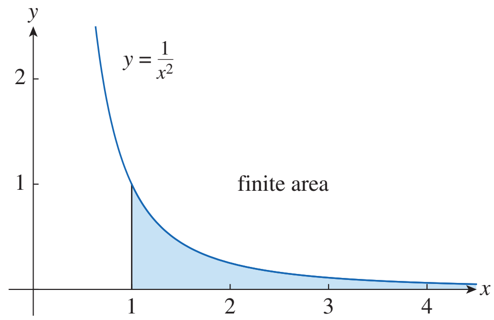


Figure 5.72

$\int_1^{\infty} \frac{1}{x^2}\,dx$ converges.

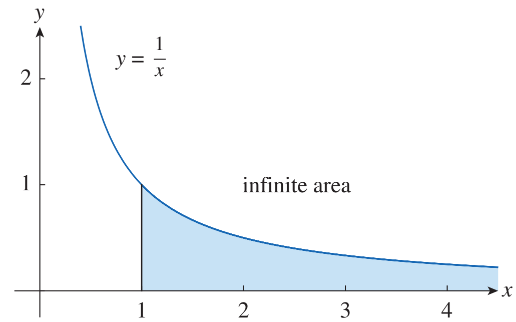


Figure 5.73

$\int_1^{\infty} \frac{1}{x}\,dx$ diverges.

Note that both $\frac{1}{x^2}$ and $\frac{1}{x}$ approach 0 as $x \to \infty$, but $\frac{1}{x^2}$ approaches 0 faster than $\frac{1}{x}$. The values of $\frac{1}{x}$ don't decrease fast enough for the integral to converge or have a finite value.

Example 2 L'Hospital's Rule with an Improper Integral

Evaluate $\int_{-\infty}^{0} xe^x\,dx$.

Solution

The interval is infinite; this is a Type 1 improper integral.

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{t \to -\infty} \int_t^0 xe^x\,dx \qquad \text{Improper integral definition.}$$

Use integration by parts to find an antiderivative.

$$u = x \qquad dv = e^x\,dx$$
$$du = dx \qquad v = \int e^x\,dx = e^x$$

$$= \lim_{t \to -\infty} \left(\left[xe^x\right]_t^0 - \int_t^0 e^x\,dx \right) \qquad \text{Integration by parts.}$$

$$= \lim_{t \to -\infty} \left(\left[0 - te^t\right] - \left[1 - e^t\right] \right) \qquad \text{FTC2.}$$

$$= \lim_{t \to -\infty} \frac{-t}{e^{-t}} - \lim_{t \to -\infty} 1 + \lim_{t \to -\infty} e^t \qquad \text{Rewrite first term; limit properties.}$$

$$= \lim_{t \to -\infty} \frac{-1}{-e^{-t}} - 1 + 0 \qquad \text{L'Hospital's Rule; evaluate limits.}$$

$$= \lim_{t \to -\infty} (e^t) - 1 = 0 - 1 = -1 \qquad \text{Evaluate limit.}$$

Therefore, the improper integral $\int_{-\infty}^{0} xe^x\,dx$ is convergent. ■

Example 3 Consider Two Separate Integrals

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Common Error

$\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx$

Correct Method

This improper integral must be split into two integrals, evaluated separately.

Solution

The interval is infinite, and in this case it is necessary to split the integral into two separate improper integrals. We can choose to split at any real number, but $a = 0$ often leads to an easier evaluation.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx \qquad \text{Improper integral definition.}$$

We need to evaluate the integrals on the right side separately.

$$\begin{aligned}
\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{t\to\infty}\int_{0}^{t} \frac{1}{1+x^2}\,dx && \text{Improper integral definition.}\\
&= \lim_{t\to\infty}\Big[\tan^{-1}x\Big]_0^t && \text{Table of Indefinite Integrals.}\\
&= \lim_{t\to\infty}\left[\tan^{-1}t - \tan^{-1}0\right] && \text{FTC2.}\\
&= \lim_{t\to\infty}\tan^{-1}t = \frac{\pi}{2} && \text{Evaluate limit.}
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{t} \frac{1}{1+x^2}\,dx &= \lim_{t\to-\infty}\int_{t}^{0} \frac{1}{1+x^2}\,dx && \text{Improper integral definition.}\\
&= \lim_{t\to-\infty}\Big[\tan^{-1}x\Big]_t^0 && \text{Table of Indefinite Integrals.}\\
&= \lim_{t\to-\infty}\left[\tan^{-1}0 - \tan^{-1}t\right] && \text{FTC2.}\\
&= -\lim_{t\to-\infty}\tan^{-1}t = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2} && \text{Evaluate limit.}
\end{aligned}$$

Since both integrals are convergent, the given integral is convergent, and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $\dfrac{1}{1+x^2} > 0$, the given improper integral can be interpreted as the area of the infinite region that lies under the graph of $y = \dfrac{1}{1+x^2}$ and above the x-axis, as shown in Figure 5.74. ■

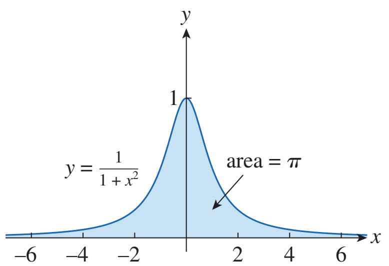


Figure 5.74
The region is of infinite extent but has finite area, equal to π.

Example 4 Improper Integral with a Parameter

For what values of p is the integral $\displaystyle\int_{1}^{\infty} \frac{1}{x^p}\,dx$ convergent?

Solution

We know from Example 1 that if $p = 1$, then the integral is divergent.
So assume $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx \qquad \text{Improper integral definition.}$$

$$= \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t \qquad \text{Power Rule.}$$

$$= \lim_{t\to\infty}\frac{1}{1-p}\left[\frac{1}{t^{p-1}}-1\right] \qquad \text{FTC2.}$$

If $p > 1$, then $p - 1 > 0$. As $t \to \infty$, $t^{p-1} \to \infty$, and $\dfrac{1}{t^{p-1}} \to 0$.

Therefore, $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$ if $p > 1$ and the integral converges.

If $p < 1$, then $p - 1 < 0$. As $t \to \infty$, $\dfrac{1}{t^{p-1}} = t^{1-p} \to \infty$, and the integral diverges. ■

We will use the solution to Example 4 often. Here is a summary of the results.

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ is convergent if } p > 1 \text{ and divergent if } p \le 1 \qquad (1)$$

Type 2: Discontinuous Integrands

Suppose that f is a positive continuous function defined on a finite interval $[a, b)$ and that the graph of f has a vertical asymptote at b. Let S be the unbounded region below the graph of f and above the x-axis between a and b. For Type 1 improper integrals the regions extend indefinitely in a horizontal direction. In this case the region is infinite in a vertical direction. The area of the part of S between a and t, as shown in Figure 5.75, is

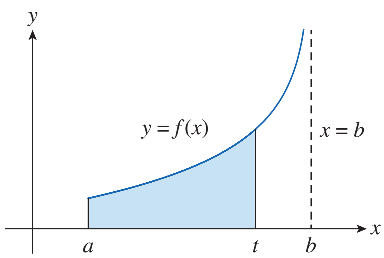


Figure 5.75
The area of the part of the region S between a and t. The region S is infinite in the vertical direction.

$$A(t) = \int_a^t f(x)\,dx$$

If $A(t)$ approaches a definite number A as $t \to b^-$, then we say that the area of the region S is A, and we express this result mathematically using the following notation.

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

This equation is used to define an improper integral of Type 2 even when f is not a positive function.

Definition • Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

if this limit exists as a finite number.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

if this limit exists as a finite number.

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

A Closer Look

1. Evaluating a Type 2 improper integral is also a two-step process. First, evaluate a definite integral (and simplify), and then evaluate the limit.
2. Parts (b) and (c) of this definition are illustrated in Figures 5.76 and 5.77 for the case in which $f(x) \geq 0$. In Figure 5.76 the graph of f has a vertical asymptote at $x = a$. In Figure 5.77, the graph of f has a vertical asymptote at $x = c$.

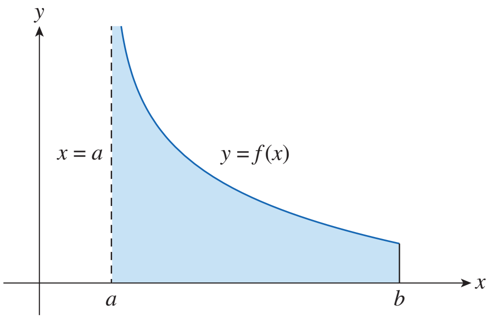


Figure 5.76
A graphical representation of the improper integral $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$.

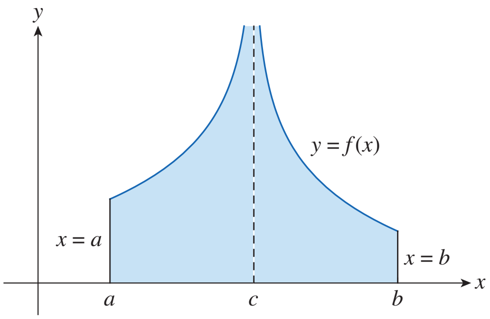


Figure 5.77
A graphical representation of the improper integral $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Example 5 Discontinuity on the Left

Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

Solution

This is an improper integral because $f(x) = \dfrac{1}{\sqrt{x-2}}$ is discontinuous at $x = 2$; the function is not defined at $x = 2$ and the graph of f has a vertical asymptote at $x = 2$.

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx \qquad \text{Improper integral definition.}$$

$$= \lim_{t\to 2^+} \left[2\sqrt{x-2}\right]_t^5 \qquad u = x - 2\text{; Power Rule.}$$

$$= \lim_{t\to 2^+} 2\left[\sqrt{3} - \sqrt{t-2}\right] = 2\sqrt{3} \qquad \text{FTC2; evaluate limit.}$$

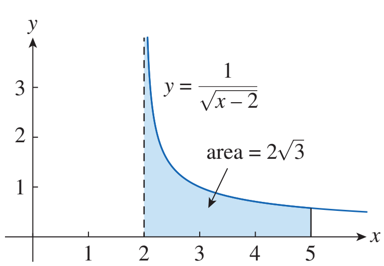


Figure 5.78
A graphical interpretation of the improper integral $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

Therefore, the given improper integral is convergent, and since the integrand is positive, we can interpret the value of the integral as the area of the shaded region in Figure 5.78. ■

Example 6 Discontinuity on the Right

Determine whether $\int_0^{\pi/2} \sec x\, dx$ converges or diverges.

Solution

This integral is improper because $\sec x$ is discontinuous at $x = \dfrac{\pi}{2}$:
$\lim\limits_{x\to(\pi/2)^-} \sec x = \infty$

$$\begin{aligned}
\int_0^{\pi/2} \sec x\, dx &= \lim_{x\to(\pi/2)^-} \int_0^t \sec x\, dx && \text{Improper integral definition.}\\
&= \lim_{x\to(\pi/2)^-} \Big[\ln|\sec x + \tan x|\Big]_0^t && \text{Table of Indefinite Integrals.}\\
&= \lim_{x\to(\pi/2)^-} \big[\ln(\sec t + \tan t) - \ln(\sec 0 + \tan 0)\big] && \text{FTC2.}\\
&= \lim_{x\to(\pi/2)^-} \big[\ln(\sec t + \tan t) - \ln 1\big] = \infty && \text{Simplify; evaluate limit; } \sec t \to \infty \text{ and } \tan t \to \infty \text{ as } t \to (\pi/2)^-.
\end{aligned}$$

Therefore, the given improper integral is divergent. ■

Example 7 Discontinuity in an Interval

Determine whether $\int_0^3 \dfrac{dx}{x-1}$ converges or diverges.

Solution

The function $f(x) = \dfrac{1}{x-1}$ is discontinuous at $x = 1$; f is not defined at $x = 1$ and the graph of f has a vertical asymptote at $x = 1$.

Therefore, we need to split the given integral into two separate improper integrals.

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1} \qquad \text{Improper integral definition.}$$

Consider the first integral, with bounds 0 and 1.

$$\begin{aligned}
\int_0^1 \frac{dx}{x-1} &= \lim_{t\to1^-} \int_0^t \frac{dx}{x-1} && \text{Improper integral definition.}\\
&= \lim_{t\to1^-} \Big[\ln|x-1|\Big]_0^t && \text{Substitution: } u = x - 1.\\
&= \lim_{t\to1^-} \big[\ln|t-1| - \ln|-1|\big] && \text{FTC2.}\\
&= \lim_{t\to1^-} \ln(1-t) = -\infty && \text{Simplify; evaluate limit; as } t\to1^-,\ 1-t\to0^+.
\end{aligned}$$

Therefore, $\int_0^1 \frac{dx}{x-1}$ is divergent. This implies that the given integral, $\int_0^3 \frac{dx}{x-1}$ is divergent. We do not need to evaluate $\int_1^3 \frac{dx}{x-1}$. ■

Note: Suppose we fail to notice the discontinuity and asymptote $x = 1$ in Example 7. This leads to an ordinary integral evaluation and an incorrect final answer:

$$\int_0^3 \frac{dx}{x-1} = \Big[\ln|x-1|\Big]_0^3 = \ln 2 - \ln 1 = \ln 2$$

This calculation is wrong because the integral is improper and must be evaluated in terms of limits.

Therefore, whenever you see an integral of the form $\int_a^b f(x)\,dx$, you need to decide whether this is an ordinary definite integral or an improper integral. This determination is made by carefully considering the function f on the interval $[a, b]$.

Example 8 L'Hospital's Rule with an Improper Integral

Evaluate $\int_0^1 \ln x\,dx$.

Solution

The function $f(x) = \ln x$ is not defined at $x \le 0$; the graph of f has a vertical asymptote at $x = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$.

Therefore, the given integral is improper.

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x\,dx \qquad \text{Improper integral definition.}$$

Use integration by parts to find an antiderivative.

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}\,dx \qquad v = \int dx = x$$

$$\int_t^1 \ln x\,dx = \big[x \ln x\big]_t^1 - \int_t^1 dx \qquad \text{Integration by parts.}$$

$$= \big[1 \ln 1 - t \ln t\big] - \big[x\big]_t^1 \qquad \text{FTC2; antiderivative.}$$

$$= -t \ln t - [1 - t] = -t \ln t - 1 + t \qquad \text{FTC2; simplify.}$$

Use l'Hospital's Rule to find the limit of the first term.

$$\lim_{t\to 0^+} t \ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} \qquad \text{Rewrite to use l'Hospital's Rule.}$$

$$= \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} (-t) = 0 \qquad \text{Derivative of the numerator and denominator; simplify, evaluate limit.}$$

Therefore,

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} (-t \ln t - 1 + t) = -0 - 1 + 0 = -1.$$

Figure 5.79 shows the geometric interpretation of this result. The area bounded by the graph of $y = \ln x$ and the x-axis between 0 and 1 is 1. ■

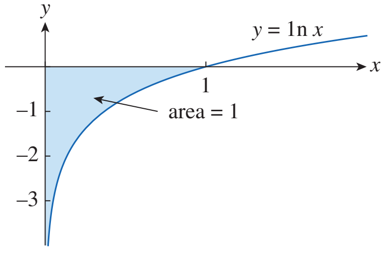


Figure 5.79
Geometric interpretation of the improper integral $\int_0^1 \ln x\,dx$.

A Comparison Test for Improper Integrals

Sometimes it is impossible to find the exact value of an improper integral but still possible to determine whether it is convergent or divergent. It seems reasonable to compare known integrals to a given integral in order to draw a conclusion. Although this result is stated for Type 1 integrals, a similar theorem is true for Type 2 integrals.

Comparison Theorem

Suppose that f and g are continuous functions such that $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\displaystyle\int_a^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\int_a^{\infty} g(x)\,dx$ is convergent.

(b) If $\displaystyle\int_a^{\infty} g(x)\,dx$ is divergent, then $\displaystyle\int_a^{\infty} f(x)\,dx$ is divergent.

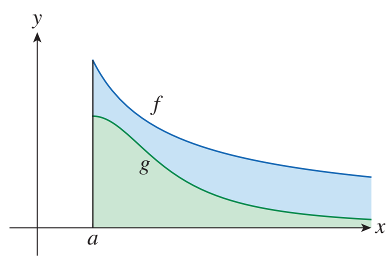


Figure 5.80
The figure suggests that if the area under the top graph of $y = f(x)$ is finite, then so is the area under the bottom graph of $y = g(x)$. And if the area under the graph of $y = g(x)$ is infinite, then so is the area under the graph of $y = f(x)$.

A Closer Look

1. Figure 5.80 illustrates the Comparison Theorem.
2. The converse of this theorem is not necessarily true.

If $\displaystyle\int_a^{\infty} g(x)\,dx$ is convergent, then $\displaystyle\int_a^{\infty} f(x)\,dx$ may or may not be convergent.

And if $\displaystyle\int_a^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\int_a^{\infty} g(x)\,dx$ may or may not be divergent. ■

Example 9 Convergent by Comparison

Show that $\displaystyle\int_0^{\infty} e^{-x^2}\,dx$ is convergent.

Solution

We cannot evaluate this integral explicitly because the antiderivative of e^{-x^2} is not an elementary function.

Start by writing the integral as two separate integrals.

$$\int_0^{\infty} e^{-x^2}\,dx = \int_0^{1} e^{-x^2}\,dx + \int_1^{\infty} e^{-x^2}\,dx$$

The first integral on the right side is an ordinary definite integral. Although the antiderivative is not an elementary function, geometrically the integral represents the area of a bounded region and therefore is finite. See Figure 5.81.

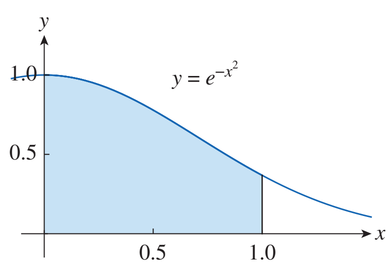


Figure 5.81
Geometrically, the integral $\displaystyle\int_0^1 e^{-x^2}\,dx$ represents the area of the shaded (bounded) region.

For the second integral,

$$x \geq 1 \quad \Rightarrow \quad x^2 \geq x \quad \Rightarrow \quad -x^2 \leq -x \quad \Rightarrow \quad e^{-x^2} \leq e^{-x}.$$

Consider the following integral:

$$\int_1^{\infty} e^{-x}\,dx = \lim_{t\to\infty}\int_1^{t} e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_1^t = \lim_{t\to\infty}\left[-e^{-t} + e^{-1}\right] = e^{-1}$$

Use the Comparison Theorem: $g(x) = e^{-x^2} \leq e^{-x} = f(x)$.

Since $\displaystyle\int_1^{\infty} e^{-x}\,dx$ is convergent, so is $\displaystyle\int_1^{\infty} e^{-x^2}\,dx$, and therefore,

$$\int_0^{\infty} e^{-x^2}\,dx \text{ is convergent.}$$

Figure 5.82 illustrates this result.

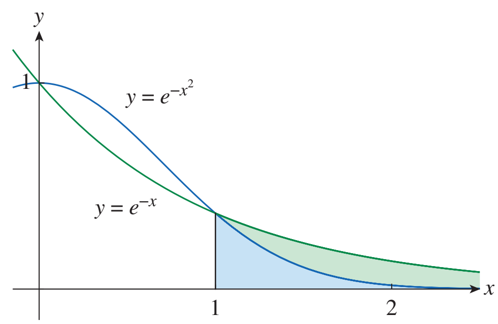


Figure 5.82
The area under the graph of $y = e^{-x}$ over the interval $[1, \infty)$ is finite. Therefore, the area under the graph of $y = e^{-x^2}$ over the interval $[0, \infty)$ is also finite.

Using Riemann sums, it can be shown that the value of $\int_0^{\infty} e^{-x^2}\,dx$ is approximately 0.8862. These sums converge very quickly because $e^{-x^2} \to 0$ very rapidly as $x \to \infty$. In probability and statistics applications, it is important to know the exact value of the improper integral. Using methods of multivariable calculus, it can be shown that this value is $\dfrac{\sqrt{\pi}}{2}$. ■

t	$\int_1^t \frac{1+e^{-x}}{x}\,dx$
2	0.8636306042
5	1.8276735512
10	2.5219648704
100	4.8245541204
500	6.4339920328
1000	7.1271392134
5000	8.7365771258
10,000	9.4297243064

Table 5.8
As x increases without bound, the value of the definite integral also increases without bound. This table suggests that the definite integral increases very slowly, but the analytical result shows that this integral does indeed diverge.

Example 10 Divergent by Comparison

Show that $\int_1^{\infty} \frac{1+e^{-x}}{x}\,dx$ is divergent.

Solution

We need to find an appropriate integral for comparison.

$$1 + e^{-x} \geq 1 \quad \Rightarrow \quad \frac{1+e^{-x}}{x} \geq \frac{1}{x}$$

$\int_1^{\infty} \frac{1}{x}\,dx$ is divergent ($p = 1$).

Therefore, $\int_1^{\infty} \frac{1+e^{-x}}{x}\,dx$ is divergent by comparison.

Table 5.8 provides some numerical evidence that this integral diverges. ■

5.10 Exercises

1. Explain why each of the following integrals is improper.

(a) $\int_1^2 \frac{x}{x-1}\,dx$ (b) $\int_0^{\infty} \frac{1}{1+x^3}\,dx$

(c) $\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx$ (d) $\int_0^{\pi/4} \cot x\,dx$

2. Which of the following integrals are improper? Why?

(a) $\int_0^{\pi/4} \tan x\,dx$ (b) $\int_0^{\pi} \tan x\,dx$

(c) $\int_{-1}^{1} \frac{dx}{x^2 - x - 2}$ (d) $\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$

3. Find the area under the graph of $y = \dfrac{1}{x^3}$ from $x = 1$ to $x = t$ and evaluate your expression for $t = 10$, 100, and 1000. Then find the total area under this graph for $x \geq 1$.

4. (a) Graph the functions $f(x) = \dfrac{1}{x^{1.1}}$ and $g(x) = \dfrac{1}{x^{0.9}}$ in the viewing rectangle $[0, 10] \times [0, 1]$ and then in $[0, 100] \times [0, 1]$.

(b) Find the area under the graphs of f and g from $x = 1$ to $x = t$, and use these expressions to find the areas for $t = 10, 100, 10^4, 10^6, 10^{10}$, and 10^{20}.

(c) Find the total area under each curve for $x \geq 1$, if it exists.

Determine whether the integral is convergent or divergent. Evaluate those that are convergent.

5. $\displaystyle\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx$

6. $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx$

7. $\displaystyle\int_1^2 \frac{dx}{x-1}$

8. $\displaystyle\int_2^3 \frac{dx}{\sqrt{x-2}}$

9. $\displaystyle\int_{-\infty}^0 \frac{1}{3-4x}\,dx$

10. $\displaystyle\int_1^\infty \frac{1}{(2x+1)^3}\,dx$

11. $\displaystyle\int_2^\infty e^{-5p}\,dp$

12. $\displaystyle\int_{-\infty}^0 2^r\,dr$

13. $\displaystyle\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}\,dx$

14. $\displaystyle\int_{-\infty}^\infty (y^3 - 3y^2)\,dy$

15. $\displaystyle\int_0^\infty xe^{-x}\,dx$

16. $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

17. $\displaystyle\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx$

18. $\displaystyle\int_0^\infty \sin^2\alpha\,d\alpha$

19. $\displaystyle\int_0^\infty \sin\theta\, e^{\cos\theta}\,d\theta$

20. $\displaystyle\int_1^\infty \frac{1}{x^2+x}\,dx$

21. $\displaystyle\int_0^\infty \frac{dx}{x^3+1}$

22. $\displaystyle\int_2^\infty \frac{dv}{v^2+2v-3}$

23. $\displaystyle\int_{-\infty}^0 ze^{2z}\,dz$

24. $\displaystyle\int_2^\infty ye^{-3y}\,dy$

25. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

26. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

27. $\displaystyle\int_{-\infty}^0 \frac{z}{z^4+4}\,dz$

28. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

29. $\displaystyle\int_0^\infty e^{-\sqrt{y}}\,dy$

30. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}+x\sqrt{x}}$

31. $\displaystyle\int_0^1 \frac{1}{x}\,dx$

32. $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,dx$

33. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

34. $\displaystyle\int_{-1}^2 \frac{x}{(x+1)^2}\,dx$

35. $\displaystyle\int_{-2}^3 \frac{1}{x^4}\,dx$

36. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

37. $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

38. $\displaystyle\int_0^5 \frac{w}{w-2}\,dw$

39. $\displaystyle\int_0^{\pi/2} \tan^2\theta\,d\theta$

40. $\displaystyle\int_0^4 \frac{dx}{x^2-x-2}$

41. $\displaystyle\int_0^1 r\ln r\,dr$

42. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta$

43. $\displaystyle\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx$

44. $\displaystyle\int_0^1 \frac{e^{1/x}}{x^3}\,dx$

45. $\displaystyle\int_0^1 x^3\ln x\,dx$

46. $\displaystyle\int_1^\infty \frac{\tan^{-1}x}{1+x^2}\,dx$

Sketch the region and find its area (if the area is finite).

47. $R = \{(x, y) \mid x \geq 1, 0 \leq y \leq e^{-x}\}$

48. $R = \{(x, y) \mid x \leq 0, 0 \leq y \leq e^{x}\}$

49. $R = \left\{(x, y) \mid x \geq 1, 0 \leq y \leq \dfrac{1}{x^3+x}\right\}$

50. $R = \{(x, y) \mid x \geq 0, 0 \leq y \leq xe^{-x}\}$

51. $R = \left\{(x, y) \mid 0 \leq x < \dfrac{\pi}{2}, 0 \leq y \leq \sec^2 x\right\}$

52. $R = \left\{(x, y) \mid -2 < x \leq 0, 0 \leq y \leq \dfrac{1}{\sqrt{x+2}}\right\}$

53. Let $g(x) = \dfrac{\sin^2 x}{x^2}$.

(a) Use technology to construct a table of values of $\int_1^t g(x)\,dx$ for $t = 2$, 5, 10, 100, 1000, and 10,000. Do you think $\int_1^\infty g(x)\,dx$ is convergent? Explain your reasoning.

(b) Use the Comparison Theorem with $f(x) = \dfrac{1}{x^2}$ to show that $\int_1^\infty g(x)\,dx$ is convergent.

(c) Illustrate part (b) by graphing f and g in the same viewing rectangle for $1 \leq x \leq 10$. Use your graph to explain intuitively why $\int_1^\infty g(x)\,dx$ is convergent.

54. Let $g(x) = \dfrac{1}{\sqrt{x}-1}$.

(a) Use technology to construct a table of values of $\int_2^t g(x)\,dx$ for $t = 5$, 10, 100, 1000, and 10,000. Do you think $\int_2^\infty g(x)\,dx$ is convergent or divergent? Explain your reasoning.

(b) Use the Comparison Theorem with $f(x) = \dfrac{1}{\sqrt{x}}$ to show that $\int_2^\infty g(x)\,dx$ is divergent.

(c) Illustrate part (b) by graphing f and g in the same viewing rectangle for $2 \leq x \leq 20$. Use your graph to explain intuitively why $\int_2^\infty g(x)\,dx$ is divergent.

Use the Comparison Theorem to determine whether the integral is convergent or divergent.

55. $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

56. $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$

57. $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$

58. $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$

59. $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$

60. $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$

61. The integral

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

is improper for two reasons: the interval $[0, \infty)$ is infinite, and the integrand has an infinite discontinuity at 0. Evaluate this integral by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows.

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$
$$= \int_0^1 \frac{1}{\sqrt{x}(1+x)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$$

62. Evaluate

$$\int_2^\infty \frac{1}{\sqrt{x^{2-4}}}\,dx$$

using the same method as in Exercise 61.

Find the values of p for which the integral converges and evaluate the integral for those values of p.

63. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

64. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$

65. $\displaystyle\int_0^1 x^p \ln x\,dx$

66. (a) Evaluate the integral $\displaystyle\int_0^\infty x^n e^{-x}\,dx$ for $n = 0, 1, 2$, and 3.

(b) Guess the value of $\displaystyle\int_0^\infty x^n e^{-x}\,dx$ when n is an arbitrary positive integer.

(c) Use mathematical induction to confirm your guess.

67. (a) Show that $\displaystyle\int_{-\infty}^{\infty} x\,dx$ is divergent.

(b) Show that

$$\lim_{t\to\infty} \int_{-t}^{t} x\,dx = 0$$

This shows that we cannot define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx$$

68. If $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ is convergent and a and b are real numbers, show that

$$\int_{-\infty}^{a} f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_b^\infty f(x)\,dx$$

69. A manufacturer of lightbulbs wants to produce bulbs that last about 700 hours, but, of course, some bulbs burn out faster than others. Let $F(t)$ be the fraction of the company's bulbs that burn out before t hours, so $F(t)$ always lies between 0 and 1.

(a) Make a rough sketch of what you think the graph of F might look like.

(b) Explain the meaning of the derivative $r(t) = F'(t)$ in the context of this problem.

(c) Find $\displaystyle\int_0^\infty r(t)\,dt$. Explain why this value makes sense in the context of this problem.

70. The *average speed* of molecules in an ideal gas is

$$\bar{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2} \int_0^\infty v^3 e^{-Mv^2/(2RT)}\,dv$$

where M is the molecular weight of the gas, R is the gas constant, T is the gas temperature, and v is the molecular speed. Show that

$$\bar{v} = \sqrt{\frac{8RT}{\pi M}}$$

71. Recall that a radioactive substance decays exponentially. The mass at time t is $m(t) = m(0)e^{kt}$, where $m(0)$ is the initial mass and k is a negative constant. The *mean life M* of an atom in the substance is

$$M = -k\int_0^\infty te^{kt}\,dt$$

For the radioactive carbon isotope, ^{14}C, used in radiocarbon dating, the value of k is -0.000121. Find the mean life of a ^{14}C atom.

72. Astronomers use a technique called stellar *stereography* to determine the density of stars in a star cluster from the observed (two-dimensional) density that can be analyzed from a photograph. Suppose that in a spherical cluster of radius R the density of stars depends only on the distance r from the center of the cluster. If the perceived star density is given by $y(s)$, where s is the observed planar distance from the center of the cluster, and $x(r)$ is the actual density, it can be shown that

$$y(s) = \int_s^R \frac{2r}{\sqrt{r^2 - s^2}}\,x(r)\,dr$$

If the actual density of stars in a cluster is $x(r) = \frac{1}{2}(R - r)^2$, find the perceived density $y(s)$.

73. In a study of the spread of illicit drug use from an enthusiastic user to a population of N users, the authors model the number of expected new users by the equation

$$\gamma = \int_0^{\infty} \frac{cN(1 - e^{-kt})}{k} e^{-\lambda t}\, dt$$

where c, k, and λ are positive constants. Evaluate this integral to express γ in terms of c, N, k, and λ.

Source: F. Hoppensteadt et al., "Threshold Analysis of a Drug Use Epidemic Model," *Mathematical Biosciences*. 53 (1981): 79–87.

74. Dialysis treatment removes urea and other waste products from patients' blood by diverting some of the blood flow externally through a machine called a dialyzer. The rate at which urea is removed from the blood (in mg/min) is often well described by the equation

$$u(t) = \frac{r}{V} C_0 e^{-rt/V}$$

where r is the rate of flow of blood through the dialyzer (in mL/min), V is the volume of the patient's blood (in mL) and C_0 is the amount of urea in the blood (in mg) at time $t = 0$. Evaluate the integral $\int_0^{\infty} u(t)\, dt$ and explain the meaning of this value in the context of this problem.

75. Determine how large the number a has to be so that

$$\int_a^{\infty} \frac{1}{1 + x^2}\, dx < 0.001$$

76. Estimate the numerical value of $\int_0^{\infty} e^{-x^2}\, dx$ by writing it as the sum of $\int_0^4 e^{-x^2}\, dx$ and $\int_4^{\infty} e^{-x^2}\, dx$. Approximate the first integral by using a trapezoidal sum with $n = 8$ and show that the second integral is smaller than $\int_4^{\infty} e^{-4x}\, dx$, which is less than 0.0000001.

77. Show that $\int_0^{\infty} x^2 e^{-x^2}\, dx = \frac{1}{2}\int_0^{\infty} e^{-x^2}\, dx$.

78. Show that $\int_0^{\infty} e^{-x^2}\, dx = \int_0^1 \sqrt{-\ln y}\, dy$ by interpreting the integrals as areas.

79. Find the value of C for which the integral

$$\int_0^{\infty} \left(\frac{1}{\sqrt{x^2 + 4}} - \frac{C}{x + 2} \right) dx$$

converges. Evaluate the integral for this value of C.

80. Find the value of C for which the integral

$$\int_0^{\infty} \left(\frac{x}{x^2 + 1} - \frac{C}{3x + 1} \right) dx$$

converges. Evaluate the integral for this value of C.

81. Suppose f is continuous on $[0, \infty)$ and $\lim_{x \to \infty} f(x) = 1$. Is it possible that $\int_0^{\infty} f(x)\, dx$ is convergent? Justify your answer.

82. Show that if $a > -1$ and $b > a + 1$, then the following integral is convergent.

$$\int_0^{\infty} \frac{x^a}{1 + x^b}\, dx$$

5 Review

Concepts and Vocabulary

1. (a) Write an expression for a Riemann sum of a function f. Explain the meaning of the notation that you use.
   (b) If $f(x) \geq 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
   (c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
2. (a) Write the definition of the definite integral of a continuous function from a to b.
   (b) What is the geometric interpretation of the definite integral $\int_a^b f(x)\,dx$ if $f(x) \geq 0$?
   (c) What is the geometric interpretation of the definite integral $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values in the interval $[a, b]$? Illustrate with a diagram.
3. Explain in your own words a left, right, and midpoint Riemann sum.
4. State both parts of the Fundamental Theorem of Calculus.
5. (a) State the Net Change Theorem.
   (b) If $r(t)$ is the rate at which water flows into a storage tank, explain the meaning of the definite integral $\int_{t_1}^{t_2} r(t)\,dt$ in this context.
6. Suppose a particle moves back and forth along a horizontal line so that its velocity at time t is given by $v(t)$, measured in feet per second, and acceleration is given by $a(t)$.
   (a) Explain the meaning of $\int_{60}^{120} v(t)\,dt$.
   (b) Explain the meaning of $\int_{60}^{120} |v(t)|\,dt$.
   (c) Explain the meaning of $\int_{60}^{120} a(t)\,dt$.
7. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.
   (b) Explain the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$.
8. Explain in your own words what is meant by the statement, "differentiation and integration are inverse processes."
9. (a) State the Substitution Rule. Explain how to use this rule in practice.
   (b) State the rule for integration by parts. Explain how to use this rule in practice.
10. State the rules for approximating the definite integral $\int_a^b f(x)\,dx$ with the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. Which would you expect to provide the best estimate? Explain how to approximate the error for each rule.
11. Define the following improper integrals.
    (a) $\int_a^\infty f(x)\,dx$ (b) $\int_{-\infty}^b f(x)\,dx$ (c) $\int_{-\infty}^\infty f(x)\,dx$
12. Define the improper integral $\int_a^b f(x)\,dx$ for each of the following cases.
    (a) f has an infinite discontinuity at a.
    (b) f has an infinite discontinuity at b.
    (c) f has an infinite discontinuity at c, where $a < c < b$.
13. State the Comparison Theorem for improper integrals.

True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. If f and g are continuous on $[a, b]$, then
$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$
2. If f and g are continuous on $[a, b]$, then
$$\int_a^b [f(x)g(x)]\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$
3. If f is continuous on $[a, b]$, then
$$\int_a^b 5f(x)\,dx = 5\int_a^b f(x)\,dx$$
4. If f is continuous on $[a, b]$, then
$$\int_a^b xf(x)\,dx = x\int_a^b f(x)\,dx$$
5. If f is continuous on $[a, b]$ and $f(x) \geq 0$, then
$$\int_a^b \sqrt{f(x)}\,dx = \sqrt{\int_a^b f(x)\,dx}$$

6. If f' is continuous on $[1, 3]$, then
$$\int_1^3 f'(v)\,dv = f(3) - f(1)$$

7. If f and g are continuous and $f(x) \geq g(x)$ for $a \leq x \leq b$, then
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

8. If f and g are differentiable and $f(x) \geq g(x)$ for $a < x < b$, then $f'(x) \geq g'(x)$ for $a < x < b$.

9. $\displaystyle\int_{-1}^{1} \left(x^5 - 6x^9 + \frac{\sin x}{(1 + x^4)^2} \right) dx = 0$

10. $\displaystyle\int_{-5}^{5} (ax^2 + bx + c)\,dx = 2\int_0^5 (ax^2 + c)\,dx$

11. $\displaystyle\int_0^3 e^{x^2}\,dx = \int_0^5 e^{x^2}\,dx + \int_5^3 e^{x^2}\,dx$

12. $\displaystyle\int_0^4 \frac{x}{x^2 - 1}\,dx = \frac{1}{2}\ln 15$

13. $\displaystyle\int_{-2}^{1} \frac{1}{x^4}\,dx = -\frac{3}{8}$

14. $\displaystyle\int_1^\infty \frac{1}{x^{\sqrt{2}}}\,dx$ is convergent.

15. $\displaystyle\int_0^2 (x - x^3)\,dx$ represents the area under the curve $y = x - x^3$ from $x = 0$ to $x = 2$.

16. If $\displaystyle\int_0^1 f(x)\,dx = 0$, then $f(x) = 0$ for $0 \leq x \leq 1$.

17. If f has a discontinuity at 0, then $\displaystyle\int_{-1}^{1} f(x)\,dx$ does not exist.

18. All continuous functions have derivatives.

19. All continuous functions have antiderivatives.

20. The Midpoint Rule is always more accurate than the Trapezoidal Rule.

21. If f is continuous, then
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t \to \infty} \int_{-t}^{t} f(x)\,dx$$

22. If f is continuous on $[0, \infty)$ and $\displaystyle\int_1^\infty f(x)\,dx$ is convergent, then $\displaystyle\int_0^\infty f(x)\,dx$ is convergent.

23. If f is a continuous, decreasing function on $[1, \infty)$ and $\displaystyle\lim_{x \to \infty} f(x) = 0$, then $\displaystyle\int_1^\infty f(x)\,dx$ is convergent.

24. If $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$ are both convergent, then $\displaystyle\int_a^\infty [f(x) + g(x)]\,dx$ is convergent.

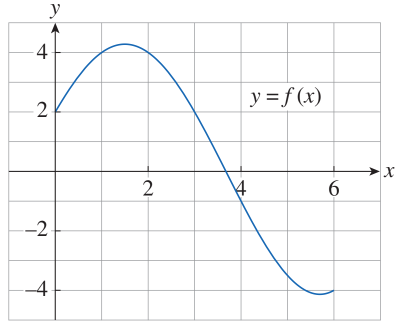
25. If $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$ are both divergent, then $\displaystyle\int_a^\infty [f(x) + g(x)]\,dx$ is divergent.

26. If $f(x) \leq g(x)$ and $\displaystyle\int_0^\infty g(x)\,dx$ diverges, then $\displaystyle\int_0^\infty f(x)\,dx$ also diverges.

27. If f is continuous on $[a, b]$, then
$$\frac{d}{dx}\left[\int_a^b f(x)\,dx\right] = f(x)$$

Exercises

1. Use the given graph of the function f to find the Riemann sum with six subintervals. Take the sample points to be (a) left endpoints and (b) right endpoints. In each case, draw a diagram and explain what the Riemann sum represents.



2. Let $f(x) = x^2 - x$ for $0 \leq x \leq 2$.

(a) Find the right Riemann sum with $n = 4$ equal subintervals. Explain, with the aid of a diagram, what this Riemann sum represents.

(b) Use the definition of a definite integral (with right endpoints) to calculate the value of the integral
$$\int_0^2 (x^2 - x)\,dx$$

(c) Use the Fundamental Theorem of Calculus to check your answer to part (b).

(d) Draw a diagram to explain the geometric meaning of the definite integral in part (b).

3. Evaluate
$$\int_0^1 \left(x + \sqrt{1 - x^2}\right) dx$$
by interpreting it in terms of areas.

4. Express

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin x_i \, \Delta x$$

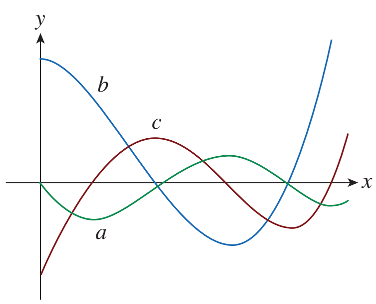
as a definite integral on the interval $[0, \pi]$ and then evaluate the integral. Assume that the interval $[0, \pi]$ is divided into n subintervals of equal width $\Delta x = \pi/n$ and the endpoints of the subintervals are $x_0 = 0, x_1, x_2, \ldots, x_n = \pi$.

5. If $\int_0^6 f(x)\,dx = 10$ and $\int_0^4 f(x)\,dx = 7$, find $\int_4^6 f(x)\,dx$.

6. (a) Write $\int_1^5 (x + 2x^5)\,dx$ as a limit of a right Riemann sum. Use technology to evaluate the sum and to compute the limit.

(b) Use the Fundamental Theorem of Calculus to check your answer to part (a).

7. The figure shows the graphs of f, f', and $\int_0^x f(t)\,dt$.



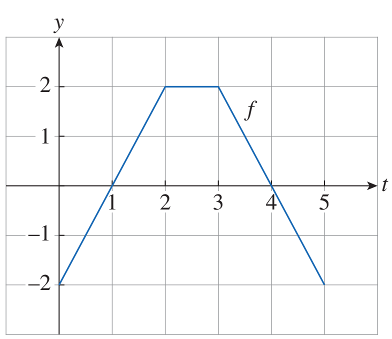
Identify each graph, and justify your choices.

8. Evaluate each expression.

(a) $\displaystyle\int_0^1 \frac{d}{dx}\left(e^{\arctan x}\right) dx$ (b) $\displaystyle\frac{d}{dx}\left[\int_0^1 e^{\arctan x}\,dx\right]$

(c) $\displaystyle\frac{d}{dx}\left[\int_0^x e^{\arctan t}\,dt\right]$

9. The graph of f consists of three line segments as shown in the figure.



If $g(x) = \int_0^x f(t)\,dt$, find $g(4)$, $g'(4)$, and $g''(4)$.

Evaluate the integral.

10. $\displaystyle\int_1^2 (8x^3 + 3x^2)\,dx$

11. $\displaystyle\int_0^T (x^4 - 8x + 7)\,dx$

12. $\displaystyle\int_0^1 (1 - x^9)\,dx$

13. $\displaystyle\int_0^1 (1 - x)^9\,dx$

14. $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du$

15. $\displaystyle\int_0^1 (\sqrt[4]{u} + 1)^2\,du$

16. $\displaystyle\int_0^1 y(y^2 + 1)^5\,dy$

17. $\displaystyle\int_0^2 y^2\sqrt{1 + y^3}\,dy$

18. $\displaystyle\int_1^5 \frac{dt}{(t - 4)^2}$

19. $\displaystyle\int_0^1 \sin(3\pi t)\,dt$

20. $\displaystyle\int_1^2 \frac{1}{2 - 3x}\,dx$

21. $\displaystyle\int_1^2 x^3 \ln x\,dx$

22. $\displaystyle\int_0^5 \frac{x}{x + 10}\,dx$

23. $\displaystyle\int_0^5 ye^{-0.6y}\,dy$

24. $\displaystyle\int_0^1 v^2\cos(v^3)\,dv$

25. $\displaystyle\int_{-1}^1 \frac{\sin x}{1 + x^2}\,dx$

26. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2 + \cos t}\,dt$

27. $\displaystyle\int_0^1 \frac{e^x}{1 + e^{2x}}\,dx$

28. $\displaystyle\int \left(\frac{1 - x}{x}\right)^2 dx$

29. $\displaystyle\int_1^{10} \frac{x}{x^2 - 4}\,dx$

30. $\displaystyle\int \frac{x + 2}{\sqrt{x^2 + 4x}}\,dx$

31. $\displaystyle\int \frac{\csc^2 x}{1 + \cot x}\,dx$

32. $\displaystyle\int \sin \pi t \cos \pi t\,dt$

33. $\displaystyle\int \sin x \cos(\cos x)\,dx$

34. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

35. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

36. $\displaystyle\int \tan x \ln(\cos x)\,dx$

37. $\displaystyle\int \frac{x}{\sqrt{1 - x^4}}\,dx$

38. $\displaystyle\int \frac{x^3}{1 + x^4}\,dx$

39. $\displaystyle\int \frac{1 + x^4}{x^3}\,dx$

40. $\displaystyle\int \frac{\sec\theta \tan\theta}{1 + \sec\theta}\,d\theta$

41. $\displaystyle\int_0^{\pi/4} (1 + \tan t)^3 \sec^2 t\,dt$

42. $\displaystyle\int_0^3 |x^2 - 4|\,dx$

43. $\displaystyle\int_0^4 |\sqrt{x} - 1|\,dx$

44. $\displaystyle\int \frac{dt}{t^2 + 6t + 8}$

45. $\displaystyle\int e^{\sqrt[3]{x}}\,dx$

46. $\displaystyle\int \tan^{-1} x\,dx$

47. $\displaystyle\int_0^4 x(\sqrt{x} - 1)\,dx$

Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative in the same viewing window (use $C = 0$).

48. $\displaystyle\int \frac{\cos x}{\sqrt{1 + \sin x}}\,dx$

49. $\displaystyle\int \frac{x^3}{\sqrt{x^2 + 1}}\,dx$

50. Use a graph to find an estimate of the area of the region bounded by the graph of $y = x\sqrt{x}$, the x-axis, and the lines $x = 0$ and $x = 4$. Then find the exact area.

51. Graph the function $f(x) = \cos^2 x \sin^3 x$ and use the graph to estimate the value of the definite integral $\int_0^{2\pi} f(x)\,dx$. Evaluate the definite integral to check your estimate.

Find the derivative of the function.

52. $F(x) = \displaystyle\int_0^x \frac{t^2}{1+t^3}\,dt$

53. $F(x) = \displaystyle\int_x^1 \sqrt{t + \sin t}\,dt$

54. $g(x) = \displaystyle\int_0^{x^4} \cos(t^2)\,dt$

55. $g(x) = \displaystyle\int_1^{\sin x} \frac{1-t^2}{1+t^4}\,dt$

56. $y = \displaystyle\int_{\sqrt{x}}^x \frac{e^t}{t}\,dt$

57. $y = \displaystyle\int_{2x}^{3x+1} \sin(t^4)\,dt$

Use the Table of Integrals on the Reference Pages to evaluate the integral.

58. $\displaystyle\int e^x\sqrt{1-e^{2x}}\,dx$

59. $\displaystyle\int \csc^5 t\,dt$

60. $\displaystyle\int \sqrt{x^2+x+1}\,dx$

61. $\displaystyle\int \frac{\cot x}{\sqrt{1+2\sin x}}\,dx$

Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule with $n = 10$ to approximate the given integral. Round your answers to six decimal places. If possible, determine whether your answers are underestimates or overestimates.

62. $\displaystyle\int_0^1 \sqrt{1+x^4}\,dx$

63. $\displaystyle\int_0^{\pi/2} \sqrt{\sin x}\,dx$

64. Estimate the errors involved in Exercises 62 and 63, parts (a) and (b). How large should n be in each case to guarantee the error is less than 0.00001?

65. Use Simpson's Rule with $n = 6$ to estimate the area under the curve $y = \dfrac{e^x}{x}$ from $x = 1$ to $x = 4$.

66. Use the Midpoint Rule with $n = 6$ to approximate

$$\int_0^3 \sin(x^3)\,dx.$$

67. Let $f(x) = \sin(\sin x)$.

(a) Use a graph to find an upper bound for $|f^{(4)}(x)|$.

(b) Use Simpson's Rule with $n = 10$ to approximate $\int_0^{\pi} f(x)\,dx$ and use part (a) to estimate the error.

(c) How large should n be to guarantee that the magnitude of the error is using S_n is less than 0.00001?

68. (a) Explain how to evaluate $\int x^5 e^{-2x}\,dx$ without using tables or a CAS. (Don't actually do the integration.)

(b) Explain how to evaluate $\int x^5 e^{-2x}\,dx$ using tables. (Don't actually do the integration.)

(c) Use a CAS to evaluate $\int x^5 e^{-2x}\,dx$.

(d) Graph the integrand and the indefinite integral on the same coordinate axes.

Use Property 8 of integrals to estimate the value of the definite integral.

69. $\displaystyle\int_1^3 \sqrt{x^2+3}\,dx$

70. $\displaystyle\int_3^5 \frac{1}{x+1}\,dx$

Use the properties of integrals to verify the inequality.

71. $\displaystyle\int_0^1 x^2\cos x\,dx \le \frac{1}{3}$

72. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \le \frac{\sqrt{2}}{2}$

73. $\displaystyle\int_0^1 e^x\cos x\,dx \le e - 1$

74. $\displaystyle\int_0^1 x\sin^{-1}x\,dx \le \frac{\pi}{4}$

Evaluate the integral or show that it is divergent.

75. $\displaystyle\int_1^\infty \frac{1}{(2x+1)^3}\,dx$

76. $\displaystyle\int_0^\infty \frac{\ln x}{x^4}\,dx$

77. $\displaystyle\int_{-\infty}^0 e^{-2x}\,dx$

78. $\displaystyle\int_0^1 \frac{1}{2-3x}\,dx$

79. $\displaystyle\int_1^e \frac{dx}{x\sqrt{\ln x}}$

80. $\displaystyle\int_2^6 \frac{y}{\sqrt{y-2}}\,dy$

81. Use the Comparison Theorem to determine whether the integral

$$\int_1^\infty \frac{x^3}{x^5+2}\,dx$$

is convergent or divergent.

82. Find the values of a such that $\int_0^\infty e^{ax}\cos x\,dx$ is convergent. Use the Table of Integrals to evaluate the integral for those values of a.

83. A particle moves along a horizontal line so that its velocity at time t is given by $v(t) = t^2 - t$, where v is measured in meters per second. Find (a) the displacement and (b) the total distance traveled by the particle during the time interval $[0, 5]$.

84. A particle moves along the x-axis so that its velocity at time t is given by $v(t) = 3t^2 - 9t + 6$. At time $t = 0$ the particle is located at $s(0) = 5$.

(a) Find the total distance traveled by the particle during the time interval $[0, 2]$.

(b) Find the position function $s(t)$ corresponding to the motion of the particle.

85. A particle moves along a horizontal line so that its acceleration function at time t is given by $a(t) = 12\cos(3t)$. At $t = 0$, its velocity is $v(0) = 2$ and its position is $s(0) = 0$.

(a) Find the function $s(t)$ corresponding to the motion of this particle.

(b) For $0 \le t \le \pi$, find the values of t for which this particle is at rest.

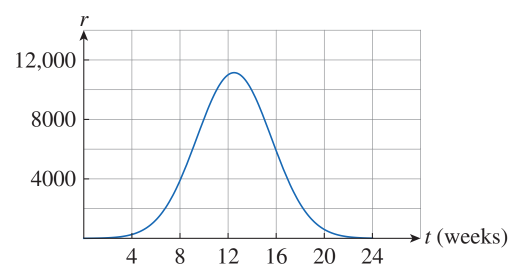
86. The speedometer reading (v) on a car for selected values of t are given in the table.

t (min)	v (mi/h)	t (min)	v (mi/h)
0	40	6	56
1	42	7	57
2	45	8	57
3	49	9	55
4	52	10	56
5	54		

Use Simpson's Rule to estimate the distance traveled by the car.

87. Let $r(t)$ be the rate at which the world's oil is consumed, where t is measured in years starting at $t = 0$ on January 1, 2000, and $r(t)$ is measured in barrels per year. Explain the meaning of $\int_0^8 r(t)\,dt$ in this context.

88. Suppose a population of honeybees increases at a rate of $r(t)$ bees per week, where the graph of r is shown in the figure.



Use Simpson's Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.

89. Suppose that the temperature of a long, thin rod placed along the x-axis is initially $\dfrac{C}{2a}$ if $|x| \le a$ and 0 if $|x| > a$. It can be shown that if the heat diffusivity of the rod is k, then the temperature of the rod at the point x at time t is

$$T(x, t) = \frac{C}{a\sqrt{4\pi kt}} \int_0^a e^{-(x-u)^2/(4kt)}\,du$$

To find the temperature distribution that results from an initial hot spot concentrated at the origin, we need to compute

$$\lim_{a \to 0} T(x, t)$$

Use l'Hospital's Rule to find this limit.

90. The Fresnel function,

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

was introduced in Section 5.4. Fresnel also used the function

$$C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt$$

in his theory of the diffraction of light waves.

(a) Find the intervals on which C is increasing.
(b) Find the intervals on which C is concave up.
(c) Use technology to solve the following equation:

$$\int_0^x \cos\left(\frac{\pi t^2}{2}\right) = 0.7$$

(d) Plot the graphs of C and S in the same viewing rectangle. How are these graphs related?

91. If f is a continuous function such that

$$\int_0^x f(t)\,dt = xe^{2x} + \int_0^x e^{-t} f(t)\,dt$$

for all x, find an explicit formula for $f(x)$.

92. Find a function f and a value of the constant a such that

$$2\int_a^x f(t)\,dt = 2\sin x - 1$$

93. If f' is continuous on $[a, b]$, show that

$$2\int_a^b f(x) f'(x)\,dx = [f(b)]^2 - [f(a)]^2$$

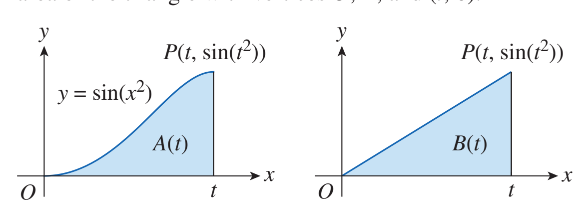
94. If n is a positive integer, prove that

$$\int_0^1 (\ln x)^n\,dx = (-1)^n n!$$

95. If f' is continuous on $[0, \infty)$ and $\lim_{x \to \infty} f(x) = 0$, show that

$$\int_0^\infty f'(x)\,dx = -f(0)$$

96. The figure shows two regions in the first quadrant: $A(t)$ is the area under the curve $y = \sin(x^2)$ from 0 to t, and $B(t)$ is the area of the triangle with vertices O, P, and $(t, 0)$.



Find $\displaystyle\lim_{t \to 0^+} \frac{A(T)}{B(T)}$.

Focus on Problem Solving

One of the principles of problem solving is *recognizing something familiar*. Try to recognize a part of the function in the next example in order to help find the limit.

Example Dissimilar Parts

Evaluate $\displaystyle\lim_{x\to 3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}\,dt\right)$.

Solution

Start by examining each *piece* of the function.

Consider the first factor, $\dfrac{x}{x-3}$, as x approaches 3.

As $x \to 3^+$, $\dfrac{x}{x-3} \to \dfrac{3}{(+)} \to \infty$. As $x \to 3^-$, $\dfrac{x}{x-3} \to \dfrac{3}{(-)} \to -\infty$.

The second factor approaches $\displaystyle\int_3^3 \frac{\sin t}{t}\,dt = 0$ as $x \to 3$.

Therefore, it is not clear what's happening to the product. It could be going to 0, it could be increasing without bound, or it could be approaching some finite number.

The second factor, $\displaystyle\int_3^x \frac{\sin t}{t}\,dt$, has an x as the upper limit and is the same type of integral that occurs in Part 1 of the Fundamental Theorem of Calculus.

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

This suggests that we might be able to use differentiation in some way in order to solve the original limit.

Thinking about differentiation, notice that the denominator in the first factor, $x - 3$, is in the form of the denominator in the (alternate) definition of the derivative.

$$F'(a) = \lim_{x\to a}\frac{F(x) - F(a)}{x - a}$$

Let $F(x) = \displaystyle\int_3^x \frac{\sin t}{t}\,dt$, and $a = 3$.

$$F(3) = \int_3^3 \frac{\sin t}{t}\,dt = 0 \text{ and } \frac{F(x) - F(3)}{x-3} = \frac{\displaystyle\int_3^x \frac{\sin t}{t}\,dt}{x-3}$$

Let's return to the original limit, factor appropriately, and use the function F.

$$\begin{aligned}
\lim_{x\to 3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}\,dt\right) &= \lim_{x\to 3} x \cdot \lim_{x\to 3}\frac{\displaystyle\int_3^x \frac{\sin t}{t}\,dt}{x-3} && \text{Factor; Limit Laws.}\\
&= 3 \cdot \lim_{x\to 3}\frac{F(x) - F(3)}{x-3} && \text{Evaluate first limit; rewrite second expression using } F.\\
&= 3\,F'(3) = 3\,\frac{\sin 3}{3} && \text{FTC1.}\\
&= \sin 3 && \text{Simplify.}
\end{aligned}$$

■

Problems

1. Let $f(x) = \dfrac{2cx - x^2}{c^3}$ for $c > 0$.

(a) Graph several members of this family of functions and consider the regions enclosed by these curves and the x-axis. Make a conjecture about how the areas of these regions are related.
(b) Prove your conjecture in part (a).
(c) Reconsider your graphs in part (a) and use them to sketch the curve traced out by the vertices (highest points) of the family of functions. What type of curve is this?
(d) Find an equation for the curve you sketched in part (c).

2. If $x \sin \pi x = \int_0^{x^2} f(t)\, dt$, where f is a continuous function, find $f(4)$.

3. If $f(x) = \int_0^x x^2 \sin(t^2)\, dt$, find $f'(x)$.

4. If f is a differentiable function such that $f(x)$ is never 0 and $\int_0^x f(t)\, dt = [f(x)]^2$ for all x, find f.

5. Evaluate $\displaystyle\lim_{x\to 0} \frac{1}{x} \int_0^x (1 - \tan 2t)^{1/t}\, dt$.

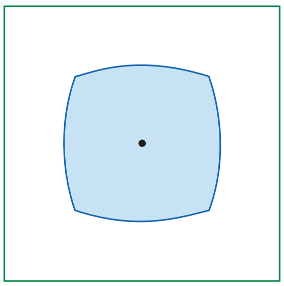
6. A circular disk of radius r is used in an evaporator and is rotated in a vertical plane. If it is to be partially submerged in the liquid so as to maximize the exposed wet area of the disk, show that the center of the disk should be positioned at a height $\dfrac{r}{\sqrt{1 + \pi^2}}$ above the surface of the liquid.

7. If $\int_0^4 e^{(x-2)^4} dx = k$, find the value of $\int_0^4 xe^{(x-2)^4}\, dx$.

8. If $f(x) = \displaystyle\int_0^{g(x)} \frac{1}{\sqrt{1 + t^3}}\, dt$, where $g(x) = \displaystyle\int_0^{\cos x} [1 + \sin(t^2)]\, dt$, find $f'(\pi/2)$.

9. Find a function f such that $f(1) = -1$, $f(4) = 7$, and $f'(x) > 3$ for all x, or prove that such a function cannot exist.

10. The figure shows a region consisting of all points inside a square that are closer to the center than to the sides of the square. Find the area of the region.



11. Find the interval $[a, b]$ for which the value of the integral $\int_a^b (2 + x - x^2)\, dx$ is a maximum.

12. Suppose f is continuous, $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$, and $\int_0^1 f(x) = \frac{1}{3}$. Find the value of the integral $\int_0^1 f^{-1}(y)\, dy$.

13. Find $\dfrac{d^2}{dx^2}\left[\displaystyle\int_0^x \left(\int_1^{\sin t} \sqrt{1 + u^4}\, du \right) dt \right]$.

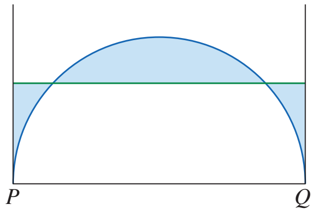
14. Use an integral to estimate the sum $\sum_{i=1}^{10,000} \sqrt{i}$.

15. Evaluate $\displaystyle\int_{-1}^{\infty} \left(\frac{x^4}{1+x^6}\right)^2 dx$.

16. Find the minimum value of the area of the region under the curve $y = x + \dfrac{1}{x}$ from $x = a$ to $x = a + 1.5$, for all $a > 0$.

17. Evaluate $\displaystyle\int_0^1 (\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3})\, dx$.

18. The figure shows a semicircle with radius 1, horizontal diameter PQ, and tangent lines at P and Q.



At what height above the diameter should the horizontal line be placed so that the shaded area is a minimum?

19. Show that

$$\int_0^1 (1-x^2)^n\, dx = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

Hint: Start by showing that if I_n denotes the integral, then

$$I_{k+1} = \frac{2k+2}{2k+3} I_k$$

20. Graph $f(x) = \sin(e^x)$ and use the graph to estimate the value of t such that $\int_t^{t+1} f(x)\, dx$ is a maximum. Then find the exact value of t that maximizes this integral.

21. A person initially standing at the point O walks along a pier pulling a rowboat by a rope of length L. The person keeps the rope straight and taut. The path followed by the boat is a curve called a *tractrix*, and it has the property that the rope is always tangent to the curve, as shown in Figure 5.83.

(a) Show that if the path followed by the boat is the graph of the function $y = f(x)$, then

$$f'(x) = \frac{dy}{dx} = \frac{-\sqrt{L - x^2}}{x}$$

(b) Determine the equation for the tractrix $y = f(x)$.

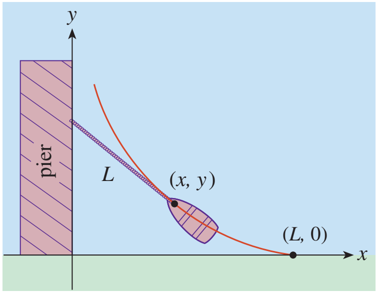
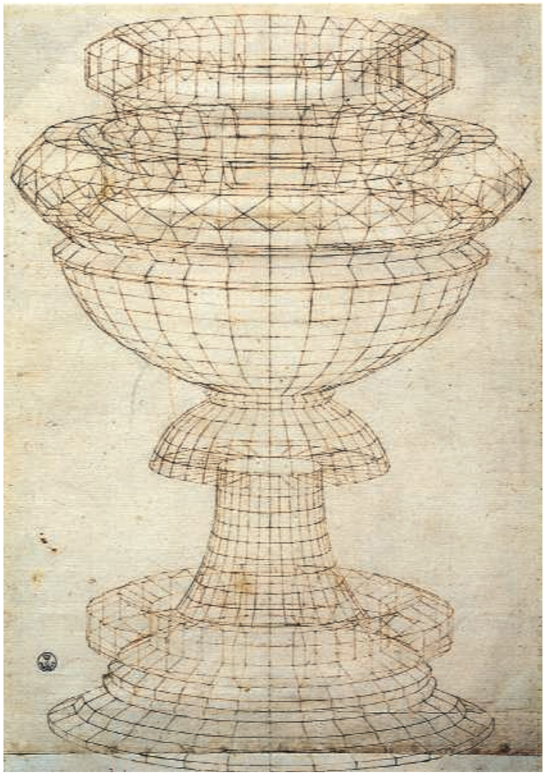


Figure 5.83
The rope is tangent to the curve.

22. For any number c, let $f_c(x)$ be the smaller of the two numbers $(x-c)^2$ and $(x-c-2)^2$. Then let $g(c) = \int_0^1 f_c(x)\, dx$. Find the maximum and minimum values of $g(c)$ if $-2 \le c \le 2$.



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Florentine painter and mathematician Paulo Uccello created this *Perspective drawing of a chalice* (3D rendering). He is remembered for his interest in scientific laws and defining perspective. This drawing and 3D rendering provide a visualization of the calculus concept of the volume of a solid of revolution.

6 Applications of Integration

Contents

In this chapter, we consider some of the applications of the definite integral, including area between curves, volumes of solids, and the length of a curve. The common theme in each application is the following general method, similar to the technique used to find the area under a curve.

First, break up a quantity Q into a large number of small parts. Next, approximate each small part by a quantity of the form $f(x_i^*)\,\Delta x$ and approximate Q by a Riemann sum. Then, take the limit and express Q as a definite integral. And finally, approximate the definite integral using Simpson's Rule, for example, or evaluate the definite integral using the Fundamental Theorem of Calculus.

6.1 More About Areas

In Chapter 5, we defined and calculated the area of a region bounded above by the graph of a function and below by the x-axis, and between the lines $x = a$ and $x = b$. In this section, we use a definite integral to find the area of a region that lies between the graphs of two functions. We also consider regions enclosed by parametric curves.

Areas Between Curves

Suppose f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$. Let R be the region bounded above by the graph of $y = f(x)$, below by the graph of $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. Figure 6.1 shows an example of this type of a region R.

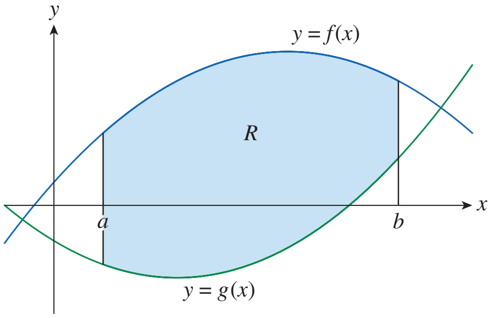


Figure 6.1
The region R is bounded above by the graph of $y = f(x)$, below by the graph of $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$.

In order to approximate the area of R, we will use the same method as in Section 5.1. Divide R into n strips of equal width and then approximate the ith strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$. See Figure 6.2 and note that we could take all of the sample points to be the right endpoints, in which case $x_i^* = x_i$.

The Riemann sum

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$$

is, therefore, an approximation to what we intuitively think of as the area of R. See Figure 6.3.

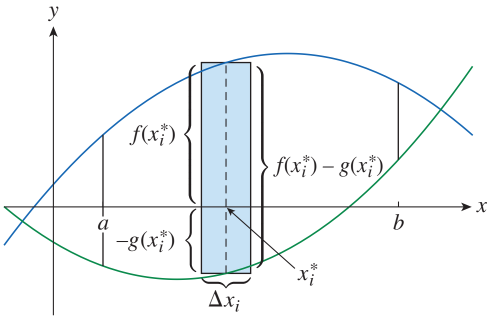


Figure 6.2
Typical approximating rectangle.

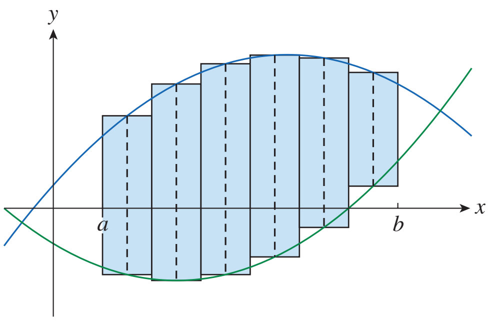


Figure 6.3
Approximating rectangles.

This approximation appears to become better and better as n increases. Therefore, we define the **area** A of the region R as the limiting value of the sum of the areas of these approximating rectangles.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x \tag{1}$$

The limit expression in Equation 1 is the definition of the definite integral of $f - g$ on the interval $[a, b]$. This leads to the formula for area.

Area Between Curves

The area A of the region bounded by the graphs of $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$ is

$$A = \int_a^b [f(x) - g(x)]\, dx \tag{2}$$

A Closer Look

1. If f and g are continuous on $[a, b]$, then $f - g$ is continuous on $[a, b]$, and therefore, $\int_a^b [f(x) - g(x)]\, dx$ exists.
2. If $f(x) \geq g(x)$ for all x in $[a, b]$ and $f(x) < 0$ and/or $g(x) < 0$ for some or all values of x in the interval $[a, b]$, then the area is given by the same integral expression

$$A = \int_a^b [f(x) - g(x)]\, dx$$

3. If $g(x) = 0$, then R is the region bounded above by the graph of $y = f(x)$, below by the x-axis, and by the lines $x = a$ and $x = b$. The area of R is

$$A = \int_a^b [f(x) - 0]\, dx = \int_a^b f(x)\, dx$$

This is the formula given in Chapter 5.

4. If $f(x) \geq g(x) \geq 0$ for all x in $[a, b]$, then, as we can see from Figure 6.4, the area of the region R is

$A = [\text{area under } y = f(x)] - [\text{area under } y = g(x)]$ — Subtract areas.

$= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$ — Use integral expressions for area.

$= \int_a^b [f(x) - g(x)]\, dx$ — Property of definite integrals.

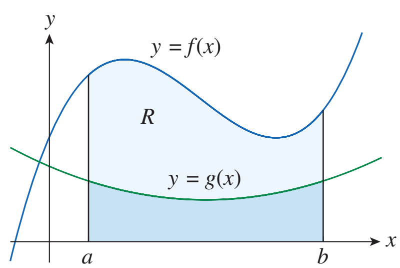


Figure 6.4
The area of the region R is
$A = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx.$

Example 1 Area Between Two Curves

Find the area of the region R bounded above by the graph of $y = e^x$, below by the graph of $y = x$, and between the lines $x = 0$ and $x = 1$.

Solution

The region R is shown in Figure 6.5, along with a typical approximating rectangle.

The upper boundary is the graph of $y = e^x$.

The lower boundary is the graph of $y = x$.

Use the area formula:

$A = \int_0^1 [e^x - x]\, dx = \left[e^x - \frac{1}{2}x^2\right]_0^1$ — Equation 2; antiderivative.

$= \left[e^1 - \frac{1^2}{2}\right] - \left[e^0 - \frac{0^2}{2}\right] = e - \frac{3}{2}$ — FTC2; simplify.

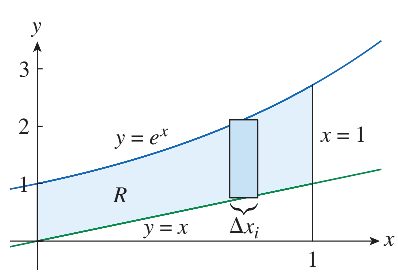


Figure 6.5
The region R and a typical approximating rectangle.

Note that the difference $y_T - y_B$ is not affected by the position of the two curves with respect to the x-axis.

Figure 6.6 shows a typical approximating rectangle with width Δx to help identify the correct integrand for finding the area between two curves. In general, when setting up an integral to find an area, it is helpful to sketch the region and a typical approximating rectangle in order to identify the top curve, y_T, and the bottom curve, y_B, as shown in Figure 6.6.

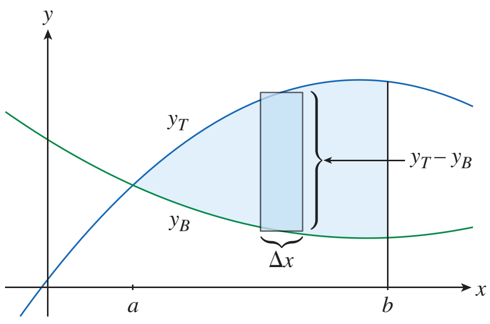


Figure 6.6
A typical approximating rectangle with width Δx and height $y_T - y_B$.

The area of a typical approximating rectangle is $(y_T - y_B)\,\Delta x$ and the equation

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (y_T - y_B)\,\Delta x = \int_a^b (y_T - y_B)\,dx$$

summarizes the procedure of adding (in the limiting case) the areas of all the typical rectangles.

Notice that in Figure 6.6, the left boundary reduces to a point. It is also possible that the right boundary reduces to a point. In Example 2, both side boundaries reduce to points. Therefore, in order to find the bounds on the area integral, it may be necessary to find a and/or b, that is, the x-coordinate(s) of the point(s) of intersection of the graphs.

Example 2 Area Between Two Curves; Both Side Boundaries Points

Find the area of the region R bounded by the graphs of the parabolas $y = x^2$ and $y = 2x - x^2$.

Solution

There are no lines given as left and right boundaries of the region R.

Find the points of intersection of the parabolas by solving the equations simultaneously.

$x^2 = 2x - x^2$ — Set the functions equal.

$2x^2 - 2x = 0$ — Move all terms to left side.

$2x(x - 1) = 0$ — Factor.

$x = 0, 1$ — Principle of Zero Products.

Use either equation to find the y-coordinates.

The points of intersection are (0, 0) and (1, 1).

Figure 6.7 shows a graph of the two parabolas, a typical approximating rectangle, and the region R.

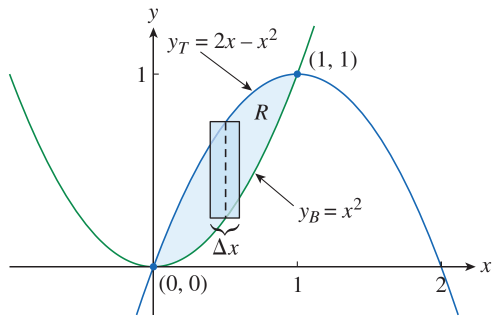


Figure 6.7
The graphs of the two parabolas, a typical approximating rectangle, and the region R.

The top curve and the bottom curve are $y_T = 2x - x^2$ and $y_B = x^2$, respectively.

The area of a typical rectangle is $(y_T - y_B)\,\Delta x = (2x - x^2 - x^2)\,\Delta x$.

The region R lies between the lines $x = 0$ and $x = 1$. The total area is

$$A = \int_0^1 (2x - 2x^2)\,dx = 2\int_0^1 (x - x^2)\,dx \qquad \text{Area formula; simplify integrand.}$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \qquad \text{Antiderivative.}$$

$$= 2\left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right)\right] = \frac{1}{3} \qquad \text{FTC2; simplify.}$$

Sometimes the points of intersection of two curves cannot be found analytically. In Example 3, we use technology to find the points of intersection and then the area of a region.

Example 3 Area Between Two Curves; Use Technology

Find the area of the region R bounded by the graphs of $y = \dfrac{x}{\sqrt{x^2 + 1}}$ and $y = x^4 - x$.

Solution

To find the exact (analytical) points of intersection, we would have to solve the equation $\dfrac{x}{\sqrt{x^2 + 1}} = x^4 - x$.

This equation is, in fact, impossible to solve analytically. So, we will use technology.

Figure 6.8 shows the graphs of the two functions and the region R. The graphs intersect at the origin $(0, 0)$; that's one point of intersection.

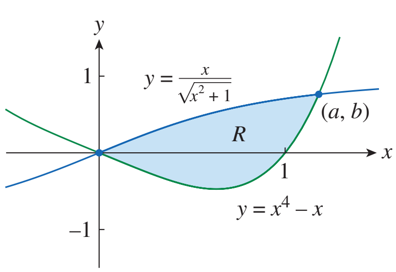


Figure 6.8
The graphs of the two functions and the region R.

Use technology to find the other point of intersection: $(a, b) \approx (1.181, 0.763)$. Note that the coordinates of the point of intersection are presented here with three digits to the right of the decimal, rounded. However, they are stored using the greatest degree of accuracy allowed by the technology.

The area of the region R is

$$a = \int_0^a \left[\frac{x}{\sqrt{x^2 + 1}} - (x^4 - x)\right] dx \approx 0.785. \qquad \text{Use technology.}$$

We can find an antiderivative for this integrand, use the FTC2, and arrive at the same answer. However, in this case, it is just as efficient to use technology to evaluate the definite integral.

Example 4 Area Between Velocity Curves

The graph of the velocity, in ft/s, for two cars, A and B, that start side by side and move along the same road is shown in Figure 6.9. A table of values showing the velocity of each car and the difference in velocities at two-second intervals of time t is given in Table 6.1.

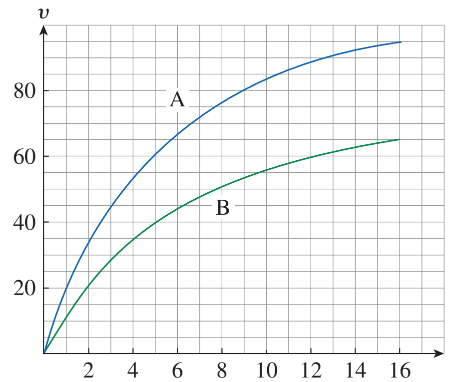


Figure 6.9
The velocity graphs for cars A and B.

t (sec)	v_A (ft/s)	v_B (ft/s)	$v_A - v_B$ (ft/s)
0	0	0	0
2	34	21	13
4	54	34	20
6	67	44	23
8	76	51	25
10	84	56	28
12	89	60	29
14	92	63	29
16	95	65	30

Table 6.1
The velocities of cars A and B at 2-second intervals.

What does the area between the curves represent? Use Simpson's Rule to estimate the area.

Solution

The area under the car A velocity curve represents the distance traveled by car A during the first 16 seconds. Similarly, the area under the car B velocity curve is the distance traveled by car B during this time period. Therefore, the area between these two curves, the difference of the areas under the curves, is the distance between the cars after 16 seconds.

Use Simpson's Rule with $n = 8$ subintervals and $\Delta t = \dfrac{16 - 0}{8} = 2$.

$$\int_0^{16} (v_A - v_B)\,dt \approx \frac{2}{3}[0 + 4(13) + 2(20) + 4(23) + 2(25) + 4(28) + 2(29) + 4(29) + 30]$$
$$= 367 \text{ ft}$$

After 16 seconds, the distance between the cars is approximately 367 ft. ■

In some cases, it is easier, or necessary, to find the area of a region by writing x as a function of y. Suppose the region R is bounded by the graphs of $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$. See Figure 6.10.

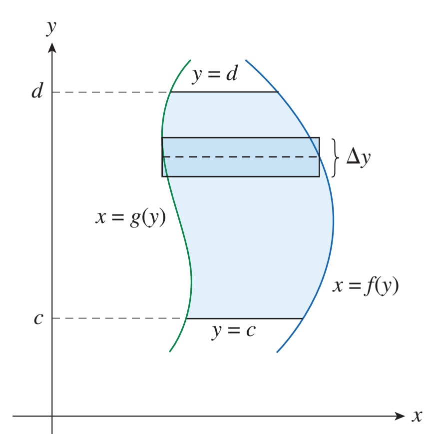


Figure 6.10
The region is bounded more naturally by a right curve and a left curve.

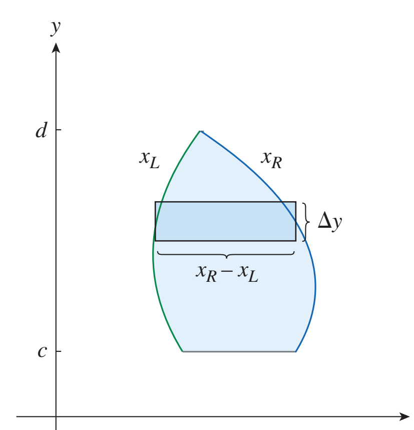


Figure 6.11
A typical approximating rectangle.

The area of this region is

$$A = \int_c^d [f(y) - g(y)]\, dy$$

The graph of $x = f(y)$ is the *right* curve and the graph of $x = g(y)$ is the *left* curve, so a way to remember this formula is

$$A = \int_c^d (x_R - x_L)\, dy$$

In this case, a typical approximating rectangle has dimensions $x_R - x_L$ and Δy, as shown in Figure 6.11.

Example 5 Area of a Region; Left and Right Curves

Find the area of the region enclosed by the graphs of the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution

Figure 6.12 shows a graph of the line and the parabola, and the region. We need to find the points of intersection to completely define the region and to specify the bounds on the definite integral used to compute the area of the region.

Solve the two equations simultaneously.

$$y^2 = 2x + 6$$
$$(x - 1)^2 = 2x + 6$$
$$x^2 - 4x - 5 = 0$$
$$(x - 5)(x + 1) = 0 \quad \Rightarrow \quad x = 5, -1$$

Use either equation to find the y-coordinates.

The points of intersection are $(-1, -2)$ and $(5, 4)$.

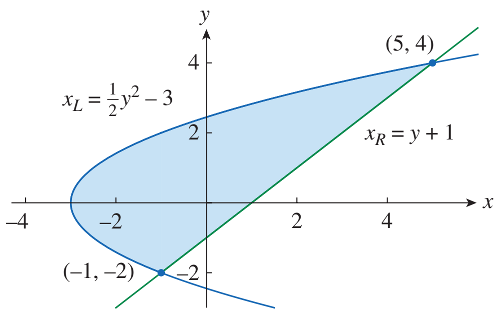


Figure 6.12
The region bounded by the line and the parabola.

It seems more natural, and easier, to find the area of the region in terms of y, that is, to use a right curve and a left curve.

Solve each equation for x in terms of y.

$$x_L = \frac{1}{2}y^2 - 3 \qquad \text{and} \qquad x_R = y + 1$$

Set up the definite integral in terms of y, with the appropriate y-values for bounds.

Common Error

Even though the integral is in terms of y, the bounds are still in terms of x.

Correct Method

When finding the area of a region using a definite integral in terms of y, the appropriate y-values must be used for the limits of integration.

$$A = \int_{-2}^{4} (x_R - x_L)\,dy = \int_{-2}^{4} \left[(y+1) - \left(\frac{1}{2}y^2 - 3 \right) \right] dy$$

$$= \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4 \right) dy \qquad \text{Simplify the integrand.}$$

$$= \left[-\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y \right]_{-2}^{4} \qquad \text{Antiderivative.}$$

$$= \left[\left(-\frac{1}{6}(64) + 8 + 16 \right) - \left(\frac{4}{3} + 2 - 8 \right) \right] = 18 \qquad \text{FTC2; simplify.}$$

Note: We can also find the area of the region by integrating with respect to x. However, the calculation is more involved because the bottom curve changes over the interval $[-3, 5]$. Therefore, we would have to split the region in two as shown in Figure 6.13.

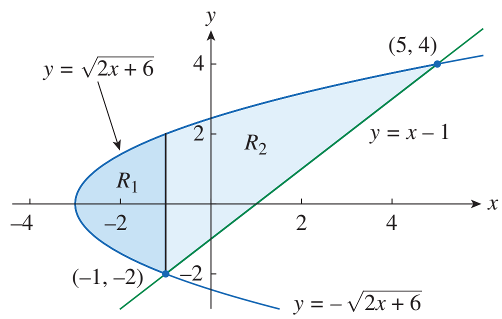


Figure 6.13
If we integrate with respect to x, then we need to split the region into two parts.

It is always a good idea to sketch the region of interest to help decide which integral to use:

$$\int_a^b (y_T - y_B)\,dx \quad \text{or} \quad \int_c^d (x_R - x_L)\,dy$$

Areas Enclosed by Parametric Curves

Recall that if $F(x) \geq 0$, then the area under the graph of $y = F(x)$ from $x = a$ to $x = b$ is $A = \int_a^b F(x)\,dx$. If the graph is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate this area using the Substitution Rule for Definite Integrals.

Use the following from the parametric equations:

$y = g(t)$, $x = f(t)$ $\Rightarrow$ $dx = f'(t)\,dt$ and assume that $a = f(\alpha)$ and $b = f(\beta)$

The area under the curve is

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)\,dt$$

If the curve is traced out once such that $a = f(\beta)$ and $b = f(\alpha)$, then

$$A = \int_a^b y\,dx = \int_\beta^\alpha g(t)\,f'(t)\,dt$$

Example 6 Area Associated with a Cycloid

Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

Solution

One arch of the cycloid is traced out for $0 \le \theta \le 2\pi$.

Use the Substitution Rule with $y = r(1 - \cos\theta)$ and $dx = r(1 - \cos\theta)\,d\theta$.

$$A = \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1-\cos\theta)\,r(1-\cos\theta)\,d\theta \qquad \text{Use expressions for } y \text{ and } dx.$$

$$= r^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta = r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta \qquad \text{Simplify; expand.}$$

$$= r^2\int_0^{2\pi}\left[1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta \qquad \text{Trigonometric identity.}$$

$$= r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \qquad \text{Simplify; integrate.}$$

$$= r^2\left[\left(\frac{3}{2}\cdot 2\pi - 2\sin 2\pi + \frac{1}{4}\sin 4\pi\right) - \left(\frac{3}{2}\cdot 0 - 2\sin 0 + \frac{1}{4}\sin 0\right)\right] \qquad \text{FTC2.}$$

$$= r^2\left(\frac{3}{2}\cdot 2\pi\right) = 3\pi r^2 \qquad \text{Simplify.}$$

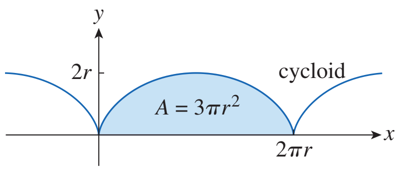


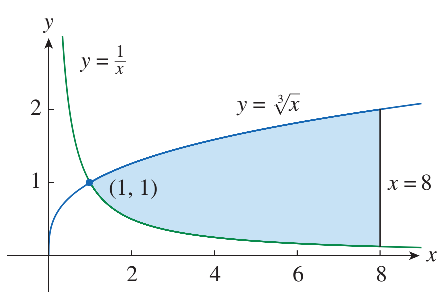
Figure 6.14
The graph of the cycloid and the region.

The area under one arch of the cycloid is three times the area of the rolling circle that generates the cycloid. See Figure 6.14. ■

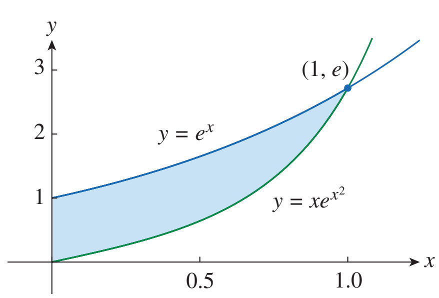
6.1 Exercises

Find the area of the shaded region.

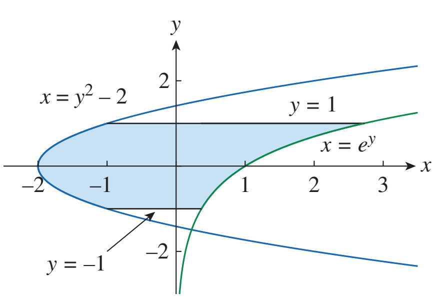
1.



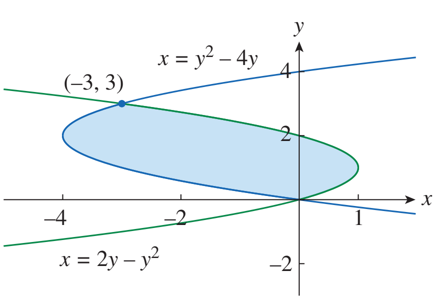
2.



3.



4.



Sketch the region enclosed by the graphs of the given functions. Determine whether it is more natural to integrate with respect to x or to y to find the area of the region. Draw a typical approximating rectangle and label the height and width. Then find the area of the region.

5. $y = e^x, \quad y = x^2 - 1, \quad x = -1, \quad x = 1$

6. $y = \ln x, \quad xy = 4, \quad x = 1, \quad x = 3$

7. $y = (x-2)^2, \quad y = x$

8. $y = x^2 - 4x, \quad y = 2x$

9. $y = \dfrac{1}{x}, \quad y = \dfrac{1}{x^2}, \quad x = 2$

10. $y = \sin x, \quad y = \dfrac{2x}{\pi}, \quad x \geq 0$

11. $x = 1 - y^2, \quad x = y^2 - 1$

12. $4x + y^2 = 12, \quad x = y$

Sketch the region enclosed by the graphs of the given functions and find the area of the region.

13. $y = 12 - x^2, \quad y = x^2 - 6$

14. $y = x^2, \quad y = 4x - x^2$

15. $y = \sec^2 x, \quad y = 8\cos x, \quad -\dfrac{\pi}{3} \leq x \leq \dfrac{\pi}{3}$

16. $y = \cos x, \quad y = 2 - \cos x, \quad 0 \leq x \leq 2\pi$

17. $x = 2y^2, \quad x = 4 + y^2$

18. $y = \sqrt{x-1}, \quad x - y = 1$

19. $y = \cos \pi x, \quad y = 4x^2 - 1$

20. $x = y^4, \quad y = \sqrt{2-x}, \quad y = 0$

21. $y = \tan x, \quad y = 2\sin x, \quad -\dfrac{\pi}{3} \leq x \leq \dfrac{\pi}{3}$

22. $y = x^3, \quad y = x$

23. $y = \sqrt[3]{2x}, \quad y = \dfrac{1}{8}x^2, \quad 0 \leq x \leq 6$

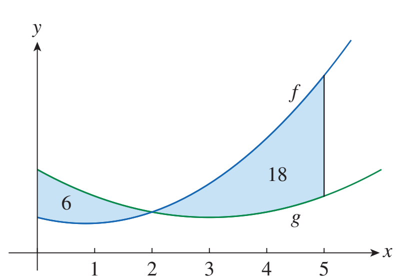
24. $y = \cos x, \quad y = 1 - \cos x, \quad 0 \leq x < \pi$

25. $y = x^4, \quad y = 2 - |x|$

26. $y = \dfrac{1}{x}, \quad y = x, \quad y = \dfrac{1}{4}x, \quad x > 0$

27. $y = \dfrac{1}{4}x^2, \quad y = 2x^2, \quad x + y = 3, \quad x \geq 0$

28. The graphs of two functions are shown in the figure with the areas of the regions between the curves indicated.



(a) What is the total area between the graphs for $0 \leq x \leq 5$?

(b) What is the value of $\displaystyle\int_0^5 |f(x) - g(x)|\,dx$?

(c) What is the value of $\displaystyle\int_0^5 [f(x) - g(x)]\,dx$?

(d) What is the value of $\displaystyle\int_0^5 [g(x) - f(x)]\,dx$?

Sketch the region enclosed by the graphs of the given functions and find the area of the region.

29. $y = \dfrac{x}{\sqrt{1+x^2}}, \quad y = \dfrac{x}{\sqrt{9-x^2}}, \quad x \geq 0$

30. $y = \dfrac{x}{1+x^2}, \quad y = \dfrac{x^2}{1+x^3}$

31. $y = \dfrac{\ln x}{x}, \quad y = \dfrac{(\ln x)^2}{x}$

Evaluate the integral and interpret it as the area of a region. Sketch the region.

32. $\displaystyle\int_0^{\pi/2} |\sin x - \cos 2x|\,dx$ **33.** $\displaystyle\int_{-1}^{1} |3^x - 2^x|\,dx$

Use technology to sketch the region bounded by the graphs of the given functions and to find the x-coordinates of the points of intersection of the curves. Then find the approximate area of the region.

34. $y = x\sin(x^2), \quad y = x^4, \quad x \geq 0$

35. $y = \dfrac{x}{(x^2+1)^2}, \quad y = x^5 - x, \quad x \geq 0$

36. $y = x^2 \ln x, \quad y = \sqrt{x-1}$

37. $y = x\cos x, \quad y = x^{10}$

38. $y = 3x^2 - 2x, \quad y = x^3 - 3x + 4$

39. $y = 1.3^x, \quad y = 2\sqrt{x}$

Use technology to sketch the region bounded by the graphs of the given functions. Find the approximate area of the region.

40. $y = \dfrac{2}{1+x^4}, \quad y = x^2$

41. $y = e^{1-x^2}, \quad y = x^4$

42. $y = \tan^2 x, \quad y = \sqrt{x}$

43. $y = \cos x, \quad y = x + 2\sin^4 x$

44. $y = e^{-x^2/2}, \quad y = \ln x, \quad y = -x + 1$

45. Let $f(x) = \sqrt{x}$.

(a) Find an equation of the line tangent to the graph of f at the point where $x = 4$.

(b) Find the area of the region bounded by the graph of f, the y-axis, and the tangent line from part (a).

46. Sketch the region in the xy-plane defined by the inequalities $4x - y^2 \geq 0$ and $y \geq (x - 2)^2$, and find its area.

47. Racing cars driven by Driver A and Driver B are side by side at the start of a race. The table shows the velocities of each car (in miles per hour) at selected times during the first 10 seconds.

t	v_A	v_B	t	v_A	v_B
0	0	0	6	69	80
1	20	22	7	75	86
2	32	37	8	81	93
3	46	52	9	86	98
4	54	61	10	90	102
5	62	71			

Use Simpson's Rule to estimate how much farther Driver B travels than Driver A does during the first 10 seconds.

48. Bicyclists A and B start out from the same place at the same time. The velocity of bicyclist A, in feet per second, is modeled by the equation $v_A(t) = \dfrac{40t}{\sqrt{t^2 + 20}}$ and the velocity of bicyclist B, in feet per second, is modeled by the equation $v_B(t) = \dfrac{36t}{\sqrt{t^2 + 10}}$, where $0 \leq t \leq 360$.

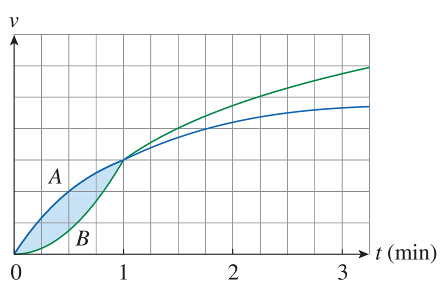
(a) At time $t = 2$, which of the two bicyclists is going faster? By how much? Indicate the units of measure.

(b) At time $t = 0$, both bicyclists have zero velocity. What is the first time after this that they have equal velocity?

(c) After 20 seconds, which bicyclist has gone farther? By how much, to the nearest foot?

(d) Let $d_A(t)$ and $d_B(t)$ represent the distance in feet that each bicyclist has covered after t seconds. Write a formula for each of these distances in terms of t.

49. Two cars, A and B, start side by side and accelerate from rest. The figure shows the graphs of their velocity functions.



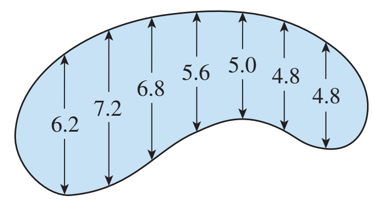
(a) Which car is ahead after 1 minute? Explain your reasoning.

(b) What is the meaning of the area of the shaded region in the context of this problem?

(c) Which car is ahead after 2 minutes? Explain your reasoning.

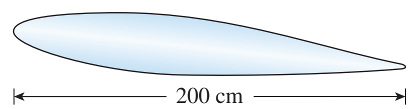
(d) Estimate the time at which the cars are again side by side.

50. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure.



Use Simpson's Rule to estimate the area of the pool.

51. A cross-section of an airplane wing is shown in the figure.

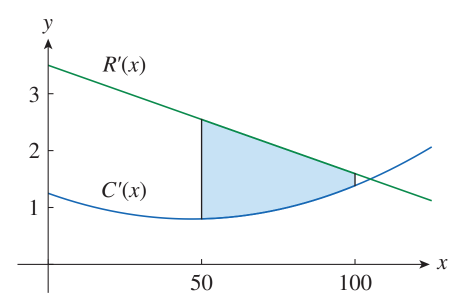


Measurements of the thickness of the wing, in centimeters, at 20-centimeter intervals are 5.8, 20.3, 26.7, 29.0, 27.6, 27.3, 23.8, 20.5, 15.1, 8.7, and 2.8. Use Simpson's Rule to estimate the area of the wing's cross-section.

52. Suppose the birth rate of a population is $b(t) = 2200\,e^{0.024t}$ people per year and the death rate is $d(t) = 1460\,e^{0.018t}$ people per year. Find the area between the graphs of b and d for $0 \leq t \leq 10$. What does this area represent in the context of this problem?

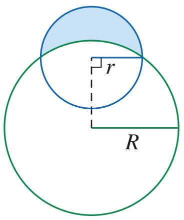
53. The rates at which rain fell, in inches per hour, in two different locations t hours after the start of a storm are given by $f(t) = 0.73t^3 - 2t^2 + t + 0.6$ and $g(t) = 0.17t^2 - 0.5t + 1.1$. Compute the area between the graphs for $0 \leq t \leq 2$ and interpret your answer in the context of the problem.

54. The figure shows the graphs of the marginal revenue function R' and the marginal cost function C' for a manufacturer.



Recall that $R(x)$ and $C(x)$ represent the revenue and cost when x units are manufactured. Assume that R and C are measured in thousands of dollars. Explain the meaning of the shaded region in this context. Use the Midpoint Rule to estimate the value of this quantity.

55. Find the area of the crescent-shaped region (called a *lune*) bounded by arcs of circles with radii r and R, as shown in the figure.



56. Let R be the region in the xy-plane defined by the inequalities $x - 2y^2 \geq 0$, $1 - x - |y| \geq 0$. Sketch the region R and find its area.

57. Use the parametric equations of an ellipse, $x = a\cos\theta$, $y = b\sin\theta$, $0 \leq \theta \leq 2\pi$, to find the area it encloses.

58. Find the area of the region enclosed by the curve described by $x = t^2 - 2t$, $y = \sqrt{t}$ and the y-axis.

59. Find the area of the region enclosed by the x-axis and the curve $x = 1 + e^t$, $y = t - t^2$.

60. Graph the astroid described by $x = a\cos^3\theta$, $y = a\sin^3\theta$ and set up an integral for the area that it encloses. Use technology to evaluate the integral.

61. Find the area bounded by the loop of the curve with parametric equations $x = t^2$, $y = t^3 - 3t$.

62. Find the area of the region enclosed by the loop of the curve with parametric equations $x = t^3 - 12t$, $y = 3t^2 + 2t + 5$.

63. Find the values of c such that the area of the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576.

64. Find the area of the region bounded by the parabola $y = x^2$, the tangent line to the parabola at (1, 1), and the x-axis.

65. Find the number b such that the line $y = b$ divides the region bounded by the graphs of $y = x^2$ and $y = 4$ into two regions of equal area.

66. (a) Find the number a such that the line $x = a$ bisects the area under the graph of $y = \dfrac{1}{x^2}$, $1 \leq x \leq 4$.

(b) Find the number b such that the line $y = b$ bisects the area described in part (a).

67. Find a positive continuous function f such that the area under the graph of f from 0 to t is $A(t) = t^3$ for all $t > 0$.

68. Suppose that $0 < c < \dfrac{\pi}{2}$. Find the value of c for which the area of the region bounded by the graphs of $y = \cos x$, $y = \cos(x - c)$, and $x = 0$ is equal to the area of the region bounded by the graphs of $y = \cos(x - c)$, $x = \pi$, and $y = 0$.

69. For what values of m do the line $y = mx$ and the graph of $y = \dfrac{x}{x^2 + 1}$ enclose a region? Find the area of this region.

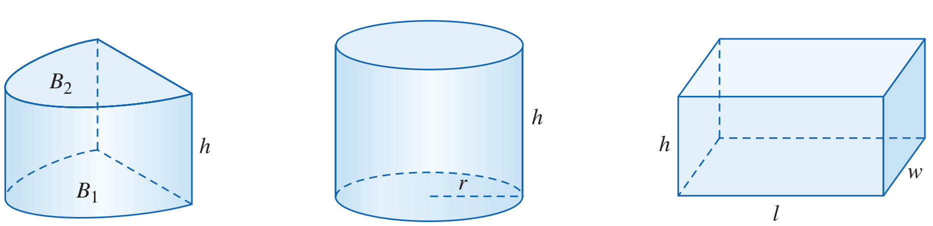
6.2 Volumes

The problem of finding the volume of certain types of solids is similar to the problem of finding the area of regions in the plane. We have an intuitive idea of what volume means, but we need to be precise by using calculus to provide an exact definition of volume.

Consider a simple common solid, a **cylinder**, or more precisely, a *right cylinder*. As shown in Figure 6.15(a), a cylinder is bounded below by a plane region B_1, called the **base**, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments that are perpendicular to the base and join B_1 to B_2. If the area of the base is A and the height of the cylinder, the distance from B_1 to B_2, is h, then the volume V of the cylinder is defined as

$$V = Ah$$

In particular, if the base is a circle with radius r, then the cylinder is a circular cylinder with volume $V = \pi r^2 h$ [see Figure 6.15(b)], and if the base is a rectangle with length l and width w, then the cylinder is a rectangular box (also called a *rectangular parallelepiped*) with volume $V = lwh$ [see Figure 6.15(c)].



(a) Cylinder $V = Ah$ (b) Circular cylinder $V = \pi r^2 h$ (c) Rectangular box $V = lwh$

Figure 6.15
Some simple common solids.

For a solid S that isn't a cylinder, we first *cut* S into pieces and approximate each piece by a cylinder. We estimate the volume of S by adding the volumes of the cylinders. We can find the exact volume of S through a familiar limiting process in which the number of pieces becomes large.

Consider the solid S shown in Figure 6.16. Pass a plane perpendicular to the x-axis through S and consider the intersection of the plane and S, a plane region called a **cross-section** of S. Let $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x, where $a \le x \le b$. Think of slicing S through x and computing the area of this slice. The cross-sectional area $A(x)$ will vary as x increases from a to b.

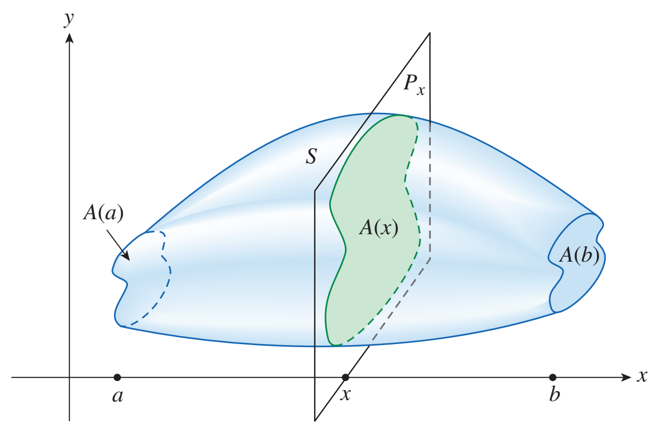


Figure 6.16
Cross-section of a solid S.

Divide S into n *slabs* of equal width Δx by using the planes $P_{x_1}, P_{x_2}, \ldots$ to slice the solid. Think of slicing a loaf of bread. Choose sample points x_i^* in $[x_{i-1}, x_i]$ and approximate the volume of the ith slice S_i (the part of S that lies between the planes $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base $A(x_i^*)$ and *height* Δx. See Figure 6.17.

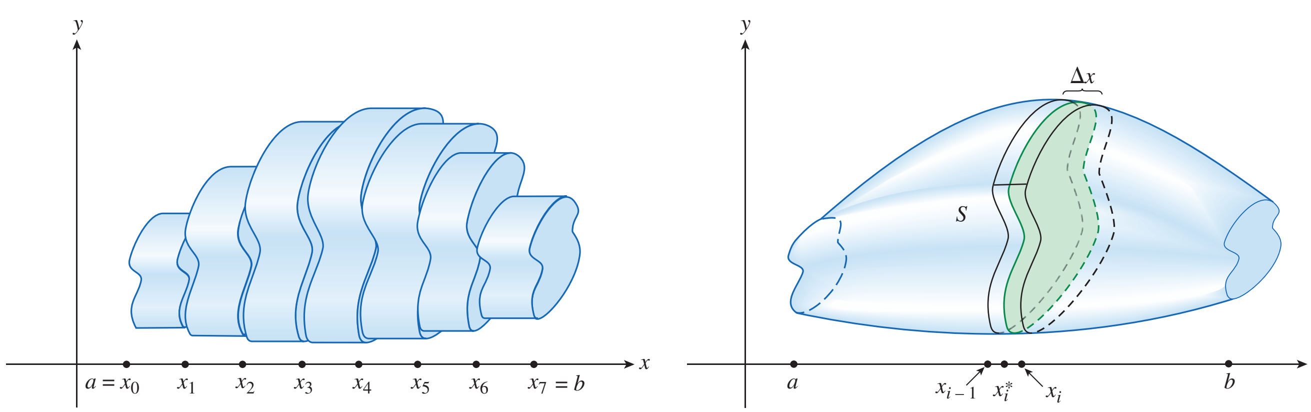


Figure 6.17
We can approximate the volume of S by adding the volumes of each slice.

The volume of the ith cylinder is $A(x_i^*)\,\Delta x$. Therefore, an approximation to our intuitive idea of the volume of the ith slice S_i is

$$V(S_i) \approx A(x_i^*)\,\Delta x$$

Adding the volumes of these slices, we get an approximation to the total volume (or what we think of intuitively as the volume):

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

This approximation certainly seems to become better and better as n, the number of slices, increases without bound, or as $n \to \infty$. Think of the slices, or cross-sections, as becoming thinner and thinner. Therefore, we *define* the volume as the limit of these sums as $n \to \infty$. But this is the limit of a Riemann sum and by definition is a definite integral.

It can be proved that this definition is independent of how S is situated with respect to the x-axis. In other words, no matter how we slice S with parallel planes, we always get the same answer for V.

Definition • Volume

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

When we use the volume formula $V = \displaystyle\int_a^b A(x)\,dx$, it is important to remember that $A(x)$ is the area of a moving cross-section obtained by slicing through x perpendicular to the x-axis.

Notice that, for a cylinder, the cross-sectional area is constant: $A(x) = A$ for all x.

Using the definition, the volume of a cylinder is $V = \displaystyle\int_a^b A\,dx = A(b - a)$. This agrees with the intuitive formula $V = Ah$.

Example 1 Volume of a Sphere

Show that the volume of a sphere of radius r is $V = \dfrac{4}{3}\pi r^3$.

Solution

Position the sphere so that its center is at the origin, as shown in Figure 6.18.

Then the plane P_x intersects the sphere in a circle with radius $y = \sqrt{r^2 - x^2}$ (by the Pythagorean Theorem).

The area of a cross-section is $A(x) = \pi y^2 = \pi(r^2 - x^2)$.

Use the definition of volume with $a = -r$ and $b = r$.

$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx \qquad \text{Definition of volume.}$$

$$= 2\pi \int_0^r (r^2 - x^2)\,dx \qquad \text{The integrand is an even function.}$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \qquad \text{Antiderivative.}$$

$$= 2\pi \left[\left(r^3 - \frac{r^3}{3} \right) - (0) \right] = \frac{4}{3}\pi r^3 \qquad \text{FTC2; simplify.}$$

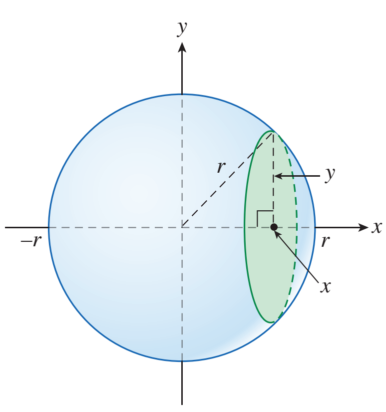


Figure 6.18
Each cross-section is a circle of radius $y = \sqrt{r^2 - x^2}$.

Example 2 Triangular Cross-Sections

The base of a solid S is a circle of radius 1. For this solid, cross-sections perpendicular to the x-axis are equilateral triangles. Find the volume of this solid.

Solution

Position the circle in the xy-plane so that it is centered at the origin. The solid, its base, and a typical cross-section are shown in Figure 6.19.

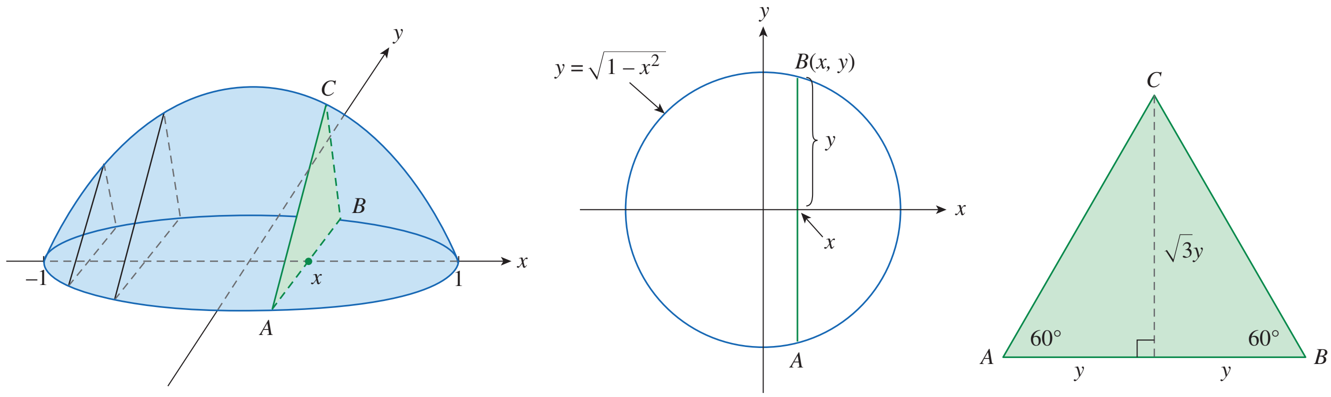


Figure 6.19
The solid, its base, and a typical cross-section, which is a triangle.

Since the point B lies on the circle, $y = \sqrt{1-x^2}$.

The base of the triangle ABC is $|AB| = 2y = 2\sqrt{1-x^2}$.

Since the triangle is equilateral, the height is $\sqrt{3}y = \sqrt{3}\sqrt{1-x^2}$.

The cross-sectional area is $A(x) = \dfrac{1}{2} \cdot 2\sqrt{1-x^2} \cdot \sqrt{3}\sqrt{1-x^2} = \sqrt{3}(1-x^2)$.

Use the definition of volume.

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \sqrt{3}(1-x^2)\,dx \qquad \text{Definition of volume.}$$

$$= 2\int_{0}^{1} \sqrt{3}(1-x^2)\,dx \qquad \text{The integrand is an even function.}$$

$$= 2\sqrt{3}\left[x - \frac{x^3}{3}\right]_0^1 \qquad \text{Antiderivative.}$$

$$= 2\sqrt{3}\left[\left(1 - \frac{1}{3}\right) - (0)\right] = \frac{4\sqrt{3}}{3} \qquad \text{FTC2; simplify.}$$

Figure 6.20 illustrates the resulting solid. ■

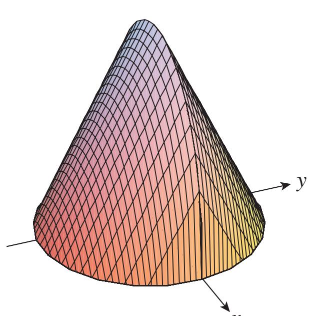


Figure 6.20
The resulting solid.

Example 3 Square Cross-Sections

Find the volume of a pyramid whose base is a square with side L and whose height is h.

Solution

Position the pyramid so that its vertex is at the origin and the central axis is along the x-axis. Figure 6.21 shows the pyramid and a typical cross-section P_x, which is a square.

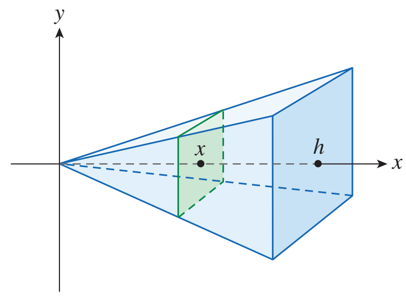


Figure 6.21
The pyramid and a typical cross-section.

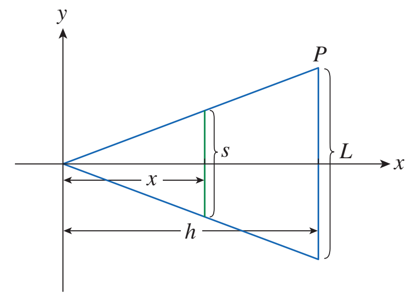


Figure 6.22
Similar triangles used to write an expression for s.

We can write the length of the side s of the square in terms of x by using similar triangles. Use Figure 6.22 to equate two ratios and solve for s.

$$\frac{x}{h} = \frac{s/2}{L/2} = \frac{s}{L} \quad \Rightarrow \quad s = \frac{Lx}{h}$$

The cross-sectional area is $A(x) = s^2 = \dfrac{L^2}{h^2}x^2$.

Use the definition of volume.

$$\begin{aligned} V &= \int_0^h A(x)\,dx = \int_0^h \frac{L^2}{h^2}x^2\,dx && \text{Definition of volume.} \\ &= \frac{L^2}{h^2}\left[\frac{x^3}{3}\right]_0^h && \text{Antiderivative.} \\ &= \frac{L^2}{h^2}\left(\frac{h^3}{3} - 0\right) = \frac{L^2 h}{3} && \text{FTC2; simplify.} \end{aligned}$$

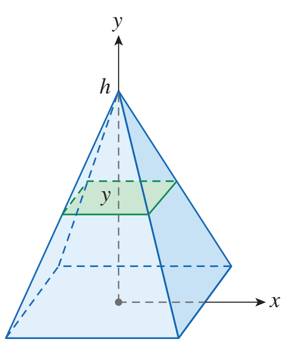


Figure 6.23
Another way to model the pyramid.

Note: We can find this volume by using other equations to model the pyramid. For example, if we place the center of the base at the origin and the vertex on the positive y-axis, as in Figure 6.23, the expression for volume is

$$V = \int_0^h \frac{L^2}{h^2}(h - y)^2\,dy = \frac{L^2 h}{3}. \quad \blacksquare$$

The Disk Method

The general method for finding the volume of a solid can be applied to a common special case: when the cross-sections are circular cylinders, or *disks*. If a region in a plane is revolved about a line in the plane, the resulting solid is a **solid of revolution**. The line is called the **axis of revolution**.

Suppose f is a continuous function on the interval $[a, b]$ such that $f(x) \geq c$ for all x in $[a, b]$, or $f(x) \leq c$ for all x in $[a, b]$. Revolve the plane region R bounded by the graphs

of $y = f(x)$, $y = c$, $x = a$, and $x = b$ about the line $y = c$ to generate a solid S. Find the volume of the solid S. Figure 6.24 shows this solid of revolution.

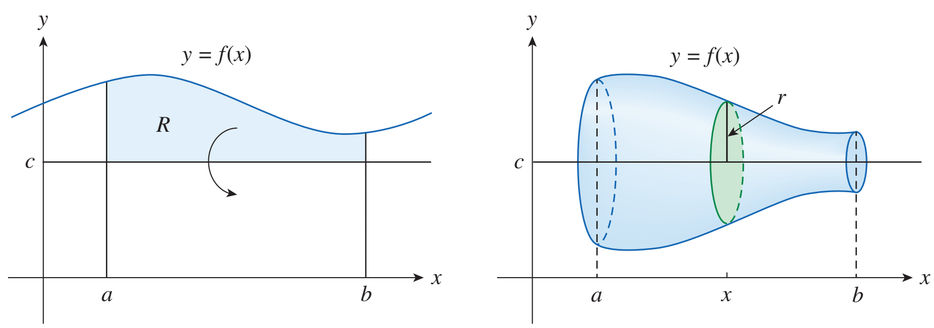


Figure 6.24
The region R is revolved about the line $y = c$ to form the solid of revolution S. A typical cross-section at x is a circle.

A cross-section of this solid at x is a circle. Therefore, the area $A(x)$ is

$$A(x) = \pi r^2 = \pi(f(x) - c)^2$$

The volume of the solid is

$$V = \int_a^b A(x)\,dx = \pi \int_a^b (f(x) - c)^2\,dx = \pi \int_a^b [r(x)]^2\,dx$$

where $r(x)$ denotes the radius of the solid at an arbitrary value x.

A similar formula can be derived if the axis of revolution is vertical.

The Disk Method

To find the volume of a solid of revolution using the **disk method**, use one of the following formulas:

Horizontal Axis of Revolution	Vertical Axis of Revolution
Volume $= V = \pi \int_a^b [r(x)]^2\,dx$	Volume $= V = \pi \int_c^d [r(y)]^2\,dy$

A Closer Look

1. $r(x)$ or $r(y)$ is the radius of a representative disk.
2. The radius is always perpendicular to the axis of revolution.
3. The region always has the axis of revolution as a bounding curve on the interval $[a, b]$ or $[c, d]$.
4. To determine the variable of integration, draw a rectangle in the region R that is rotated about the axis of revolution to produce a representative disk. If the width is Δx, then integrate with respect to x. If the width is Δy, then integrate with respect to y.

   Another way to determine the variable of integration: draw the radius of an arbitrary (circular) cross-section. If the radius is perpendicular to the x-axis,

then integrate with respect to x. If the radius is perpendicular to the y-axis, then integrate with respect to y.

5. A straightforward application of the disk method involves a plane region bounded by the graph of $y = f(x) \geq 0$ and the x-axis. If the axis of revolution is the x-axis, then $r(x) = f(x)$.

Example 4 Volume Using the Disk Method

The region bounded by the graph of $y = \sqrt{x}$ and the x-axis between 0 and 1 is rotated about the x-axis. Find the volume of the resulting solid.

Solution

The region and the resulting solid of revolution are shown in Figure 6.25.

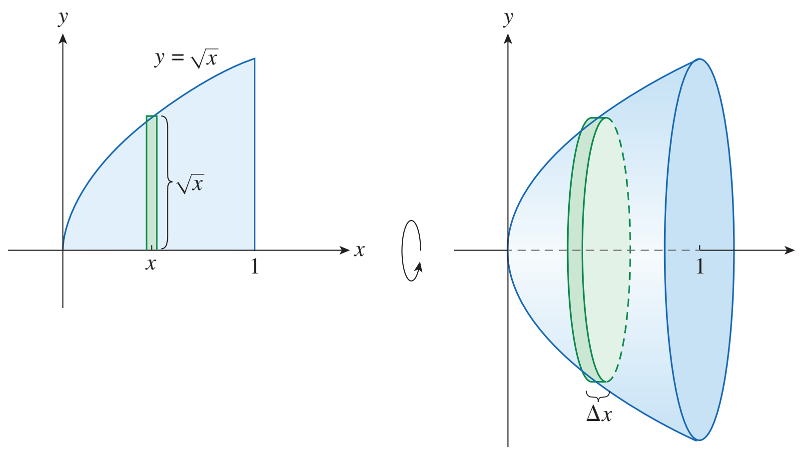


Figure 6.25
The plane region, a representative disk, and the resulting solid of revolution.

When we slice through the solid at x, we obtain a disk with radius $\sqrt{x}$.

The solid lies between $x = 0$ and $x = 1$, so the volume is

$$V = \pi \int_0^1 [r(x)]^2\,dx \qquad \text{The disk method.}$$

$$= \pi \int_0^1 [\sqrt{x}]^2\,dx = \pi \int_0^1 x\,dx \qquad \text{The radius of a disk is } \sqrt{x}.$$

$$= \pi \left[\frac{x^2}{2}\right]_0^1 = \pi\left[\frac{1}{2} - 0\right] = \frac{\pi}{2} \qquad \text{Antiderivative; FTC2; simplify.}$$

Example 5 Horizontal Line as the Axis of Revolution

The region bounded by the graph of $y = x^3$, $y = 1$, and $x = 0$ is rotated about the line $y = 1$. Find the volume of the resulting solid.

Solution

The plane region and the resulting solid are shown in Figure 6.26.

The width of an arbitrary disk is Δx, so we integrate with respect to x.

The radius of a disk at x is $1 - x^3$.

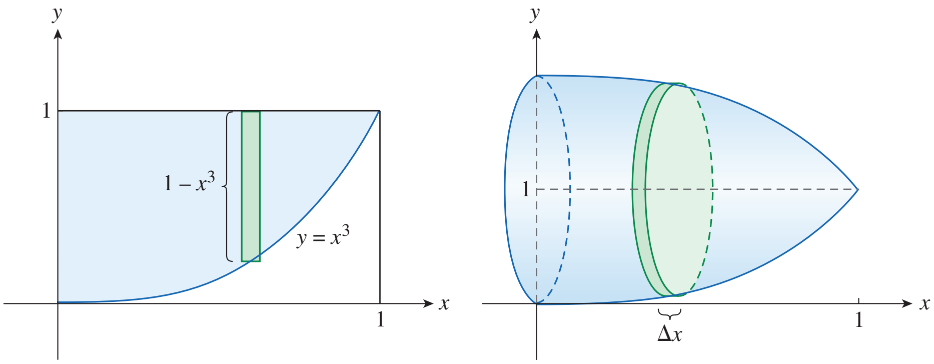


Figure 6.26
The plane region, a representative disk, and the resulting solid of revolution.

The solid lies between $x = 0$ and $x = 1$, so the volume is

$$V = \pi \int_0^1 [r(x)]^2\, dx \qquad \text{The disk method.}$$

$$= \pi \int_0^1 [1 - x^3]^2\, dx = \pi \int_0^1 (1 - 2x^3 + x^6)\, dx \qquad \text{The radius of a disk is } 1 - x^3.$$

$$= \pi \left[x - \frac{x^4}{2} + \frac{x^7}{7} \right]_0^1 \qquad \text{Antiderivative.}$$

$$= \pi \left[\left(1 - \frac{1^4}{2} + \frac{1^7}{7} \right) - (0) \right] = \frac{9\pi}{14} \qquad \text{FTC2; simplify.}$$

Example 6 Revolve About a Vertical Line

The region bounded by the graphs of $y = x^3$, $y = 8$, and $x = 0$ is rotated about the y-axis. Find the volume of the resulting solid.

Solution

The plane region and the resulting solid are shown in Figure 6.27.

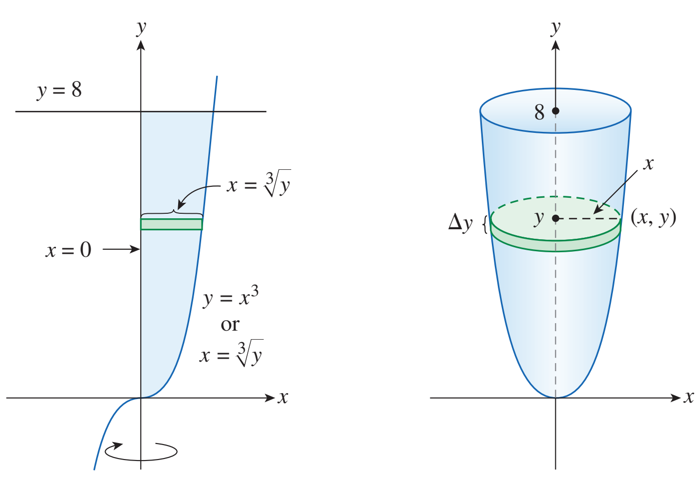


Figure 6.27
The plane region, a representative disk, and the resulting solid of revolution.

Because the region is rotated about the y-axis, it makes sense to slice the solid perpendicular to the y-axis. We still obtain circular cross-sections, but each of width Δy. Therefore, we integrate with respect to y.

The radius of an arbitrary disk is $x = \sqrt[3]{y}$.

The solid lies between $y = 0$ and $y = 8$, so the volume is

$$\begin{aligned} V &= \pi \int_0^8 [r(y)]^2\, dy && \text{The disk method.} \\ &= \pi \int_0^8 [y^{1/3}]^2\, dy = \pi \int_0^8 y^{2/3}\, dy && \text{The radius of a disk is } y^{1/3}. \\ &= \pi \left[\frac{3}{5} y^{5/3}\right]_0^8 && \text{Antiderivative.} \\ &= \frac{3\pi}{5}[8^{5/3} - 0] = \frac{96\pi}{5} && \text{FTC2; simplify.} \end{aligned}$$

The Washer Method

Suppose the axis of revolution does not form a bounding curve to the region (being revolved) throughout the interval of integration. In this case, we can express the volume of the resulting solid as a difference of the volumes of two solids such that the regions revolved to generate the solids each has the axis of revolution as a bounding curve throughout the interval of integration.

The volume of the resulting solid is $V = V_o - V_i$, where V_o is the outer volume related to the bounding curve farther from the axis of revolution, and V_i is the inner volume related to the bounding curve closer to the axis of revolution.

Consider the shaded region in Figure 6.28, bounded by an **outer radius** r_o and an **inner radius** r_i. Revolving this region about the line $y = c$ results in the solid shown in Figure 6.29. A cross-section perpendicular to the axis of revolution, in the plane P_x, has the shape of a *washer*, an annular ring, with outer radius r_o and inner radius r_i.

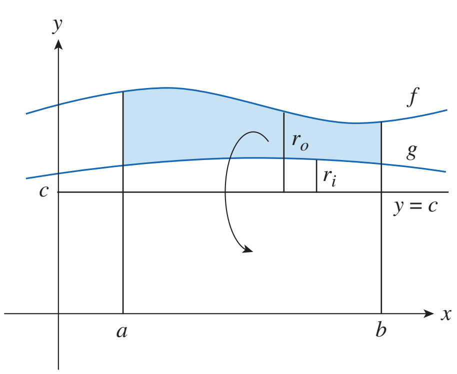


Figure 6.28
The shaded region is revolved about the line $y = c$.

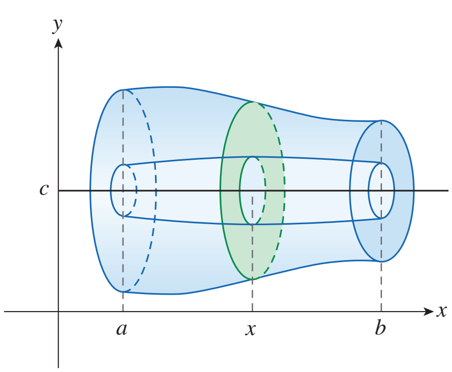


Figure 6.29
The resulting solid of revolution.

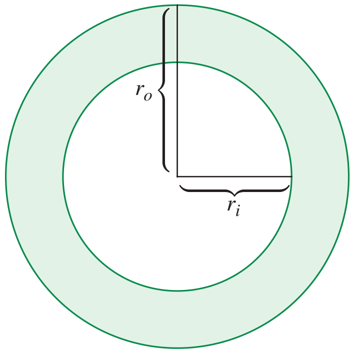


Figure 6.30
A cross-section has the shape of a washer with outer radius r_o and inner radius r_i.

The area of the washer is obtained by subtracting the area of the inner disk from the area of the outer disk. See Figure 6.30.

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 = \pi(r_o^2 - r_i^2)$$

Use the definition of volume to find the volume of the solid of revolution.

$$V = \int_a^b A(x)\,dx = \pi \int_a^b (r_o^2 - r_i^2)\,dx$$

This technique for finding volume is called the **washer method**.

Example 7 The Washer Method

The region bounded by the graphs of $y = x$ and $y = x^2$ is revolved about the x-axis. Find the volume of the resulting solid.

Solution

The graphs of $y = x$ and $y = x^2$ intersect at the points (0, 0) and (1, 1). The region bounded by these graphs is shown in Figure 6.31, and the resulting solid of revolution is shown in Figure 6.32.

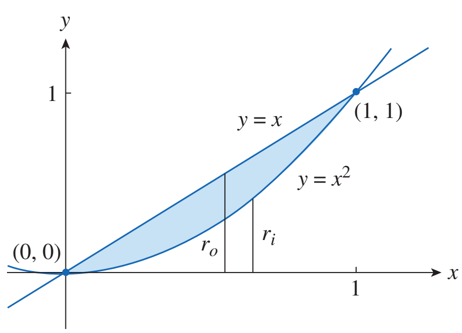


Figure 6.31
The region enclosed by the graphs, the outer radius, and the inner radius.

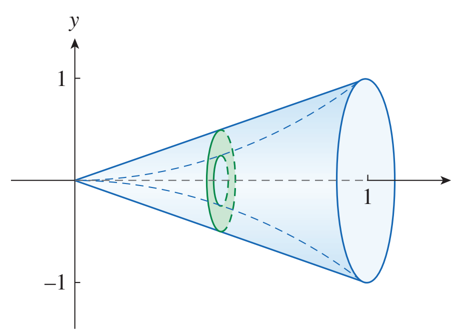


Figure 6.32
The solid of revolution.

Draw an outer radius: a straight line from the axis of revolution to the outer bounding curve. Draw an inner radius: a straight line from the axis of revolution to the inner bounding curve. Since the inner and outer radii are perpendicular to the x-axis, integrate with respect to x.

Find the length of each radius at an arbitrary value x.

$$r_o = x - 0 = x \qquad r_i = x^2 - 0 = x^2$$

The volume is

$$V = \pi \int_0^1 (r_0^2 - r_i^2)\,dx = \pi \int_0^1 (x^2 - x^4)\,dx \qquad \text{The washer method.}$$

$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 \qquad \text{Antiderivative.}$$

$$= \pi\left[\left(\frac{1^3}{3} - \frac{1^5}{5}\right) - (0)\right] = \frac{2\pi}{15} \qquad \text{FTC2; simplify.}$$

Common Error

$$V = \pi \int_a^b (r_o - r_i)^2\,dx$$

Correct Method

The integrand is the difference of the squared radii, not the square of a difference in the radii.

$$V = \pi \int_a^b (r_o^2 - r_i^2)\,dx$$

Example 8 Washer Method and a Horizontal Line

The region bounded by the graphs of $y = x$ and $y = x^2$ is revolved about the line $y = 2$. Find the volume of the resulting solid.

Solution

This is the same region as in Example 7. However, this time we revolve the region about the line $y = 2$, as shown in Figure 6.33. The resulting solid is shown in Figure 6.34.

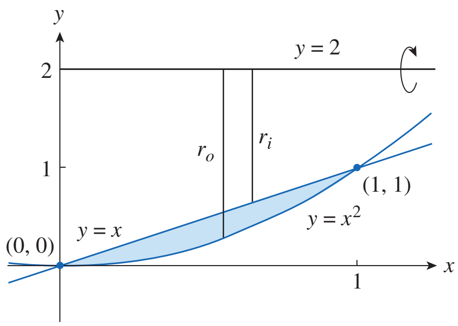


Figure 6.33
The region enclosed by the graphs, the outer radius, and the inner radius.

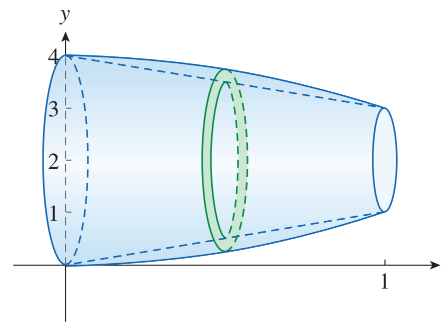


Figure 6.34
The solid of revolution.

Draw an outer radius and an inner radius. Each is perpendicular to the x-axis, so we integrate with respect to x.

Find the length of each radius at an arbitrary value x (see Figure 6.33).

$r_o = 2 - x^2 \qquad r_i = 2 - x$

The volume is

$$V = \pi \int_0^1 (r_o^2 - r_i^2)\,dx = \pi \int_0^1 [(2 - x^2)^2 - (2 - x)^2]\,dx \qquad \text{The washer method.}$$

$$= \pi \int_0^1 (x^4 - 5x^2 + 4x)\,dx \qquad \text{Expand and combine terms.}$$

$$= \pi \left[\frac{x^5}{5} - 5 \cdot \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} \right]_0^1 \qquad \text{Antiderivative.}$$

$$= \pi \left[\left(\frac{1}{5} - 5 \cdot \frac{1}{3} + 4 \cdot \frac{1}{2} \right) - (0) \right] = \frac{8\pi}{15} \qquad \text{FTC2; simplify.}$$

■

Example 9 Washer Method and a Vertical Line

The region bounded by the graphs of $y = x$ and $y = x^2$ is revolved about the line $x = -1$. Find the volume of the resulting solid.

Solution

This is the same region as in Examples 7 and 8. However, this time we revolve the region about the line $x = -1$. See Figures 6.35 and 6.36.

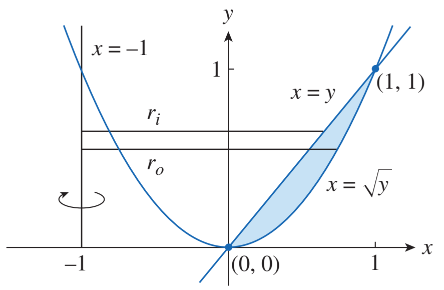


Figure 6.35
The region enclosed by the graphs, the outer radius, and the inner radius.

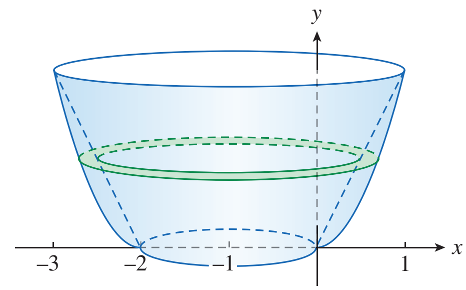


Figure 6.36
The solid of revolution.

Draw outer and inner radii. Each radius is perpendicular to the y-axis, so we integrate with respect to y.

Find the length of each radius at an arbitrary value y.

$$r_o = \sqrt{y} - (-1) = \sqrt{y} + 1 \qquad r_i = y - (-1) = y + 1$$

The volume is

$$V = \pi \int_0^1 (r_o^2 - r_i^2)\, dy = \pi \int_0^1 [(\sqrt{y} + 1)^2 - (y + 1)^2]\, dy \qquad \text{The washer method.}$$

$$= \pi \int_0^1 (2\sqrt{y} - y - y^2)\, dy \qquad \text{Expand; collect terms.}$$

$$= \pi \left[\frac{4y^{2/3}}{3} - \frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 \qquad \text{Antiderivative.}$$

$$= \pi \left[\left(\frac{4}{3} - \frac{1}{2} - \frac{1}{3} \right) - (0) \right] = \frac{\pi}{2} \qquad \text{FTC2; simplify.}$$

6.2 Exercises

Find the volume of the solid obtained by rotating the region bounded by the graphs of the given expressions about the specified line. Sketch the region, the solid, and a typical disk or washer.

1. $y = x + 1, \quad y = 0, \quad x = 0, \quad x = 2;$ about the x-axis

2. $y = 1 - x^2, \quad y = 0;$ about the x-axis

3. $y = \sqrt{x - 1}, \quad y = 0, \quad x = 5;$ about the x-axis

4. $y = e^x, \quad y = 0, \quad x = -1, \quad x = 1;$ about the x-axis

5. $y = \ln x, \quad y = 1, \quad y = 2, \quad x = 0;$ about the y-axis

6. $x = 2\sqrt{y}, \quad x = 0, \quad y = 9;$ about the y-axis

7. $2x = y^2, \quad x = 0, \quad y = 4;$ about the y-axis

8. $y = x^3, \quad y = x, \quad x \geq 0;$ about the x-axis

9. $y = \frac{1}{4}x^2, \quad y = 5 - x^2;$ about the x-axis

10. $y = 6 - x^2, \quad y = 2;$ about the x-axis

11. $y^2 = x, \quad x = 2y;$ about the y-axis

12. $x = 2 - y^2, \quad x = y^4;$ about the y-axis

13. $y = \frac{1}{4}x^2, \quad x = 2, \quad y = 0;$ about the y-axis

14. $y = x, \quad y = \sqrt{x};$ about $y = 1$

15. $y = e^{-x}, \quad y = 1, \quad x = 2;$ about $y = 2$

16. $y = 1 + \sec x, \quad y = 3, \quad -\frac{\pi}{3} \leq x \leq \frac{\pi}{3};$ about $y = 1$

The region bounded by the given curves is rotated about the specified line. Find the volume of the resulting solid.

17. $y = \dfrac{1}{x}, \quad x = 1, \quad x = 2, \quad y = 0;$ about the x-axis

18. $x = 2y - y^2, \quad x = 0;$ about the y-axis

19. $x - y = 1, \quad y = x^2 - 4x + 3;$ about $y = 3$

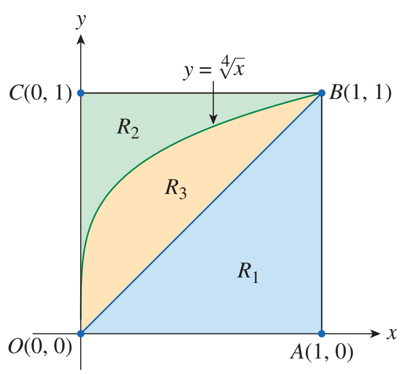
20. $x = y^2, \quad x = 1;$ about $x = 1$

21. $y = x^3, \quad y = \sqrt{x};$ about $x = 1$

22. $y = x^3, \quad y = \sqrt{x};$ about $y = 1$

23. $y = \sin x, \quad y = \cos x, \quad 0 \le x \le \dfrac{\pi}{4};$ about $y = -1$

Three regions are defined in the figure. Find the volume generated by rotating the given region about the specified line.



24. R_1 about OA

25. R_1 about OC

26. R_1 about AB

27. R_1 about BC

28. R_2 about OA

29. R_2 about OC

30. R_2 about AB

31. R_2 about BC

32. R_3 about OA

33. R_3 about OC

34. R_3 about AB

35. R_3 about BC

Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

36. $x^2 - y = 1, \quad x = 3;$ about $x = -2$

37. $y = \cos x, \quad y = 2 - \cos x, \quad 0 \le x \le 2\pi;$ about $y = 4$

Use technology to find the volume of the solid obtained by rotating the region bounded by the graphs of the given expressions about the specified line.

38. $y = e^{-x^2}, \quad y = 0, \quad x = -1, \quad x = 1$
(a) About the x-axis
(b) About $y = -1$

39. $y = \cos^2 x, \quad y = 0, \quad x = -\dfrac{\pi}{2}, \quad x = \dfrac{\pi}{2}$
(a) About the x-axis
(b) About $y = 1$

40. $x^2 + 4y^2 = 4, \quad y = 0$
(a) About $y = 2$
(b) About $x = 2$

Use technology to find the x-coordinates of the points of intersection of the given curves and to find the approximate volume of the solid obtained by rotating the region bounded by the curves about the x-axis.

41. $y = 2 + x^2 \cos x, \quad y = x^4 + x + 1$

42. $y = 3\sin(x^2), \quad y = e^{x/2} + e^{-2x}$

43. $y = \ln(x^6 + 2), \quad y = \sqrt{3 - x^3}$

44. $y = 1 + xe^{-x^3}, \quad y = \arctan(x^2)$

Use a computer algebra system to find the exact volume of the solid obtained by rotating the region bounded by the graphs of the given expressions about the specified line.

45. $y = \sin^2 x, \quad y = 0, \quad 0 \le x \le \pi;$ about $y = -1$

46. $y = x, \quad y = xe^{1 - x/2};$ about $y = 3$

The definite integral represents the volume of a solid. Describe the solid.

47. $\pi \displaystyle\int_0^{\pi} \sin x\, dx$

48. $\pi \displaystyle\int_0^{\pi/2} \cos^2 x\, dx$

49. $\pi \displaystyle\int_{-1}^{1} (1 - y^2)^2\, dy$

50. $\pi \displaystyle\int_0^{1} (y^4 - y^8)\, dy$

51. $\pi \displaystyle\int_1^{4} [3^2 - (3 - \sqrt{x})^2]\, dx$

52. $\pi \displaystyle\int_0^{\pi/2} [(1 + \cos x)^2 - 1^2]\, dx$

53. A CAT scan produces equally spaced cross-sectional views of a human organ that provide information about the organ otherwise obtained only by surgery. Suppose that a CAT scan of a human liver shows cross-sections spaced 1.5 cm apart. The liver is 15 cm long and the cross-sectional areas, in square centimeters, are 0, 18, 58, 79, 94, 106, 117, 128, 63, 39, and 0. Use the Midpoint Rule to estimate the volume of the liver.

54. Let R be the region in the first quadrant bounded by the y-axis and the graphs of $y = x^2 + 1$ and $y = 5 - e^x$.
(a) Find the area of the region R.
(b) Suppose the region R is rotated about the x-axis. Find the volume of the resulting solid.
(c) The solid S has base R. Each cross-section perpendicular to the x-axis is a semicircle whose diameter lies in R. Find the volume of the solid S.

(d) The vertical line $x = k$ divides the region R into two regions of equal areas. Write, but do not solve, an equation involving one or more integral expressions that could be used to determine the value of k.

55. Let R be the region bounded by the graphs of $f(x) = 6 \sin 2x$ and the x-axis from $x = 0$ to $x = \dfrac{\pi}{2}$.

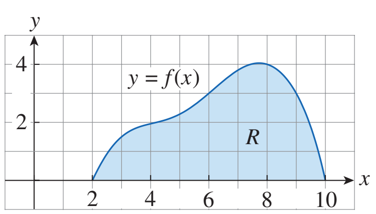
(a) Find the area of the region R. Find the volume of the solid that results from revolving the region R about the x-axis.
(b) The solid S has base R. Each cross-section of this solid perpendicular to the x-axis is an equilateral triangle with a side in R. Find the volume of the solid S.
(c) Find the volume of the solid obtained by revolving the region R about the horizontal line $y = -3$.

56. A log 10 m long is cut at 1-meter intervals and the cross-sectional areas A of each piece (at a distance x from the end of the log) are given in the table.

x (m)	A (m^2)	x (m)	A (m^2)
0	0.68	6	0.53
1	0.65	7	0.55
2	0.64	8	0.52
3	0.61	9	0.50
4	0.58	10	0.48
5	0.59		

Use the Midpoint Rule with $n = 5$ to estimate the volume of the log.

57. Let R be the shaded region in the figure, bounded by the graph of $y = f(x)$, and the x-axis.



(a) The region R is revolved about the x-axis to form a solid S_1. Use Simpson's Rule with $n = 8$ equal subintervals to estimate the volume of the solid S_1.
(b) The region R is revolved about the y-axis to form a solid S_2. Use Simpson's Rule with $n = 4$ equal subintervals to estimate the volume of the solid S_2.

58. Let R be the region in the first quadrant bounded by the graph of $y = x^2$ and the line $y = 4$.
(a) Find the area of the region R.
(b) A solid S has base R. Each cross-section of this solid perpendicular to the y-axis is a semicircle, with its diameter in R. Find the volume of the solid S.
(c) Another solid is formed by revolving R about the line $y = 7$. Find the volume of this new solid.

59. A model for the shape of a bird's egg is obtained by revolving about the x-axis the region under the graph of

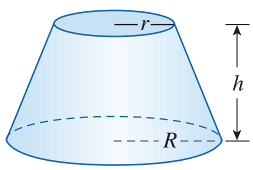
$$f(x) = (ax^3 + bx^2 + cx + d)\sqrt{1 - x^2}$$

(a) Use technology to find the volume of an egg using this model.
(b) For a red-throated loon, $a = -0.06$, $b = 0.04$, $c = 0.1$ and $d = 0.54$. Graph f and find the volume of an egg of this species.

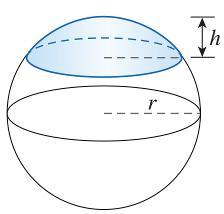
Find the volume of the described solid S.

60. A right circular cone with height h and base radius r

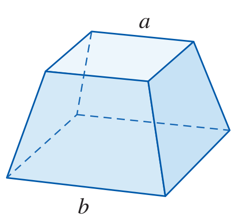
61. A frustum of a right circular cone with height h, lower base radius R, and top radius r



62. A cap of a sphere with radius r and height h



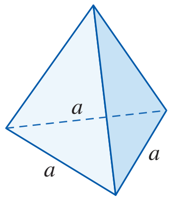
63. A frustum of a pyramid with square base of side b, square top of side a, and height h



What happens if $a = b$? What happens if $a = 0$?

64. A pyramid with height h and rectangular base with dimensions b and $2b$

65. A pyramid with height h and base an equilateral triangle with side a (a tetrahedron)



66. A tetrahedron with three mutually perpendicular faces and three mutually perpendicular edges with lengths 3 cm, 4 cm, and 5 cm.

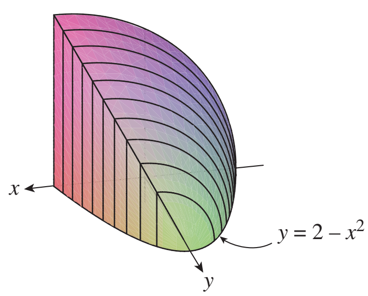
67. The base of a solid S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are squares.

68. The base of a solid S is an elliptical region with boundary curve $9x^2 + 4y^2 = 36$. Cross-sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base.

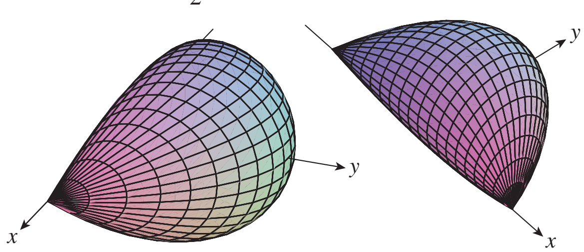
69. The base of a solid S is the triangular region with vertices at the points (0, 0), (1, 0), and (0, 1).
(a) Cross-sections perpendicular to the y-axis are equilateral triangles.
(b) Cross-sections perpendicular to the x-axis are squares.

70. The base of a solid S is the region enclosed by the graph of the parabola $y = 1 - x^2$ and the x-axis.
(a) Cross-sections perpendicular to the y-axis are squares.
(b) Cross-sections perpendicular to the x-axis are isosceles triangles with height equal to the base.

71. The base of a solid S is the region enclosed by the graph of $y = 2 - x^2$ and the x-axis. Cross-sections perpendicular to the y-axis are quarter circles.

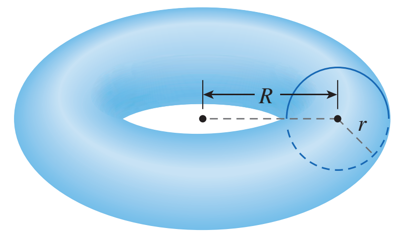


72. The solid S is bounded by the circles that are perpendicular to the x-axis, intersect the x-axis, and have centers on the graph of the parabola $y = \frac{1}{2}(1 - x^2)$, $-1 \leq x \leq 1$.



73. The base of a solid S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are isosceles triangles with height h and unequal side in the base.
(a) Set up, but do not evaluate, a definite integral for the volume of S.
(b) Interpret the integral as an area to find the volume of S.

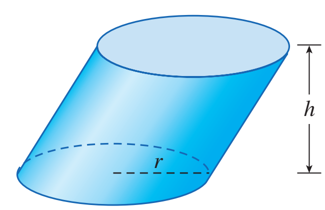
74. A *torus* with radii r and R is a donut-shaped solid as shown in the figure.



(a) Set up, but do not evaluate, a definite integral for the volume of the torus.
(b) Interpret the integral as an area to find the volume of the torus.

75. A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

76. (a) Cavalieri's Principle states that if a family of parallel planes gives equal cross-sectional areas for two solids S_1 and S_2, then the volumes of S_1 and S_2 are equal. Prove this principle.
(b) Use Cavalieri's Principle to find the volume of the oblique cylinder shown in the figure.



77. Find the volume common to two circular cylinders, each with radius r, if the axes of the cylinders intersect at right angles.

78. Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

79. A hole of radius r is bored through the middle of a cylinder of radius $R > r$ at right angles to the axis of the cylinder. Set up, but do not evaluate, an integral for the volume cut out.

80. A hole of radius r is bored through the center of a sphere of radius $R > r$. Find the volume of the remaining portion of the sphere.

81. Some of the pioneers of calculus, such as Kepler and Newton, were inspired by the problem of finding the volumes of wine barrels. (In fact, Kepler published a book *Stereometria doliorum* in 1615 devoted to methods for finding the volumes of barrels.) They often approximated the shape of the sides by parabolas.

(a) A barrel with height h and maximum radius R is constructed by rotating about the x-axis the graph of the parabola $y = R - cx^2$, $-\frac{h}{2} \le x \le \frac{h}{2}$, where c is a positive constant. Show that the radius of each end of the barrels is $r = R - d$, where $d = \frac{ch^2}{4}$.

(b) Show that the volume enclosed by the barrel is

$$V = \frac{1}{3}\pi h\left(2R^2 + r^2 - \frac{2}{5}d^2\right)$$

82. Suppose that a region R has area A and lies above the x-axis. When R is rotated about the x-axis, it sweeps out a solid with volume V_1. When R is rotated about the line $y = -k$ (where k is a positive number), it sweeps out a solid with volume V_2. Express V_2 in terms of V_1, k, and A.

Discovery Project | Rotating on a Slant

We can now find the volume of a solid of revolution obtained by rotating a region about a horizontal or vertical line. But what if we rotate a region about a slanted line, that is, a line that is neither horizontal nor vertical? The purpose of this project is to discover a formula for the volume of a solid of revolution when the axis of revolution is a slanted line.

Let C be the portion, or arc, of the curve $y = f(x)$ between the points $P(p, f(p))$ and $Q(q, f(q))$, and let R be the region bounded by the curve C, by the line $y = mx + b$ (which lies entirely below C), and by the perpendiculars to the line from P and Q. See Figure 6.37.

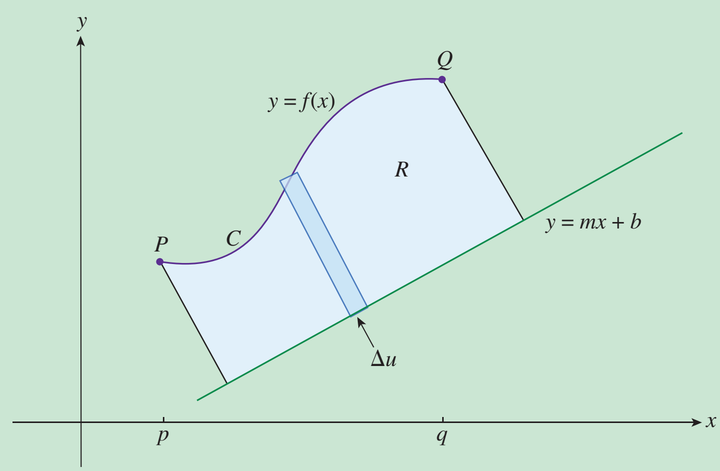


Figure 6.37
The region R is revolved about the line $y = mx + b$ to form a solid of revolution.

1. Show that the area of R is

$$\frac{1}{1 + m^2}\int_p^q [f(x) - mx - b][1 + mf'(x)]\,dx$$

Hint: This formula can be verified by subtracting areas, but it will be helpful throughout the project to derive it by first approximating the area using rectangles perpendicular to the line, as shown in Figure 6.38. Use this figure to help express in Δu in terms of Δx.

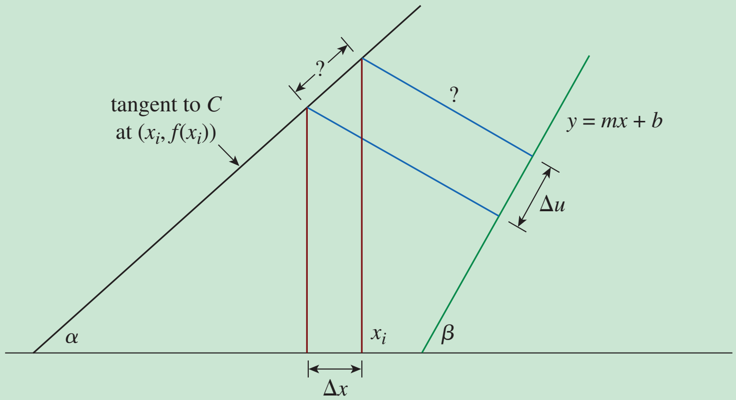


Figure 6.38
Use this figure to help express Δu in terms of Δx.

2. Find the area of the region R in Figure 6.39.

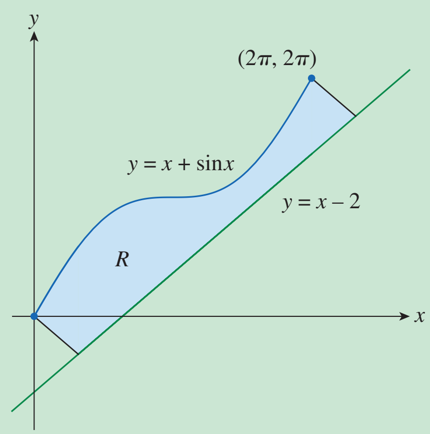


Figure 6.39
Consider the solid S obtained by rotating the region R about the line $y = x - 2$.

3. Find a formula (similar to the one in Problem 1) for the volume of the solid obtained by rotating the region R (in Figure 6.39) about the line $y = mx + b$.

4. Find the volume of the solid obtained by rotating the region R (in Figure 6.39) about the line $y = x - 2$.

6.3 Volumes by Cylindrical Shells

Some volume problems are very difficult to solve by using the disk method or the washer method. For example, consider the problem of finding the volume of the solid obtained by revolving the region bounded by the graphs of $y = 2x^2 - x^3$ and $y = 0$ about the y-axis. See Figure 6.40. If we slice perpendicular to the y-axis, we get a familiar washer. However, to find the inner and the outer radius of the washer, we would have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y; not an easy task.

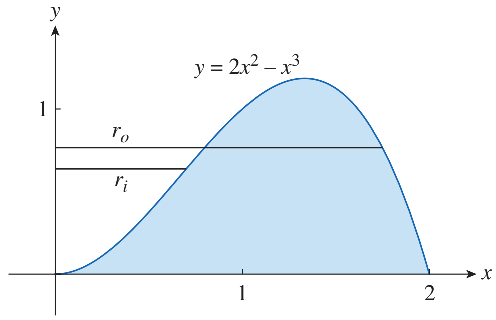


Figure 6.40
Consider the volume of the solid obtained by rotating the shaded region about the y-axis. Finding explicit expressions for the inner and the outer radii is tricky.

In cases like this, we can often use the **method of cylindrical shells** to find the volume of the resulting solid. To derive the general formula for the volume of the solid, first consider a typical cylindrical shell with inner radius r_1 and outer radius r_2, and height h, as shown in Figure 6.41.

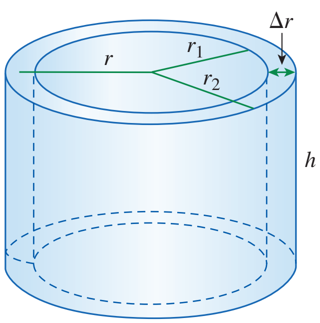


Figure 6.41
A typical cylindrical shell.

The volume V of this cylindrical shell is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder.

$$\begin{aligned} V &= V_2 - V_1 \\ &= \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h && \text{Volume of a cylinder.} \\ &= \pi(r_2 + r_1)(r_2 - r_1)h && \text{Factor; difference of two squares.} \\ &= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) && \text{Multiply by 1 in a convenient form, } \tfrac{2}{2}. \end{aligned}$$

Let $\Delta r = r_2 - r_1$, the thickness of the shell, and let $r = \dfrac{1}{2}(r_2 + r_1)$, the average radius of the shell. Then this formula for the volume of a cylindrical shell becomes

$$V = 2\pi\, r\, h\, \Delta r \tag{1}$$

A practical way to remember and use this formula is

$$V = 2\pi \times \begin{pmatrix}\text{radius of a} \\ \text{typical shell}\end{pmatrix} \times \begin{pmatrix}\text{height of a} \\ \text{typical shell}\end{pmatrix} \times \begin{pmatrix}\text{thickness of a} \\ \text{typical shell}\end{pmatrix}$$

Let R be the region bounded by the graphs of $y = f(x)$, where $f(x) \geq 0$, and $y = 0$ and the lines $x = a$ and $x = b$, where $b > a \geq 0$. Let s be the solid obtained by revolving the region R about the y-axis. See Figure 6.42.

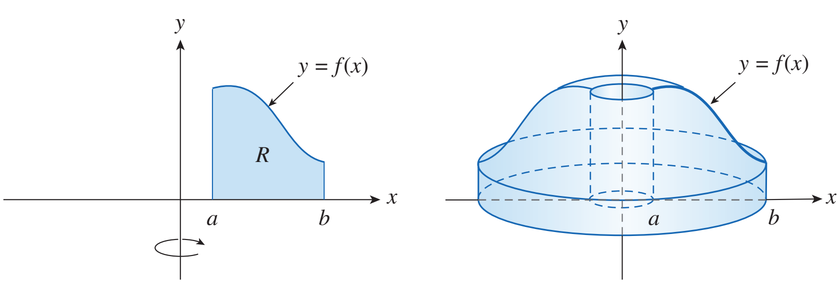


Figure 6.42
Rotate the region R about the y-axis to obtain the solid S.

Divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width Δx and let $\bar{x}_i$ be the midpoint of the ith subinterval. Rotate the rectangle with base $[x_{i-1}, x_i]$ and height $f(\bar{x}_i)$ about the y-axis. The result is a cylindrical shell with average radius $\bar{x}_i$, height $f(\bar{x}_i)$, and thickness Δx. See Figure 6.43.

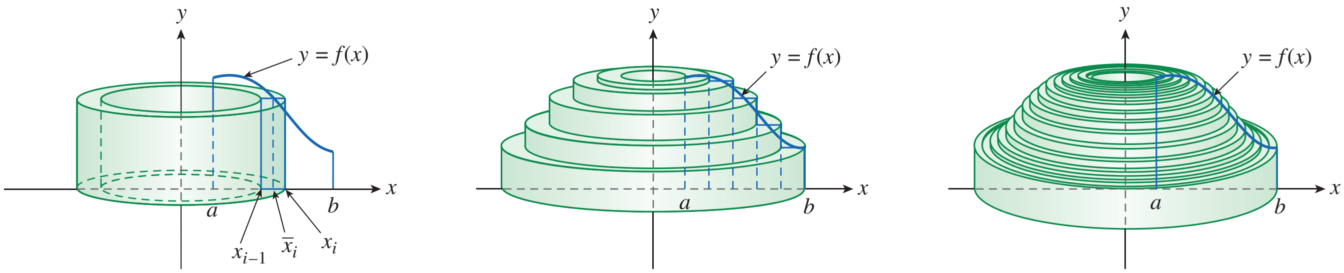


Figure 6.43
A typical cylindrical shell and how these shells are used to approximate the volume of the solid S.

Using Equation 1, the volume of this cylindrical shell is

$$V_i = 2\pi\bar{x}_i f(\bar{x}_i)\,\Delta x$$

An approximation to the volume V of the solid S is given by the sum of the volumes of these cylindrical shells.

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi\bar{x}_i f(\bar{x}_i)\,\Delta x$$

It seems reasonable, and Figure 6.43 suggests, that this approximation becomes better as the number of subintervals n increases. Using the definition of a definite integral, we can write

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\bar{x}_i f(\bar{x}_i)\,\Delta x = \int_a^b 2\pi x f(x)\,dx$$

This leads to the following general formula.

The Method of Cylindrical Shells

The volume of the solid obtained by rotating about the y-axis the region bounded by the graphs of $y = f(x) \geq 0$ and the y-axis from $x = a$ to $x = b$ is

$$V = \int_a^b 2\pi\, x f(x)\,dx \qquad \text{where } 0 \leq a \leq \text{ or } = b \tag{2}$$

A Closer Look

1. The argument used to develop the method of cylindrical shells certainly seems reasonable, and it can be shown more precisely using integration by parts.
2. A way to remember and use this formula is to draw a typical shell with radius x, height $f(x)$, and thickness Δx or dx. See Figure 6.44.

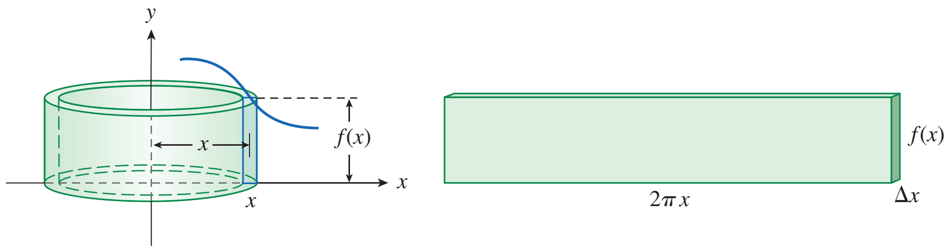


Figure 6.44
A typical cylindrical shell, radius x, height $f(x)$, and thickness Δx or dx.

Here are two ways to write and interpret the formula for volume.

$$V = \int_a^b 2\pi \underbrace{x}_{\text{radius}} \underbrace{f(x)}_{\text{height}} \underbrace{dx}_{\text{thickness}} = \int_a^b \underbrace{2\pi x}_{\text{circumference}} \underbrace{f(x)}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$

A more general way to remember this is

$$V = \int_a^b 2\pi \underbrace{r}_{\text{radius}} \underbrace{h}_{\text{height}} \underbrace{t}_{\text{thickness}}$$

This reasoning will be helpful in other applications, for example, if we rotate about lines other than the y-axis. ■

Example 1 Use the Shell Method

Let R be the region bounded by the graphs of $y = 2x^2 - x^3$ and $y = 0$. Find the volume of the solid S obtained by revolving the region R about the y-axis.

Solution

Sketch the graph of $y = 2x^2 - x^3$ and the region R. See Figure 6.45.

Draw a line at an arbitrary value x in the region R representing the height of a typical shell.

Calculate the height: $h = f(x) - 0 = 2x^2 - x^3$.

Draw a line representing the radius of a typical shell, perpendicular to the height and to the axis of revolution.

Calculate the radius: $r = x - 0 = x$.

Since the height is perpendicular to the x-axis, the thickness is $t = dx$.

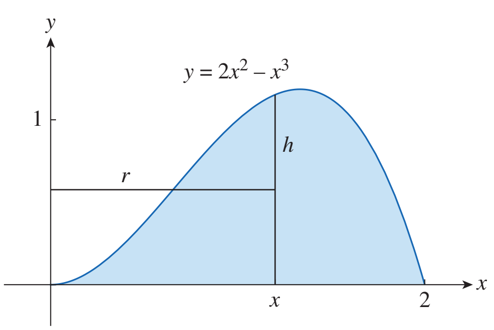


Figure 6.45
A typical shell has radius x, height $2x^2 - x^3$, and thickness dx.

Use the method of cylindrical shells to compute the volume.

$$V = \int_0^2 2\pi rht = \int_0^2 2\pi x(2x^2 - x^3)\,dx$$ Use expressions for the radius, height, and thickness.

$$= 2\pi \int_0^2 (2x^3 - x^4)\,dx = 2\pi\left[\frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2$$ Antiderivative.

$$= 2\pi\left[\left(8 - \frac{32}{5}\right) - (0)\right] = \frac{16}{5}\pi$$ FTC2; simplify.

Figure 6.46 is an illustration of the solid generated in this example.

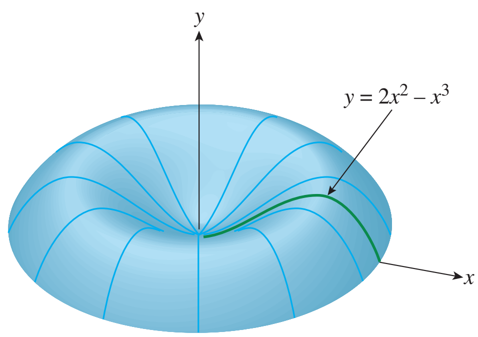


Figure 6.46
A visualization of the solid S.

Note: In this example, the method of cylindrical shells is much easier than the washer method. We did not have to find the coordinates of the local maximum, and we did not have to solve the equation that describes the curve for x in terms of y. It takes practice, pattern recognition, and good problem-solving skills to use the best, quickest, most appropriate method. ■

Example 2 A Solid Cup

Let R be the region in the first quadrant bounded by the graphs of $y = x$ and $y = x^2$. Find the volume of the solid S obtained by revolving R about the y-axis.

Solution

Sketch the graphs of $y = x$ and $y = x^2$, the region R, and the height and radius of a typical shell. The graphs intersect at the points (0, 0) and (1, 1). See Figure 6.47.

The resulting solid S is shown in Figure 6.48.

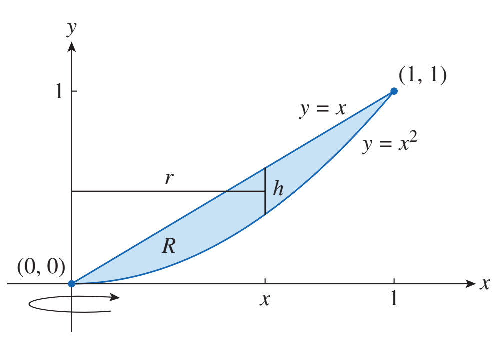


Figure 6.47
The region R and a typical shell with height h and radius r.

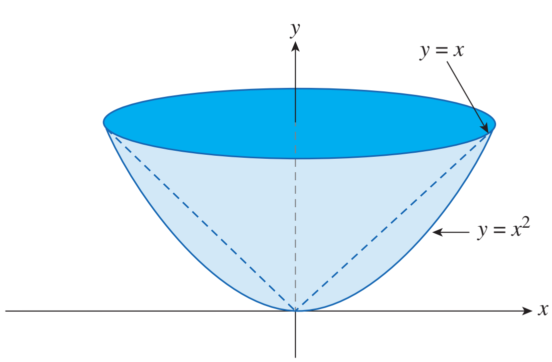


Figure 6.48
A visualization of the solid S.

The radius: $r = x - 0 = x$

The height: $h = x - x^2$

The height is perpendicular to the x-axis; the thickness is $t = dx$.

The volume of the solid S is

$$V = \int_0^1 2\pi x(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx \qquad \text{Method of cylindrical shells; simplify integrand.}$$

$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 \qquad \text{Antiderivative.}$$

$$= 2\pi\left[\left(\frac{1}{3} - \frac{1}{4}\right) - (0)\right] = \frac{\pi}{6} \qquad \text{FTC2; simplify.}$$

The method of cylindrical shells can also be used to compute the volume of a solid obtained by rotating a region about the x-axis. Follow the same strategy: identify the radius, height, and thickness of a typical shell.

Example 3 Rotation About the x-Axis

Let R be the region in the first quadrant bounded by the graph of $y = \sqrt{x}$, the x-axis, and the lines $x = 0$ and $x = 1$. Use the method of cylindrical shells to find the volume of the solid S obtained by revolving the region R about the x-axis.

Solution

Figure 6.49 shows a graph of $y = \sqrt{x}$, the region R, and the height and radius of a typical shell. Figure 6.50 shows a graph of the resulting solid S.

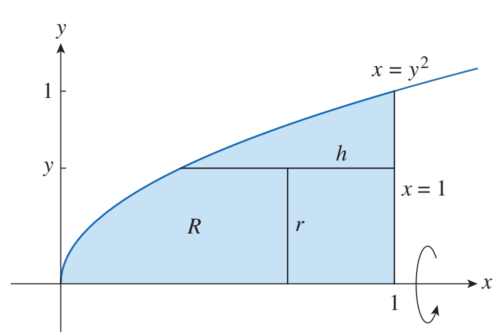


Figure 6.49
The region R, and the height and radius of a typical shell.

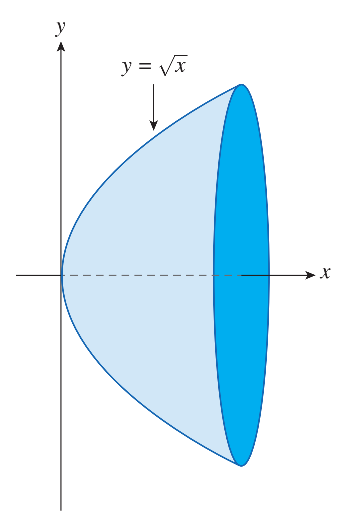


Figure 6.50
A visualization of the solid S.

The radius: $r = y - 0 = y$

The height: $y = 1 - y^2$

The height is perpendicular to the y-axis; the thickness is $t = dy$.

The volume of the solid S is

$$V = \int_0^1 2\pi y(1 - y^2)\, dy = 2\pi \int_0^1 (y - y^3)\, dy \qquad \text{Method of cylindrical shells; simplify integrand.}$$

$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 \qquad \text{Antiderivative.}$$

$$= 2\pi \left[\left(\frac{1}{2} - \frac{1}{4} \right) - (0) \right] = \frac{\pi}{2} \qquad \text{FTC2; simplify.}$$

Note: This problem provides a basic example of the shell method when revolving a region about a horizontal line. However, the disk method is probably easier to use in this case.

Example 4 Rotation About a Vertical Axis

Let R be the region in the first quadrant bounded by the graph of $y = x - x^2$ and the x-axis. Find the volume of the solid S obtained by rotating R about the line $x = 2$.

Solution

Figure 6.51 shows the graph of $y = x - x^2$, the region R, the axis of revolution, and the height and radius of a typical shell.

Figure 6.52 shows the solid of revolution S.

The radius: $r = 2 - x$

The height: $h = (x - x^2) - 0 = x - x^2$

The height is perpendicular to the x-axis; the thickness is $t = dx$.

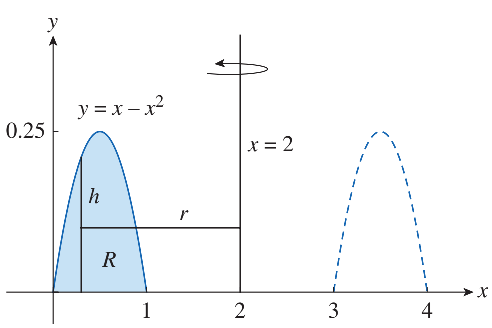


Figure 6.51
The region R, and the height and radius of a typical shell.

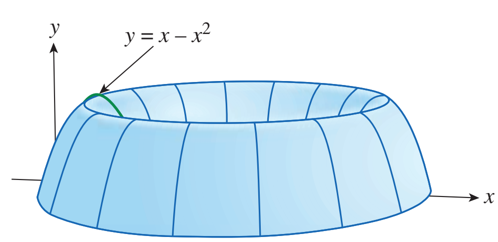


Figure 6.52
A visualization of the solid S.

The volume of the solid S is

$$V = \int_0^1 2\pi(2 - x)(x - x^2)\,dx \qquad \text{Method of cylindrical shells.}$$

$$= 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\,dx \qquad \text{Expand; simplify the integrand.}$$

$$= 2\pi\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 \qquad \text{Antiderivative.}$$

$$= 2\pi\left[\left(\frac{1}{4} - 1 + 1\right) - (0)\right] = \frac{\pi}{2} \qquad \text{FTC2; simplify.}$$

Disks and Washers versus Cylindrical Shells

When computing the volume of a solid of revolution, one method is often more appropriate or less complicated than another: disks (or washers) or cylindrical shells. Here are some things to think about when making a choice.

(1) Determine whether the region is more easily described by top and bottom curves of the form $y = f(x)$ or by left and right boundaries of the form $x = g(y)$.

(2) Decide whether the limits of integration are easier to find for one variable versus the other.

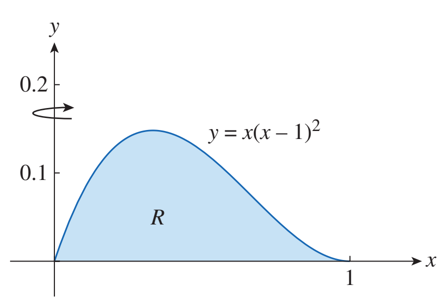
(3) Consider whether the region requires two separate integrals when using one variable versus the other.

(4) Consider whether the resulting integral can be evaluated.

The easier variable to work with generally dictates the method. Draw a sample rectangle in the region, corresponding to a cross-section of the solid. Remember that the thickness of the rectangle, Δx or Δy, corresponds to the variable of integration.

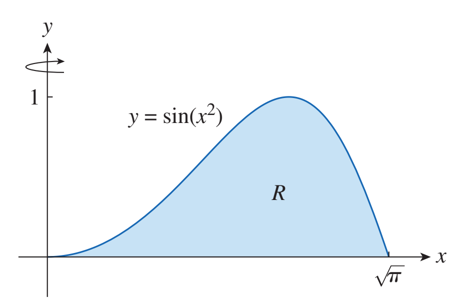
6.3 Exercises

1. Let S be the solid obtained by revolving the region R shown in the figure about the y-axis.



Explain why it is difficult to use washers to find the volume V of S. Sketch the height and radius of a typical shell. Use the method of cylindrical shells to find the volume of S.

2. Let S be the solid obtained by revolving the region R shown in the figure about the y-axis.



Sketch the height and radius of a typical shell. Use the method of cylindrical shells to find the volume of S. Do you think this method is preferable to washers? Explain your reasoning.

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the graphs of the given equations about the y-axis.

3. $y = \sqrt[3]{x}, \quad y = 0, \quad x = 1$

4. $y = x^3, \quad y = 0, \quad x = 1, \quad x = 2$

5. $y = e^{-x^2}, \quad y = 0, \quad x = 0, \quad x = 1$

6. $y = 4x - x^2, \quad y = x$

7. $y = x^2, \quad y = 6x - 2x^2$

8. Let R be the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$. Let V be the volume of the solid obtained by rotating R about the y-axis. Find V both by washers and by the method of cylindrical shells.

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the graphs of the given equations about the x-axis.

9. $xy = 1, \quad x = 0, \quad y = 1, \quad y = 3$

10. $y = \sqrt{x}, \quad x = 0, \quad y = 2$

11. $y = x^{3/2}, \quad y = 8, \quad x = 0$

12. $x = -3y^2 + 12y - 9, \quad x = 0$

13. $x = 1 + (y - 2)^2, \quad x = 2$

14. $x + y = 4, \quad x = y^2 - 4y + 4$

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the graphs of the given equations about the specified axis.

15. $y = x^3, \quad y = 8, \quad x = 0;$ about $x = 3$

16. $y = 4 - 2x, \quad y = 0, \quad x = 0;$ about $x = -1$

17. $y = 4x - x^2, \quad y = 3;$ about $x = 1$

18. $y = \sqrt{x}, \quad x = 2y;$ about $x = 5$

19. $x = 2y^2, \quad y \geq 0, \quad x = 2;$ about $y = 2$

20. $x = 2y^2, \quad x = y^2 + 1;$ about $y = -2$

In each of the following exercises:
(a) Set up an integral for the volume of the solid obtained by rotating the region bounded by the graphs of the given equations about the specified axis.
(b) Use technology to evaluate the integral.

21. $y = xe^{-x}, \quad y = 0, \quad x = 2;$ about the y-axis

22. $y = \tan x, \quad y = 0, \quad x = \dfrac{\pi}{4};$ about $x = \dfrac{\pi}{2}$

23. $y = \cos^4 x, \quad y = -\cos^4 x, \quad -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2};$ about $x = \pi$

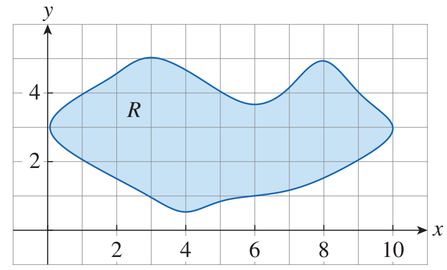
24. $y = x, \quad y = \dfrac{2x}{1 + x^3};$ about $x = -1$

25. $x = \sqrt{\sin y}, \quad 0 \leq y \leq \pi, \quad x = 0;$ about $y = 4$

26. $x^2 - y^2 = 7, \quad x = 4;$ about $y = 5$

27. Use Simpson's Rule with $n = 10$ equal subintervals to estimate the volume obtained by rotating about the y-axis the region under the graph of $y = \sqrt{1 + x^3}, 0 \leq x \leq 1$.

28. The region R shown in the figure is rotated about the y-axis to form a solid of revolution.



Use Simpson's Rule with $n = 10$ equal subintervals to estimate the volume of the solid.

Each integral represents the volume of a solid of revolution. Describe the solid.

29. $\displaystyle\int_0^3 2\pi x^5\, dx$

30. $\displaystyle\int_1^3 2\pi y \ln y\, dy$

31. $\displaystyle 2\pi \int_1^4 \frac{y + 2}{y^2}\, dy$

32. $\displaystyle\int_0^1 2\pi(2 - x)(3^x - 2^x)\, dx$

Use technology to estimate the x-coordinates of the points of intersection of the graphs of the given functions. Use this information to estimate the volume of the solid obtained by rotating the region bound by the graphs of the given functions about the y-axis.

33. $y = x^2 - 2x, \quad y = \dfrac{x}{x^2 + 1}$

34. $y = e^{\sin x}, \quad y = x^2 - 4x + 5$

Use technology to find the volume of the solid obtained by rotating the region bounded by the graphs of the given curves about the specified line.

35. $y = \sin^2 x, \quad y = \sin^4 x, \quad 0 \leq x \leq \dfrac{\pi}{2};$ about $x = \dfrac{\pi}{2}$

36. $y = x^3 \sin x, \quad y = 0, \quad 0 \leq x \leq \pi;$ about $x = -1$

The region bounded by the graphs of the given expressions is rotated about the specified line. Find the volume of the resulting solid by any method.

37. $y = -x^2 + 6x - 8 \quad y = 0;$ about the y-axis

38. $y = -x^2 + 6x - 8, \quad y = 0;$ about the x-axis

39. $y^2 - x^2 = 1, \quad y = 2;$ about the x-axis

40. $y^2 - x^2 = 1, \quad y = 2;$ about the y-axis

41. $x^2 + (y - 1)^2 = 1;$ about the y-axis

42. $x = (y - 3)^2, \quad x = 4;$ about $y = 1$

43. $x = (y - 1)^2, \quad x - y = 1;$ about $x = -1$

44. Let R be the region bounded by the graph of $y = e^{-x} - 0.5$ and the x- and y-axes. Use technology to find the exact volume of the solid S obtained by revolving the region R about the y-axis.

45. Let T be the triangular region with vertices (0, 0), (1, 0), and (1, 2) and let V be the volume of the solid generated when T is rotated about the line $x = a$, where $a > 1$. Find an expression for a in terms of V.

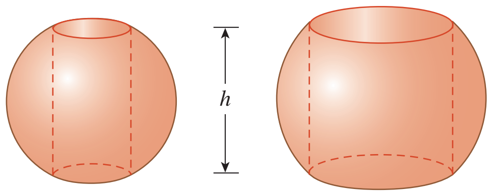
Use cylindrical shells to find the volume of the solid.

46. A sphere of radius r.

47. A right circular cone with height h and base radius r.

48. A solid torus with radii r and R.

49. A certain napkin ring is made by drilling holes with different diameters through two wooden balls (which also have different diameters). Both napkin rings have the same height h, as shown in the figure.



(a) Guess which ring has more wood in it.

(b) Check your guess: use cylindrical shells to compute the volume of a napkin ring created by drilling a hole with radius r through the center of a sphere of radius R and express the answer in terms of h.

6.4 Arc Length

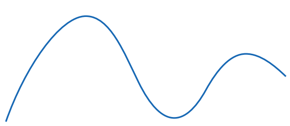


Figure 6.53
We could use a string to estimate the length of a curve.

In this section, we will use a definite integral to find the length of a curve over an interval $a \le x \le b$. We first need to consider what is meant by the length of a curve. Practically, think of fitting a piece of string to the curve in Figure 6.53. Lift the string and pull it straight, and then measure the length of the string with a ruler. This process seems logical but could be difficult to do with much accuracy, even if the curve was quite simple. We need a precise definition for the length of an arc of a curve, in the same spirit as the definitions we developed for area and volume.

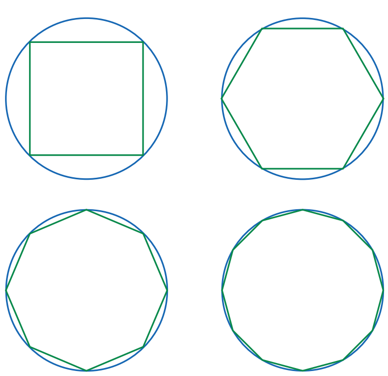


Figure 6.54
We can use polygons to find the circumference of a circle.

If a curve consists of only straight line segments, then we can find its length: we use the distance formula to find the length of each segment, and just add the lengths of the line segments that form the curve. Therefore, it seems reasonable to define the length of a general curve by first approximating it by straight line segments and then taking the limit as the number of segments increases. This process can be used to find the circumference of a circle by taking the limit of the lengths of inscribed polygons. See Figure 6.54.

We can apply this method to measure the length of a more general curve. Suppose the curve C is described by the parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \le t \le b$$

Let's assume that C is **smooth**, that is, the derivatives $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero for $a < t < b$. (This condition ensures that C has no sudden change in direction.)

Divide the parameter interval $[a, b]$ into n equal subintervals with endpoints $a = t_0, t_1, t_2, \ldots, t_n = b$ and width Δt. Let $x_i = f(t_i)$ and $y_i = g(t_i)$. The point $P_i(x_i, y_i)$

lies on C and the polygonal path with vertices $P_0, P_1, \ldots, P_n$, illustrated in Figure 6.55, is an approximation to C.

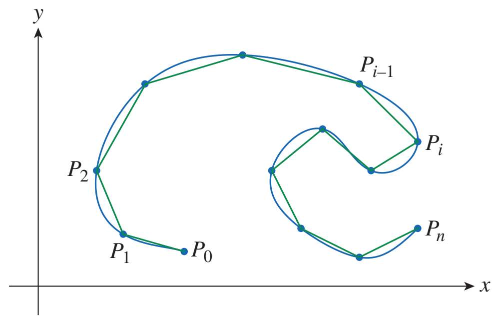


Figure 6.55
The polygonal path with vertices $P_0, P_1, P_2, \ldots, P_n$ is an approximation to C.

The **length** L of the curve C is approximately the length of this polygonal path and the approximation gets better and better as n increases. For example, in Figure 6.56, the segment of the curve between P_{i-1} and P_i has been magnified and the approximations with more line segments and successively smaller values of Δt are shown. The length L of the curve C is defined to be the limit of the lengths of these polygonal paths (if the limit exists):

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

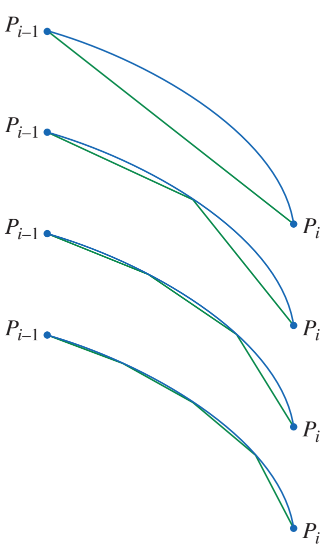


Figure 6.56
The approximation gets better as n, the number of line segments, increases.

This procedure for defining arc length is very similar to the method used to define area and volume: we start by dividing the curve into a large number of small parts. We found the approximate lengths of the small parts and added the lengths together. Finally, we took the limit as $n \to \infty$.

This definition of arc length is very reasonable but is not very practical. We need an expression involving a definite integral that yields the value L. Let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, then the length of the ith segment is

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

From the definition of a derivative, we know that

$$f'(t_i) \approx \frac{\Delta x_i}{\Delta t}$$

if Δt is small. Note that we could have used any sample point t_i^* in place of t_i. Therefore,

$$\Delta x_i \approx f'(t_i)\,\Delta t \qquad \Delta y_i \approx g'(t_i)\,\Delta t$$

Use these results to write the length of each line segment.

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &\approx \sqrt{[f'(t_i)\,\Delta t]^2 + [g'(t_i)\,\Delta t]^2} \\ &= \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2}\,\Delta t \qquad \text{Since } \Delta t > 0. \end{aligned}$$

Use this expression in the definition of the length L of a curve C.

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2}\,\Delta t$$

This is the limit of a Riemann sum and by definition is a definite integral equal to

$$\int_a^b \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2}\, dt$$

This formula is indeed true, provided that the curve is traced out only once for $a \le t \le b$.

Arc Length Formula

If a smooth curve with parametric equations $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed exactly once as t increases from a to b, then its length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{1}$$

Example 1 Length of a Parametric Curve

Find the length of the arc of the curve $x = t^2$, $y = t^3$ that lies between the points (1, 1) and (4, 8).

Solution

The graph of the curve and the portion between (1, 1) and (4, 8) are shown in Figure 6.57.

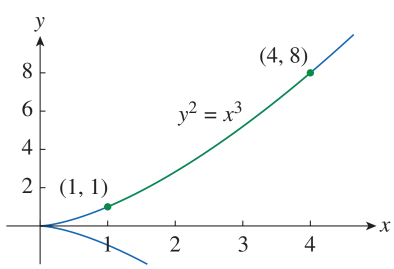


Figure 6.57
Graph of the parametric curve.

The portion of the curve between (1, 1) and (4, 8) corresponds to the parameter interval $1 \le t \le 2$.

Use the arc length formula.

$$L = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt$$

$$= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt = \int_1^2 t\sqrt{4 + 9t^2}\, dt$$

Use the method of substitution.

$u = 4 + 9t^2 \qquad t = 1 : u = 13$

$du = 18t\, dt \qquad t = 2 : u = 40$

$$dt = \frac{du}{18t}$$

$$L = \int_{13}^{40} t\sqrt{u}\,\frac{du}{18t} = \frac{1}{18}\int_{13}^{40} \sqrt{u}\, du \qquad \text{Change of variables.}$$

$$= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} \qquad \text{Antiderivative.}$$

$$= \frac{1}{27}[40^{3/2} - 13^{3/2}] = \frac{1}{27}[80\sqrt{10} - 13\sqrt{13}] \qquad \text{FTC2; simplify.}$$

Since the curve is very close to a straight line on the interval $[1, 4]$, the arc length should be just slightly larger than the distance from $(1, 1)$ to $(4, 8)$, which is $\sqrt{58} \approx 7.615773$. Using the arc length formula, we find $L = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13}) \approx 7.633705$. ■

Curve Defined in Terms of *x* and *y*

If a curve C is defined by the equation $y = f(x)$, $a \leq x \leq b$, then we can regard x as a parameter. Then the parametric equations are $x = x$, $y = f(x)$, and Equation 1 becomes

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{2}$$

Similarly, if a curve is defined by the equation $x = f(y)$, $a \leq y \leq b$, we regard y as the parameter and the length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy \tag{3}$$

Because Equations 1, 2, and 3 contain a square root, calculating arc length often involves an integral that is very difficult or even impossible to evaluate explicitly. Therefore, we often have to resort to finding an approximation to the length of a curve, as illustrated in the next example.

Example 2 Approximation of an Arc Length Using Simpson's Rule

Estimate the length of the portion of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $(2, \frac{1}{2})$.

Solution

Solve the expression for y in terms of x.

$$y = \frac{1}{x} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x^2}$$

Use Equation 2 to write an expression for the length.

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx$$

It is impossible to evaluate this integral exactly. Use Simpson's Rule with $a = 1$, $b = 2$, $n = 10$, $\Delta x = 0.1$, and $f(x) = \sqrt{1 + 1/x^4}$.

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx \\ &\approx \frac{\Delta x}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \cdots + 2f(1.8) + 4f(1.9) + f(2)] \\ &\approx 1.1321 \end{aligned}$$

■

Using a CAS, we find that $L \approx 1.13209039$.

Example 3 Length of a Parabola

Find the length of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

Solution

Since the curve is defined as x in terms of y, use Equation 3.

$$x = y^2 \quad \Rightarrow \quad \frac{dx}{dy} = 2y$$

The arc length is

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy = \int_0^1 \sqrt{4y^2 + 1}\, dy.$$

There is no basic antidifferentiation formula for this integrand. However, using either a CAS or the Table of Integrals (Formula 21 with $u = 2y$), we find that

$$L = \int_0^1 \sqrt{1 + 4y^2}\, dy = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5} + 2)}{4} \approx 1.478943.$$

Figure 6.58 shows the arc of the parabola and the polygonal approximations for $n = 1$ and $n = 2$ line segments. For $n = 1$, the approximate length is $L_1 = \sqrt{2}$, the diagonal of a square.

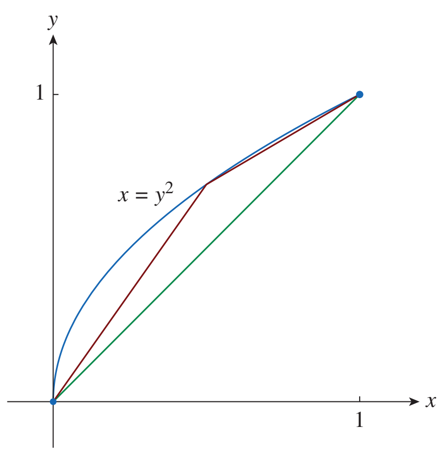


Figure 6.58
Arc of the parabola from $(0, 0)$ to $(1, 1)$ and two polygonal approximations.

n	L_n
1	1.414214
2	1.445496
4	1.463531
8	1.472447
16	1.476368
32	1.477962
64	1.478578

Table 6.2
Approximations to the length of the parabola.

Table 6.2 shows the approximations L_n found by dividing the interval $[0, 1]$ into n equal subintervals. Notice that as n increases, the approximations are closer to the exact length. ■

Example 4 Length of a Cycloid

Find the length of one arch of the cycloid.

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

Solution

From Example 7 in Section 1.6, we know that one arch of the cycloid is described by the parameter interval $0 \leq \theta \leq 2\pi$.

$$\frac{dx}{d\theta} = r(1 - \cos\theta) \qquad \frac{dy}{d\theta} = r\sin\theta$$

The arc length is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\, d\theta = r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

We can evaluate this integral using trigonometric identities. However, using a computer algebra system we have

$$L = r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta = 8r.$$

Therefore, the length of one arch of a cycloid is eight times the radius of the generating circle. See Figure 6.59.

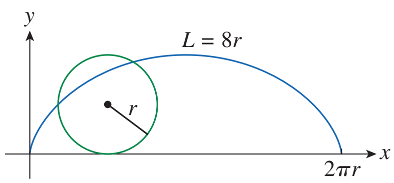


Figure 6.59
The length of the arch is $8r$. This was first proved in 1658 by Sir Christopher Wren, who later became the architect of St. Paul's Cathedral in London.

6.4 Exercises

1. Use the arc length formula to find the length of the curve described by $y = 2x - 5$, $-1 \leq x \leq 3$. Check your answer by recognizing that the curve is a line segment and calculating its length by the distance formula.

2. In Example 2 in Section 1.6, we showed that the parametric equations $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$, represent the unit circle. Use these equations to show that the length of the unit circle is 2π.

3. In Example 3 in Section 1.6, we showed that the equations $x = \sin 2t$, $y = \cos 2t$, $0 \leq t \leq 2\pi$, represent the unit circle. Use these equations to evaluate the integral in Equation 1 and explain why this result is not the length of the unit circle.

4. Use the arc length formula to find the length of the curve described by $y = \sqrt{2 - x^2}$, $0 \leq x \leq 1$. Check your answer by recognizing that the curve is part of a circle.

5. The definite integral $\displaystyle\int_1^4 \sqrt{1 + \frac{x^2}{4}}\, dx$ represents the length of an arc along a specific curve over a particular interval. Identify and sketch the curve and the coordinates of the endpoints of the arc.

Set up an integral that represents the length of the curve and use technology to find the length.

6. $y = \sin x$, $\quad 0 \leq x \leq \pi$

7. $y = xe^{-x}$, $\quad 0 \leq x \leq 2$

8. $y = x - \ln x$, $\quad 1 \leq x \leq 4$

9. $x = y^2 - 2y$, $\quad 0 \leq x \leq 2$

10. $x = \sqrt{y} - y$, $\quad 1 \leq y \leq 4$

11. $y^2 = \ln x$, $\quad -1 \leq y \leq 1$

12. $x = t + \cos t$, $\quad y = t - \sin t$, $\quad 0 \leq t \leq 2\pi$

13. $x = t\cos t$, $\quad y = t\sin t$, $\quad 0 \leq t \leq 2\pi$

Find the exact length of the curve.

14. $y = 1 + 6x^{3/2}$, $\quad 0 \leq x \leq 1$

15. $y = \ln(\cos x)$, $\quad 0 \leq x \leq \dfrac{\pi}{3}$

16. $y = \ln(\sec x)$, $\quad 0 \leq x \leq \dfrac{\pi}{4}$

17. $y^2 = 4(x+4)^3, \quad 0 \le x \le 2, \quad y > 0$

18. $x = y^{3/2}, \quad 0 \le y \le 1$

19. $y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$

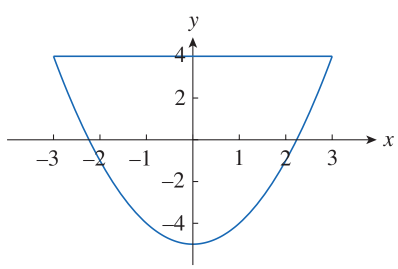
20. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x, \quad 1 \le x \le 2$

21. $x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \le t \le 1$

22. $x = a(\cos\theta + \theta\sin\theta), \quad y = a(\sin\theta - \theta\cos\theta), \quad 0 \le \theta \le \pi$

23. Find the length of the arc of the curve described by $x^2 = (y-4)^2$ from $P(1, 5)$ to $Q(8, 8)$.

24. Consider the region bounded by the graphs of $y = x^2 - 5$ and $y = 4$ as shown in the figure.



Find the perimeter of this region.

Graph the curve and visually estimate its length. Use technology to find the length and compare it to your estimate.

25. $y = x^2 + x^3, \quad 1 \le x \le 2$

26. $y = x + \cos x, \quad 0 \le x \le \frac{\pi}{2}$

Use Simpson's Rule with $n = 10$ to estimate the arc length of the curve. Compare your answer with the value of the integral obtained using technology.

27. $y = xe^{-x}, \quad 0 \le x \le 5$

28. $y = x\sin x, \quad 0 \le x \le 2\pi$

29. $y = \ln(1 + x^3), \quad 0 \le x \le 5$

30. $x = y + \sqrt{y}, \quad 1 \le y \le 2$

31. $x = \sin t, \quad y = t^2, \quad 0 \le t \le 2\pi$

32. Find the length of the loop of the curve $x = 3t - t^3$, $y = 3t^2$.

33. Consider the curve C described by $y = x\sqrt[3]{4 - x}$, $0 \le x \le 4$.
(a) Graph the curve C.
(b) Compute the lengths of polygonal paths with $n = 1, 2,$ and 4 sides using equal subintervals. Illustrate by sketching these polygonal paths.
(c) Set up an integral for the length of the curve.
(d) Use technology to evaluate the integral in part (c).

34. Consider the curve C described by $y = x + \sin x$, $0 \le x \le 2\pi$.
(a) Graph the curve C.
(b) Compute the lengths of polygonal paths with $n = 1, 2,$ and 4 sides using equal subintervals. Illustrate by sketching these polygonal paths.
(c) Set up an integral for the length of the curve.
(d) Use technology to evaluate the integral in part (c).

Use either a CAS or the Table of Integrals to find the exact length of the curve.

35. $x = t^3, \quad y = t^4, \quad 0 \le t \le 1$

36. $y^2 = 4x, \quad 0 \le y \le 2$

37. $y = \ln x, \quad 1 \le x \le \sqrt{3}$

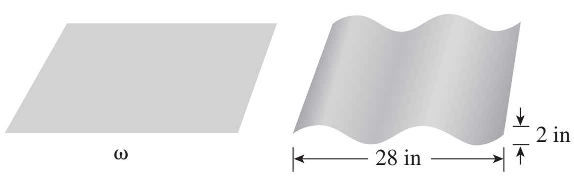
38. A steady wind blows a kite due west. The kite's height above the ground from horizontal position $x = 0$ ft to $x = 80$ ft is given by $y = 150 - \frac{1}{40}(x - 50)^2$. Find the distance traveled by the kite.

39. A hawk flying at 15 m/s at an altitude of 180 accidentally drops its prey. The parabolic trajectory of the falling prey is described by the equation

$$y = 180 - \frac{x^2}{45}$$

until it hits the ground, where y is its height above the ground and x is the horizontal distance traveled in meters. Calculate the distance traveled by the prey from the time it is dropped until the time it hits the ground.

40. A manufacturer of corrugated metal roofing wants to produce panels that are 28 inches wide and 2 inches high by processing flat sheets of metal, as shown in the figure.



The profile of the roofing takes the shape of a sine wave and the sine curve has equation $y = \sin\left(\frac{\pi x}{7}\right)$. Find the width ω of a flat metal sheet that is needed to make a 28-inch panel.

41. Find the total length of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$, where $a > 0$.

42. Show that the total length of the ellipse $x = a\sin\theta$, $y = b\cos\theta$, $a > b > 0$, is

$$L = 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,d\theta$$

where e is the eccentricity of the ellipse; $e = \frac{c}{a}$, where $c = \sqrt{a^2 - b^2}$.

43. The curves described by the equations $x^n + y^n = 1$, $n = 4, 6, 8, \ldots$, are called **fat circles**. Graph the curves with $n = 2, 4, 6, 8$, and 10 to see why. Set up an integral for the length L_{2k} of the fat circle with $n = 2k$. Without attempting to evaluate this integral, state the value of $\lim_{k\to\infty} L_{2k}$.

44. (a) Graph the **epitrochoid** with equations

$$x = 11\cos t - 4\cos(11t/2)$$
$$y = 11\sin t - 4\sin(11t/2)$$

Find the parameter interval that produces the complete curve.

(b) Use technology to find the approximate length of this curve.

45. The curve described by the equations

$$x = C(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du$$

$$y = S(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du$$

where C and S are the Fresnel functions (introduced in Section 5.4) is called **Cornu's spiral**.

(a) Graph this curve. Explain what happens to the curve as $t \to \infty$ and as $t \to -\infty$.

(b) Find the length of Cornu's spiral from the origin to the point with parameter value t.

Discovery Project | Arc Length Contest

The curves shown in Figure 6.60 are all examples of continuous functions f that have the following properties.

(1) $f(0) = 0$ and $f(1) = 0$

(2) $f(x) \geq 0$ for $0 \leq x \leq 1$

(3) The area under the graph of f from 0 to 1 is equal to 1.

The length L of each curve, however, is different.

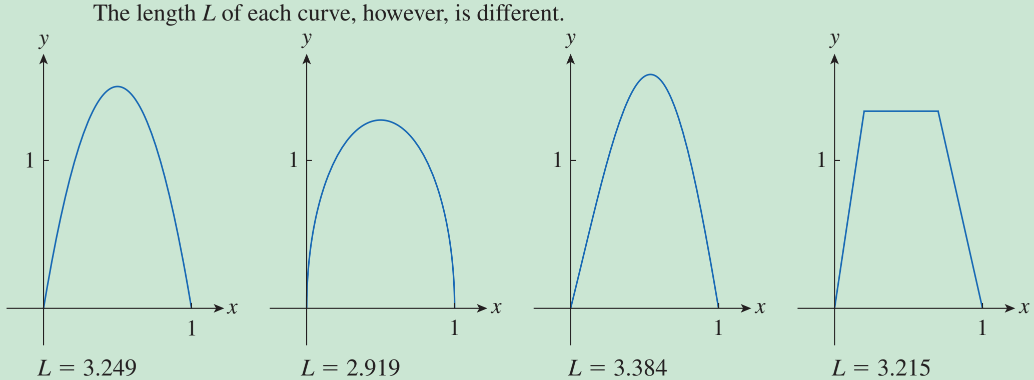


Figure 6.60
Graphs of functions with the stated properties, all with different arc lengths.

Try to find a function that satisfies properties (1), (2), and (3) and compute the arc length. The winning entry is the one with the smallest arc length.

6.5 Average Value of a Function

We can calculate the arithmetic mean, or *average* value, of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

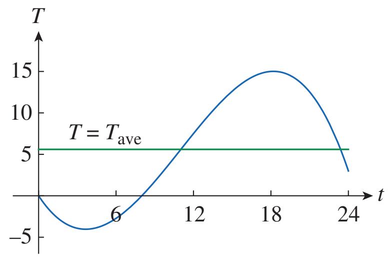


Figure 6.61
The value $T = T_{\text{ave}}$ appears to be the average temperature over the entire interval [0, 24].

But how do we compute the average temperature during a day, for example, if infinitely many temperature readings are possible? Figure 6.61 shows the graph of a temperature function $T(t)$, where t is measured in hours and T in °C, and a guess at the average temperature, T_{ave}.

Let's try to compute the average value of a function $y = f(x)$, $a \le x \le b$. Start by dividing the interval $[a, b]$ into n equal subintervals, each of width $\Delta x = \dfrac{b-a}{n}$. Select values $x_1^*, x_2^*, \ldots, x_n^*$ in successive subintervals and compute the average of the numbers $f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$:

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}$$

For example, if f represents a temperature function and $n = 24$, then we would take temperature readings every hour and find the average.

Since $\Delta x = \dfrac{b-a}{n}$, we can solve for n: $n = \dfrac{b-a}{\Delta x}$. Use this expression for n to rewrite the average value.

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{\dfrac{b-a}{\Delta x}} = \frac{1}{b-a}\left[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)\right]\Delta x \quad \text{Rewrite fraction.}$$

$$= \frac{1}{b-a}\left[f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\right] \quad \text{Distribute } \Delta x.$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x \quad \text{Use summation notation.}$$

As n increases, we are averaging a larger number of values. For example, instead of taking temperature readings every hour, we might be averaging observations taken every minute, or even every second.

And, as n increases, it seems reasonable that we would obtain a better approximation to the true average value of the function over the interval. Notice that this summation is a Riemann sum, and therefore, the limiting value is the definition of a definite integral.

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Therefore, we define the **average value of f** on the interval $[a, b]$ as

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

For a positive function, we can interpret this definition as

$$\text{average height} = \frac{\displaystyle\int_a^b f(x)\,dx}{b-a} = \frac{\text{area}}{\text{width}}$$

Common Error

The average value is $\dfrac{f(a)+f(b)}{2}$ or the average value is $\dfrac{f(b)-f(a)}{b-a}$.

Correct Method

The first expression is the average of two function values. The second expression is the average rate of change of the function f over the interval $[a, b]$. The average value of a function over an interval involves a definite interval.

Example 1 Average Value of a Function

Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

Solution

The interval is $[-1, 2]$. Therefore, $a = -1$ and $b = 2$.

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)\,dx \qquad \text{Definition of average value.}$$

$$= \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2} \qquad \text{Antiderivative.}$$

$$= \frac{1}{3}\left[\left(2 + \frac{2^3}{3}\right) - \left(-1 + \frac{(-1)^3}{3}\right)\right] = 2 \qquad \text{FTC2; simplify.}$$

The graph of f and the line $y = f_{\text{ave}}$ are shown in Figure 6.62. Note that the area of the region bounded by the graph of f, the x-axis, and the lines $x = -1$ and $x = 2$ appears to be the same as the area of the rectangle with height f_{ave} and width $2 - (-1) = 3$.

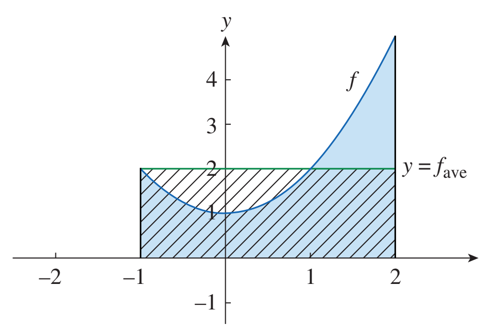


Figure 6.62
The area of the rectangle is the same as the area under the graph of f.

Mean Value Theorem for Integrals

If $T(t)$ is the temperature at time t, it seems reasonable to ask if there is a specific time (in the interval) when the temperature is the exact same as the average temperature. For the temperature function shown in Figure 6.61, there appear to be two such times, just before noon and just before midnight. For a continuous function, there is always at least one number c at which the value of the function f is exactly equal to the average value of the function, that is, $f(c) = f_{\text{ave}}$.

The Mean Value Theorem for Integrals

If f is a continuous function on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

The Mean Value Theorem for Integrals is a consequence of the Mean Value Theorem for Derivatives and the Fundamental Theorem of Calculus. The proof is outlined in Exercise 30.

The geometric interpretation of the Mean Value Theorem for Integrals is that, for positive functions f, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b.

Figure 6.63 illustrates this geometric interpretation.

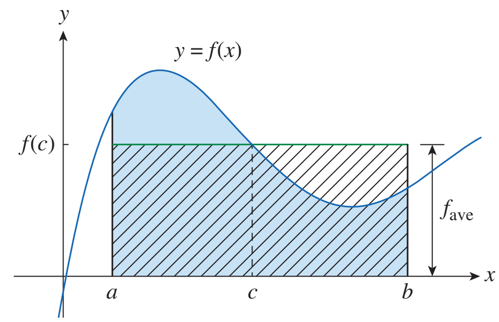


Figure 6.63
You can always chop off the top of a (two-dimensional) mountain at a certain height and use it to fill in the valleys so that the mountaintop becomes completely flat.

Example 2 MVT for Integrals

The function $f(x) = 2 + x - \frac{x^2}{4}$ is continuous on the interval $[1, 5]$. Therefore, the Mean Value Theorem for Integrals says there is a number c in $[1, 5]$ such that

$$\int_1^5 \left(2 + x - \frac{x^2}{4}\right) dx = f(c)[5 - 1].$$

Let's try to find the value of c.

$$\int_1^5 \left(2 + x - \frac{x^2}{4}\right) dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{12}\right]_1^5 \qquad \text{Antiderivative.}$$

$$= \left[2(5) + \frac{5^2}{2} - \frac{5^3}{12}\right] - \left[2(1) + \frac{1^2}{2} - \frac{1^3}{12}\right] = \frac{29}{3} \qquad \text{FTC2; simplify.}$$

Use this value of the definite integral to solve for c.

$$\frac{29}{3} = \left(2 + c - \frac{c^2}{4}\right)(4) \qquad \text{Use the value of the definite integral.}$$

$$c^2 - 4c + \frac{5}{3} = 0 \qquad \text{Simplify; bring all terms to one side.}$$

$$c_1 = \frac{1}{3}(6 - \sqrt{21}) \approx 0.472,\ c_2 = \frac{1}{3}(6 + \sqrt{21}) \approx 3.528 \qquad \text{Quadratic formula.}$$

In this case there are two solutions for c, but only one is in the interval [1, 5] that satisfies the conclusion of the Mean Value Theorem for Integrals.

Figure 6.64 illustrates this conclusion.

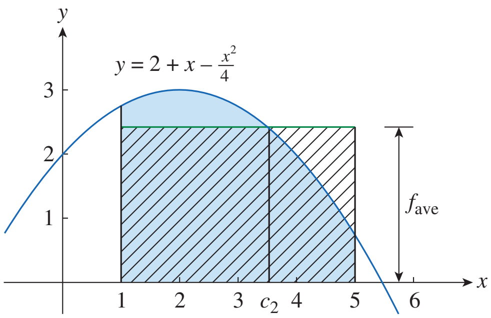


Figure 6.64
In this case there is one value of c that satisfies the conclusion of the Mean Value Theorem for Integrals.

Example 3 Average Velocity

Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average value of its velocity function, $v(t)$, during the trip.

Solution

Let $s(t)$ be the position of the car at time t.

Then, by definition, the average velocity of the car over the interval is

$$\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

The average value of the velocity function on the interval $[t_1, t_2]$ is

$$v_{ave} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} s'(t)\,dt \qquad \text{Definition of average value; } s'(t) = v(t).$$

$$= \frac{1}{t_2 - t_1}\left[s(t)\right]_{t_1}^{t_2} \qquad \text{FTC; et Change Theorem.}$$

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \text{average velocity} \qquad \text{Simplify.}$$

Common Error

The average value of the velocity function is found using a difference quotient: $\frac{v(b) - v(a)}{b - a}$.

Correct Method

The average value of the velocity function involves the position function: $\frac{s(b) - s(a)}{b - a}$.

6.5 Exercises

Find the average value of the function on the given interval.

1. $f(x) = 3x^2 + 8x, \quad [-1, 2]$

2. $f(x) = 4x - x^2, \quad [0, 4]$

3. $f(x) = \sqrt{x}, \quad [0, 4]$

4. $g(x) = 3\cos x, \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

5. $g(t) = \frac{t}{\sqrt{3 + t^2}}, \quad [1, 3]$

6. $f(t) = e^{\sin t}\cos t, \quad \left[0, \frac{\pi}{2}\right]$

7. $f(x) = \frac{x^2}{(x^3 + 3)^2}, \quad [-1, 1]$

8. $f(\theta) = \sec^2\left(\frac{\theta}{2}\right), \quad \left[0, \frac{\pi}{2}\right]$

9. $h(x) = \cos^4 x \sin x, \quad [0, \pi]$

10. $h(u) = \frac{\ln u}{u}, \quad [1, 5]$

For each function:
(a) Find the average value of f on the given interval.
(b) Find c such that $f_{ave} = f(c)$.
(c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

11. $f(x) = (x - 3)^2, \quad [2, 5]$

12. $f(x) = \frac{1}{x}, \quad [1, 3]$

13. $f(x) = \ln x, \quad [1, 3]$

14. $f(x) = 2\sin x - \sin 2x, \quad [0, \pi]$

15. $f(x) = 2xe^{-x^2}, \quad [0, 2]$

16. $f(x) = \dfrac{2x}{(1 + x^2)^2}, \quad [0, 2]$

---

17. If f is continuous and $\int_1^3 f(x)\,dx = 8$, show that f takes on the value 4 at least once in the interval $[1, 3]$.

18. Find all numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

19. Let

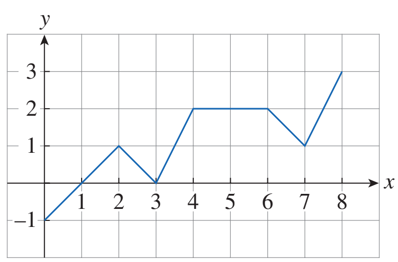
$$f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 2x & \text{if } x > 2 \end{cases}$$

What is the average value of f on $[1, 4]$?

20. For a continuous function f, the table gives values of $f(x)$ for selected values of x. Use Simpson's Rule to estimate the average value of f on the interval $[20, 50]$.

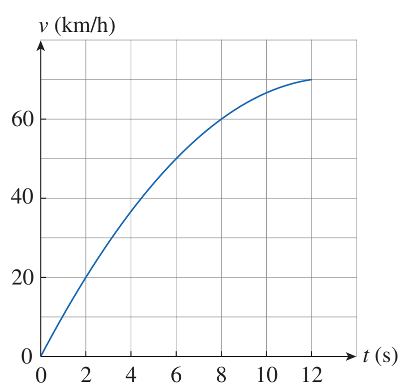
x	20	25	30	35	40	45	50
$f(x)$	42	38	31	29	35	48	60

21. The graph of a function f is shown in the figure.



Find the average value of f on the interval $[0, 8]$.

22. The graph of the velocity (in km/h) of a car that is accelerating is shown in the figure, where t is measured in seconds.



(a) Use the Midpoint Rule with $n = 6$ equal subintervals to estimate the average velocity of the car during the first 12 seconds.

(b) At what time was the instantaneous velocity equal to the average velocity?

23. In a certain city the temperature, in °F, t hours after 9 AM was modeled by the function

$$T(t) = 50 + 14 \sin\left(\frac{\pi t}{12}\right)$$

Find the average temperature during the period from 9 AM to 9 PM.

24. A cup of coffee with temperature 95°C is placed in a room with temperature 20°C. The temperature of the coffee after t minutes is given by the equation

$$T(t) = 20 + 75e^{-0.02t}$$

What is the average temperature of the coffee during the first half hour?

25. The linear density in a rod 8 m long is $\dfrac{12}{\sqrt{x+1}}$ kg/m, where x is measured in meters from one end of the rod. Find the average density of the rod.

26. If a freely falling body starts from rest, then its position is given by $s(t) = \frac{1}{2}gt^2$ where g is the acceleration due to gravity and t is a measure of elapsed time. Let the velocity after a time T be represented by v_T. Show that the average value of the velocity function is $v_{\text{ave}} = \frac{1}{2}v_T$.

Find an expression for the velocity v as a function of position s. Show that the average value of this velocity function, with respect to s, is $v_{\text{ave}} = \frac{2}{3}v_T$.

27. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 seconds. The function $f(t) = \frac{1}{2}\sin\left(\frac{2\pi t}{5}\right)$ has often been used to model the rate of air flow into the lungs, in L/s. Use this model to compute the average volume of inhaled air in the lungs in one respiratory cycle.

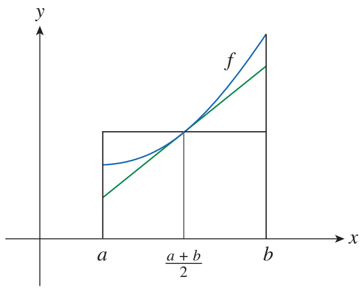
28. The velocity v of blood that flows in a blood vessel with radius R and length l at a distance r from the central axis is

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood. Find the average velocity, with respect to r, over the interval $0 \le r \le R$. Compare the average velocity with the maximum velocity.

29. Use the figure to show that if f is concave up on the interval (a, b), then

$$f_{\text{ave}} > f\left(\frac{a+b}{2}\right)$$



30. Prove the Mean Value Theorem for Integrals by applying the Mean Value Theorem for Derivatives to the function $F(x) = \int_a^x f(t)\,dt$.

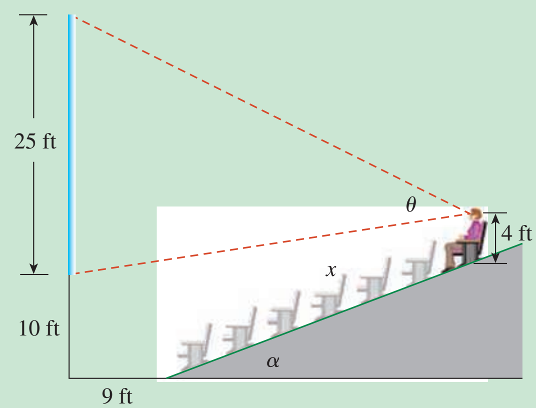
31. Let $f_{\text{ave}}[a, b]$ denote the average value of f on the interval $[a, b]$ and $a < c < b$. Show that

$$f_{\text{ave}}[a, b] = \frac{c-a}{b-a} \cdot f_{\text{ave}}[a, c] + \frac{b-c}{b-a} \cdot f_{\text{ave}}[c, b]$$

Applied Project | The Best Seat at the Movies

A movie theater has a screen that is positioned 10 ft off the floor and is 25 ft high. The first row of seats is placed 9 ft from the screen and the rows are set 3 ft apart. The floor of the seating area is inclined at an angle of $\alpha = 20°$ above the horizontal and the distance up the incline that you sit is x. The theater has 21 rows of seats, so $0 \leq x \leq 60$.

Suppose you decide that the best place to sit is in the row where the angle θ subtended by the (top and bottom of the) screen at your eyes is a maximum. In addition, suppose that your eyes are 4 ft above the floor, as shown in the figure.



1. Show that

$$\theta = \arccos\left(\frac{a^2 + b^2 - 625}{2ab}\right)$$

where $\quad a^2 = (9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2$

and $\quad b^2 = (9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2$

2. Use a graph of θ as a function of x to estimate the value of x that maximizes θ. In which row would you sit? What is the viewing angle θ in this row?

3. Use technology to differentiate θ and find a numerical value for the root of the equation $\dfrac{d\theta}{dx} = 0$. Does this value confirm your result in Problem 2?

4. Use the graph of θ to estimate the average value of θ on the interval $0 \leq x \leq 60$. Use technology to compute the average value. Compare with the maximum and minimum values of θ.

6.6 Applications to Physics and Engineering

In this section, we will consider three applications of integration related to physics and engineering: work, force due to water pressure, and centers of mass. Similar to the previous applications, the strategy is to break up the (continuous) physical quantity into a large number of (discrete) small parts, approximate each small part, add the results, consider a limit, recognize a Riemann sum, and define the resulting definite integral.

Work

The term *work* is used in everyday language to mean the total amount of effort required to perform a certain task. In physics, work has a technical meaning that depends on the idea of *force.* Intuitively, you can think of a force as describing a push or pull of an object, for example, a horizontal push of a calculus book across a table, or the downward pull of Earth's gravity on a tennis ball.

In general, if an object moves along a straight line with position function $s(t)$, then the **force** F on the object (in the same direction) is defined by Newton's Second Law of Motion as the product of its mass m and its acceleration:

$$F = m\frac{d^2s}{dt^2} \tag{1}$$

If $\dfrac{d^2s}{dt^2} = c$, a constant, then $F = m \cdot c$ is also a constant.

In the SI metric system, the mass is measured in kilograms (kg), the displacement in meters (m), the time in seconds (s), and the force in newtons ($\text{N} = \text{kg} \cdot \text{m}/\text{s}^2$). Therefore, a force of 1 N acting on a mass of 1 kg produces an acceleration of $1\ \text{m}/\text{s}^2$. Note that in the U.S. Customary system, the units for displacement, time, and force are feet (ft), pounds (lb), and seconds (s), respectively.

If the acceleration is constant, then the force F is also constant and the work done is defined to be the product of the force F and the distance d, that the object moves.

$$W = Fd \qquad \text{work} = \text{force} \times \text{distance} \tag{2}$$

If F is measured in newtons and d in meters, then the unit for W is a newton-meter, which is called a joule (J). If F is measured in pounds and d in feet, then the unit for W is a foot-pound (ft-lb), which is about 1.36 J.

For example, suppose you lift a 1.2-kg book off the floor to place on a desk that is 0.7 m high. The force you exert is equal and opposite to that exerted by gravity. Using Equation 1, the force is

The acceleration due to gravity is $-9.8\ \text{m}/\text{s}^2$.

$$F = mg = (1.2)(9.8) = 11.76\ \text{N}$$

Using Equation 2, we can find the work done in lifting the book:

$$W = Fd = (11.76)(0.7) = 8.232\ \text{J}$$

Suppose a 20-lb weight is lifted 6 ft off the ground, then the force is given as $F = 20\ \text{lb}$, so the work done is

$$W = Fd = 20 \cdot 6 = 120\ \text{ft-lb}$$

In this case, we don't need to multiply by g (the acceleration due to gravity) because we are given the *weight* (a force) and not the mass.

Equation 2 defines work if the force is constant, but let's consider the case in which the force is variable. Suppose that an object moves along the x-axis in the positive direction, from $x = a$ to $x = b$, and at each point x between a and b a force $f(x)$ acts on the object, where f is a continuous function.

Divide the interval $[a, b]$ into n equal subintervals with endpoints $a = x_0, x_1, \ldots, x_n = b$, each of width Δx. Choose a sample point x_i^* in the ith subinterval $[x_{i-1}, x_i]$. Then the force at that point is $f(x_i^*)$. If n is large, then Δx is small, and since f is continuous, the values of f don't change much over the interval $[x_{i-1}, x_i]$. Therefore, f is approximately constant on each subinterval and (using Equation 1) the work W_i that is done in moving the object from x_{i-1} to x_i is approximately

$$W_i \approx f(x_i^*)\Delta x$$

The total work done is approximated by

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} f(x_i^*)\,\Delta x \tag{3}$$

It seems reasonable that this approximation becomes better and better as n becomes larger. Therefore, we define the **work done in moving the object from *a* to *b*** as the limit of this quantity as $n \to \infty$. The right side of Equation 3 is a Riemann sum and the limit is a definite integral. Therefore, the work W is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx \tag{4}$$

Example 1 Work Done by a Variable Force

An object is moving along the x-axis. The force acting on the object x feet from the origin is given by $f(x) = x^2 + 2x$. Find the work done in moving the object from $x = 1$ to $x = 3$.

Solution

Use the expression for $f(x)$ and the formula for work.

$$W = \int_1^3 f(x)\;dx = \int_1^3 (x^2 + 2x)\;dx \qquad \text{Formula for work.}$$

$$= \left[\frac{x^3}{3} + x^2\right]_1^3 \qquad \text{Antiderivative.}$$

$$= \left[\frac{3^3}{3} + 3^2\right] - \left[\frac{1^3}{3} + 1^2\right] = \frac{50}{3} \qquad \text{FTC2; simplify.}$$

The work done in moving the object from $x = 1$ to $x = 3$ is $16\frac{2}{3}$ ft-lb.

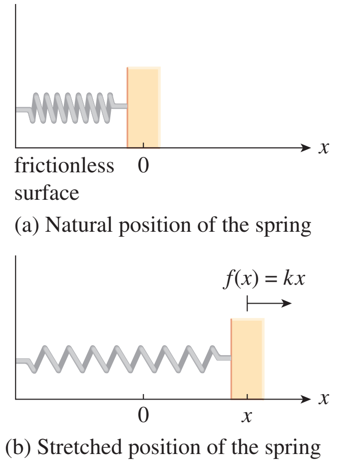


Figure 6.65
Illustration of Hooke's Law.

In the next example, we use a law from physics: **Hooke's Law** states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x: $f(x) = kx$, where k is a positive constant (called the **spring constant**). Hooke's Law is illustrated in Figure 6.65 and holds provided that x is not too large.

Example 2 Work Needed to Stretch a Spring

A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

Solution

Using Hooke's Law, the force required to hold the spring stretched x meters beyond its natural length is $f(x) = kx$.

When the spring is stretched from 10 cm to 15 cm, the amount stretched is $5\ \text{cm} = 0.05\ \text{m}$.

Therefore, $f(0.05) = 40 \quad \Rightarrow \quad 0.05k = 40 \quad \Rightarrow \quad k = \dfrac{40}{0.05} = 800.$

This means $f(x) = 800x$. Use this expression to determine the work done in stretching the spring from 15 cm to 18 cm.

$$\begin{aligned} W &= \int_{0.05}^{0.08} 800x\ dx && \text{Formula for work.} \\ &= \left[800\frac{x^2}{2}\right]_{0.05}^{0.08} && \text{Antiderivative.} \\ &= 400\left[(0.08)^2 - (0.05)^2\right] = 1.56\ \text{J} && \text{FTC2; simplify.} \end{aligned}$$

Example 3 Work Needed to Lift a Cable

A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

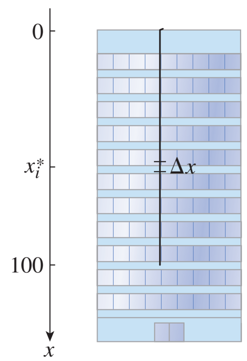


Figure 6.66
Cable hanging from a building.

Solution

There is no formula given for the force function. However, we can construct an expression for work using the general argument applied to this specific case.

Place the origin at the top of the building and the x-axis pointing downward as in Figure 6.66.

Divide the cable into small parts each with length Δx.

Let x_i^* be a point in the ith subinterval. All the points in this interval are lifted by approximately the same amount, namely x_i^*.

The cable weighs 2 pounds per foot. So, the weight of the ith part of the cable is $2\,\Delta x$.

Therefore, the work done on the ith part, in foot-pounds, is

$$\underbrace{(2\,\Delta x)}_{\text{force}} \cdot \underbrace{x_i^*}_{\text{distance}} = 2x_i^*\ \Delta x.$$

The total work is obtained by adding all of these approximations and letting n increase without bound, equivalently, $\Delta x \to 0$.

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^*\ \Delta x = \int_0^{100} 2x\ dx && \text{Definition for work.} \\ &= \left[x^2\right]_0^{100} = 10{,}000\ \text{ft-lb} && \text{Antiderivative; FTC2; simplify.} \end{aligned}$$

Example 4 Work Needed to Empty a Tank

A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. Note: The density of water is $1000\ \text{kg/m}^3$.

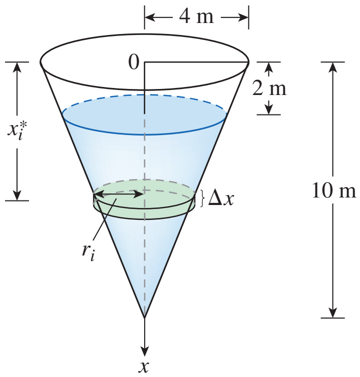


Figure 6.67
The water tank and coordinate system.

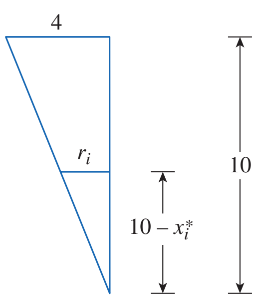


Figure 6.68
Use similar triangles to find an expression for r_i.

Solution

Use a vertical coordinate line to measure depths from the top of the tank. See Figure 6.67.

The water extends from a depth of 2 m to a depth of 10 m. Divide the interval [2, 10] into n equal subintervals with endpoints $2 = x_0, x_1, \ldots, x_n = 10$. Choose x_i^* in the ith subinterval.

This process divides the water into n layers. The ith layer is approximated by a circular cylinder with radius r_i and height Δx.

Use similar triangles to find an expression for r_i. Using Figure 6.68:

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10} \quad \Rightarrow \quad r_i = \frac{2}{5}(10 - x_i^*)$$

An approximation to the volume of water in the ith layer is

$$V_i \approx \pi r_i^2 \, \Delta x = \frac{4\pi}{25}(10 - x_i^*)^2 \, \Delta x.$$

The mass of the ith layer is

$$m_i = \text{density} \times \text{volume} \qquad \text{Definition of mass.}$$

$$\approx 1000 \cdot \frac{4\pi}{25}(10 - x_i^*)^2 \, \Delta x \qquad \text{Use density of water and expression for volume.}$$

$$= 160\pi(10 - x_i^*)^2 \, \Delta x \qquad \text{Simplify.}$$

The force required to raise this layer must counteract, or balance, the force of gravity. Therefore,

$$F_i = m_i g \approx (9.8)160\pi(10 - x_i^*)^2 \, \Delta x \approx 1568\pi(10 - x_i^*)^2 \, \Delta x.$$

Each particle in the layer must travel a distance of approximately x_i^*.
The work done to raise this layer to the top is approximately the product of the force F_i and the distance x_i^*:

$$W_i \approx F_i x_i^* \approx 1568\pi x_i^*(10 - x_i^*)^2 \, \Delta x$$

To find the total work required to empty the tank, add the contributions of each of the n layers, and then take the limit as $n \to \infty$.

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} 1568\pi x_i^*(10 - x_i^*)^2 \, \Delta x \qquad \text{Total work to empty the tank.}$$

$$= \int_2^{10} 1568\pi x(10 - x)^2 \, dx \qquad \text{Riemann sum; definition of a definite integral.}$$

$$= 1568\pi \int_2^{10} (100x - 20x^2 + x^3) \, dx \qquad \text{Simplify integrand.}$$

$$= 1568\pi \left[50x^2 - \frac{20x^3}{3} + \frac{x^4}{4} \right]_2^{10} \qquad \text{Antiderivative.}$$

$$= 1568\pi \left(\frac{2048}{3} \right) \approx 3.4 \times 10^6 \text{ J} \qquad \text{FTC2; simplify.}$$

Hydrostatic Pressure and Force

Deep-sea divers are well aware that water pressure increases as they dive deeper. The reason is because the weight of the water above them increases. Let's try to model this experience in general.

Suppose a thin horizontal plate with area A square meters is submerged in a fluid of density ρ kilograms per cubic meter at a depth d meters below the surface of the fluid. See Figure 6.69.

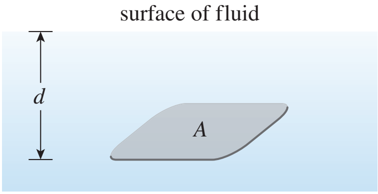


Figure 6.69
Horizontal plate submerged in a fluid.

The fluid directly above the plate has volume $V = Ad$. So, the mass of the fluid above the plate is $m = \rho V = \rho A d$. The force exerted by the fluid on the plate is therefore,

$$F = mg = \rho g A d$$

where g is the acceleration due to gravity. The pressure P on the plate is defined to be the force per unit area:

$$P = \frac{F}{A} = \rho g d$$

When using U.S. Customary units, we write $P = \rho g d = \delta d$, where $\delta = \rho g$ is the weight density (as opposed to ρ, which is the mass density). For example, the weight density of water is $\delta = 62.5 \text{ lb/ft}^3$.

The SI unit for measuring pressure is newtons per square meter, which is called a pascal (abbreviation: $1 \text{ N/m}^2 = 1 \text{ Pa}$). Since this is a small unit, the kilopascal (kPa) is often used. For example, because the density of water is $\rho = 1000 \text{ kg/m}^3$, the pressure at the bottom of a swimming pool 2 m deep is

$$\begin{aligned} P &= \rho g d = 1000 \text{ kg/m}^3 \times 9.8 \text{ m/s}^2 \times 2 \text{ m} \\ &= 19{,}600 \text{ Pa} = 19.6 \text{ kPa} \end{aligned}$$

An important principle of fluid pressure is the experimentally verified fact that at *any point in a liquid, the pressure is the same in all directions.* For example, a diver feels the same pressure on their nose and both ears. Thus, the pressure in any direction at a depth in a fluid with mass density ρ is given by

$$P = \rho g d = \delta d \tag{5}$$

This concept helps us determine the hydrostatic force against a *vertical* plate or wall or dam in a fluid. This is not a straightforward problem because the pressure is not constant, but increases as the depth increases.

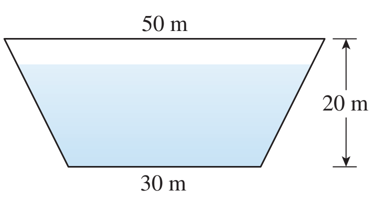


Figure 6.70
The shape of the dam is a trapezoid.

Example 5 Hydrostatic Force on a Dam

A dam has the shape of the trapezoid shown in Figure 6.70. The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.

Solution

Let the x-axis be vertical with origin at the surface of the water, as shown in Figure 6.71.

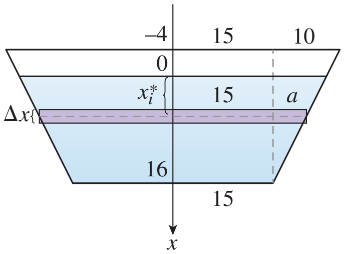


Figure 6.71
Draw the x-axis vertical and consider a horizontal strip with height Δx.

The depth of the water is 16 m. Divide the interval [0, 16] into n equal subintervals of length Δx with endpoints $0 = x_0, x_1, x_2, \ldots, x_n = 16$. Let x_i^* be an arbitrary value in the i subinterval $[x_{i-1}, x_i]$.

The ith horizontal strip of the dam is approximated by a rectangle with height Δx and width w_i. Use similar triangles, as illustrated in Figure 6.72, to find an expression for a.

$$\frac{a}{16 - x_i^*} = \frac{10}{20} \quad \Rightarrow \quad a = \frac{16 - x_i^*}{2} = 8 - \frac{x_i^*}{2}$$

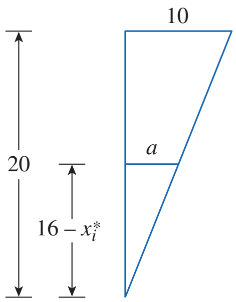


Figure 6.72
Use similar triangles to find an expression for a.

The width of the ith horizontal strip is

$$w_i = 2(15 + a) = 2\left(15 + 8 - \frac{1}{2}x_i^*\right) = 46 - x_i^*.$$

If A_i is the area of the ith strip, then

$$A_i \approx w_i \Delta x = (46 - x_i^*)\,\Delta x.$$

If Δx is small, then the pressure P_i on the ith strip is almost constant, and we can use Equation 5 to write $P_i \approx 1000\,g x_i^*$.

The hydrostatic force F_i acting on the ith strip is the product of the pressure and the area:

$$F_i = P_i A_i \approx 1000\,g x_i^*\,(46 - x_i^*)\,\Delta x$$

To obtain the total hydrostatic force on the dam, add these forces and consider the limit as $n \to \infty$.

$$\begin{aligned}
F &= \lim_{n\to\infty} \sum_{i=1}^{n} 1000\,g x_i^*\,(46 - x_i^*)\,\Delta x && \text{Total force on the dam.}\\
&= \int_0^{16} 1000\,g x\,(46 - x)\,dx && \text{Riemann sum; definition of a definite integral.}\\
&= 1000(9.8)\int_0^{16} (46x - x^2)\,dx && \text{Simplify integrand; use } g = 9.8.\\
&= 9800\left[23x^2 - \frac{x^3}{3}\right]_0^{16} && \text{Antiderivative.}\\
&= 9800\left[\left(23 \cdot 16^2 - \frac{16^3}{3}\right) - (0)\right] \approx 4.43 \times 10^7 \text{ N} && \text{FTC2; simplify.}
\end{aligned}$$

Moments and Centers of Mass

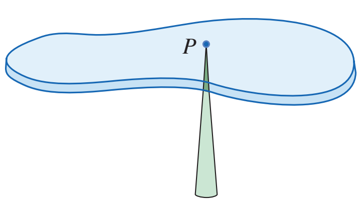


Figure 6.73
Center of mass illustration.

Given a thin plate of any given shape, we would like to find a point P on which the plate balances horizontally. See Figure 6.73. The point P is called the **center of mass** (or center of gravity) of the plate.

We first consider a simpler situation illustrated in Figure 6.74. Two objects with masses m_1 and m_2 are attached to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum. The rod will balance if

$$m_1 d_1 = m_2 d_2 \tag{6}$$

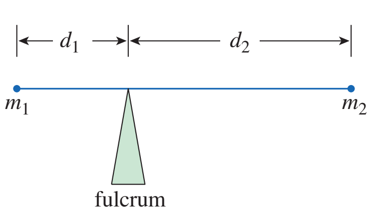


Figure 6.74
The rod is balanced if the fulcrum is in this position.

This is an experimental fact discovered by Archimedes and is called the Law of the Lever. This law seems reasonable if you think about two people balancing on a seesaw. The lighter person has to sit farther away from the center for the seesaw to balance.

Let's model this situation more precisely. Suppose that the rod lies along the x-axis such that an object with mass m_1 is at x_1 and an object with mass m_2 is at x_2. And, suppose the center of mass is at $\bar{x}$. Compare Figures 6.74 and 6.75: $d_1 = \bar{x} - x_1$ and $d_2 = x_2 - \bar{x}$.

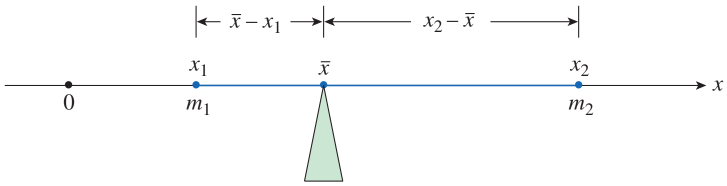


Figure 6.75
Suppose the rod lies along the x-axis and the center of mass is at $\bar{x}$.

Use these expressions for d_1 and d_2 in Equation 6:

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$
$$m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$
$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{7}$$

The numbers m_1x_1 and m_2x_2 are called the **moments** of the masses m_1 and m_2 (with respect to the origin). Equation 7 shows that the center of mass $\bar{x}$ is obtained by adding the moments of the masses and dividing by the total mass $m = m_1 + m_2$.

In general, suppose a system of n objects with masses $m_1, m_2, \ldots, m_n$ is located at the points $x_1, x_2, \ldots, x_n$ on the x-axis. In a similar manner, it can be shown that the center of mass of the system is located at

$$\bar{x} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_ix_i}{m} \tag{8}$$

where $m = \sum_{i=1}^{n} m_i$ is the total mass of the system. The sum of the individual moments, $M = \sum_{i=1}^{n} m_ix_i$ is called the **moment of the system about the origin**. Using this notation, Equation 8 can be written as $m\bar{x} = M$. This shows that if the total mass were concentrated at the center of mass $\bar{x}$, then its moment would be the same as the moment of the system.

Now consider a situation in which a system of n objects with masses $m_1, m_2, \ldots, m_n$ are located at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane. Figure 6.76 shows an example with $n = 3$ objects.

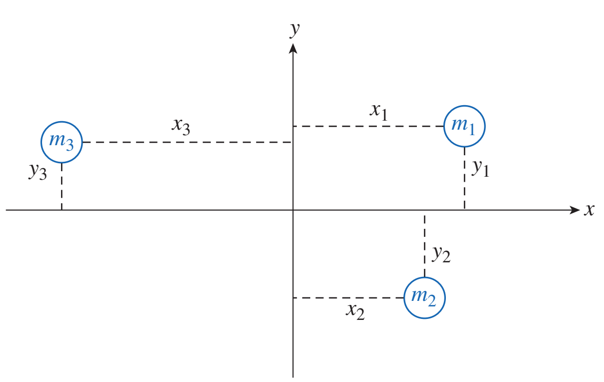


Figure 6.76
A system of objects in the xy-plane.

Analogous to the one-dimensional case, we define the **moment of the system about the *x*-axis** and the **moment of the system about the *y*-axis** as

$$M_x = \sum_{i=1}^{n} m_i y_i \qquad M_y = \sum_{i=1}^{n} m_i x_i \tag{9}$$

The value M_x measures the tendency of the system to rotate about the x-axis and M_y measures the tendency to rotate about the y-axis.

As in the one-dimensional case, the coordinates of the center of mass $(\bar{x}, \bar{y})$ are given in terms of the moments by the formulas

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m} \tag{10}$$

where $m = \sum_{i=1}^{n} m_i$ is the total mass. Since $m\bar{x} = M_y$ and $m\bar{y} = M_x$, the center of mass $(\bar{x}, \bar{y})$ is the point where a single object of mass m would have the same moments as the system.

Example 6 Objects in the Plane

Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points $(-1, 1)$, $(2, -1)$, and $(3, 2)$.

Solution

Compute the moments.

$M_x = 3(1) + 4(-1) + 8(2) = 15$

$M_y = 3(-1) + 4(2) + 8(3) = 29$

The total mass is $m = 3 + 4 + 8 = 15$.

Use Equation 10 to find the coordinates of the center of mass.

$$\bar{x} = \frac{M_y}{m} = \frac{29}{15} \qquad \bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1$$

Figure 6.77 shows the objects in the plane and the center of mass, $\left(\frac{29}{15}, 1\right)$.

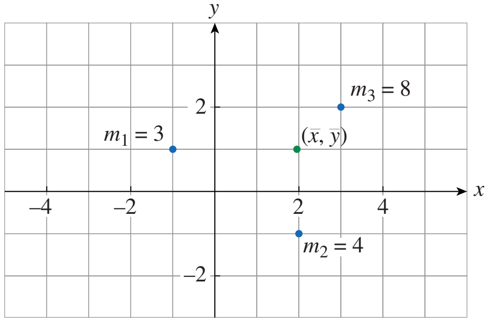


Figure 6.77

The center of mass is $\left(\frac{29}{15}, 1\right)$.

To generalize this procedure even more, consider a flat plate (called a *lamina*) with uniform density ρ that fills a region R in the plane. The goal is to find the center of mass of the plate, called the **centroid** of R. To find the center of mass, we will use the

symmetry principle which says that if R is symmetric about a line l, then the centroid of R lies on l. If R is reflected about l, then R remains the same, so its centroid remains fixed. And the only fixed points lie on l. Therefore, the centroid of a rectangle, for example, is its center.

Moments should be defined so that if the entire mass of a region is concentrated at the center of mass, then its moments remain unchanged. In addition, the moment of the union of two nonoverlapping regions should be the sum of the moments of the individual regions.

Suppose the region R is bounded above by the continuous function f, below by the x-axis, and by the lines $x = a$ and $x = b$, as shown in Figure 6.78. Divide the interval $[a, b]$ into n equal subintervals with endpoints $a = x_0, x_1, x_2, \ldots, x_n = b$ and width Δx. Choose the sample point x_i^* to be the midpoint $\bar{x}_i$ of the ith subinterval, that is, $x_i^* = \dfrac{x_{i-1} + x_i}{2}$. The n rectangles $R_1, R_2, \ldots, R_n$ in which R_i has width Δx and height $f(\bar{x}_i)$ together approximate the region R as shown in Figure 6.79.

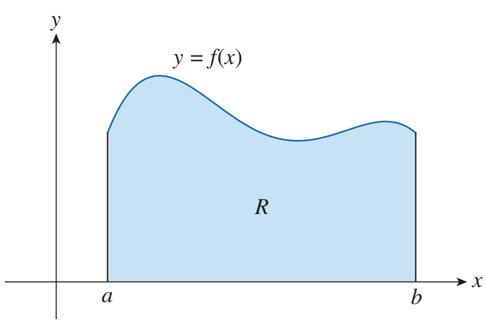


Figure 6.78
The region R in the plane.

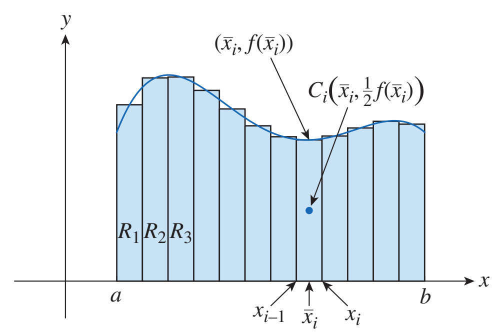


Figure 6.79
The n rectangles that approximate the region R.

The centroid of the ith approximating rectangle R_i is its center $C_i\left(\bar{x}_i, \frac{1}{2}f(\bar{x}_i)\right)$. Its area is $f(\bar{x}_i)\Delta x$ and, therefore, the mass of the rectangle R_i is $\rho f(\bar{x}_i)\Delta x$.

The moment of R_i about the y-axis is the product of its mass and the distance from C_i to the y-axis, which is $\bar{x}_i$. Therefore

$$M_y(R_i) = [\rho f(\bar{x}_i)\,\Delta x]\bar{x}_i = \rho \bar{x}_i f(\bar{x}_i)\,\Delta x$$

We can add these moments to obtain the moment of the polygonal approximation to R. If we take the limit as $n \to \infty$, we obtain the moment of R about the y-axis.

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i)\Delta x = \rho \int_a^b x f(x)\,dx$$

In a similar manner, we can compute the moment of R_i about the x-axis as the product of its mass and the distance from C_i to the x-axis.

$$M_x(R_i) = [\rho f(\bar{x}_i)\Delta x]\frac{1}{2}f(\bar{x}_i) = \rho \cdot \frac{1}{2}[f(\bar{x}_i)]^2\Delta x$$

Add these moments and take the limit as $n \to \infty$ to obtain the moment of R about the x-axis:

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \frac{1}{2}[f(\bar{x}_i)]^2\Delta x = \rho \int_a^b \frac{1}{2}[f(x)]^2\,dx$$

Just as for systems of particles, or objects, the center of mass of the plate is defined so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. The mass of the plate is the product of its density and its area:

$$m = \rho A = \rho \int_a^b f(x)\, dx$$

Finally, putting all these formulas together,

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b xf(x)\, dx}{\rho \int_a^b f(x)\, dx} = \frac{\int_a^b xf(x)\, dx}{\int_a^b f(x)\, dx}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2\, dx}{\rho \int_a^b f(x)\, dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2\, dx}{\int_a^b f(x)\, dx}$$

Note that ρ cancels in each quotient and does not appear in the final expression. This means that the location of the center of mass is independent of the density.

In summary, the center of mass of the plate (or the centroid of R) is located at the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\, dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\, dx \tag{11}$$

Example 7 Semicircular Plate

Find the center of mass of a semicircular plate of radius r.

Solution

In order to use Equation 11, consider the following model: $f(x) = \sqrt{r^2 - x^2}$, $a = -r$, and $b = r$. The semicircular plate is illustrated in Figure 6.80.

We don't need to use the formula for $\bar{x}$ because, by the symmetry principle, the center of mass must lie on the y-axis. Therefore, $\bar{x} = 0$.

The area of the semicircle is $A = \frac{1}{2}\pi r^2$.

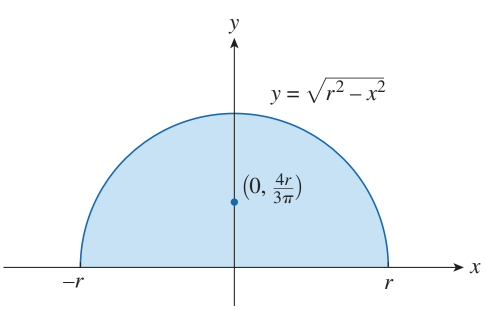


Figure 6.80
A semicircular plate of radius r.

$$\bar{y} = \frac{1}{A}\int_{-r}^{r} \frac{1}{2}[f(x)]^2\,dx \qquad \text{Formula for } \bar{y}.$$

$$= \frac{1}{\frac{1}{2}\pi r^2} \cdot \frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx \qquad \text{Use expressions for } A \text{ and } f(x).$$

$$= \frac{2}{\pi r^2}\int_0^r (r^2 - x^2)\,dx = \frac{2}{\pi r^2}\left[r^2 x - \frac{x^3}{3}\right]_0^r \qquad f \text{ is an even function; Antiderivative.}$$

$$= \frac{2}{\pi r^2}\left[\left(r^2 \cdot r - \frac{r^3}{3}\right) - (0)\right] = \frac{4r}{3\pi} \qquad \text{FTC2; simplify.}$$

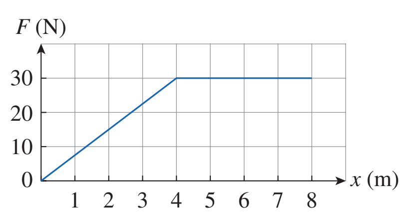
The center of mass is located at the point $\left(0, \dfrac{4r}{3\pi}\right)$. ■

6.6 Exercises

1. A particle is moved along the x-axis by a force that measures $\dfrac{10}{(1+x)^2}$ pounds at a point x feet from the origin. Find the work done in moving the particle from the origin to a distance of 9 ft.

2. When a particle is located a distance x meters from the origin, a force of $\cos\left(\dfrac{\pi x}{3}\right)$ newtons acts on it. How much work is done in moving the particle from $x = 1$ to $x = 2$? Interpret your answer by considering the work done from $x = 1$ to $x = 1.5$ and from $x = 1.5$ to $x = 2$.

3. The graph of a force function (in newtons) that increases to its maximum value and then remains constant is shown in the figure.



How much work is done by the force in moving an object a distance of 8 m?

4. The table gives values of a force function $f(x)$ measured in newtons for selected values of x measured in meters. Use Simpson's Rule to estimate the work done by the force in moving an object a distance of 18 m.

x	0	3	6	9	12	15	18
$f(x)$	9.8	9.1	8.5	8.0	7.7	7.5	7.4

5. A force of 10 lb is required to hold a spring stretched 4 inches beyond its natural length. How much work is done in stretching it from its natural length to 6 inches beyond its natural length?

6. A spring has a natural length of 20 cm. If a 25-N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

7. Suppose that 2 J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm.
(a) How much work is needed to stretch the spring from 35 cm to 40 cm?
(b) How far beyond its natural length will a force of 30 N keep the spring stretched?

8. If the work required to stretch a spring 1 ft beyond its natural length is 12 ft-lb, how much work is needed to stretch it 9 inches beyond its natural length?

9. A spring has natural length of 20 cm. Compare the work W_1 done in stretching the spring from 20 cm to 30 cm with the work W_2 done in stretching it from 30 cm to 40 cm. How are W_1 and W_2 related?

10. If 6 J of work is needed to stretch a spring from 10 cm to 12 cm and another 10 J is needed to stretch it from 12 cm to 14 cm, what is the natural length of the spring?

Show how to approximate the required work by a Riemann sum. Then express the work as an integral and evaluate it.

11. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft high.
(a) How much work is done in pulling the rope to the top of the building?
(b) How much work is done in pulling half the rope to the top of the building?

12. A chain lying on the ground is 10 m long and its mass is 80 kg. How much work is required to raise one end of the chain to a height of 6 m?

13. A cable that weighs 2 lb/ft is used to lift 800 lb of coal up a mine shaft 500 ft deep. Find the work done.

14. A bucket that weighs 4 lb and a rope of negligible weight are used to draw water from a well that is 80 ft deep. The bucket is filled with 40 lb of water and is pulled up at a rate of 2 ft/s. Water leaks out of a hole in the bucket at a rate of 0.2 lb/s. Find the work done in pulling the bucket to the top of the well.

15. A 10-kg bucket is lifted from the ground to a height of 12 m at a constant speed with a rope that weighs 0.8 kg/m. Initially, the bucket contains 36 kg of water. However, water leaks from the bucket at a constant rate beginning when the bucket is lifted and finishes draining just as the bucket reaches the 12 m level. How much work is done?

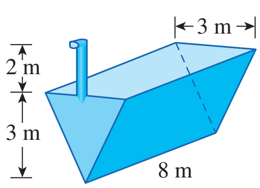
16. A 10-ft chain weighs 25 lb and hangs from a ceiling. Find the work done in lifting the lower end of the chain to the ceiling so that it's level with the upper end.

17. An aquarium 2 m long, 1 m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. Use the fact that the density of water is 1000 kg/m^3.

18. A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high, and the depth of the water is 4 ft. How much work is required to pump all of the water out over the side? Use the fact that water weighs 62.5 lb/ft^3.

A tank is full of water. Find the work required to pump the water out of the spout. Use the fact that water weighs 62.5 lb/ft^3.

19.



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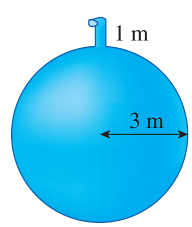
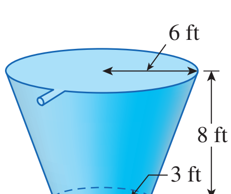


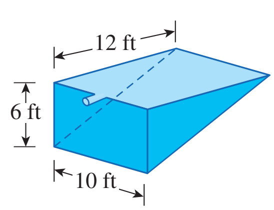
sphere

21.



frustum of a cone

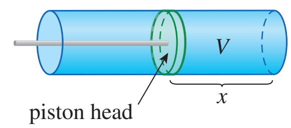
22.



23. Suppose that for the tank in Exercise 19, the pump breaks down after 4.7×10^5 J of work has been done. What is the depth of the water remaining in the tank?

24. Solve Exercise 20 if the tank is half full of oil that has a density of 900 kg/m^3.

25. When gas expands in a cylinder with radius r, the pressure at any given time is a function of the volume: $P = P(V)$. The force exerted by the gas on the piston (shown in the figure) is the product of the pressure and the area: $F = \pi r^2 P$.



Show that the work done by the gas when the volume expands from volume V_1 to volume V_2 is

$$W = \int_{V_1}^{V_2} P\, dV$$

26. In a steam engine, the pressure P and volume V of steam satisfy the equation $PV^{1.4} = k$, where k is a constant. This is true for adiabatic expansion, that is, expansion in which there is no heat transfer between the cylinder and its surroundings. Use Exercise 25 to calculate the work done by the engine during a cycle when the steam starts at a pressure of 160 lb/in^2 and a volume of 100 in^3 and expands to a volume of 800 in^3.

27. (a) Newton's Law of Gravitation states that two bodies with masses m_1 and m_2 attract each other with a force

$$F = G\frac{m_1 m_2}{r^2}$$

where r is the distance between the bodies and G is the gravitational constant. If one of the bodies is fixed, find the work needed to move the other from $r = a$ to $r = b$.

(b) Compute the work required to launch a 1000-kg satellite vertically to an orbit 1000 km high. Assume that Earth's mass is 5.98×10^{24} kg and is concentrated at its center. Use the radius of Earth as 6.37×10^6 m and $G = 6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$.

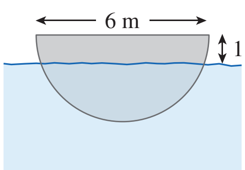
28. (a) Use an improper integral and information from Exercise 27 to find the work needed to propel a 1000-kg satellite out of Earth's gravitational field.

(b) Find the *escape velocity* v_0 that is needed to propel a rocket of mass m out of the gravitational field of a planet with mass M and radius R. Use the fact that the initial kinetic energy of $\frac{1}{2}mv_0^2$ supplies the needed work.

29. An aquarium 5 ft long, 2 ft wide, and 3 ft deep is full of water. Find (a) the hydrostatic pressure on the bottom of the aquarium, (b) the hydrostatic force on the bottom, and (c) the hydrostatic force on one end of the aquarium.

30. A tank is 8 m long, 4 m wide, 2 m high, and contains kerosene with density 820 kg/m^3 to a depth of 1.5 m. Find (a) the hydrostatic pressure on the bottom of the tank, (b) the hydrostatic force on the bottom, and (c) the hydrostatic force on one end of the tank.

A vertical plate is submerged (or partially submerged) in water and has the indicated shape. Explain how to approximate the hydrostatic force against one side of the plate by a Riemann sum. Then express the force as an integral and evaluate it.

31.


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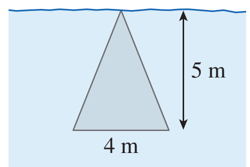

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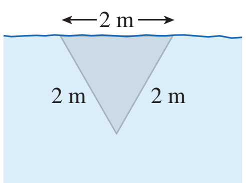

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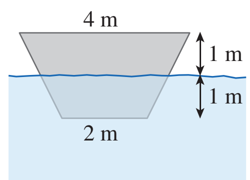
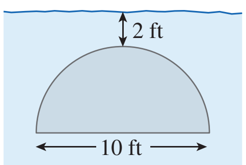
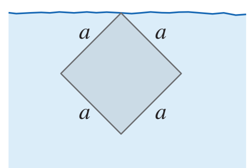


35.


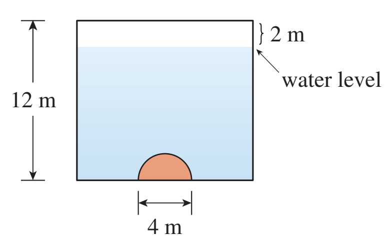
36.


37. A trough is filled with a liquid of density 840 kg/m^3. The ends of the trough are equilateral triangles with sides 8 m long and vertex at the bottom. Find the hydrostatic force on one end of the trough.

38. A large tank is designed with ends in the shape of the region between the curves $y = \frac{1}{2}x^2$ and $y = 12$, measured in feet. Find the hydrostatic force on one end of the tank if it is filled to a depth of 8 ft with gasoline. Assume the gasoline's density is 42.0 lb/ft^3.

39. A swimming pool is 20 ft wide and 40 ft long and its bottom is an inclined plane, the shallow end having a depth of 3 ft and the deep end, 9 ft. If the pool is full of water, estimate the hydrostatic force on (a) the shallow end, (b) the deep end, (c) one of the sides, and (d) the bottom of the pool.

40. A vertical dam has a semicircular gate as shown in the figure.

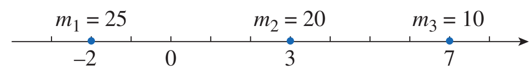


Find the hydrostatic force against the gate.

41. A vertical, irregularly shaped plate is submerged in water. The table shows measurements of its width, taken at the indicated depths. Use Simpson's Rule to estimate the force of the water against the plate.

Depth (m)	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Plate width (m)	0	0.8	1.7	2.4	2.9	3.3	3.6

42. Point masses m_i are located on the x-axis as shown in the figure.



Find the moment M of the system about the origin and the center of mass $\bar{x}$.

The masses m_i are located at the points P_i. Find the moments M_x and M_y and the center of mass of the system.

43. $m_1 = 6, \quad m_2 = 5, \quad m_3 = 10$
$P_1(1, 5), \quad P_2(3, -2), \quad P_3(-2, -1)$

44. $m_1 = 6, \quad m_2 = 5, \quad m_3 = 1, \quad m_4 = 4$
$P_1(1, -2), \quad P_2(3, 4), \quad P_3(-3, -7), \quad P_4(6, -1)$

Sketch the region bounded by the curves, and visually estimate the location of the centroid. Then find the exact coordinates of the centroid.

45. $y = 4 - x^2, \quad y = 0$

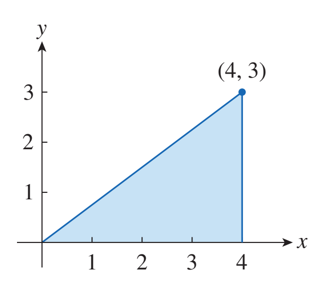
46. $3x + 2y = 6, \quad y = 0, \quad x = 0$

47. $y = e^x, \quad y = 0, \quad x = 0, \quad x = 1$

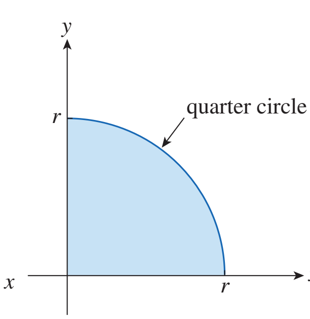
48. $y = \dfrac{1}{x}, \quad y = 0, \quad x = 1, \quad x = 2$

Calculate the moments M_x and M_y and the center of mass of a lamina with the given density and shape.

49. $\rho = 10$



50. $\rho = 2$



51. (a) Let R be the region that lies between the two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ and $a \leq x \leq b$. Use the same sort of reasoning that led to the formulas in Equation 11 to show that the centroid of R is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)\,dx$$

(b) Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.

52. Let R be the region that lies between the curves $y = x^m$ and $y = x^n$, $0 \leq x \leq 1$, where m and n are integers with $0 \leq n < m$.

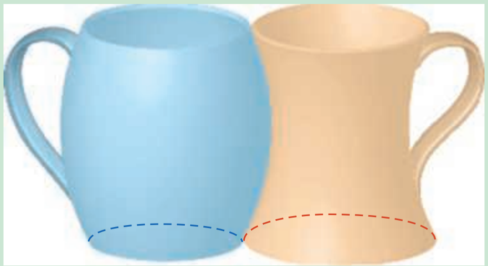
(a) Sketch the region R.

(b) Find the coordinates of the centroid of R.

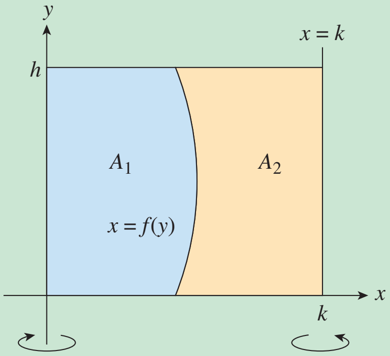
(c) Try to find values of m and n such that the centroid lies *outside* of the region R.

Discovery Project | Complementary Coffee Cups

Suppose you have a choice of two coffee cups of the type shown in the figure. Cup A bends outward and Cup B bends inward; they have the same height and their shapes fit together perfectly.



Cup A Cup B



It seems reasonable to ask which cup holds more coffee. Of course you could fill one cup with water and pour it into the other one. However, it also seems like there should be a more exact mathematical approach involving calculus.

Notice that if we ignore the handles, the volume of coffee in each cup can be equated to the volume of a solid of revolution.

1. Suppose the cups have height h, cup A is formed by rotating the curve $x = f(y)$ about the y-axis, and cup B is formed by rotating the same curve about the line $x = k$. Find the value of k such that the two cups hold the same amount of coffee.

2. What does your result from Problem 1 indicate about the areas A_1 and A_2 shown in the figure?

3. Based on your own measurements and observations, suggest a value for h and an equation for $x = f(y)$, and calculate the amount of coffee that each cup holds.

6.7 Applications to Economics and Biology

In this section, we will consider some applications of integration related to economics (consumer surplus) and biology (blood flow, cardiac output). Other applications are described in the exercises.

Consumer Surplus

In Section 4.6, we learned that the demand function $p(x)$ is the price that a company has to charge in order to sell x units of a commodity. Usually, selling larger quantities requires a lower price, so the demand function is a decreasing function. The graph of a typical demand function, called a **demand curve**, is shown in Figure 6.81. If X is the amount of the commodity that is currently available, then $P = p(X)$ is the current selling price.

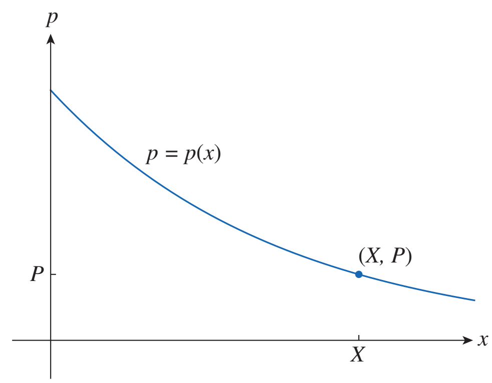


Figure 6.81
The graph of a typical demand function.

Let's consider the area of a region associated with this graph related to saving some money. Divide the interval $[0, X]$ into n equal subintervals with endpoints $0 = x_0, x_1, \ldots, x_n = X$, each of width $\Delta x = \dfrac{X}{n}$. Let $x_i^* = x_i$ be the right endpoint of the ith subinterval.

After the first x_{i-1} units are sold, suppose an additional Δx units become available. The additional units can be sold at a price of $p(x_i)$ for a total of $p(x_i)\,\Delta x$ additional dollars. Therefore, the total amount of money paid by consumers who are willing to pay at least $p(x_i)$ is approximately

$$p(x_1)\,\Delta x + p(x_2)\,\Delta x + \cdots + p(x_n)\,\Delta x = \sum_{i=1}^{n} p(x_i)\,\Delta x$$

This sum corresponds to the area enclosed by the rectangles in Figure 6.82.

If we let $n \to \infty$, this Riemann sum approaches the integral

$$\int_0^X p(x)\,dx$$

which is the total area under the demand curve from 0 to X.

The total amount of money actually spent by consumers is $X \cdot P$, which is the area of the rectangle illustrated in Figure 6.83. The total amount of money consumers are willing to pay minus the amount actually spent is called the **consumer surplus**. This amount can be written as

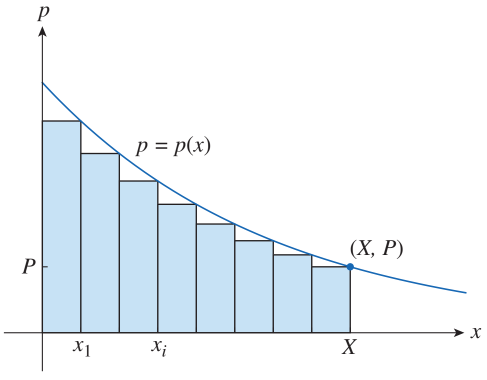


Figure 6.82
Divide $[0, X]$ into n equal subintervals.

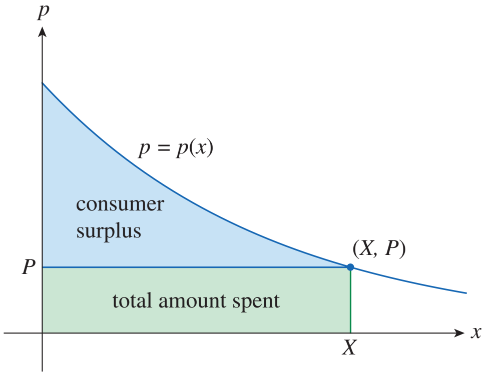


Figure 6.83
The consumer surplus is represented by the area under the demand curve and above the line $p = P$.

$$\begin{aligned}
\text{consumer surplus} &= \int_0^X p(x)\,dx - P \cdot X && \text{Difference between what consumers are willing to pay and the actual amount paid.}\\
&= \int_0^X p(x)\,dx - \int_0^X P\,dx && \text{Rewrite } X \cdot P \text{ as a definite integral.}\\
&= \int_0^X [p(x) - P]\,dx && \text{Property of definite integrals.}
\end{aligned}$$

The consumer surplus represents the amount of money saved by consumers in purchasing at price P, corresponding to an amount demanded of X. Figure 6.83 shows a graphical interpretation of the consumer surplus as the area under the demand curve and above the line $p = P$.

Example 1 Consumer Surplus

Suppose the demand for certain product, in dollars, is given by the function $p(x) = 1200 - 0.2x - 0.0001x^2$. Find the consumer surplus when the sales level is 500.

Solution

The number of units sold is $X = 500$; the corresponding price is

$$p = 1200 - (0.2)(500) - (0.0001)(500)^2 = 1075.$$

The consumer surplus is

$$\begin{aligned}
\int_0^{500} [p(x) - P]\,dx &= \int_0^{500} (1200 - 0.2x - 0.0001x^2 - 1075)\,dx && \text{Definition of consumer surplus.}\\
&= \int_0^{500} (125 - 0.2x - 0.0001x^2)\,dx && \text{Combine terms.}\\
&= \left[125x - 0.1x^2 - (0.0001)\left(\frac{x^3}{3}\right)\right]_0^{500} && \text{Antiderivative.}\\
&= \left[\left((125)(500) - (0.1)(500)^2 - \frac{(0.0001)(500)^3}{3}\right) - (0)\right]\\
&= 33{,}333.33 \text{ dollars} && \text{FTC2; simplify.}
\end{aligned}$$

Blood Flow

In Section 3.8, we introduced the law of laminar flow:

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

This function returns the velocity v of blood that flows along a blood vessel with radius R and length l at a distance r from the central axis. The constant P is the pressure difference between the ends of the vessel and η is the viscosity of the blood.

In order to compute the rate of blood flow, or *flux* (volume per unit time), first consider smaller, equally spaced radii $r_1, r_2, \ldots$. As in the derivation of the method of cylindrical shells, the approximate area of the ring (or washer) with inner radius r_{i-1} and outer radius r_i is

$$2\pi r_i \Delta r \quad \text{where} \quad \Delta r = r_i - r_{i-1}$$

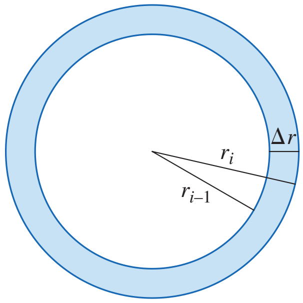


Figure 6.84
A ring in a blood vessel with inner radius r_{i-1} and outer radius r_i.

This ring and area are illustrated in Figure 6.84.

If Δr is small, then the velocity is almost constant throughout this ring and can be approximated by $v(r_i)$. Therefore, the volume of blood per unit time that flows across the ring is approximately

$$(2\pi r_i \Delta r)\ v(r_i) = 2\pi r_i v(r_i)\,\Delta r$$

An approximation to the total volume of blood that flows across a cross-section per unit time is the sum of the volume flowing across each ring:

$$\sum_{i=1}^{n} 2\pi r_i v(r_i)\,\Delta r$$

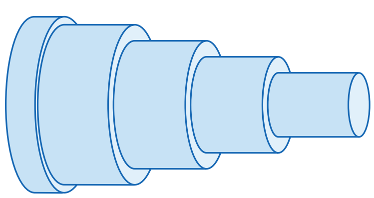


Figure 6.85
A visualization of the approximate blood flow across a cross-section.

This approximation is illustrated in Figure 6.85. Notice that the velocity and, therefore, the volume per unit time, increases toward the center of the blood vessel.

As usual, this approximation gets better as n increases. The limit of this sum, as $n \to \infty$, is the exact value of the **flux** (or *discharge*), which is the volume of blood that passes a cross-section per unit time.

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r_i v(r_i)\,\Delta r = \int_0^R 2\pi r\, v(r)\,dr \qquad \text{Definition of a definite integral.}$$

$$= \int_0^R 2\pi r \frac{P}{4\eta l}(R^2 - r^2)\,dr \qquad \text{Use the expression for } v(r).$$

$$= \frac{\pi P}{2\eta l}\int_0^R (R^2 r - r^3)\,dr = \frac{\pi P}{2\eta l}\left[R^2\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=R} \qquad \text{Rearrange terms; Antiderivative.}$$

$$= \frac{\pi P}{2\eta l}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{\pi P R^4}{8\eta l} \qquad \text{FTC2; simplify.}$$

The expression for flux

$$F = \frac{\pi P R^4}{8\eta l} \tag{1}$$

is called **Poiseuille's Law**; it shows that the flux is proportional to the fourth power of the radius of the blood vessel.

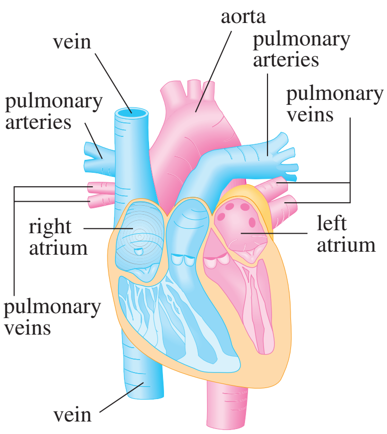


Figure 6.86
The human cardiovascular system.

Cardiac Output

Figure 6.86 illustrates the human cardiovascular system. Blood returns from the body through the veins, enters the right atrium of the heart, and is pumped to the lungs through the pulmonary arteries for oxygenation. It then flows back into the left atrium through the pulmonary veins and then out to the rest of the body through the aorta. The **cardiac output** of the heart is the volume of blood pumped by the heart per unit time, that is, the rate of flow into the aorta.

The *dye dilution method* is an approach to measure cardiac output. Dye is injected into the right atrium and flows through the heart into the aorta. A probe inserted into the aorta measures the concentration of the dye leaving the heart at equally spaced times over a time interval $[0, T]$, until the dye has cleared.

Let $c(t)$ be the concentration of the dye measured at time t. Divide the interval $[0, T]$ into n equal subintervals with endpoints $0 = t_0, t_1, t_2, \ldots, t_n = T$, each of width Δt. The amount of dye that flows past the measuring point during the subinterval from $t = t_{i-1}$ to $t = t_i$ is approximately

$$(\text{concentration})(\text{volume}) = c(t_i)(F\,\Delta t)$$

where F is the rate of flow that we would like to determine. An approximation to the total amount of dye that flows past the measuring point is a sum of the measurements associated with each subinterval:

$$\sum_{i=1}^{n} c(t_i)\ F\,\Delta t = F\sum_{i=1}^{n} c(t_i)\ \Delta t$$

Let $n \to \infty$ and use the definition of a definite integral to obtain:

$$A = F\int_0^T c(t)\,dt$$

Therefore, the cardiac output is given by

$$F = \frac{A}{\displaystyle\int_0^T c(t)\ dt} \qquad (2)$$

where the amount of dye A is known and the integral can be approximated from the concentration readings.

Example 2 Cardiac Output

A 5 mg bolus of dye is injected into a patient's right atrium. The concentration of the dye (in milligrams per liter) is measured in the aorta at 1-second intervals and the resulting values are shown in Table 6.3. Estimate the cardiac output.

t	$c(t)$	t	$c(t)$
0	0	6	6.1
1	0.4	7	4.0
2	2.8	8	2.3
3	6.5	9	1.1
4	9.8	10	0
5	8.9		

Table 6.3
Dye concentration measurements.

Solution

The amount of dye injected is $A = 5$.

Using the information in the table: $\Delta t = 1$ and $T = 10$.

Use Simpson's Rule to approximate the integral of the concentration:

$$\begin{aligned}\int_0^{10} c(t)\ dt &\approx \frac{1}{3}[0 + 4(0.4) + 2(2.8) + 4(6.5) + 2(9.8) + 4(8.9) \\ &\qquad + 2(6.1) + 4(4.0) + 2(2.3) + 4(1.1) + 0] \\ &\approx 41.87\end{aligned}$$

Using Equation 2, the approximate cardiac output is

$$F = \frac{A}{\int_0^{10} c(t)\,dt} \approx \frac{5}{41.87} \approx 0.12 \text{ L/s} = 7.2 \text{ L/min}.$$

6.7 Exercises

1. The marginal cost function $C'(x)$ is the derivative of the cost function (Sections 3.8 and 4.6). If the marginal cost of manufacturing x meters of a fabric is $C'(x) = 5 - 0.008x + 0.000009x^2$ (measured in dollars per meter) and the fixed start-up cost is $C(0) = \$20{,}000$, use the Net Change Theorem to find the cost of producing the first 2000 units.

2. The marginal revenue from the sale of x units of a product is $12 - 0.0004x$. If the revenue from the sale of the first 1000 units is \$12,400, find the revenue from the sale of the first 5000 units.

3. The marginal cost of producing x units of a certain product is $74 + 1.1x - 0.002x^2 + 0.00004x^3$ (in dollars per unit). Find the increase in cost if the production level is raised from 1200 units to 1600 units.

4. The demand function for a certain commodity is $p(x) = 20 - 0.05x$. Find the consumer surplus when the sales level is 300. Illustrate by drawing the demand curve and identifying the consumer surplus as an area of a plane region.

5. A demand curve is given by $p(x) = \dfrac{450}{x + 8}$. Find the consumer surplus when the selling price is \$10.

6. The **supply function** $p_S(x)$ for a commodity gives the relation between the selling price and the number of units that manufacturers will produce at that price. For a higher price, it seems reasonable that manufacturers will produce more units, so p_S is an increasing function of x. Let X be the amount of the commodity currently produced and let $P = p_S(X)$ be the current price. Some producers would be willing to make and sell the commodity for a lower selling price but by selling the commodity at the current price P, they are receiving more than their minimal price. The excess is called the **producer surplus**. An argument similar to that for consumer surplus shows that the producer surplus is given by the definite integral

$$\int_0^X [P - p_S(x)]\,dx$$

Find the producer surplus for the supply function $p_S(x) = 3 + 0.01x^2$ at a sales level of $X = 10$. Illustrate this result by sketching the supply curve and identifying the producer surplus as an area of a plane region.

7. Suppose a supply curve is modeled by the function $p_S(x) = 200 + 0.2x^{3/2}$. Find the producer surplus when the selling price is \$400.

8. For a given commodity and pure competition, the number of units produced and the price per unit are determined as the coordinates of the point of intersection of the supply and demand curves. Suppose a demand function and a supply function are given by

$$p(x) = 50 - \frac{1}{20}x \qquad p_S(x) = 20 + \frac{1}{10}x$$

Find the equilibrium point and then the consumer surplus and the producer surplus. Illustrate these results by sketching the supply and demand curves and identifying the surpluses as areas of plane regions.

9. A company selling specialty kitchen cabinet handles modeled the demand curve for its product (in dollars) by the equation

$$p(x) = \frac{800{,}000e^{-x/5000}}{x + 20{,}000}$$

Use technology to estimate the sales level when the selling price is \$16. Find the consumer surplus for this sales level.

10. A movie theater has been charging \$10.50 per person and selling about 500 tickets on a typical weeknight. After surveying their customers, the theater estimates that for every 25 cents that they lower the price, the number of moviegoers will increase by 20 per night. Find the demand function and calculate the consumer surplus when the tickets are priced at \$8.00.

11. If the amount of capital that a company has at time t is $f(t)$, then the derivative, $f'(t)$, is called the *net investment flow*. Suppose that the net investment flow is $\sqrt{t}$ million dollars per year (where t is measured in years). Find the increase in capital (the *capital formation*) from the fourth year to the eighth year.

12. Suppose revenue flows into a company at a rate of $f(t) = 9000\sqrt{1 + 2t}$, where t is measured in years and $f(t)$ is measured in dollars per year. Find the total revenue obtained in the first 4 years.

13. *Pareto 's Law of Income* states that the number of people with incomes between $x = a$ and $x = b$ is $N = \int_a^b Ax^{-k}\,dx$, where A and k are constants with $A > 0$ and $k > 1$. The average income for this group of people is given by

$$\bar{x} = \frac{1}{N}\int_a^b Ax^{1-k}\,dx$$

Find the value of $\bar{x}$.

14. Suppose an unusually hot and wet summer is causing a huge increase in the mosquito population near a lake resort area. The number of mosquitos is increasing at an estimated rate of $r(t) = 2200 + 10e^{0.8t}$ mosquitos per week. Find the change in the mosquito population between the fifth and ninth weeks of the summer.

15. Use Poiseuille's Law to calculate the rate of flow in a small human artery if $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000\ \text{dynes/cm}^2$.

16. High blood pressure results from constriction of the arteries. To maintain a normal flow rate (flux), the heart has to pump harder, thus increasing the blood pressure. Use Poiseuille's Law to show that if R_0 and P_0 are normal values of the radius and pressure in an artery and the constricted values are R and P, then for the flux to remain constant, P and R are related by the equation

$$\frac{P}{P_0} = \left(\frac{R_0}{R}\right)^4$$

Use this expression to show that if the radius of an artery is reduced to three-fourths of its former value, then the pressure is more than tripled.

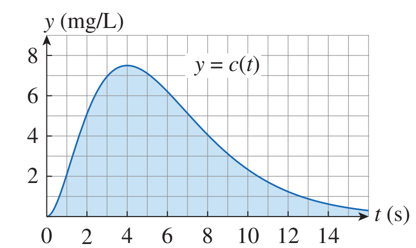
17. Suppose the dye dilution method is used to measure a patient's cardiac output with 6 mg of dye. The dye concentrations, in mg/L, are modeled by the function $c(t) = 20te^{-0.6t}$, $0 \le t \le 10$, where t is measured in seconds. Find the cardiac output.

18. Suppose the dye dilution method is used to measure a patient's cardiac output. After an 8 mg injection of dye, the dye concentration measurements, in mg/L, at 2-second intervals are shown in the table.

t	$c(t)$	t	$c(t)$
0	0	12	3.9
2	2.4	14	2.3
4	5.1	16	1.6
6	7.8	18	0.7
8	7.6	20	0
10	5.4		

Use Simpson's Rule to estimate the cardiac output.

19. Suppose the dye dilution method is used to measure a patient's cardiac output. The graph of the concentration function $c(t)$ is shown after a 7 mg injection of dye.



Use Simpson's Rule to estimate the cardiac output.

6.8 Probability

In this section, we will consider some applications of integration related to probability and continuous random variables.

Continuous Random Variables

A **continuous random variable** is a function often used to model a population that can be any number in some interval. For example, consider the oxygen concentration in a randomly selected area of the Gulf of Mexico, the weight of a randomly selected sea turtle, or the lifetime of a randomly selected car battery. Each of these values can be some number in a (continuous) interval. The length of the interval doesn't matter, only that we can theoretically select some number in an interval.

Even though we do not have a measurement device that is infinitely accurate, we assume we can select any number in an interval.

Suppose we can model the time, in minutes, it takes to walk the Freedom Trail in Boston by a continuous random variable X. We might want to know the probability that it takes between 90 and 120 minutes to complete the walk. This probability statement is written as $P(90 \le X \le 120)$.

The probability of an event is defined as the long-run, or limiting, relative frequency of occurrence of the event. Therefore, the probability of any event is a number between 0 and 1. A probability near 0 means the event is unlikely to occur, and a probability near 1 means the event is likely to occur.

A continuous random variable X is completely characterized by a **probability density function** f. This function is defined so that the probability that X takes on any value between a and $b\,(a < b)$ is the area under the density curve. That is,

$$\mathrm{P}(a \le X \le b) = \int_a^b f(x)\,dx \tag{1}$$

The WAC Bennet Dam is one of the highest earth-filled dams and controls the flow of water from Williston Lake.

For example, suppose X represents the water discharge rate, in cubic feet per second, at the WAC Bennet Dam on the Peace River on a randomly selected day. Figure 6.87 shows a graph of the probability density function f. The shaded region in this figure represents the probability that the discharge rate is between 1400 and 1475 ft^3/s on a randomly selected day.

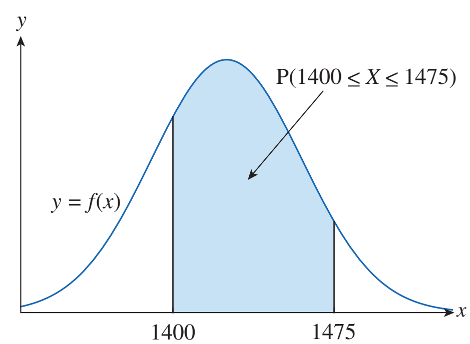


Figure 6.87
The shaded area represents $\mathrm{P}(1400 \le X \le 1475)$.

A probability density function is defined for all real numbers. Although the graphs of probability density functions can vary, each must satisfy the following properties.

(1) The probability density function must be defined such that the total area under the curve is exactly 1, because the total probability associated with any random variable is exactly 1. This means that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{2}$$

Note that the value of the probability density function $f(x)$ may be greater than 1 for some numbers, but the total area under the curve must still be exactly 1.

(2) For all values of x, $f(x) \ge 0$. This means that the graph of any probability density function always lies on or above the x-axis.

Here is an example to illustrate some of these concepts.

Example 1 Probability Concepts

Suppose the amount of fat, in grams, in a randomly selected 8-ounce cup of frozen chocolate yogurt is a random variable X with probability density function given by

$$f(x) = \begin{cases} -0.03125(x-8) & 0 \le x \le 8 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Sketch the graph of $y = f(x)$.

(b) Verify that f is a valid probability density function.

(c) Find the probability that a randomly selected cup of frozen chocolate yogurt has between 4 and 6 grams of fat.

Solution

(a) The graph of f is a straight line between $x = 0$ and $x = 8$, and 0 everywhere else. Figure 6.88 show a graph of the probability density function.

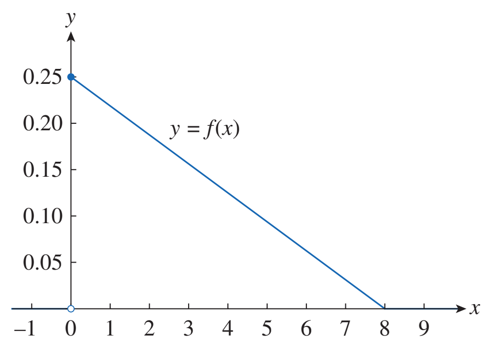


Figure 6.88
Graph of the probability density function.

(b) Verify that f satisfies the two properties for a probability density function. $f(x) \geq 0$ by examining the algebraic definition of f.

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^8 f(x)\, dx$$ There is no area, equivalently no probability, under the curve to the left of $x = 0$ or to the right of $x = 8$.

$$= \int_0^8 -0.03125\,(x-8)\, dx$$ Use the expression for $f(x)$.

$$= \left[-0.03125\left(\frac{x^2}{2} - 8x\right)\right]_0^8$$ Antiderivative.

$$= -0.03125\left[\left(\frac{8^2}{2} - 8 \cdot 8\right) - (0)\right] = 1$$ FTC2; simplify.

f is a valid probability distribution.

(c) Translate the question into a probability statement, and then find the value of the appropriate definite integral.

$$\mathrm{P}(4 \leq X \leq 6) = \int_4^6 f(x)\, dx$$ Probability statement; definite integral.

$$= \int_4^6 -0.03125(x-8)\, dx$$ Use the expression for $f(x)$.

$$= \left[-0.03125\left(\frac{x^2}{2} - 8x\right)\right]_4^6$$ Antiderivative.

$$= -0.03125\left[\left(\frac{6^2}{2} - 8 \cdot 6\right) - \left(\frac{4^2}{2} - 8 \cdot 4\right)\right] = 0.1875$$ FTC2; simplify.

The probability that a randomly selected cup of frozen chocolate yogurt has between 4 and 6 grams of fat is 0.1875. Figure 6.89 illustrates this result. Note that we could also compute this definite integral by finding the area of a trapezoid.

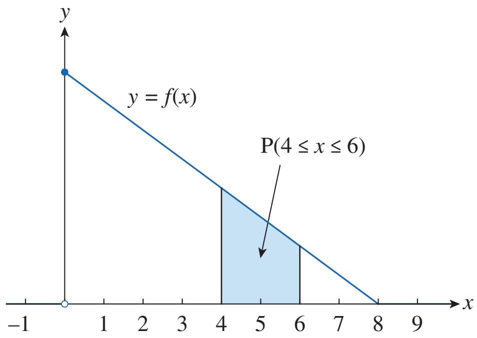


Figure 6.89
The area of the shaded region is $P(4 \le X \le 6)$.

Example 2 Probability Density Function for Waiting Times

The length of time waiting, for example, in a checkout line or a doctor's office and the time until a piece of equipment fails, for example a washer or automobile, are often modeled using a decreasing exponential function. Find the exact form of such a probability density function.

Solution

Suppose you decide to call a company for some technical assistance with a product. Let the random variable T represent the time, in minutes, one waits on hold before an agent of a company answers your call.

If f is the probability density function and your call is placed at time $t = 0$, then $\int_0^2 f(t)\,dt$ represents the probability that an agent answers some time in the interval [0, 2], or within the first 2 minutes, and $\int_4^5 f(t)\,dt$ is the probability that your call is answered during the fifth minute.

For $t < 0$, $f(t) = 0$; the agent can't answer before you place the call.

For $t > 0$, a decreasing exponential function has the form $f(t) = Ae^{-ct}$, where A and c are positive constants. Therefore,

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ Ae^{-ct} & \text{if } t \ge 0 \end{cases}$$

Use Equation 2 to determine the value of A.

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_0^{\infty} f(t)\,dt \qquad \text{Type I improper integral: infinite interval.}$$

$$= \int_0^{\infty} A^{-ct}\,dt = \lim_{b\to\infty} \int_0^b Ae^{-ct}\,dt \qquad \text{First integral 0; definition of improper integral.}$$

$$= \lim_{b\to\infty} \left[-\frac{A}{c} e^{-ct} \right]_0^b = \lim_{b\to\infty} \frac{A}{c}(1 - e^{-cb}) \qquad u\text{-substitution; FTC2.}$$

$$= \frac{A}{c} \qquad \text{Evaluate limit.}$$

Therefore, $\dfrac{A}{c} = 1 \;\Rightarrow\; A = c.$

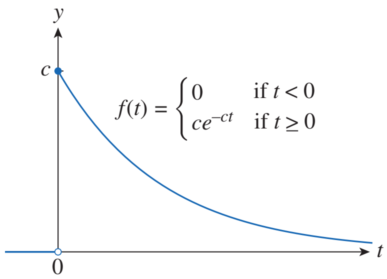


Figure 6.90
Graph of an exponential probability density function.

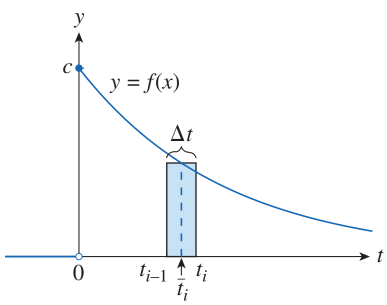


Figure 6.91
Visualization of the approximate probability of a call answered between t_{i-1} and t_i minutes.

Thus, every exponential probability density function has the form

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ ce^{-ct} & \text{if } t \geq 0 \end{cases} \quad \text{where } c > 0.$$

Figure 6.90 shows the graph of a typical exponential probability density function. ■

Average Values

Consider the waiting time for a company to answer a phone call. As you are placed on hold, it seems reasonable to ask how long is the expected wait time. Let the random variable T represent the time, in minutes, you wait on hold and let f be the corresponding probability density function.

Suppose a sample of N people call the company. Hopefully, no one has to wait more than an hour, so let's restrict our attention to the interval $0 \leq t \leq 60$. Divide the interval into n equal subintervals with endpoints $0 = t_0, t_1, t_2, \ldots, t_n = 60$ and width Δt. Think of Δt as lasting a minute, or half minute, or 10 seconds, or even a second. The probability that someone's call is answered during the time period from t_{i-1} to t_i is the area under the curve $y = f(t)$ from t_{i-1} to t_i, which is approximately equal to $f(\bar{t}_i)\,\Delta t$. This is the area of the approximating rectangle shown in Figure 6.91, where $\bar{t}_i$ is the midpoint of the interval.

The long-term proportion of calls that are answered during the time period from t_{i-1} to t_i is $f(\bar{t}_i)\,\Delta t$. So, we expect that, out of our sample of N callers, the number of calls answered in that time period is approximately $Nf(\bar{t}_i)\,\Delta t$ and the time that each waited is about $\bar{t}_i$. Therefore, the total time waited by these N callers is approximately $\bar{t}_i[Nf(\bar{t}_i)\,\Delta t]$.

If we add over all such intervals, we get the approximate total of everybody's waiting times:

$$\sum_{i=1}^{n} N\,\bar{t}_i f(\bar{t}_i)\,\Delta t$$

If we divide by the number of callers N, we get the approximate *average* waiting time:

$$\sum_{i=1}^{n} \bar{t}_i f(\bar{t}_i)\,\Delta t$$

This expression is a Riemann sum for the function $tf(t)$. As the number of subintervals increases without bound ($n \to \infty$), equivalently the time interval shrinks ($\Delta t \to 0$); this Riemann sum approaches the integral

$$\int_0^{60} tf(t)\,dt$$

This integral is called the *mean waiting time*.

We traditionally denote the mean by the Greek letter μ (mu).

In general, the **mean** of a random variable X with probability density function f is defined to be

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx$$

The mean can be interpreted as the long-run average value of the random variable X. It can also be interpreted as a measure of centrality of the probability density function.

The expression for the mean is similar to the one we have seen before.

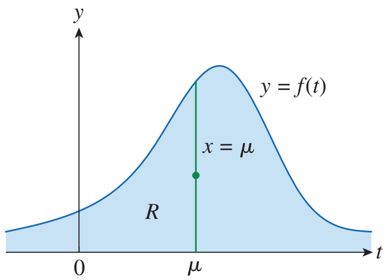


Figure 6.92
The centroid for the region R is at a point on the line $x = \mu$.

If R is the region that lies under the graph of f, we know from Equation 6.6.11 that the x-coordinate of the centroid of R is

$$\bar{x} = \frac{\displaystyle\int_{-\infty}^{\infty} xf(x)\,dx}{\displaystyle\int_{-\infty}^{\infty} f(x)\,dx} = \int_{-\infty}^{\infty} xf(x)\,dx = \mu$$

because the integral expression in the denominator is 1. So, a thin plate in the shape of R balances at a point on the vertical line $x = \mu$. See Figure 6.92.

Example 3 Mean Waiting Time

Find the mean of the exponential distribution of Example 2:

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ ce^{-ct} & \text{if } t \geq 0 \end{cases}$$

Solution

Use the definition of the mean.

$$\mu = \int_{-\infty}^{\infty} tf(t)\,dt = \int_0^{\infty} tce^{-ct}\,dt$$

Evaluate the integral using integration by parts with $u = t$ and $dv = ce^{-ct}$.

$$\int_0^{\infty} tce^{-ct}\,dt = \lim_{b\to\infty} \int_0^b tce^{-ct}\,dt \qquad \text{Type I improper integral.}$$

$$= \lim_{b\to\infty} \left(\left[-te^{-ct}\right]_0^b + \int_0^b e^{-ct}\,dt \right) \qquad \text{Integration by parts.}$$

$$= \lim_{b\to\infty} \left(-be^{-cb} + \frac{1}{c} - \frac{e^{-cb}}{c} \right) = \frac{1}{c} \qquad \text{FTC2; evaluate limit.}$$

Therefore, $\mu = \dfrac{1}{c}$ and we can rewrite the probability density function as

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mu^{-1}e^{-1/\mu} & \text{if } t \geq 0 \end{cases}$$

Example 4 Waiting Time Probabilities

Suppose the mean waiting time for a customer's call to be answered by a company representative is 5 minutes.

(a) Find the probability that a randomly selected call is answered during the first minute.

(b) Find the probability that a randomly selected customer waits more than 5 minutes to be answered.

Solution

(a) Let the random variable T be the waiting time. T has an exponential distribution with mean $\mu = 5$.

The probability density function for T is

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ 0.2e^{-t/5} & \text{if } t \geq 0 \end{cases}$$

The probability that a call is answered during the first minute is

$$P(0 \le T \le 1) = \int_0^1 f(t)\,dt = \int_0^1 0.2e^{-t/5}\,dt \qquad \text{Equation 1.}$$

$$= \Big[0.2(-5)e^{-t/5}\Big]_0^1 \qquad u\text{-substitution.}$$

$$= 1 - e^{-1/5} \approx 0.1813 \qquad \text{FTC2.}$$

So, about 18% of customers' calls are answered during the first minute.

(b) The probability that a customer waits more than 5 minutes is

$$P(T > 5) = \int_5^\infty f(t)\,dt = \int_5^\infty 0.2e^{-t/5}\,dt \qquad \text{Equation 1.}$$

$$= \lim_{b\to\infty}\int_5^b 0.2e^{-t/5}\,dt = \lim_{b\to\infty}(e^{-1} - e^{-b/5}) \qquad \text{Improper integral; } u\text{-substitution.}$$

$$= \frac{1}{e} \approx 0.3679 \qquad \text{Evaluate limit.}$$

About 37% of customers wait more than 5 minutes before their calls are answered. ■

In Example 4(b), we might expect that the mean is a waiting time such that 50% of the callers wait more than that time, and 50% wait less. However, notice that the mean waiting time is 5 minutes, and only 37% of callers wait more than 5 minutes. Because some callers have to wait much longer, for example 10 or 15 minutes, these large, outlying times *pull* the mean in their direction.

Another measure of centrality, or central tendency, of a random variable X is the *median*. In Example 4, the median is a number m such that half the callers have a waiting time less than m and the other callers have a waiting time longer than m. In general, the **median** of a random variable X with probability density function f is the number m such that

$$\int_m^\infty f(x)\,dx = \frac{1}{2}$$

This means that half the area under the graph of f lies to the right of m (and half to the left of m). In Exercise 11, you are asked to show that the median waiting time for the company described in Example 4 is approximately 3.5 minutes.

Normal Distributions

The normal distribution is one of the most important probability distributions in probability and statistics. A normal distribution is used to model many real-world phenomena—for example, birth weights, heights of individuals from a homogeneous population, annual rainfall at a given location, and retirement ages—and is used extensively in statistical inference. The probability density function for a normal random variable X is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}, \qquad -\infty < \mu < \infty, \qquad \sigma > 0 \tag{3}$$

The standard deviation is denoted by the lowercase Greek letter σ (sigma).

It can be shown that the mean of X is indeed μ. The positive constant σ is called the **standard deviation**; it is a measure of the spread of the distribution of X.

The graph of the probability density function for any normal random variable is bell-shaped, symmetric, and centered at μ. Figure 6.93 shows some examples. For small values of σ, the values of X are clustered closer to the mean, whereas for larger values of σ, the distribution is more spread out. Statisticians use various methods to estimate μ and σ from the data in a sample.

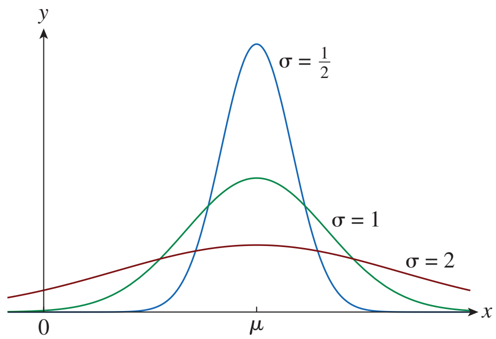


Figure 6.93
The value of σ determines the spread, or variability, in a normal distribution.

The function defined in Equation 3 is indeed a valid probability density function. For all values of x, $f(x) > 0$, and it can be shown using methods from multivariable calculus that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\, dx = 1$$

Example 5 IQ Scores

Suppose Intelligence Quotient (IQ) scores are normally distributed with mean 100 and standard deviation 15. Figure 6.94 shows a graph of the corresponding probability density function.

(a) Find the probability that a randomly selected person has an IQ score between 85 and 115.
(b) Find the proportion of the population that has an IQ score above 140.

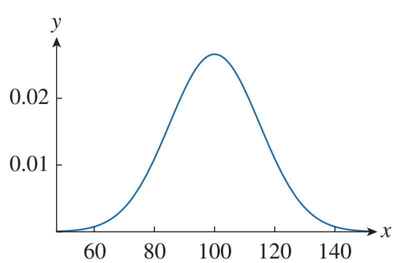


Figure 6.94
Graph of the probability density function associated with IQ scores.

Solution

(a) Let the random variable X be the IQ score of a randomly selected person. The probability distribution for X is given by Equation 3 with $\mu = 100$ and $\sigma = 15$. The probability that X is between 85 and 115 is

$$P(85 \le X \le 115) = \int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/(2\cdot 15)^2}\, dx.$$

Recall from Section 5.8 that the function $y = e^{-x^2}$ does not have an elementary antiderivative, so we cannot evaluate the integral exactly. However, we can use technology (or the Midpoint Rule or Simpson's Rule) to estimate the integral.

$P(85 \le X \le 115) \approx 0.6827$ Use technology.

So, approximately 68% of the population has an IQ score between 85 and 115, that is, within one standard deviation of the mean. Figure 6.95 illustrates this probability calculation.

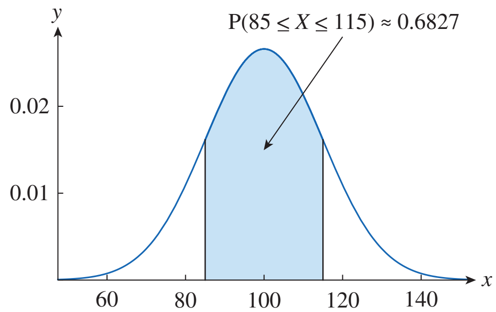


Figure 6.95
The shaded area represents $P(85 \le X \le 115)$.

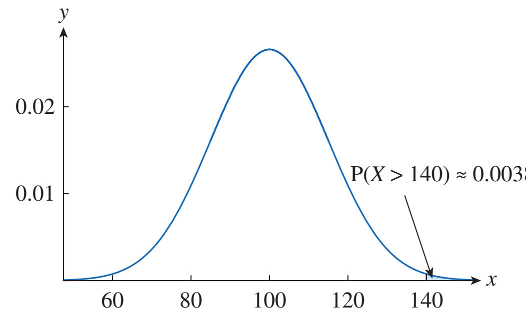


Figure 6.96
The shaded area represents $P(X > 140)$.

(b) The probability that a randomly selected person has an IQ score of more than 140 is

$$P(X > 140) = \int_{140}^{\infty} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450}\,dx.$$

Use technology to estimate this integral. Note that we can use an upper bound of 200 (or greater) since people with an IQ of over 200 are extremely rare; equivalently, there is very little area under the probability density function greater than 200.

$P(X > 140) \approx 0.0038$ Use technology.

Therefore, the proportion of the population with an IQ over 140 is approximately 0. 004. Figure 6.96 illustrates this probability calculation. ■

6.8 Exercises

1. Let $f(x)$ be the probability density function for the random variable X, the lifetime of a manufacturer's highest quality car tier, measured in miles. Explain the mean of each integral.

(a) $\int_{30{,}000}^{40{,}000} f(x)\,dx$ (b) $\int_{25{,}000}^{\infty} f(x)\,dx$

2. Let $f(x)$ be the probability density function for the random variable X, the weight of a randomly selected small mouth bass caught on the Susquehanna River, measured in pounds. Explain the meaning of each integral.

(a) $\int_{4}^{\infty} f(x)\,dx$ (b) $\int_{5}^{8} f(x)\,dx$

3. Let $f(t)$ be the probability density function for the random variable T, the time it takes for a randomly selected student to drive to school in the morning, measured in minutes. Write the following probabilities as integrals.
(a) The probability that a randomly selected student drives to school in less than 15 minutes.
(b) The probability that a randomly selected student takes more than half an hour to drive to school.

4. Let $f(y)$ be the probability density function for the random variable Y, the total amount of rainfall on the island of Hilo during the month of May, measured in inches. Write the following probabilities as integrals.
(a) The probability that the total amount of rain in May is less than 7 inches.
(b) The probability that the total amount of rain in May is more than 9 inches.

5. Let X be a random variable with probability density function f defined by

$$f(x) = \begin{cases} \dfrac{3}{64}x\sqrt{16 - x^2} & \text{if } 0 \le x \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Verify that f is a valid probability density function.
(b) Find $P(X < 2)$.
(c) Sketch the graph of f and explain why $m = 2$ is not the median.

6. Let X be a random variable with probability density function f defined by

$$f(x) = \begin{cases} xe^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(a) Verify that f is a valid probability density function.
(b) Find $P(1 \leq X \leq 2)$.

7. Let X be a random variable with probability density function f defined by $f(x) = \dfrac{c}{1 + x^2}$.

(a) Find the value of c such that f is a valid probability density function.
(b) Use the value of c from part (a) to find $P(-1 \leq X \leq 1)$.

8. Let X be a random variable with probability density function f defined by

$$f(x) = \begin{cases} kx^2(1 - x) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

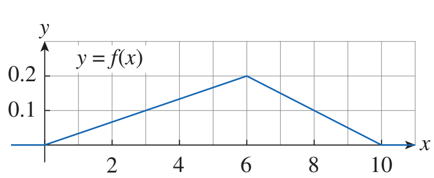
(a) Find the value of k such that f is a valid probability density function.
(b) Use the value of k from part (a) to find $P(X \geq \frac{1}{2})$. Sketch a graph to illustrate this probability.
(c) Find the mean of X.

9. The probability density function for X, the amount of time (in hours) a randomly selected car is parked in a mall parking lot, is given by

$$f(x) = \begin{cases} -\dfrac{2}{25}(x - 5) & \text{if } 0 \leq x \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Verify that f is a valid probability density function.
(b) Find the probability that a randomly selected car is parked for more than 3 hours.
(c) Find the probability that a randomly selected car is parked for less than 2 hours.

10. Let X be a random variable with probability density function f whose graph is shown in the figure.



(a) Verify that f is a valid probability density function.
(b) Use the graph to find the following probabilities:
(i) $P(X < 3)$ (ii) $P(3 \leq X \leq 8)$
(c) Find the mean of X.

11. Show that the mean waiting time for a phone call to the company described in Example 4 is approximately 3.5 minutes.

12. The time until failure of a certain type of lightbulb is modeled by an exponential distribution with mean $\mu = 1000$ hours.

(a) Find the probability a randomly selected bulb fails within the first 200 hours.
(b) Find the probability a randomly selected bulb lasts for more than 800 hours.
(c) Find the median lifetime of these lightbulbs.

13. The time that a customer waits for service at a fast-food restaurant is a random variable X that has an exponential distribution with mean 2.5 minutes.

(a) Find the probability that a randomly selected customer has to wait more than 4 minutes.
(b) Find the probability that a randomly selected customer is served within the first 2 minutes.
(c) The manager wants to advertise that anybody who isn't served within k minutes gets a free hamburger. However, they do not want to give away free hamburgers to more than 2% of the customers. Find the value of k.

14. According to the National Health Survey, the heights of adult males in the United States are normally distributed with mean 69.0 inches and standard deviation 2.8 inches.

(a) Find the probability that a randomly selected adult male is between 65 inches and 73 inches tall.
(b) Find the proportion of adult males who are more than 6 ft tall.

15. The "Garage Project" at the University of Arizona reports that the amount of paper discarded by households per week is normally distributed with mean 9.4 lb and standard deviation 4.2 lb.

(a) Find the probability that a randomly selected household throws out at least 10 lb of paper per week.
(b) Find the probability that a randomly selected household throws out less than 5 lb of paper per week.

16. Suppose the daily release volume from the Glen Canyon Dam is normally distributed with mean 20.8 KAF (thousand acre-feet) and standard deviation 5.5 KAF.

(a) Find the probability that the release volume is between 20 and 23 KAF on a randomly selected day.
(b) Find the probability that the release volume is more than 25 KAF on a randomly selected day.
(c) Find the value r such that the probability the release volume is between $20.8 - r$ and $20.8 + r$ is 0.95.

17. Suppose the amount of cereal packaged by a machine in a randomly selected box is normally distributed with mean μ g and standard deviation 12 g.

(a) If $\mu = 500$ g, find the probability that a randomly selected box has less than 480 g of cereal.
(b) Find the value of μ such that no more than 5% of cereal boxes contain less than 500 g.

18. Suppose the speed limit on a certain highway is 100 km/h and the actual speed of a randomly selected vehicle on this highway is normally distributed with mean 112 km/h and standard deviation 8 km/h.

(a) Find the probability that a randomly selected vehicle on this highway is traveling at a legal speed.

(b) Suppose police are instructed to ticket motorists driving 125 km/h or more. Find the proportion of motorists who will receive a ticket.

19. Show that the probability density function for a normal random variable has inflection points at $x = \mu \pm \sigma$.

20. For any normal random variable X, find the probability that X lies within two standard deviations of the mean.

21. The standard deviation for a random variable with probability density function f and mean μ is defined by

$$\sigma = \left[\int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx \right]^{1/2}$$

Find the standard deviation for an exponential random variable with mean μ.

22. Suppose Z is a normal random variable with mean 0 and standard deviation 1.

(a) Find $P(-1 \le Z \le 1)$, $P(-2 \le Z \le 2)$, and $P(-3 \le Z \le 3)$.

(b) The probabilities in part (a) are the 68–95–99.7 rule, or the Empirical Rule. Explain this rule applied to any normal distribution with mean μ and standard deviation σ.

(c) The quartiles divide a distribution into four equal parts. Find the value of Q_1, the first quartile, such that $P(Z \le Q_1) = 0.25$. Use this result to find Q_3, the third quartile, such that $P(Z \ge Q_3) = 0.25$. What is the value of Q_2, the second quartile?

6 Review

Concepts and Vocabulary

1. (a) Draw the graphs of two representative functions $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ for $a \leq x \leq b$. Show how to approximate the area between these curves using a Riemann sum and sketch the corresponding approximating rectangles. Write a definite integral in terms of $f(x)$, $g(x)$, a, and b that gives the exact area.
   (b) Explain how this approximation changes if the graphs have equations $x = f(y)$ and $x = g(y)$, where $f(y) \geq g(y)$ for $c \leq y \leq d$.
2. Suppose that Person A runs faster than Person B throughout a 1500-meter race. What is the physical meaning of the area between the graphs of their velocity functions for the first minute of the race?
3. (a) Suppose S is a solid with known cross-sectional areas. Explain how to approximate the volume of S using a Riemann sum. Write an expression involving an integral that gives the exact volume.
   (b) If S is a solid of revolution, how do you find the cross-sectional areas?
4. (a) What is the volume of a typical cylindrical shell?
   (b) Explain how to use cylindrical shells to find the volume of a solid of revolution.
   (c) Explain a situation in which it is easier to use cylindrical shells instead of slicing to find the volume of a solid of revolution.
5. (a) Write an expression for the length of a smooth curve with parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$.
   (b) How does the expression in part (a) simplify if the curve is described by an expression for y in terms of x, $y = f(x)$, $a \leq x \leq b$? What if x is given in terms of y?
6. (a) What is the average value of a function f on an interval $[a, b]$?
   (b) Explain the Mean Value Theorem for Integrals in your own words. What is its geometric interpretation?
7. Suppose that you push a book across a 6-meter-long table by exerting a force $f(x)$ at each point from $x = 0$ to $x = 6$. What does the definite integral $\int_0^6 f(x)\,dx$ represent? If $f(x)$ is measured in newtons, what are the units for the definite integral?
8. In your own words, describe how to find the hydrostatic force against a vertical wall submersed in a fluid.
9. (a) Explain the physical significance of the center of mass of a thin plate.
   (b) If the plate lies between $y = f(x)$ and $y = 0$, where $a \leq x \leq b$, write expressions for the coordinates of the center of mass.
10. Given a demand function $p(x)$, explain the meaning of consumer surplus when the amount of a commodity currently available is X and the current selling price is P. Illustrate this concept with a sketch.
11. (a) Explain the concept of cardiac output of the heart.
    (b) Explain how the cardiac output can be measured by the dye dilution method.
12. Suppose X is a random variable with probability density function f.
    (a) What properties does f have?
    (b) Explain how f is used to compute $P(a \leq X \leq b)$.
13. Let f be the probability density function for the random variable X, the weight of an adult lion in lb.
    (a) Explain the meaning of the expression $\int_{300}^{400} f(x)\,dx$ in the context of the problem.
    (b) Write an expression involving an integral that represents the mean weight of adult lions.
    (c) Write an expression involving an integral that could be used to find the median weight of adult lions.
14. Suppose X is a normal random variable with mean μ, standard deviation σ, and probability density function f.
    (a) Sketch a graph of f.
    (b) Explain the significance of the standard deviation σ.
15. Explain the general method used to derive a formula involving a definite integral in the applications of integration presented in this chapter.

True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. The area bounded by any two continuous functions f and g on $[a, b]$ is given by

$$A = \int_a^b |f(x) - g(x)|\,dx$$

2. If the region bounded by the graph of $y = kx$, the x-axis, and the line $x = a$ is revolved about the x-axis and about the y-axis, the volumes of the resulting solids are the same.
3. If the region bounded by the graphs of $y = \frac{1}{x}$, $x = 1$, and $x = k$, for $k > 1$, is revolved about the y-axis, the volume of the resulting solid is $2\pi(k - 1)$.

4. For $a < b < c$, if m_1 is the average value of f on $[a, b]$ and m_2 is the average value of f on $[b, c]$, then the average value of f on $[a, c]$ is given by $\dfrac{m_1 + m_2}{2}$.

5. If f is a continuous function on the interval $[a, b]$, then the length L of f from $x = a$ to $x = b$ is greater than or equal to the distance between $(a, f(a))$ and $(b, f(b))$.

Exercises

Find the area of the region bounded by the graphs of the given equations.

1. $y = x^2 \quad y = 4x - x^2$

2. $y = \sqrt{x}, \quad y = -\sqrt[3]{x}, \quad y = x - 2$

3. $y = 1 - 2x^2, \quad y = |x|$

4. $x + y = 0, \quad x = y^2 + 3y$

5. $y = \sin\left(\dfrac{\pi x}{2}\right), \quad y = x^2 - 2x$

6. $y = \sqrt{x}, \quad y = x^2, \quad x = 2$

7. The curve traced out by a point at a distance 1 m from the center of a circle of radius 2 m as the circle rolls along the x-axis is called a *trochoid* and has parametric equations

$$x = 2\theta - \sin\theta \qquad y = 2 - \cos\theta$$

One arch of the trochoid is given by the parameter interval $0 \le \theta \le 2\pi$. Find the area under one arch of this trochoid.

8. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^{-2x}$, $y = 1 + x$, and $x = 1$.

9. Let R be the region bounded by the graphs of $y = \tan(x^2)$, $x = 1$, and $y = 0$. Use the Midpoint Rule with $n = 4$ equal subintervals to estimate the following quantities.
(a) The area of R
(b) The volume of the solid obtained by rotating R about the x-axis

10. Let R be the region in the first quadrant bounded by the graphs of $y = x^3$ and $y = 2x - x^2$.
(a) Find the area of R.
(b) Find the volume of the solid obtained by rotating R about the x-axis.
(c) Find the volume of the solid obtained by rotating R about the y-axis.

11. Find the volume of the solid obtained by rotating the region bounded by the graphs of $y = x$ and $y = x^2$ about the given line.
(a) The x-axis
(b) The y-axis
(c) The line $y = 2$

Find the volume of the solid obtained by rotating the region bounded by the graphs of the given equations about the specified axis.

12. $y = 2x, \quad y = x^2; \qquad$ about the x-axis

13. $x = 1 + y^2, \quad y = x - 3; \qquad$ about the y-axis

14. $x = 0, \quad x = 9 - y^2; \qquad$ about $x = -1$

15. $y = x^2 + 1, \quad y = 9 - x^2; \qquad$ about $y = -1$

16. $x^2 - y^2 = a^2, \quad x = a + h; \qquad$ (where $a > 0$, $h > 0$); about the y-axis

Use technology to find the volume of the solid obtained by rotating the region bounded by the graphs of the given equation about the specified axis.

17. $y = \tan x, \quad y = x, \quad x = 0, \quad x = \dfrac{\pi}{3}; \qquad$ about the y-axis

18. $y = \sqrt{x}, \quad y = x^2; \qquad$ about $y = 2$

19. $y = \cos^2 x, \quad |x| \le \dfrac{\pi}{2}, \quad y = \dfrac{1}{4}; \qquad$ about $x = \dfrac{\pi}{2}$

20. Let R be the region bounded by the graphs of $y = 1 - x^2$ and $y = x^6 - x + 1$. Use technology to find the following quantities.
(a) The x-coordinates of the points of intersection on the graphs
(b) The area of R
(c) The volume of the solid obtained by rotating R about the x-axis
(d) The volume of the solid obtained by rotating R about the y-axis

The integral represents the volume of a solid. Describe the solid.

21. $\displaystyle\int_0^{\pi/2} 2\pi x \cos x \, dx$

22. $\displaystyle\int_0^{\pi/2} 2\pi \cos^2 x \, dx$

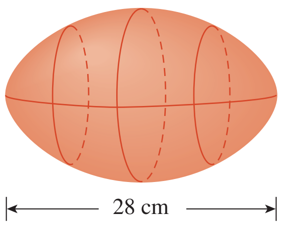
23. $\displaystyle\int_0^{\pi} \pi(2 - \sin x)^2 \, dx$

24. $\displaystyle\int_0^{4} 2\pi(6 - y)(4y - y^2) \, dy$

25. $\displaystyle\int_0^{1} \pi[(2 - x^2)^2 - (2 - \sqrt{x})^2] \, dx$

26. $\displaystyle\int_1^{3} \pi\left[(\ln x + 2)^2 - \left(\cos\left(\frac{\pi x}{2}\right) + 2\right)^2\right] dx$

27. A typical American football is 28 cm long with circumference at its widest point 53 cm. The circumference 7 cm from each end is 45 cm. Use Simpson's Rule to estimate the volume of a football.



28. The base of a solid is a circular disk with radius 3. Find the volume of the solid if parallel cross-sections perpendicular to the base are isosceles right triangles with hypotenuse lying along the base.

29. The base of a solid is the region bounded by the graphs of $y = x^2$ and $y = 2 - x^2$. Find the volume of the solid if the cross-sections perpendicular to the x-axis are squares with one side lying along the base.

30. The height of a monument is 20 m. A horizontal cross-section at a distance x meters from the top is an equilateral triangle with side $\frac{1}{4}x$ meters. Find the volume of the solid.

31. (a) The base of a solid is a square with vertices located at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. Each cross-section perpendicular to the x-axis is a semicircle. Find the volume of the solid.

(b) Show that the volume of the solid in part (a) can be computed more simply by first cutting the solid and rearranging it to form a cone.

32. Find the length of the curve with parametric equations $x = 3t^2$, $y = 2t^3$, $0 \le t \le 2$.

33. Use Simpson's Rule with $n = 10$ to estimate the length of the arc of the curve $y = \dfrac{1}{x^2}$ from $(1, 1)$ to $\left(2, \dfrac{1}{4}\right)$.

34. Find the length of the curve $y = \frac{1}{6}(x^2 + 4)^{3/2}$, $0 \le x \le 3$.

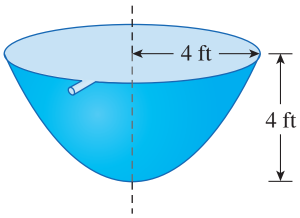
35. Find the length of the curve

$$y = \int_1^x \sqrt{\sqrt{t} - 1}\, dt \qquad 1 \le x \le 16$$

36. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

37. A 1600-lb elevator is suspended by a 200-ft cable that weighs 10 lb/ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft?

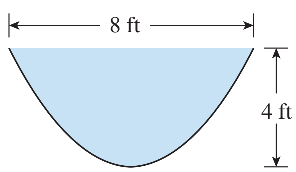
38. A tank full of water has the shape of a paraboloid of revolution as shown in the figure; that is, its shape is obtained by rotating a parabola about a vertical axis.



(a) If its height is 4 ft and the radius at the top is 4 ft, find the work required to pump the water out of the tank.

(b) After 4000 ft-lb of work has been done, what is the depth of the water remaining in the tank?

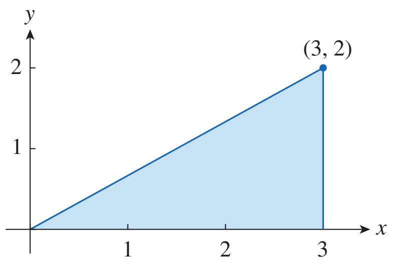
39. A trough is filled with water and its vertical ends have the shape of the parabolic region in the figure.



Find the hydrostatic force on one end of the trough.

40. A gate in an irrigation canal is constructed in the form of a trapezoid 3 ft wide at the bottom, 5 ft wide at the top, and 2 ft high. It is placed vertically in the canal so that the water just covers the gate. Find the hydrostatic force on one side of the gate.

41. Find the centroid of the shaded region in the figure.



42. The demand function for a commodity is given by

$$p(x) = 2000 - 0.1x - 0.01x^2$$

Find the consumer surplus when the sales level is 100.

43. Find the average value of the function $f(x) = x^2\sqrt{1 + x^3}$ on the interval $[0, 2]$.

44. Suppose f is a continuous function. Find the limit as $h \to 0$ of the average value of f on the interval $[x, x + h]$.

45. Suppose the dye dilution method is used to measure a patient's cardiac output. After a 6-mg injection of dye, the dye concentration measurements, in mg/L, at 2-second intervals are shown in the table.

t	$c(t)$	t	$c(t)$
0	0	14	4.7
2	1.9	16	3.3
4	3.3	18	2.1
6	5.1	20	1.1
8	7.6	22	0.5
10	7.1	24	0
12	5.8		

Use Simpson's Rule to estimate the cardiac output.

46. Suppose X is a random variable with probability density function f defined by

$$f(x) = \begin{cases} \frac{\pi}{20}\sin\left(\frac{\pi x}{10}\right) & \text{if } 0 \le x \le 10 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Show that f is a valid probability density function.
(b) Find $P(X < 4)$.
(c) Find the mean of X. Is this the value you expected? Why or why not?

47. The output of a microwave oven, measured in watts, is normally distributed with mean 1000 watts and standard deviation 50 watts. Suppose a microwave oven is selected at random.
(a) Find the probability that the output is between 1100 and 1150 watts.
(b) Find the probability that the output is less than 875 watts.

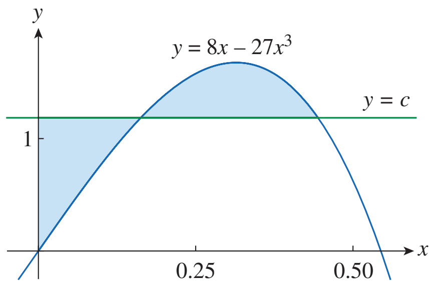
48. The length of time spent waiting in line at a certain bank is modeled by an exponential distribution with mean 8 minutes.
(a) Find the probability that a randomly selected customer waits at most 3 minutes.
(b) Find the probability that a randomly selected customer has to wait more than 10 minutes.
(c) Find the median waiting time.

49. Let R_1 be the region bounded by the graphs of $y = x^2$, $y = 0$, and $x = b$, where $b > 0$. Let R_2 be the region bounded by the graphs of $y = x^2$, $x = 0$, and $y = b^2$.
(a) Find a value b, if it exists, such that R_1 and R_2 have the same area.
(b) Find a value b, if it exists, such that R_1 sweeps out the same volume when rotated about the x-axis and the y-axis.
(c) Find a value b if it exists, such that R_1 and R_2 sweep out the same volume when rotated about the y-axis.

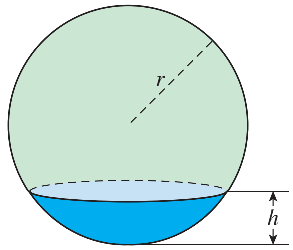
Focus on Problem Solving

1. Let S be the solid obtained by rotating the region under the graph of $y = f(x)$, where f is a positive function and $x \geq 0$, about the x-axis. The volume of S for $0 \leq x \leq b$ is b^2 for all $b > 0$. Find the function f.

2. The figure shows the graph of a horizontal line $y = c$ that intersects the graph of $y = 8x - 27x^3$. Find the number c such that the two shaded regions have equal area.



3. The figure shows a segment of height h of a sphere of radius r.



(a) Show that the volume of this segment is $V = \frac{1}{3}\pi h^2(3r - h)$.

(b) Suppose a sphere of radius 1 is sliced by a plane at a distance x from the center in such a way that the volume of one segment is twice the volume of the other. Show that x is a solution of the equation

$$3x^3 - 9x + 2 = 0$$

where $0 < x < 1$. Use Newton's method to find x accurate to four decimal places.

(c) Using the formula for the volume of a segment of a sphere, it can be shown that the depth to which a floating sphere of radius r sinks in water is a root of the equation

$$x^3 - 3rx^2 + 4r^3s = 0$$

where s is the specific gravity of the sphere. Suppose a wooden sphere of radius 0.5 m has specific gravity of 0.75. Use technology to find the depth to which the sphere will sink.

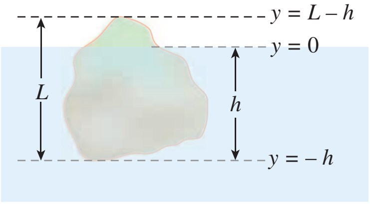
(d) A hemispherical bowl has radius 5 inches and water is running into the bowl at the rate of 0.2 in^3/s.

(i) How fast is the water level in the bowl rising at the instant the water is 3 inches deep?

(ii) At a certain instant, the water is 4 inches deep. How long will it take to fill the bowl?

4. Archimedes' Principle states that the buoyant force on an object partially or fully submerged in a fluid is equal to the weight of the fluid that the object displaces. Therefore, an object (as shown in the figure) of density ρ_0 floating partly submerged in a fluid of density ρ_f has

buoyant force given by $F = \rho_f g \int_{-h}^{0} A(y)\, dy$, where g is the acceleration due to gravity and $A(y)$ is the area of a typical cross-section of the object.



The weight of the object is given by

$$W = \rho_0 g \int_{-h}^{L-h} A(y)\, dy$$

(a) Show that the percentage of the volume of the object above the surface of the liquid is

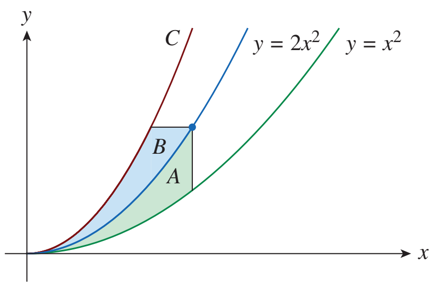
$$100\,\frac{\rho_f - \rho_0}{\rho_f}$$

(b) The density of ice is $917\ \mathrm{kg/m^3}$ and the density of seawater is $1030\ \mathrm{kg/m^3}$. What percentage of volume of an iceberg is above water?
(c) An ice cube floats in a glass filled to the brim with water. Does the water overflow when the ice melts? Justify your answer.
(d) A sphere of radius 0.4 m with negligible weight is floating in a large freshwater lake. How much work is required to completely submerge the sphere? The density of the water is $1000\ \mathrm{kg/m^3}$.

5. Water in an open bowl evaporates at a rate proportional to the area of the surface of the water. (This means that the rate of decrease of the volume is proportional to the area of the surface.) Show that the depth of the water decreases at a constant rate, regardless of the shape of the bowl.

6. A sphere of radius 1 overlaps a smaller sphere of radius r in such a way that their intersection is a circle of radius r. That is, they intersect in a great circle of the small sphere. Find r such that the volume inside the small sphere and outside the large sphere is as large as possible.

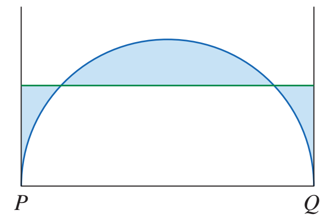
7. The figure shows a curve C with the property that, for every point P on the middle curve $y = 2x^2$, the areas of the regions A and B are equal.



Find an equation for the curve C of the form $y = kx^2$.

8. Let P be a pyramid with a square base of side $2b$ and suppose that S is a sphere with its center on the base of P and S is tangent to all eight edges of P. Find the height of P. Then find the volume of the intersection of S and P.

9. The figure shows a semicircle with radius 1, horizontal diameter PQ, and tangent lines at P and Q.



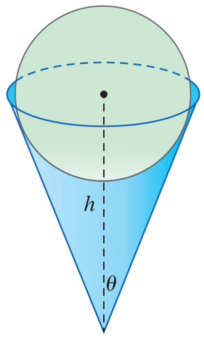
At what height above the diameter should the horizontal line be placed such that the shaded area is a minimum?

10. A curve is defined by the parametric equations

$$x = \int_1^t \frac{\cos u}{u}\,du \qquad y = \int_1^t \frac{\sin u}{u}\,du$$

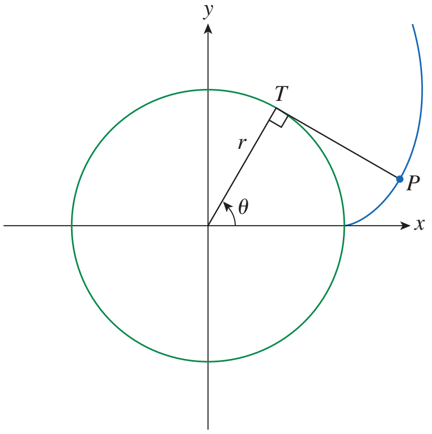
Find the length of the arc of the curve from the origin to the nearest point where there is a vertical tangent line.

11. A paper drinking cup filled with water has the shape of a cone with height h and semivertical angle θ as shown in the figure.



A ball is placed carefully in the cup, thereby displacing some of the water and making it overflow. What is the radius of the ball that causes the greatest volume of water to spill out of the cup?

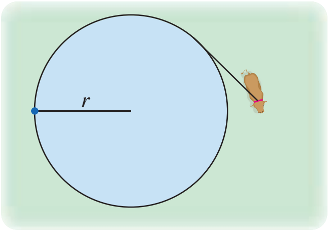
12. A string is wound around a circle and then unwound while being held taut. The curve traced out by the point P at the end of the string is called the involute of the circle.



If the circle has radius r and center at the origin and the initial position of P is $(r, 0)$, and if the parameter θ is chosen as in the figure, show that the parametric equations of the involute are

$$x = r(\cos\theta + \theta\sin\theta) \qquad y = r(\sin\theta - \theta\cos\theta)$$

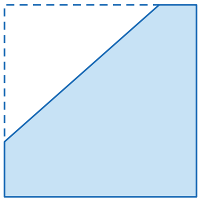
13. A cow is tied to a silo with radius r by a rope just long enough to reach the opposite side of the silo, as shown in the figure.



Find the area available for grazing by the cow.

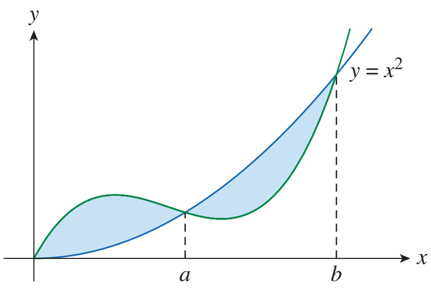
14. A uniform disk with radius 1 m is to be cut by a line so that the center of mass of the smaller piece lies halfway along a radius. How close to the center of the disk should the cut be made?

15. A triangle with area 30 cm^2 is cut from a corner of a square with side 10 cm, as shown in the figure.



If the centroid of the remaining region is 4 cm from the right side of the square, how far is it from the bottom of the square?

16. Suppose the graph of a cubic polynomial intersects the parabola $y = x^2$ when $x = 0$, $x = a$, and $x = b$, where $0 < a < b$ as shown in the figure.



If the two regions between the curves have the same area, how is b related to a?

stocker1970/Shutterstock.com

The Clifton Suspension Bridge is an iconic symbol for the city of Bristol, UK. It spans the Avon Gorge and attracts visitors from all over the world. The shape of the main cable can be modeled by a differential equation involving the weight of the bridge deck, the loads transferred to the main cable, and the tension of the cable at its lowest point.

7 Differential Equations

Contents

Differential equations are used in a wide variety of applications. In biology, differential equations are used to model population growth, the spread of a disease, and predator-prey systems. Geologists are able to model some earthquake ground motion using differential equations. Differential equations can even be used to model the price of a stock and to help assess risk and make predictions. So, it is important to be able to solve differential equations and, in some cases, at least approximate a solution.

7.1 Modeling with Differential Equations

In Section 1.2, we described the process of developing a mathematical model. We discussed the idea of formulating a mathematical model of a real-world problem either through intuitive reasoning about the phenomenon or from a physical law based on experimental evidence. A mathematical model often takes the form of a *differential equation*, that is, an equation that contains an unknown function and one or more of its derivatives. This seems reasonable because in a real-world problem we can often measure changes that occur and we would like to predict future behavior based on how current values change. We will begin by examining several examples of how differential equations arise when we model physical phenomena.

Models of Population Growth

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population. That is a reasonable assumption for a population of bacteria or animals under ideal conditions (unlimited environment, adequate nutrition, absence of predators, immunity from disease). The variables in such a model are

$t =$ time (the independent variable)
$P =$ the number of individuals in the population (the dependent variable)

The rate of growth of the population is the derivative $\dfrac{dP}{dt}$. The assumption that the rate of growth of the population is proportional to the population size is written mathematically as the equation

$$\frac{dP}{dt} = kP \tag{1}$$

where k is the proportionality constant. This is a differential equation because it contains an unknown function P and its first derivative $\dfrac{dP}{dt}$.

Let's interpret this mathematical model and consider the real-world implications. If we exclude a population of 0, then $P(t) > 0$ for all t. If $k > 0$, then Equation 1 shows that $P'(t) > 0$ for all t. This means that the population is always increasing. In fact, as $P(t)$ increases, Equation 1 shows that $\dfrac{dP}{dt}$ becomes larger. In other words, the growth rate increases as the population increases.

Let's consider a solution of Equation 1. We need to find a function whose derivative is a constant multiple of itself. We know that exponential functions have that property. In fact, if we let $P(t) = Ce^{kt}$, then

$$P'(t) = C(ke^{kt}) = k(Ce^{kt}) = kP(t)$$

Therefore, any exponential function of the form $P(t) = Ce^{kt}$ is a solution of Equation 1. In Section 7.4, we will study this equation in detail and learn that there is no other solution.

If we allow C to vary through all the real numbers, we get a *family* of solutions $P(t) = Ce^{kt}$ whose graphs are shown in Figure 7.1. As population has only positive values, we are interested in solutions in which $C > 0$. And we are also only concerned with values of t greater than the initial time $t = 0$. Figure 7.2 shows the physically meaningful solutions. If we let $t = 0$, then $P(0) = Ce^{k(0)} = C$. This means that the constant C is the initial population, $P(0)$.

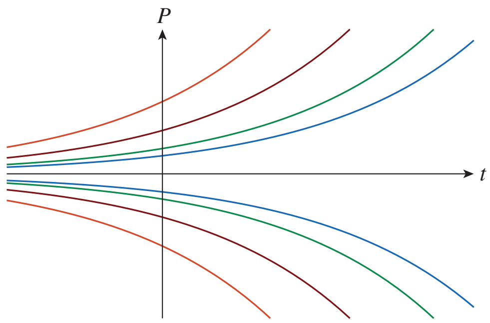


Figure 7.1
Graphs of several members of the family of solutions of $\dfrac{dP}{dt} = kP$.

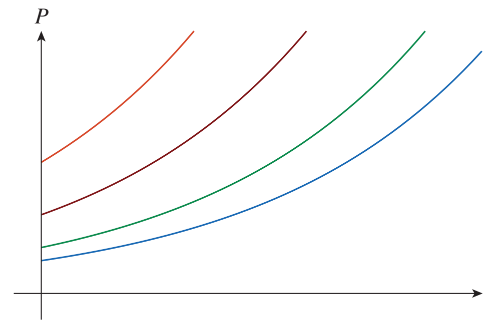


Figure 7.2
Graphs of several members of the family of solutions $P(t) = Ce^{kt}$ with $C > 0$ and $t \geq 0$.

Equation 1 is appropriate for modeling population growth under ideal conditions, but a more realistic model must allow for the fact that a given environment has limited resources. Many populations start by increasing in an exponential manner, but the population levels off when it approaches its *carrying capacity* M (or decreases toward M if it ever exceeds M). A model that accounts for these features has two assumptions:

$$\frac{dP}{dt} \approx kP \qquad \text{if } P \text{ is small (initially, the growth rate is proportional to } P.)$$

$$\frac{dP}{dt} < 0 \qquad \text{if } P > M \ (P \text{ decreases if it ever exceeds } M.)$$

An expression that incorporates both assumptions is given by the equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \tag{2}$$

Notice that if P is small compared with M, then P/M is close to 0 and, therefore, $dP/dt \approx kP$. If $P > M$, then $1 - P/M$ is negative and, therefore, $dP/dt < 0$.

Equation 2 is called the *logistic differential equation* and was proposed by the Dutch biologist Pierre-François Verhulst in the 1840s as a model for world population growth. We will develop techniques that enable us to find explicit solutions of the logistic equation in Section 7.5. For now, let's consider some qualitative characteristics of the solutions directly from Equation 2.

First note that the constant functions $P(t) = 0$ and $P(t) = M$ are solutions to the differential equation because, in either case, one of the factors on the right side of Equation 2 is zero. And this certainly makes physical sense: if the population is ever either 0 or at the carrying capacity, it remains at that value. These two constant solutions are called *equilibrium solutions.*

If the initial population $P(0)$ lies between 0 and M, then the right side of Equation 2 is positive, so $dP/dt > 0$ and the population increases. But if the population exceeds the carrying capacity, $P > M$, then $1 - P/M$ is negative, so $dP/dT < 0$ and the population decreases. Notice that in either case, if the population approaches the carrying capacity ($P \to M$), then $dP/dT \to 0$, which means the population levels off. So, we expect that the solutions to the logistic differential equation have graphs similar to those shown in Figure 7.3. Notice that these graphs move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = M$.

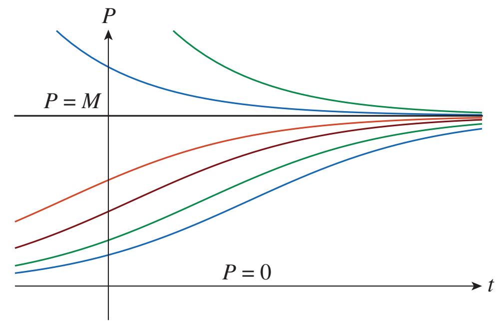


Figure 7.3
Possible graphs of solutions to the logistic equation.

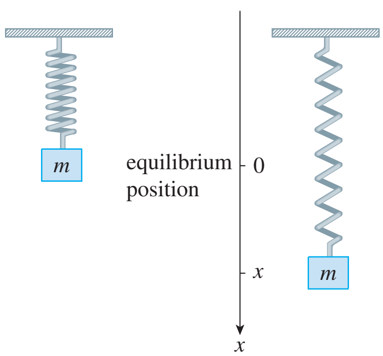


Figure 7.4
An object with mass m at the end of vertical spring.

A Model for the Motion of a Spring

Let's look at a model from the physical sciences. Consider the motion of an object with mass m at the end of a vertical spring, as in Figure 7.4. In Section 6.6, we discussed Hooke's Law, which says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:

$$\text{restoring force} = -kx$$

where k is a positive constant (called the *spring constant*). If we ignore any external resisting forces, for example air resistance and friction, then, by Newton's Second Law of Motion (force equals mass times acceleration), we can express this mathematically with the equation

$$m\frac{d^2x}{dt^2} = -kx \tag{3}$$

This is an example of a *second-order differential equation* because it involves a second derivative. Let's try to guess the form of the solution directly from the differential equation. We can rewrite Equation 3 in the form

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

which is interpreted as the second derivative of x is proportional to x but has the opposite sign. We are very familiar with two functions that exhibit this property, the sine and cosine functions. In fact, all solutions of Equation 3 can be written as combinations of certain sine and cosine functions (see Exercise 5). This seems reasonable since we expect the spring to oscillate about its equilibrium position and it is natural to think trigonometric functions are involved.

General Differential Equations

In general, a **differential equation** is an equation that contains an unknown function and one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that occurs in the equation. Therefore, Equations 1 and 2 are first-order differential equations and Equation 3 is a second-order differential equation. In these three equations, the independent variable is t, which represents time. In general, the variables (independent and dependent) can be represented by any symbols, but are usually associated with symbols relevant to the model as in Equation 3. In addition, the independent and the dependent variables are implied by a differential equation. For example, consider the differential equation

$$y' = xy \tag{4}$$

It is understood that y is an unknown function of x.

A function f is a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation. For example, f is a solution of Equation 4 if

$$f'(x) = xf(x)$$

for all values of x in some specified interval.

To *solve* a differential equation means to find all possible solutions of the equation. We have already solved some routine differential equations, for example, those of the form

$$y' = f(x)$$

For instance, we know that the general solution of the differential equation

$$y' = x^3$$

is given by

$$y = \frac{x^4}{4} + C$$

where C is an arbitrary constant.

In general, it isn't easy to solve a differential equation. There is no systematic technique that enables us to solve all differential equations. However, in Section 7.2 we will learn how to draw rough graphs of solutions even when we have no explicit formula. We will also learn how to find numerical approximations to solutions.

Example 1 Verify a Solution of a Differential Equation

Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$.

Solution

Use the expression for y to find y'.

$$y' = \frac{(1 - ce^t)(ce^t) - (1 + ce^t)(-ce^t)}{(1 - ce^t)^2} \qquad \text{Quotient Rule.}$$

$$= \frac{ce^t - c^2e^{2t} + ce^t + c^2e^{2t}}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2} \qquad \text{Simplify.}$$

Consider the right side of the differential equation.

$$\frac{1}{2}(y^2 - 1) = \frac{1}{2}\left[\left(\frac{1 + ce^t}{1 - ce^t}\right)^2 - 1\right] \qquad \text{Use the expression for } y.$$

$$= \frac{1}{2}\left[\frac{(1 + ce^t)^2 - (1 - ce^t)^2}{(1 - ce^t)^2}\right] \qquad \text{Common denominator.}$$

$$= \frac{1}{2}\frac{4ce^t}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2} \qquad \text{Simplify.}$$

Therefore, for every value of c, the given function is a solution of the differential equation.

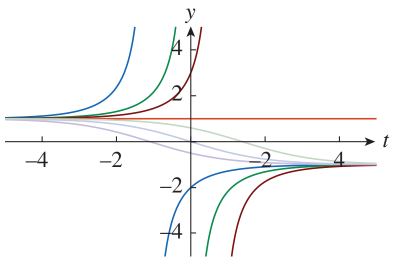


Figure 7.5
Graphs of seven members of the family of solutions.

Figure 7.5 shows the graphs of several members of this family of solutions. The differential equation shows that if $y \approx \pm 1$, then $y' \approx 0$. This observation is demonstrated graphically by the flatness of the graphs near $y = 1$ and $y = -1$. ■

When solving a differential equation, we are usually not as interested in finding a family of solutions (the *general solution*) as we are in finding a solution that satisfies some additional requirement. In many physical problems, we need to find the particular solution that satisfies a condition of the form $y(t_0) = y_0$. This is called an **initial condition**, and the problem of finding a solution of this differential equation that satisfies the initial condition is called an **initial-value problem**.

Geometrically, when we impose an initial condition, we consider the family of solution curves and choose the one that passes through the point (t_0, y_0). Physically, this corresponds to measuring the state of a system at time t_0 and using the solution of the initial-value problem to predict the future behavior of the system.

Example 2 Initial-Value Problem

Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$.

Solution

From Example 1, the family of solutions of this differential equation is

$$y = \frac{1 + ce^t}{1 - ce^t}.$$

Let $t = 0$ and $y = 2$, and solve for c.

$$2 = \frac{1 + ce^0}{1 - ce^0} = \frac{1 + c}{1 - c} \quad \Rightarrow \quad 2 - 2c = 1 + c \quad \Rightarrow \quad c = \frac{1}{3}$$

The solution of the initial-value problem is

$$y = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{3 + e^t}{3 - e^t}.$$ ■

7.1 Exercises

1. Show that $y = \frac{2}{3}e^x + e^{-2x}$ is a solution of the differential equation $y' + 2y = 2e^x$.

2. Show that $y = e^x + \cos 2x$ is a solution of the differential equation

$$y - y' = \cos 2x + 2 \sin 2x$$

3. Verify that $y = -t \cos t - t$ is a solution of the initial-value problem

$$t\frac{dy}{dt} = y + t^2 \sin t \qquad y(\pi) = 0$$

4. Let $y = e^{rx}$.
(a) Find the values of r such that y satisfies the differential equation $2y'' + y' - y = 0$.
(b) Suppose r_1 and r_2 are the values found in part (a). Show that every member of the family of functions $y = ae^{r_1 x} + be^{r_2 x}$ is also a solution of the differential equation.

5. Let $y = \cos kt$.
(a) Find the values of k such that y satisfies the differential equation $4y'' = -25y$.
(b) For the values of k found in part (a), show that every member of the family of functions $y = A \sin kt + B \cos kt$ is also a solution of the differential equation.

6. Determine which of the following functions are solutions of the differential equation $y'' + y = \sin x$.
(A) $y = \sin x$ (B) $y = \cos x$
(C) $y = \frac{1}{2}x \sin x$ (D) $y = -\frac{1}{2}x \cos x$

7. Let $y = \dfrac{\ln x + C}{x}$.
(a) Show that y is a solution of the differential equation $x^2y' + xy = 1$.
(b) Illustrate part (a) by graphing several members of the family of solutions on the same coordinate axes.
(c) Find a solution of the differential equation that satisfies the initial condition $y(1) = 2$.
(d) Find a solution of the differential equation that satisfies the initial condition $y(2) = 1$.

8. (a) Consider the differential equation $y' = -y^2$. What can you conclude about the nature of the solution of this differential equation?
(b) Verify that every member of the family of functions $y = \dfrac{1}{x + C}$ is a solution of the differential equation in part (a).
(c) Can you think of a solution of the differential equation $y' = -y^2$ that is not a member of the family of functions in part (b)?
(d) Find a solution of the initial-value problem

$$y' = -y^2 \qquad y(0) = 0.5$$

9. (a) Consider the differential equation $y' = xy^3$. What can you conclude about the graph of a solution of this differential equation when x is close to 0? What if x is large?
(b) Verify that every member of the family of functions $y = (c - x^2)^{-1/2}$ is a solution of the differential equation in part (a).
(c) Graph several members of the family of solutions on the same coordinate axes. Do these graphs confirm your conclusions in part (a)?
(d) Find a solution of the initial-value problem

$$y' = xy^3 \qquad y(0) = 2$$

10. Suppose a certain population is modeled by the differential equation

$$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

(a) For what values of P is the population increasing?
(b) For what values of P is the population decreasing?
(c) What are the equilibrium solutions to this differential equation?

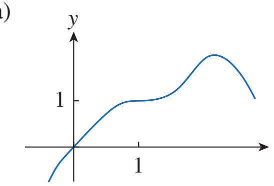
11. Suppose the function $y(t)$ satisfies the differential equation

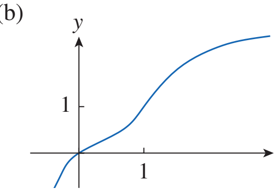
$$\frac{dy}{dt} = y^4 - 6y^3 + 5y^2$$

(a) Find the constant solutions of this differential equation.
(b) For what values of y is y increasing?
(c) For what values of y is y decreasing?

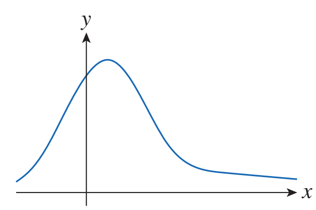
12. Explain why the functions with the given graphs cannot be solutions of the differential equation

$$\frac{dy}{dt} = e^t(y - 1)^2$$

(a)


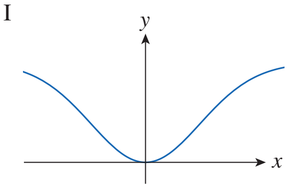
(b)


13. The graph of the function $y = f(x)$ is shown in the figure.



Which of the following differential equations could have y as a solution? Justify your answer.
(A) $y' = 1 + xy$ (B) $y' = -2xy$ (C) $y' = 1 - 2xy$

14. Match each differential equation with the graph of a solution labeled I–IV. Give a reason for each answer.
(a) $y' = 1 + x^2 + y^2$ (b) $y' = xe^{-x^2-y^2}$
(c) $y' = \dfrac{1}{1 + e^{x^2+y^2}}$ (d) $y' = \sin(xy)\cos(xy)$

I


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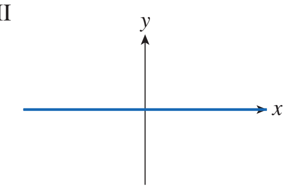
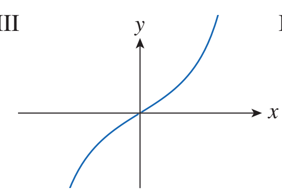
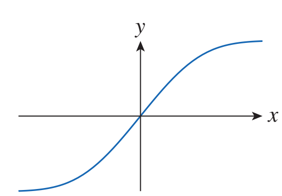


III


IV


15. Suppose a cup of freshly brewed coffee with temperature 95°C is placed in a room where the temperature is 20°C.

(a) When does the coffee cool most quickly? What happens to the rate of cooling as time goes by? Explain your reasoning.

(b) **Newton's Law of Cooling** states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. Write a differential equation that expresses Newton's Law of Cooling for this particular situation. What is the initial condition? Are your answers in part (a) consistent with this differential equation?

(c) Draw a rough sketch of the graph of the solution of the initial-value problem in part (b).

16. Psychologists interested in learning theory study **learning curves**. A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of the training time t. The derivative dP/dt represents the rate at which performance improves.

(a) When do you think P increases most rapidly? What happens to dP/dt as t increases? Explain your reasoning.

(b) If M is the maximum level of performance of which the learner is capable, explain why the differential equation

$$\frac{dP}{dt} = k(M - P) \quad k \text{ a positive constant}$$

is a reasonable model for learning.

(c) Draw a rough sketch of a possible solution of this differential equation.

7.2 Slope Fields and Euler's Method

In general, we cannot find an explicit solution for most differential equations. However, we can still learn a lot about the solution using a graphical approach (slope fields) or a numerical approach (Euler's method).

Slope Fields

Suppose we need to sketch the graph of the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

We don't have a formula for the solution, so we need another way to sketch its graph. Think about what the differential equation means—how to interpret the expression. The equation $y' = x + y$ is a formula for the slope at any point (x, y) on the graph of the solution (called the *solution curve*) which, in this example, is equal to the sum of the x- and y-coordinates of the point on the curve. Figure 7.6 is a graphical interpretation of this idea.

In particular, because the curve passes through the point $(0, 1)$, its slope at this point must be $y' = 0 + 1 = 1$. Because the solution curve is differentiable, it is locally linear. So, near the point $(0, 1)$, the solution curve looks linear, like a straight line segment through $(0, 1)$ with slope 1. See Figure 7.7.

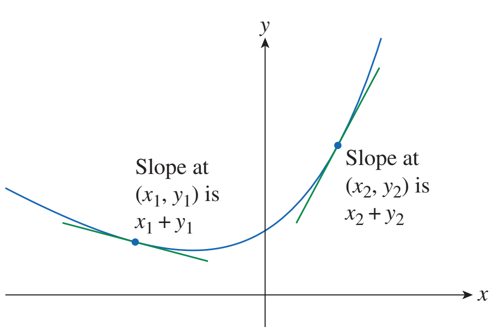


Figure 7.6
The graph of a solution of $y' = x + y$.

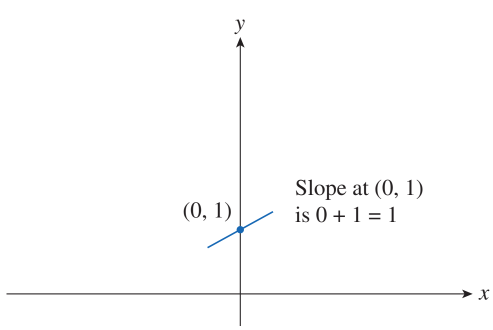


Figure 7.7
Near the point $(0, 1)$, the graph of the solution of the DE is locally linear; it looks like a straight line.

To sketch more of the solution curve, we can draw short line segments at several points (x, y) with slope $x + y$. The result is called a *slope field*, or *direction field*, and is shown in Figure 7.8. For example, the line segment at the point $(1, 2)$ has slope $1 + 2 = 3$ and the line segment at the point $(-2, -1)$ has slope $-2 + (-1) = -3$. The slope field allows us to visualize the general shape of the solution curves by indicating the direction of the curve at each point.

Now we can sketch the solution curve through the point $(0, 1)$ by following the slope field as in Figure 7.9. The solution curve is drawn so that it is parallel to the nearby line segments.

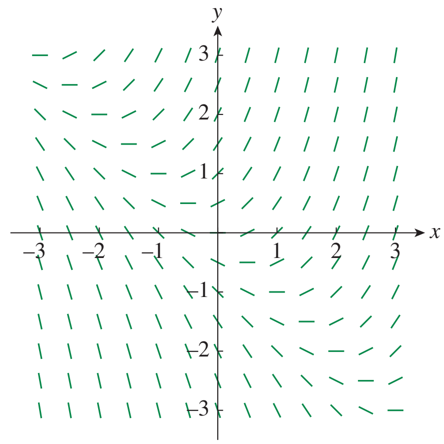


Figure 7.8
Slope field for the DE $y' = x + y$.

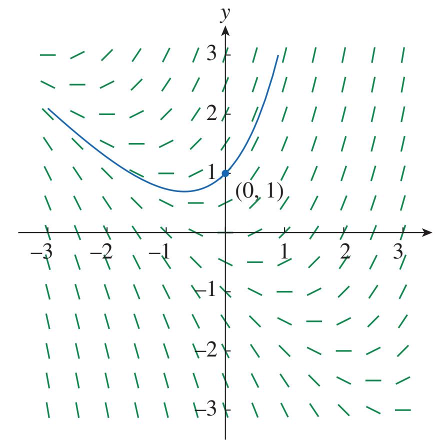


Figure 7.9
The solution curve for the DE through the point $(0, 1)$.

In general, suppose a first-order differential equation is of the form

$$y' = F(x, y)$$

where $F(x, y)$ is an expression in x and y. The differential equation can be interpreted as the slope of a solution curve at a point (x, y) on the curve is $F(x, y)$. If we draw short line segments with slope $F(x, y)$ at several points (x, y), the result is called a **slope field**, or **direction field**. These line segments indicate the direction in which the solution curve is heading. The slope field is a graphical display that shows the flow of tangent lines to the family of solutions of a differential equation and, therefore, helps us visualize the general shape of these curves. The slope field can provide information about the nature of the solution curve, for example, polynomial, exponential, logarithmic, trigonometric, etc.

Example 1 Use a Slope Field to Sketch a Solution Curve

(a) Sketch the slope field for the differential equation $y' = x^2 + y^2 - 1$.
(b) Use part (a) to sketch the solution curve that passes through the origin.

Solution

(a) Start by finding the slope of the solution curve at several points, as in the table.

x-values \ y-values	-2	-1	0	1	2
-2	7	4	3	4	7
-1	4	1	0	1	4
0	3	0	-1	0	3
1	4	1	0	1	4
2	7	4	3	4	7

Draw short line segments with these slopes at the corresponding points. The resulting slope field, based on a larger table, is shown in Figure 7.10.

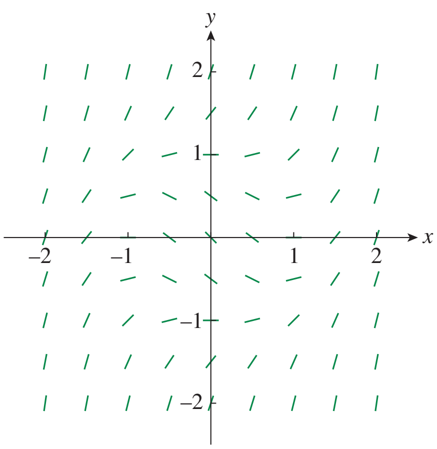


Figure 7.10
Slope field for the DE $y' = x^2 + y^2 - 1$.

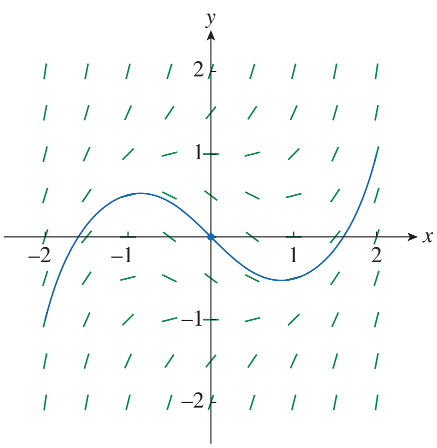


Figure 7.11
The solution curve for the DE through the origin.

(b) Start at the origin and move to the right in the direction of the line segment, which has slope -1.

Continue to draw the solution curve so that it is parallel to the nearby line segments.

Return to the origin, and draw the solution curve to the left in a similar manner. The resulting solution curve is shown in Figure 7.11. ■

The more line segments that are drawn in the slope field, the clearer the picture becomes. It's a little tedious to compute slopes and draw line segments for a large number of points. Technology is very helpful here. Figure 7.12 shows a more detailed slope field for the differential equation in Example 1. It enables us to draw, with reasonable accuracy, the solution curves shown in Figure 7.13 with y-intercepts -2, -1, 0, and 2.

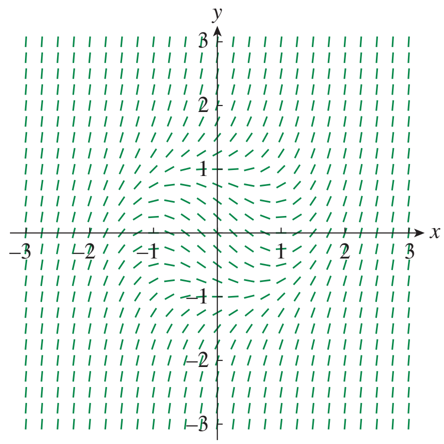


Figure 7.12
A more detailed slope field for the DE
$y' = x^2 + y^2 - 1$.

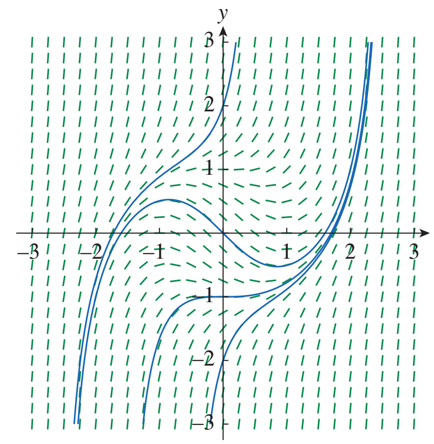


Figure 7.13
Several solution curves.

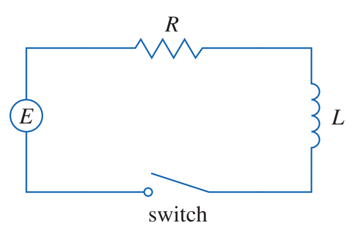


Figure 7.14
Simple electric circuit.

Slope fields can provide insight into many physical situations. Consider the simple electric circuit shown in Figure 7.14, which contains an electromotive force (usually a battery or generator) that produces a voltage of $E(t)$ volts (V) and a current of $I(t)$ amperes (A) at time t. The circuit also contains a resistor with a resistance of R ohms (Ω) and an inductor with an inductance of L henries (H).

Ohm's Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is $L\dfrac{dI}{dt}$. One of Kirchhoff's Laws says that the sum of the voltage drops is equal to the supplied voltage $E(t)$. This leads to the equation

$$L\frac{dI}{dt} + RI = E(t) \tag{1}$$

which is a first-order differential equation that models the current I at time t.

Example 2 Simple Circuit

Suppose that in the simple circuit of Figure 7.14, the resistance is 12 Ω, the inductance is 4 H, and a battery gives a constant voltage of 60 V.

(a) Draw a slope field for Equation 1 with these values.

(b) What is the limiting value of the current $I(t)$?

(c) Identify any equilibrium solutions.

(d) If the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, use the slope field to sketch the solution curve.

Solution

(a) Use $L = 4$, $R = 12$, and $E(t) = 60$ in Equation 1.

$$4\frac{dI}{dt} + 12I = 60 \quad \text{or} \quad \frac{dI}{dt} = 15 - 3I$$

The slope field for this differential equation is shown in Figure 7.15.

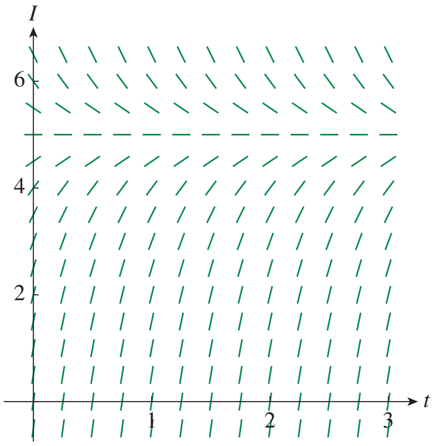


Figure 7.15

Slope field for the DE $\dfrac{dI}{dt} = 15 - 3I$.

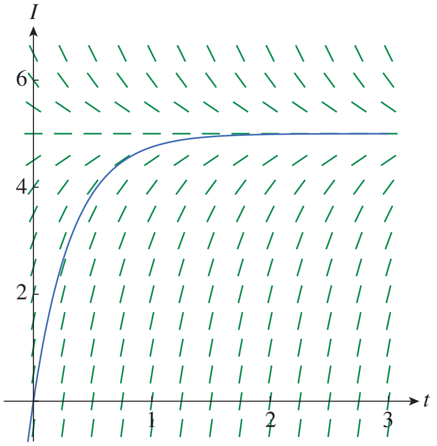


Figure 7.16

The solution curve that passes through (0, 0).

(b) It appears from the slope field that all solutions approach the value 5. Therefore, $\lim_{t\to\infty} I(t) = 5$.

(c) It appears that the constant function $I(t) = 5$ is an equilibrium solution.

We can verify this directly from the differential equation $\dfrac{dI}{dt} = 15 - 3I$.

If $I(t) = 5$, then the left side is $\dfrac{dI}{dt} = 0$ and the right side is $15 - 3(5) = 0$.

(d) Use the slope field to sketch the solution curve that passes through (0, 0), as shown in Figure 7.16. ■

Notice that in Figure 7.15 the line segments along any horizontal line are parallel. That is because the independent variable t does not occur on the right side of the equation $I' = 15 - 3I$. In general, a differential equation of the form $y' = f(y)$ in which the independent variable does not occur on the right side, is called **autonomous**. In an autonomous equation, the slopes corresponding to two different points with the same y-coordinate must be equal. This means, given one solution of an autonomous differential equation, that we can obtain infinitely many others just by shifting the graph of the known solution to the right or left.

Euler's Method

The basic concept used to generate slope fields can be used to find a numerical approximation to the solution of a differential equation. Consider the following initial value problem, used to introduce slope fields:

$$y' = x + y \qquad y(0) = 1$$

Using the differential equation: $y'(0) = 0 + 1$. Therefore, the solution curve has slope 1 at the point (0, 1). As a first approximation to the solution, we could use the linear approximation $L(x) = 1 + (1)(x - 0) = 1 + x$. That is, we could use the tangent line to the graph of the solution at the point (0, 1) as a rough approximation to the solution curve. See Figure 7.17.

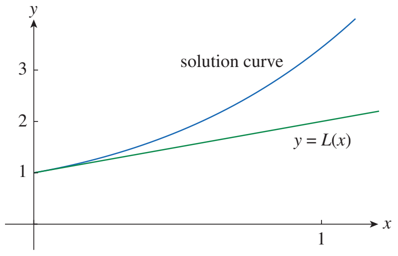


Figure 7.17
Graph of the solution curve and the first Euler approximation, the tangent line to the solution curve at the initial value.

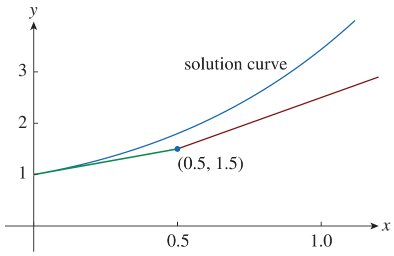


Figure 7.18
The second step in an Euler approximation.

Euler's method improves on this approximation by proceeding only a short distance along this tangent line and then by changing direction as indicated by the slope field. Figure 7.18 shows what happens if we start out along the tangent line but stop when $x = 0.5$ (and change direction). This horizontal distance traveled is called the *step size*.

Since $L(0.5) = 0.5 + 1 = 1.5$, then $y(0.5) \approx 1.5$. Take (0.5, 1.5) as the starting point for a new line segment, that is, a new tangent line. Using the differential equation, the slope of the new line is $y'(0.5) = 0.5 + 1.5 = 2$. Use the new linear function

$$y = 1.5 + 2(x - 0.5) = 0.5 + 2x$$

as an approximation to the solution for $x > 0.5$ (the line segment to the right of the point (0.5, 1.5) in Figure 7.18). Note that if we decrease the step size from 0.5 to 0.25, it seems reasonable that Euler's method would produce a better approximation.

In general, Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the slope field. Stop after a short distance, look at the slope at the new location, and proceed in that new direction. Continue stopping and changing direction according to the slope field.

Euler's method does not produce the exact solution of an initial-value problem; it provides an approximation. By decreasing the step size, and therefore increasing the number of direction changes, we obtain better approximations to the exact solution. Figure 7.19 illustrates Euler's method with a step size of 0.25.

Here's how we generalize this method. Consider the first-order initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$. The goal is to find approximate values for the solution at equally spaced numbers x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h, \ldots,$ where h is the step size. The differential equation is used to find the slope at (x_0, y_0): $y' = F(x_0, y_0)$.

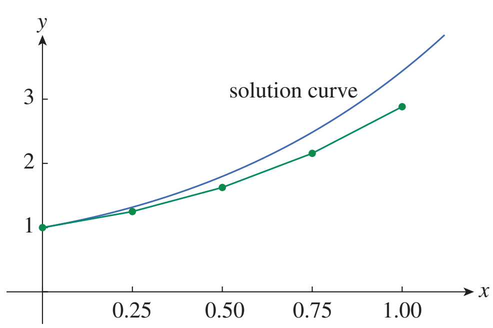


Figure 7.19
Euler approximations with step size 0.25.

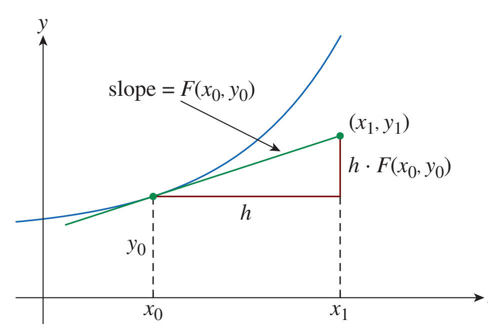


Figure 7.20
The first step in Euler's method.

Figure 7.20 shows that the approximate value of the solution when $x = x_1$ is

$$y_1 = y_0 + h \cdot F(x_0, y_0)$$

Similarly, $\quad y_2 = y_1 + h \cdot F(x_1, y_1)$

And, in general $\quad y_n = y_{n-1} + h \cdot F(x_{n-1}, y_{n-1})$

Euler's Method

Approximate values for the solution of the initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, with step size h, at $x_n = x_{n-1} + h$, are

$$y_n = y_{n-1} + h \cdot F(x_{n-1}, y_{n-1}) \qquad n = 1, 2, 3, \ldots$$

A Closer Look

1. It is possible for the step size to be negative in Euler's method. For example, h could be -0.25. In this case, Euler's method is used to approximate values for the solution of the initial-value problem to the left of the initial value.
2. For more accurate approximations, decrease the step size.
3. Using alternative notation may make it easier to remember and utilize Euler's method. Let the step size $h = \Delta x$, and use Leibniz notation to represent the derivative: $y_n = y_{n-1} + \Delta x \cdot \dfrac{dy}{dx}$.

Example 3 Euler's Method with Step Size 0.1

Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

Solution

We are given $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x + y$.

Use Euler's method.

$y_1 = y_0 + h \cdot F(x_0, y_0) = 1 + 0.1 \cdot (0 + 1) = 1.1$

$y_2 = y_1 + h \cdot F(x_1, y_1) = 1.1 + 0.1 \cdot (0.1 + 1.1) = 1.22$

$y_3 = y_2 + h \cdot F(x_2, y_2) = 1.22 + 0.1 \cdot (0.2 + 1.22) = 1.362$

Here's the interpretation: If $y(x)$ is the exact solution of the initial-value problem, then $y(0.3) \approx 1.362$.

Continue to use Euler's method to obtain Table 7.1.

n	x_n	y_n	n	x_n	y_n
1	0.1	1.100000	6	0.6	1.943122
2	0.2	1.220000	7	0.7	2.197434
3	0.3	1.362000	8	0.8	2.487178
4	0.4	1.528200	9	0.9	2.815895
5	0.5	1.721020	10	1.0	3.187485

Table 7.1
Approximate values of $y(x)$ using Euler's method with step size $h = 0.1$.

For a more accurate table of values in Example 3, we could use a smaller step size. But for a large number of small steps, the amount of computation is considerable, and therefore, it is best to use technology. Table 7.2 shows the results of applying Euler's method with decreasing step size to the initial-value problem in Example 6.

Notice that Euler's method estimates in Table 7.2 seem to be approaching limits, namely, the true values of $y(0.5)$ and $y(1)$. Figure 7.21 shows the graphs of Euler

Step size	Euler estimate of $y(0.5)$	Euler estimate of $y(1)$
0.500	1.500000	2.500000
0.250	1.625000	2.882813
0.100	1.721020	3.187485
0.050	1.757789	3.306595
0.020	1.781212	3.383176
0.010	1.789264	3.409628
0.005	1.793337	3.423034
0.001	1.796619	3.433848

Table 7.2
Approximate values of $y(x)$ using Euler's method with decreasing step size.

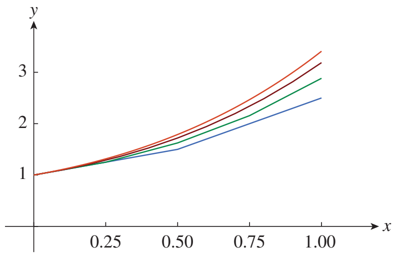


Figure 7.21
As the step size decreases, the graph of the Euler approximation approaches the exact solution curve.

approximations with step sizes 0.5, 0.25, 0.1, and 0.01, These graphs are approaching the exact solution curve as the step size h approaches 0.

Example 4 Use Euler's Method to Estimate Current

In Example 2, we discussed a simple electric circuit with resistance 12 Ω, inductance 4 H, and a battery with voltage 60 V. If the switch is closed when $t = 0$, we modeled the current I at time t by the initial-value problem

$$\frac{dI}{dt} = 15 - 3I \qquad I(0) = 0$$

Estimate the current in the circuit half a second after the switch is closed.

Solution

Use Euler's method with $F(t, I) = 15 - 3I$, $t_0 = 0$, $I_0 = 0$, and step size $h = 0.1$ second:

$$I_1 = 0 + 0.1(15 - 3 \cdot 0) = 1.5$$

$$I_2 = 1.5 + 0.1(15 - 3 \cdot 1.5) = 2.55$$

$$I_3 = 2.55 + 0.1(15 - 3 \cdot 2.55) = 3.285$$

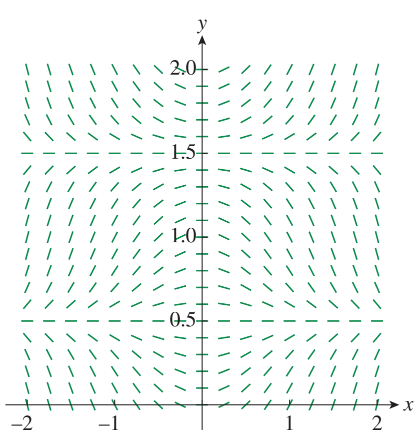
$$I_4 = 3.285 + 0.1(15 - 3 \cdot 3.285) = 3.7995$$

$$I_5 = 3.7995 + 0.1(15 - 3 \cdot 3.7995) = 4.15965$$

Therefore, the current after 0.5 s is approximately $I(0.5) \approx 4.16$ A. ■

7.2 Exercises

1. A slope field for the differential equation $y' = x \cos \pi y$ is shown in the figure.

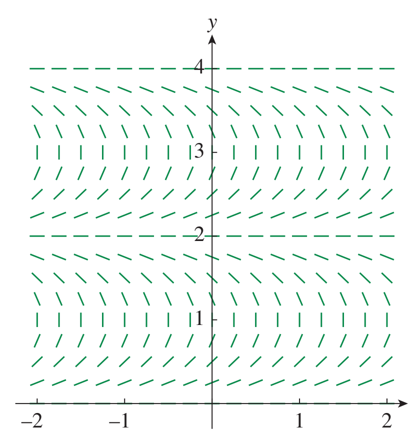


(a) Sketch the graph of the solution that satisfies the given condition.

(i) $y(0) = 0$ (ii) $y(0) = 0.5$
(iii) $y(0) = 1$ (iv) $y(0) = 1.6$

(b) Find all equilibrium solutions.

2. A slope field for the differential equation $y' = \tan\left(\dfrac{\pi y}{2}\right)$ is shown in the figure.

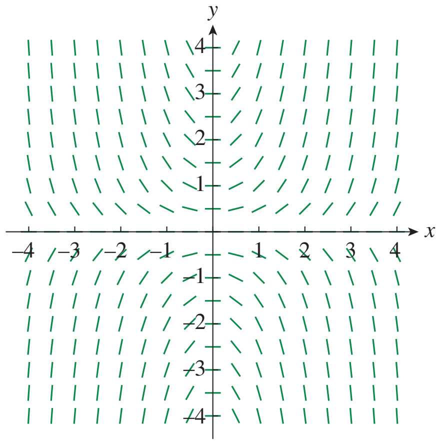


(a) Sketch the graph of the solution that satisfies the given condition.

(i) $y(0) = 1.5$ (ii) $y(0) = 0.2$
(iii) $y(0) = 2$ (iv) $y(0) = 2.5$

(b) Find all equilibrium solutions.

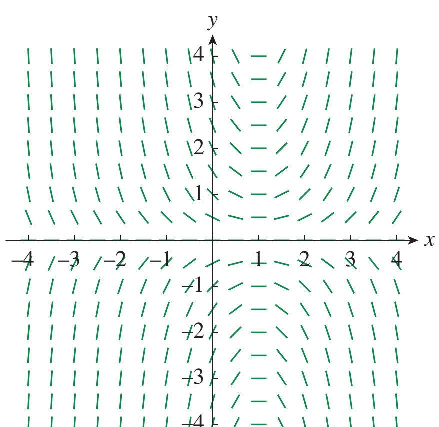
3. A slope field for the differential equation $\dfrac{dy}{dx} = F(x, y)$ is shown in the figure.



Which of the following statements is true?

(A) F does not depend on x.
(B) F does not depend on y.
(C) $F > 0$ for all $x > 0$.
(D) $F = 0$ for some x in the interval $(-1, 1)$.

4. A slope field for a differential equation is shown in the figure.

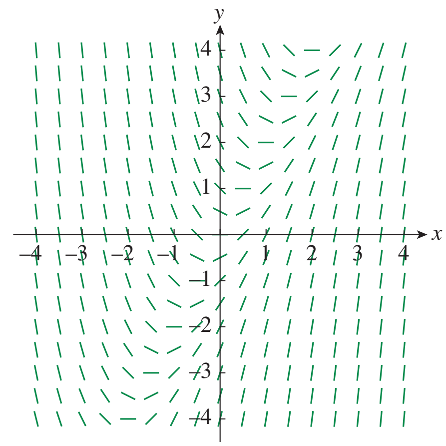


Which of the following could be the general solution? Explain your reasoning.

(A) $y = (x-1)^2 + C$ (B) $y = Ce^{x^2/2 - x}$

(C) $y = \dfrac{x-1}{x} + C$ (D) $y = x^2 - x + C$

5. A slope field for a differential equation is shown in the figure.



Which of the following differential equations could have this slope field? Explain your reasoning.

(A) $\dfrac{dy}{dx} = 2 - x$ (B) $\dfrac{dy}{dx} = x - 2y$

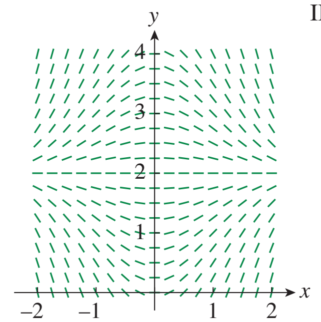
(C) $\dfrac{dy}{dx} = 2x - y$ (D) $\dfrac{dy}{dx} = y - 2x$

Match the differential equation with its slope field (labeled I–IV). Give reasons for your answers.

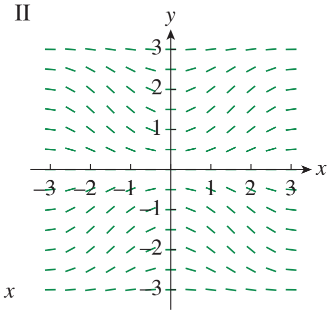
6. $y' = 2 - y$ **7.** $y' = x(2 - y)$

8. $y' = x + y - 1$ **9.** $y' = \sin x \sin y$

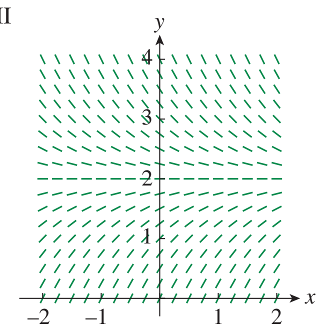
I



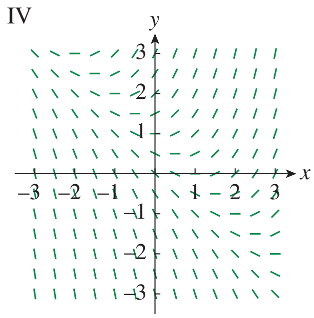
II



III



IV



10. Use the slope field labeled II (in Exercises 6–9) to sketch the graph of the solution that satisfies the given initial condition.
(a) $y(0) = 1$ (b) $y(0) = 2$ (c) $y(0) = -1$

11. Use the slope field labeled IV (in Exercises 6–9) to sketch the graph of the solution that satisfies the given initial condition.
(a) $y(0) = -1$ (b) $y(0) = 0$ (c) $y(0) = 1$

Sketch a slope field for the differential equation. Use the slope field to sketch three solution curves.

12. $y' = \frac{1}{2}y$ **13.** $y' = x - y + 1$

Sketch the slope field for the differential equation. Use the slope field to sketch a solution curve that passes through the given point.

14. $y' = y - 2x$, $(1, 0)$ **15.** $y' = xy - x^2$, $(0, 1)$

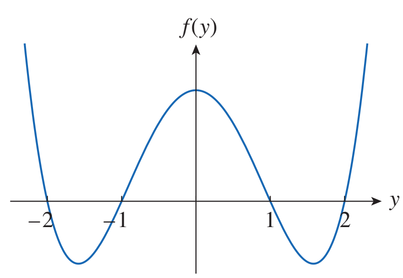
16. $y' = y + xy$, $(0, 1)$ **17.** $y' = x + y^2$, $(0, 0)$

Use technology to draw a slope field for the given differential equation. Sketch the solution curve that passes through (0, 1) on the slope field. Use technology to draw the solution curve and compare it with your sketch.

18. $y' = x^2y - \frac{1}{2}y^2$ **19.** $y' = \cos(x + y)$

20. Use technology to draw a slope field for the differential equation $y' = y^3 - 4y$. Sketch the solution curve that satisfies the initial condition $y(0) = c$ for various values of c on the slope field. For what values of c does $\lim_{t\to\infty} y(t)$ exist? What are the possible values for this limit?

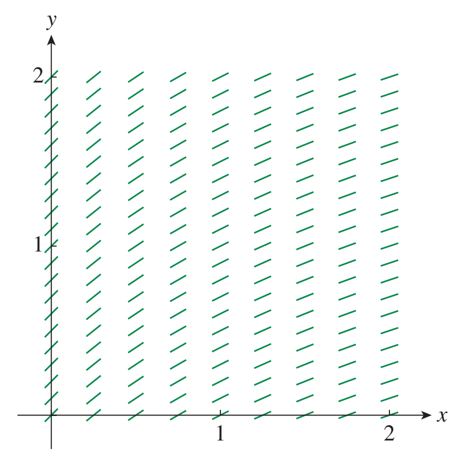
21. Make a rough sketch of a slope field for the autonomous differential equation $y' = f(y)$, where the graph of f is shown in the figure.



How does the limiting behavior of the solutions depend on the value of $y(0)$?

22. Consider the initial-value problem $y' = y$, $y(0) = 1$.
(a) Use Euler's method with each of the following step sizes to estimate the value of $y(0.4)$.
(i) $h = 0.4$ (ii) $h = 0.2$ (iii) $h = 0.1$
(b) The exact solution of the initial-value problem in part (a) is $y = e^x$. Draw the graph of $y = e^x$, $0 \le x \le 0.4$, together with the Euler approximations using the step sizes in part (a). Use your sketches to determine whether your estimates in part (a) are underestimates or overestimates.
(c) The error in Euler's method is the difference between the exact value and the approximate value. Find the errors made in part (a) in using Euler's method to estimate the true value of $y(0.4)$, namely, $e^{0.4}$. What happens to the error each time the step size is halved?

23. A slope field for a differential equation is shown in the figure.



Draw the graph of the Euler approximation to the solution curve that passes through the origin. Use step sizes $h = 1$ and $h = 0.5$. Will the Euler estimates be underestimates or overestimates? Explain your reasoning.

24. Use Euler's method with step size 0.5 to compute the approximate y-values y_1, y_2, y_3, and y_4 of the solution of the initial value problem $y' = y - 2x$, $y(1) = 0$.

25. Use Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial-value problem $y' = xy - x^2$, $y(0) = 1$.

26. Use Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial-value problem
$y' = x^2y - \frac{1}{2}y^2$, $y(0) = 1$.

27. Use Euler's method with step size 0.1 to estimate $y(0.5)$, where $y(x)$ is the solution of the initial-value problem $y' = y + xy$, $y(0) = 1$.

28. Let $y = f(x)$ be the solution of the differential equation $\frac{dy}{dx} = x + y$ with initial condition $f(0) = 1$. Use Euler's method with two equal steps to estimate the value of $f(1)$.

29. Let $y = f(x)$ be the solution of the differential equation $\dfrac{dy}{dx} = 4x(1 + y^2)$ with initial condition $f(1) = 0$. Use Euler's method with two equal steps to estimate the value of $f\left(\dfrac{1}{2}\right)$.

30. Let $y = f(x)$ be the solution of the differential equation $\dfrac{dy}{dx} = \dfrac{0.36x}{y}$ with initial condition $f(0) = 6$. Use Euler's method with three equal steps to estimate the value of $f(0.6)$.

31. (a) Let $y(x)$ be the solution of the initial-value problem $y' = \cos(x + y)$, $y(0) = 0$. Use Euler's method with step size 0.2 to estimate $y(0.6)$.
(b) Repeat part (a) with step size 0.1.

32. Let $y(x)$ be the solution of the initial-value problem

$$\frac{dy}{dx} + 3x^2y = 6x^2 \qquad y(0) = 3$$

(a) Use Euler's method and technology to estimate $y(1)$ with each step size.
(i) $h = 1$ (ii) $h = 0.1$
(iii) $h = 0.01$ (iv) $h = 0.001$
(b) Verify that $y = 2 + e^{-x^3}$ is the exact solution of the differential equation.
(c) Find the errors in using Euler's method to compute $y(1)$ with the step sizes in part (a). What happens to the error when the step size is divided by 10?

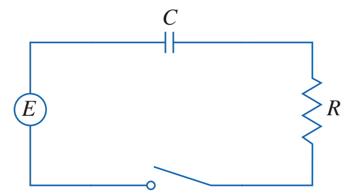
33. Let $y(x)$ be the solution of the initial-value problem

$$y' = x^3 - y^3 \qquad y(0) = 1$$

(a) Use Euler's method with step size 0.01 to estimate $y(2)$.
(b) Use technology to draw the solution curve.

34. Suppose $y = f(x)$ is the solution of the initial-value problem $\dfrac{dy}{dx} = 2x - y$, $f(0) = 2$.
(a) The function f has a critical value at $x = \ln 2$. What is the y-coordinate corresponding to this critical value?
(b) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y. Use $\dfrac{d^2y}{dx^2}$ to determine whether the critical value in part (a) corresponds to a relative minimum, a relative maximum, or neither. Justify your answer.
(c) Use Euler's method with two steps of equal size to approximate $f(-0.4)$. Is this approximation an underestimate or an overestimate? Explain your reasoning?

35. The figures shows a circuit containing an electromotive force, a capacitor with capacitance of C farads (F), and a resistor with resistance R ohms (Ω).



The voltage drop across the capacitor is Q/C, where Q is the charge (in coulombs). Using Kirchhoff's Law, we have

$$RI + \frac{Q}{C} = E(t)$$

But $I = \dfrac{dQ}{dt}$, so the equation can be rewritten as

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Suppose the resistance is 5 Ω, the capacitance is 0.05 F, and a battery provides a constant voltage of 60V.
(a) Draw a slope field for this differential equation.
(b) What is the limiting value of the charge?
(c) Is there an equilibrium solution?
(d) If the initial charge is $Q(0) = 0$ C, use the slope field to sketch the solution curve.
(e) If the initial charge is $Q(0) = 0$ C, use Euler's method with step size 0.1 to estimate the charge after half a second.

36. Suppose you have just poured a cup of freshly brewed coffee with temperature 95°C in a room where the temperature is 20°C.
(a) Write a differential equation that expresses Newton's Law of Cooling for this particular situation. What is the initial condition?
(b) Suppose it is known that the coffee cools at a rate of 1°C per minute when its temperature is 70°C. What does the differential equation become in this case?
(c) Sketch a slope field and use it to sketch the solution curve of the initial-value problem. What is the limiting value of the temperature?
(d) Use Euler's method with step size $h = 2$ minutes to estimate the temperature of the coffee after 10 minutes.

7.3 Separable Equations

We have studied first-order differential equations from a geometric point of view (slope fields) and from a numerical point of view (Euler's method). In this section, we will consider a symbolic perspective. It would be ideal to have an explicit formula for a solution of a differential equation. Unfortunately, this is not always possible. But in this section, we examine a certain type of differential equation that *can* be solved explicitly.

A **separable equation** is a first-order differential equation in which the expression for $\dfrac{dy}{dx}$ can be factored as a function of x times a function of y. That is, the differential equation can be written in the form

$$\frac{dy}{dx} = g(x)\, f(y)$$

The term *separable* follows from the fact that the expression on the right side can be separated into a function of x and a function of y. If $f(y) \neq 0$, then we can write the differential equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \tag{1}$$

where $h(y) = \dfrac{1}{f(y)}$. To solve this equation, we rewrite it in the differential form

Treat dx and dy as differentials, just like any other variable.

$$h(y)\, dy = g(x)\, dx$$

so that all y's are on one side of the equation and all x's are on the other side. Then we integrate both sides of the equation.

$$\int h(y)\, dy = \int g(x)\, dx \tag{2}$$

Equation 2 defines y implicitly as a function of x. However, in some cases, we may be able to solve explicitly for y in terms of x.

The chain rule is used to justify this procedure. If h and g satisfy Equation 2, then

$$\frac{d}{dx}\left(\int h(y)\, dy\right) = \frac{d}{dx}\left(\int g(x)\, dx\right) \qquad \text{Differentiate both sides with respect to } x.$$

so $$\frac{d}{dy}\left(\int h(y)\, dy\right) \cdot \frac{dy}{dx} = g(x) \qquad \text{Chain Rule; FTC.}$$

and $$h(y)\,\frac{dy}{dx} = g(x) \qquad \text{FTC.}$$

Therefore, Equation 1 is satisfied.

Example 1 Solve a Separable Equation

(a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Solution

(a) Separate the variables, write the equation in terms of differentials, and integrate both sides.

$$y^2\,dy = x^2\,dx$$ Separate the variables.

$$\int y^2\,dy = \int x^2\,dx$$ Integrate both sides.

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$ Integrate the left side with respect to y and the right side with respect to x, as indicated by the differentials.

where C is an arbitrary constant.

Note that we could use a constant C_1 on the left side and another constant C_2 on the right side. But we could then combine these constants by writing $C = C_2 - C_1$.

Solve this expression explicitly for y: $y = \sqrt[3]{x^3 + 3C}$.

We could leave the solution in this form, or we could write it in the form $y = \sqrt[3]{x^3 + K}$ where $K = 3C$. Since C is an arbitrary constant, so is K.

(b) Let $x = 0$ in the general solution in part (a): $y(0) = \sqrt[3]{K} = 2$.

Solve this equation for K: $K = 2^3 = 8$.

Therefore, the solution of the initial-value problem is $y = \sqrt[3]{x^3 + 8}$.

Note: It is often easier to determine the constant of integration before writing the solution as an explicit function for y. For example,

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$ Integrate both sides; include constant of integration.

$$\frac{1}{3}(2)^3 = \frac{1}{3}(0)^3 + C$$ Use the initial condition: $x = 0$ and $y = 2$.

$$\frac{8}{3} = C$$ Solve for C.

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + \frac{8}{3}$$ Use the value of C in the equation connecting x and y.

$$y = \sqrt[3]{x^3 + 8}.$$ Solve explicitly for y.

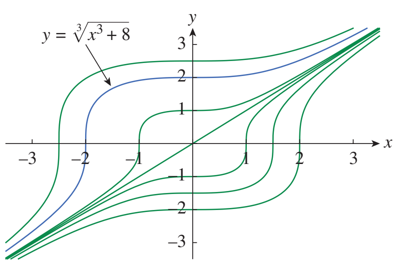


Figure 7.22
The graph of the solution of the initial-value problem in part (b) is shown in blue. Other graphs are of members of the family of solutions.

Figure 7.22 shows the graphs of several members of the family of solutions of the differential equation. ■

Example 2 A Separable Equation with an Implicit Solution

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$.

Solution

Write the equation in differential form, separate the variables, and integrate both sides.

$$(2y + \cos y)\,dy = 6x^2\,dx$$ Separate the variables.

$$\int (2y + \cos y)\,dy = \int 6x^2\,dx$$ Integrate both sides.

$$y^2 + \sin y = 2x^3 + C$$ Antiderivatives.

In this final equation, C is a constant and y is defined implicitly. In this case, it is not possible to explicitly solve for y in terms of x.

Some technology applications can plot curves defined by an implicit equation. Figure 7.23 shows the graphs of several members of the family of solutions to this differential equation.

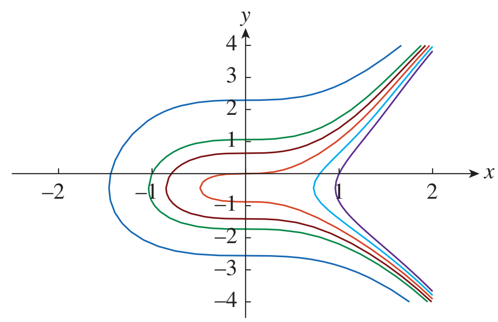


Figure 7.23
The graphs of several members of the family of solutions to the differential equation.

Example 3 Implicit to Explicit Solution

Solve the differential equation $y' = x^2 y$.

Solution

Use Leibniz notation, write the equation in differential form, separate the variables, and integrate both sides.

$$\frac{dy}{dx} = x^2 y \qquad \text{Leibniz notation.}$$

$$\frac{dy}{y} = x^2\, dx \qquad \text{Separate the variables; } y \neq 0.$$

$$\int \frac{dy}{y} = \int x^2\, dx \qquad \text{Integrate both sides.}$$

$$\ln|y| = \frac{x^3}{3} + C \qquad \text{Antiderivatives.}$$

This equation defines y implicitly as a function of x. But in this case, we can solve explicitly for y:

$$|y| = e^{\ln|y|} = e^{(x^3/3)+C} = e^C e^{x^3/3} \quad \Rightarrow \quad y = \pm e^C e^{x^3/3}.$$

We can verify that the function $y = 0$ is also a solution of the given differential equation. So, we can write the general solution in the form

$y = Ae^{x^3/3}$, where A is an arbitrary constant ($A = e^C$, or $A = -e^C$, or $A = 0$).

Figure 7.24 shows the slope field for the differential equation and Figure 7.25 shows the graphs of several solutions for various values of A.

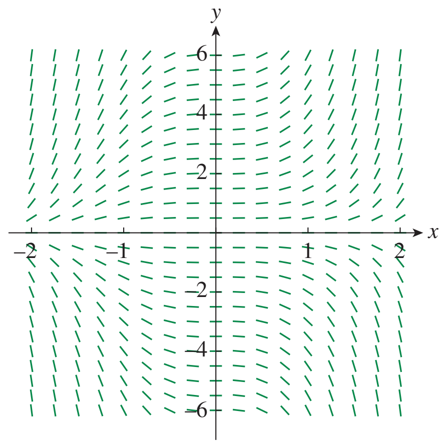


Figure 7.24
The slope field for the differential equation.

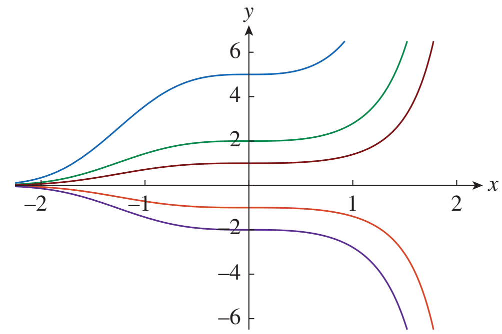


Figure 7.25
Graphs of several solutions.

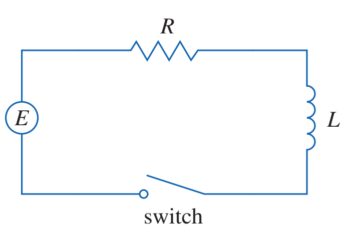


Figure 7.26
Simple electric circuit.

Example 4 Find the Current in a Circuit by Solving a Separable Equation

In Section 7.2, we modeled the current $I(t)$ in the electric circuit shown in Figure 7.26 by the differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

Find an expression for the current in a circuit where the resistance is 12 Ω, the inductance is 4 H, a battery provides a constant voltage of 60 V, and the switch is turned on when $t = 0$. What is the limiting value of the current?

Solution

Use $L = 4$, $R = 12$, and $E(t) = 60$ in the differential equation.

$$4\frac{dI}{dt} + 12I = 60 \quad \Rightarrow \quad \frac{dI}{dt} = 15 - 3I$$

The initial-value problem is $\dfrac{dI}{dt} = 15 - 3I \quad I(0) = 0$.

This is a separable differential equation; solve this in the usual way.

$$\int \frac{dI}{15 - 3I} = \int dt$$ Separate the variables; integrate both sides.

$$-\frac{1}{3}\ln|15 - 3I| = t + C$$ u-substitution; antiderivatives.

Because $I(0) = 0$ and $I(t)$ is continuous, $15 - 3I \geq 0$ and $|15 - 3I| = 15 - 3I$.

$$-\frac{1}{3}\ln|15 - 3(0)| = 0 + C \quad \Rightarrow \quad C = -\frac{1}{3}\ln 15$$ Use initial condition; solve for C.

$$-\frac{1}{3}\ln(15 - 3I) = t - \frac{1}{3}\ln 15$$ Use the value for C.

$$\ln(15 - 3I) = -3t + \ln 15$$ Multiply by -3.

$$I = 5 - 5e^{-3t}$$ Solve for I.

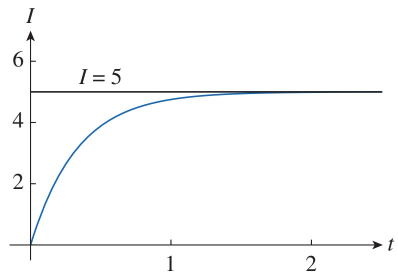


Figure 7.27
Graph of the solution curve as it approaches its limiting value.

Find the limiting current.

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty}(5 - 5e^{-3t}) = 5 - 5\lim_{t\to\infty} e^{-3t} = 5 - 0 = 5$$

Figure 7.27 shows a graph of this solution curve. ■

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family at right angles, that is, orthogonally. (See Figure 7.28.) For example, each member of the family $y = mx$ of straight lines through the origin is an orthogonal trajectory of the family $x^2 + y^2 = r^2$ of concentric circles with center at the origin (see Figure 7.29). The two families are orthogonal trajectories of each other.

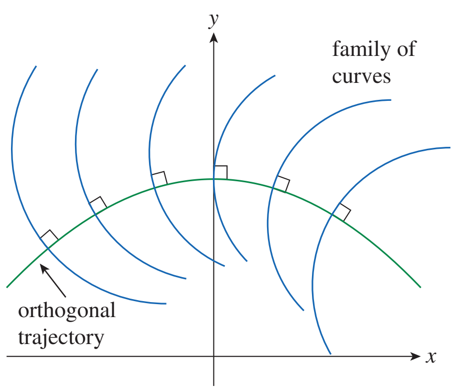


Figure 7.28
An example of a family of curves and an orthogonal trajectory.

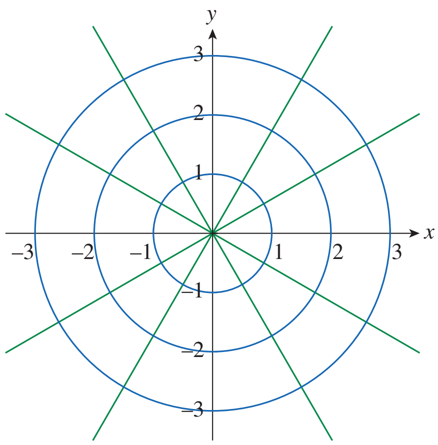


Figure 7.29
$y = mx$ is orthogonal to circles centered at the origin.

Example 5 Orthogonal Trajectories to Parabolas

Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitrary constant.

Solution

The curves described by the equation $x = ky^2$ form a family of parabolas. Each parabola is symmetric about the x-axis.

We need to find a single differential equation that is satisfied by all members of the family. Differentiate $x = ky^2$ with respect to x.

$$1 = 2ky\frac{dy}{dx} \quad\Rightarrow\quad \frac{dy}{dx} = \frac{1}{2ky}$$

This differential equation depends on k, but we need an equation that is valid for all values of k simultaneously. To eliminate k, use the original equation involving x and y, and solve for k.

$$x = ky^2 \quad\Rightarrow\quad k = \frac{x}{y^2}$$

Use this expression for k in the differential equation.

$$\frac{dy}{dx} = \frac{1}{2ky} = \frac{1}{2\frac{x}{y^2}y} = \frac{y}{2x}$$

The slope of the tangent line at any point (x, y) on one of the parabolas is $y' = \frac{y}{2x}$. On an orthogonal trajectory, the slope of the tangent line must be the negative reciprocal of this slope.

Therefore, the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = -\frac{2x}{y}.$$

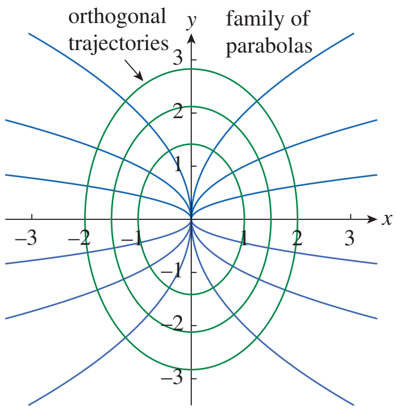


Figure 7.30
Graphs of the parabolas and the orthogonal trajectories.

Solve this separable differential equation.

$$\int y\,dy = -\int 2x\,dx \qquad \text{Separate variables; integrate both sides.}$$

$$\frac{y^2}{2} = -x^2 + C \qquad \text{Antiderivatives.}$$

$$x^2 + \frac{y^2}{2} = C \qquad \text{Rearrange terms.}$$

Therefore, the orthogonal trajectories are the family of ellipses described by $x^2 + \frac{y^2}{2} = C$, where C is an arbitrary constant. This solution is illustrated in Figure 7.30. ■

Orthogonal trajectories occur in various branches of physics. For example, in an electrostatic field, the lines of force are orthogonal to the lines of constant potential. Also, the streamlines in aerodynamics are orthogonal trajectories of the velocity-equipotential curves.

Mixing Problems

A typical mixing problem involves a tank of fixed capacity filled with a thoroughly mixed solution of some substance, for example salt. A solution of a given concentration enters the tank at a fixed rate and the mixture, thoroughly stirred, leaves at a fixed rate, which may be different from the entering rate.

Let $y(t)$ denote the amount of the substance in the tank at time t. Then $y'(t)$ is the rate at which the substance is being added minus the rate at which it is being removed. The mathematical description of this situation often leads to a first-order differential equation. We can use this same type of reasoning to model a variety of phenomena including chemical reactions, discharge of pollutants into a lake, and injection of a drug into the bloodstream.

Example 6 Remaining Salt in a Tank

A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

Solution

Let $y(t)$ be the amount of salt (in kilograms) in the tank after t minutes. We are given $y(0) = 20$; we need to find $y(30)$.

Consider the differential equation involving $y(t)$.

Note that $\dfrac{dy}{dt}$ is the rate of change of the amount of salt in the tank.

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

where (rate in) is the rate at which salt enters the tank and (rate out) is the rate at which salt leaves the tank.

The (rate in) is

$$\text{rate in} = \left(0.03\,\frac{\text{kg}}{\text{L}}\right)\left(25\,\frac{\text{L}}{\text{min}}\right) = 0.75\,\frac{\text{kg}}{\text{min}}.$$

The tank always contains 5000 L of liquid, so the concentration at time t is $\dfrac{y(t)}{5000}$ measured in kilograms per liter.

The brine flows out at a rate of 25 L/min, therefore

$$\text{rate out} = \left(\frac{y(t)}{5000}\,\frac{\text{kg}}{\text{L}}\right)\left(25\,\frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\,\frac{\text{kg}}{\text{min}}.$$

Use these expressions for (rate in) and (rate out) in the differential equation.

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

Solve this separable differential equation.

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$ Separate variables; integrate both sides.

$$-\ln|150 - y| = \frac{t}{200} + C$$ Antiderivatives.

$$-\ln|150 - 20| = \frac{0}{200} + C \quad \Rightarrow \quad C = -\ln 130$$ Use $y(0) = 20$; solve for C.

$$-\ln(150 - y) = \frac{t}{200} - \ln 130$$ Because $y(0) = 20$ and $y(t)$ is continuous, $150 - y > 0$ and $|150 - y| = 150 - y$.

$$y(t) = 150 - 130e^{-t/200}$$ Solve explicitly for $y(t)$.

The amount of salt in the tank after 30 min is

$$y(30) = 150 - 130e^{-30/200} \approx 38.108 \text{ kg}.$$

Figure 7.31 shows the graph of the function $y(t)$. Notice that as t increases, the amount of salt in the tank approaches 150 kg. ■

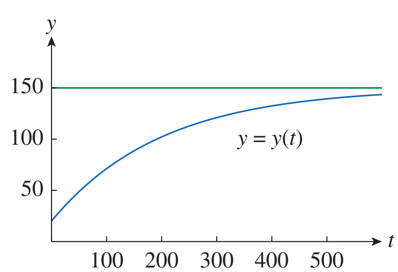


Figure 7.31
Graph of the solution curve.

7.3 Exercises

Solve the differential equation.

1. $\dfrac{dy}{dx} = 3x^2y^2$

2. $\dfrac{dy}{dx} = xe^{-y}$

3. $(x^2 + 1)y' = xy$

4. $\dfrac{dy}{dx} = x\sqrt{y}$

5. $xyy' = x^2 + 1$

6. $(y^2 + xy^2)y' = 1$

7. $(e^y - 1)y' = 2 + \cos x$

8. $\dfrac{du}{dt} = \dfrac{1 + t^4}{ut^2 + u^4t^2}$

9. $\dfrac{d\theta}{dt} = \dfrac{t \sec \theta}{\theta e^{t^2}}$

10. $\dfrac{dH}{dR} = \dfrac{RH^2\sqrt{1 + R^2}}{\ln H}$

11. $\dfrac{dp}{dt} = t^2p - p + t^2 - 1$

12. $\dfrac{dz}{dt} + e^{t+z} = 0$

Find the solution of the differential equation that satisfies the given initial condition.

13. $\dfrac{dy}{dx} = xe^y, \quad y(0) = 0$

14. $\dfrac{dy}{dx} = \dfrac{x \sin x}{y}, \quad y(0) = -1$

15. $\dfrac{dy}{dx} = (y - 1)(2x + 1), \quad y(-1) = 3$

16. $\dfrac{dy}{dx} = y\sqrt{x}, \quad y(0) = 50$

17. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}, \quad u(0) = -5$

18. $x + 3y^2\sqrt{x^2 + 1}\,\dfrac{dy}{dx} = 0, \quad y(0) = 1$

19. $x \ln x = y(1 + \sqrt{3 + y^2})y', \quad y(1) = 1$

20. $\dfrac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2$

21. $y' \tan x = a + y, \quad y\left(\dfrac{\pi}{3}\right) = a, \quad 0 < x < \dfrac{\pi}{2}$

22. $\dfrac{dL}{dt} = kL^2 \ln t, \quad L(1) = -1$

23. Find an equation of the curve that passes through the point (0, 2) and whose slope at (x, y) is $\dfrac{x}{y}$.

24. Find the function f such that $f'(x) = xf(x) - x$ and $f(0) = 2$.

25. Solve the differential equation $y' = x + y$ by making a change of variables: $u = x + y$.

26. Solve the differential equation $xy' = y + xe^{y/x}$ by making a change of variables: $v = \dfrac{y}{x}$.

27. (a) Solve the differential equation $y' = 2x\sqrt{1 - y^2}$.
(b) Solve the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 0$, and graph the solution.
(c) Does the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 2$, have a solution? Explain your reasoning.

28. Solve the differential equation $e^{-y}y' + \cos x = 0$ and graph several members of the family of solutions. How does the solution curve change as the constant of integration C varies?

29. Suppose $y = f(x)$ is a solution of the initial-value problem $\dfrac{dy}{dx} = x \sin x$, $f(1) = 3$. Find $f(2)$.

30. Suppose $y = f(x)$ is a solution of the initial-value problem $\dfrac{dy}{dx} = 2xy^2$, $f(0) = 2$. Find $f(2)$.

31. Solve the initial-value problem $y' = \dfrac{\sin x}{\sin y}$, $y(0) = \dfrac{\pi}{2}$, and graph the solution curve.

32. Solve the differential equation $y' = \dfrac{x\sqrt{x^2 + 1}}{ye^y}$ and graph several members of the family of solutions. Explain how the solution curve changes as the constant of integration C varies.

For each expression:

(a) Use technology to draw a slope field for the differential equation. Sketch several solution curves without solving the differential equation.

(b) Solve the differential equation.

(c) Use technology to draw several members of the family of solutions obtained in part (b). Compare with the curves from part (a).

33. $y' = y^2$

34. $y' = xy$

Find the orthogonal trajectories of the family of curves. Use technology to sketch several members of each family in the same viewing rectangle.

35. $x^2 + 2y^2 = k^2$

36. $y^2 = kx^3$

37. $y = \dfrac{k}{x}$

38. $y = \dfrac{1}{1 + kx}$

An **integral equation** is an equation that contains an unknown function $y(x)$ and an integral that involves $y(x)$. Solve the given integral equation.
Hint: Use an initial condition obtained from the integral equation.

39. $y(x) = 2 + \int_2^x [t - ty(t)]\, dt$

40. $y(x) = 2 + \int_1^x \frac{dt}{ty(t)}, \quad x > 0$

41. $y(x) = 4 + \int_0^x 2t\sqrt{y(t)}\, dt$

42. Consider the differential equation $\frac{dy}{dx} - 2xy = x$.

(a) Explain why any solution curve to this differential equation must have a horizontal tangent line at its y-intercept.
(b) Find the general solution of this differential equation.
(c) Find the equation of the solution curve to this differential equation that passes through the origin.

43. Consider the differential equation $\frac{dy}{dx} = 4 - y$. Let $y = f(x)$ be the particular solution of this differential equation such that $f(0) = 1$.

(a) Find an equation of the line tangent to the graph of the solution curve $y = f(x)$ at the point (0, 1). Use this line to approximate $f(1.5)$.
(b) Evaluate $\lim_{x\to 0} \frac{f(x)}{6x}$.
(c) Find the particular solution of the differential equation $\frac{dy}{dx} = 4 - y$ such that $f(0) = 1$.

44. Find a function f such that $f(3) = 2$ and

$$(t^2 + 1)f'(t) + [f(t)]^2 + 1 = 0, \quad t \neq 1$$

45. Solve the initial-value problem presented in Exercise 35, Section 7.2

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t) \qquad Q(0) = 0$$

to find an expression for the charge at time t. Find the limiting value of the charge.

46. In Exercise 36, Section 7.2, we discussed a differential equation that models the temperature of a 95°C cup of coffee in a 20°C room. Solve the differential equation to find an expression for the temperature of the coffee at time t.

47. A model for learning presented in Section 7.1 is described by the differential equation

$$\frac{dP}{dt} = k(M - P)$$

where $P(t)$ measures the performance of someone learning a skill after a training time t, M is the maximum level of performance, and k is a positive constant. Solve this differential equation to find an expression for $P(t)$. What is the limit of this expression as t increases without bound?

48. A model for the weight of a lizard is described by the differential equation

$$\frac{dW}{dt} = \frac{1}{5}(80 - W)$$

where $W(t)$ measures the weight of the lizard, in grams, at time t, in weeks. At time $t = 0$, the lizard weighs 30 grams.

(a) Use the tangent line to the graph of $y = W(t)$ at the point (0, 30) to approximate the weight of the lizard at time $t = 1$.
(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether the approximation found in part (a) is an overestimate or an underestimate. Explain your reasoning.
(c) Find the particular solution $y = W(t)$ with initial condition $W(0) = 30$.

49. Consider the differential equation $\frac{dy}{dx} = \frac{2}{xy}$. Let $y = f(x)$ be a particular solution of this differential equation such that $f(1) = 3$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.
(b) Write an equation for the line tangent to the graph of f at (1, 3). Use this tangent line to approximate the value of $f(1.2)$.
(c) Given that $f(x) > 0$ on the interval $1 \le x \le 1.5$, determine whether the approximation in part (b) is an underestimate or an overestimate. Explain your reasoning.
(d) Find the solution of the differential equation $\frac{dy}{dx} = \frac{2}{xy}$ such that $f(1) = 3$.

50. In an elementary chemical reaction, single molecules of two reactants A and B form a molecule of the product C: A + B → C. The law of mass action states that the rate of reaction is proportional to the product of the concentrations of A and B:

$$\frac{d[C]}{dt} = k[A][B]$$

Thus, if the initial concentrations are $[A] = a$ moles/L and $[B] = b$ moles/L and we write $x = [C]$, then we have

$$\frac{dx}{dt} = k(a - x)(b - x)$$

(a) Assuming that $a \neq b$, find x as a function of t. Use the fact that the initial concentration of C is 0.
(b) Find $x(t)$ assuming that $a = b$. How does this expression for $x(t)$ simplify if it is known that $[C] = \frac{1}{2}a$ after 20 seconds?

51. In contrast to the situation in Exercise 50, experiments show that the reaction $H_2 + Br_2 \to 2HBr$ satisfies the rate law

$$\frac{d[\mathrm{HBr}]}{dt} = k[\mathrm{H_2}][\mathrm{Br_2}]^{1/2}$$

and so, for this reaction, the differential equation becomes

$$\frac{dx}{dt} = k(a - x)(b - x)^{1/2}$$

where $x = [\mathrm{HBr}]$ and a and b are the initial concentrations of hydrogen and bromine.

(a) Find x as a function of t in the case where $a = b$. Use the fact that $x(0) = 0$.

(b) If $a > b$, find t as a function of x.
Hint: In performing the integration, make the substitution $u = \sqrt{b - x}$.

52. A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate that is proportional to the concentration at that time. Thus, a model for the concentration $C = C(t)$ of the glucose solution in the bloodstream is

$$\frac{dC}{dt} = r - kC$$

where k is a positive constant.

(a) Suppose that the concentration at time $t = 0$ is C_0. Determine the concentration at any time t by solving the differential equation.

(b) Assuming that $C_0 < \dfrac{r}{k}$, find $\lim_{t\to\infty} C(t)$ and interpret your answer in the context of this problem.

53. A certain small country has \$10 billion in paper currency in circulation, and each day \$50 million comes into the country's banks. The government decides to introduce new currency by having the banks replace old bills with new ones whenever old currency comes into the banks. Let $x = x(t)$ denote the amount of new currency in circulation at time t, with $x(0) = 0$.

(a) Formulate a mathematical model in the form of an initial-value problem that represents the *flow* of the new currency into circulation.

(b) Solve the initial-value problem found in part (a).

(c) How long will it take for the new bills to account for 90% of the currency in circulation?

54. A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after t minutes and (b) after 20 minutes?

55. The air in a room with volume 180 m^3 contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of 2 m^3/min and the mixed air flows out at the same rate. Find the percentage of carbon dioxide in the room as a function of time.

56. A vat with 500 gallons of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after 1 hour?

57. A tank contains 1000 L of pure water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 5 L/min. Brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. How much salt is in the tank (a) after t minutes and (b) after 1 hour?

58. When a raindrop falls, it increases in size, so its mass at time t is a function of t, namely, $m(t)$. The rate of growth of the mass is $km(t)$ for some positive constant k. When we apply Newton's Law of Motion to the raindrop, we get $(mv)' = gm$, where v is the velocity of the raindrop (directed downward) and g is the acceleration due to gravity. The *terminal velocity* of the raindrop is $\lim_{t\to\infty} v(t)$. Find an expression for the terminal velocity in terms of g and k.

59. An object of mass m is moving horizontally through a medium which resists the motion with a force that is a function of the velocity; that is,

$$m\frac{d^2s}{dt^2} = m\frac{dv}{dt} = f(v)$$

where $v = v(t)$ and $s = s(t)$ represent the velocity and position of the object at time t, respectively. For example, think of a boat moving through the water.

(a) Suppose that the resisting force is proportional to the velocity, that is, $f(v) = -kv$, where k is a positive constant. (This model is appropriate for small values of v.) Let $v(0) = v_0$ and $s(0) = s_0$ be the initial values of v and s. Determine v and s at any time t. What is the total distance that the object travels from time $t = 0$?

(b) For larger values of v, a better model is obtained by assuming that the resisting force is proportional to the square of the velocity, that is, $f(v) = -kv^2$, where $k > 0$. (This model was first proposed by Newton.) Let v_0 and s_0 be the initial values of v and s. Determine v and s at any time t. What is the total distance that the object travels in this case?

60. *Allometric growth* in biology refers to relationships between sizes of parts of an organism (skull length and body length, for instance). If $L_1(t)$ and $L_2(t)$ are the sizes of two organs in an organism of age t, then L_1 and L_2 satisfy an allometric law if their specific growth rates are proportional:

$$\frac{1}{L_1}\frac{dL_1}{dt} = k\frac{1}{L_2}\frac{dL_2}{dt}$$

where k is a constant.

(a) Use the allometric law to write a differential equation relating L_1 and L_2 and solve it to express L_1 as a function of L_2.
(b) In a study of several species of unicellular algae, the proportionality constant in the allometric law relating B (cell biomass) and V (cell volume) was found to be $k = 0.0794$. Write B as a function of V.

61. A model for tumor growth is given by the Gompertz equation

$$\frac{dV}{dt} = a(\ln b - \ln V)V$$

where a and b are positive constants and V is the volume of the tumor measured in mm^3.
(a) Find a family of solutions for tumor volume as a function of time.
(b) Find the solution that has an initial tumor volume of $V(0) = 1 \text{ mm}^3$.

62. *Homeostasis* refers to a state in which the nutrient content of a consumer is independent of the nutrient content of its food. In the absence of homeostasis, a model proposed by Sterner and Elser is given by

$$\frac{dy}{dx} = \frac{1}{\theta}\frac{y}{x}$$

where x and y represent the nutrient content of the food and the consumer, respectively, and θ is a constant such that $\theta \geq 1$.
(a) Solve the differential equation.
(b) What happens when $\theta = 1$? What happens as $\theta \to \infty$?

63. Let $A(t)$ be the area of a tissue culture at time t and let M be the final area of the tissue when growth is complete. Most cell divisions occur on the periphery of the tissue, and the number of cells on the periphery is proportional to $\sqrt{A(t)}$. So, a reasonable model for the growth of tissue is obtained by assuming that the rate of growth of the area is jointly proportional to $\sqrt{A(t)}$ and $M - A(t)$.
(a) Formulate a differential equation and use it to show that the tissue grows fastest when $A(t) = \frac{1}{3}M$.
(b) Solve the differential equation to find an expression for $A(t)$. Use technology to perform the integration.

64. According to Newton's Law of Universal Gravitation, the gravitational force on an object of mass m that has been projected vertically upward from Earth's surface is

$$F = \frac{mgR^2}{(x + R)^2}$$

where $x = x(t)$ is the object's distance above the surface at time t, R is Earth radius, and g is the acceleration due to gravity. Also, by Newton's Second Law, $F = ma = m\frac{dv}{dt}$ and so

$$m\frac{dv}{dt} = -\frac{mgR^2}{(x + R)^2}$$

(a) Suppose a rocket is fired vertically upward with an initial velocity v_0. Let h be the maximum height above the surface reached by the object. Show that

$$v_0 = \sqrt{\frac{2gRh}{R + h}}$$

Hint: By the Chain Rule, $m\frac{dv}{dt} = mv\frac{dv}{dx}$.
(b) Calculate $v_e = \lim_{h\to\infty} v_0$. This limit is called the *escape velocity* for Earth.
(c) Use $R = 3960$ mi and $g = 32 \text{ ft/s}^2$ to calculate v_e in feet per second and in miles per second.

Applied Project | How Fast Does a Tank Drain?

If water (or another liquid) drains from a tank, we expect that the flow rate will be greatest at first (when the water depth is greatest) and will gradually decrease as the water level decreases. A more precise mathematical description of how the flow rate decreases is necessary in order to answer the kinds of questions that engineers ask, for example, how long does it take for a tank to drain completely? How much water should a tank hold in order to guarantee a certain minimum water pressure for a sprinkler system?

Let $h(t)$ and $V(t)$ be the height and volume of water in a tank at time t. If water drains through a hole with area a at the bottom of the tank, then Torricelli's Law states

$$\frac{dV}{dt} = -a\sqrt{2gh} \tag{1}$$

where g is the acceleration due to gravity. An interpretation of this equation is that the rate at which water flows from the tank is proportional to the square root of the water height.

1. (a) Suppose the tank is cylindrical with height 6 ft and radius 2 ft, and the hole is circular with radius 1 inch. Let $g = -32 \text{ ft/s}^2$ and show that h satisfies the differential equation

$$\frac{dh}{dt} = -\frac{1}{72}\sqrt{h}$$

(b) Solve this differential equation to find the height of the water at time t, assuming the tank is full at time $t = 0$.

(c) How long will it take for the water to drain completely?

2. The theoretical model given in Equation 1 does not account for the rotation and viscosity of the liquid. Another, more accurate, model is given by

$$\frac{dh}{dt} = k\sqrt{h} \tag{2}$$

where the constant k depends on the physical properties of the liquid and is determined using data obtained from actually draining the tank.

(a) Suppose that a hole is drilled in the side of a cylindrical bottle and the height h of water (above the hole) decreases from 10 cm to 3 cm in 68 seconds. Use Equation to find an expression for $h(t)$. Evaluate $h(t)$ for $t = 10, 20, 30, 40, 50, 60$.

(b) Drill a 4-mm hole near the bottom of the cylindrical part of a two-liter plastic soft-drink bottle. Attach a strip of masking tape marked in centimeters from 0 to 10, with 0 corresponding to the top of the hole. With one finger over the hole, fill the bottle to the 10-cm mark. Then take your finger off the hole and record the values of $h(t)$ for $t = 10, 20, 30, 40, 50, 60$ seconds. It should take approximately 68 seconds for the water level to decrease to $h = 3$ cm. Compare your data with the values of $h(t)$ from part (a). How well did the model predict the actual values?

3. The water from sprinkler systems in some large hotels and hospitals is supplied by gravity from cylindrical tanks on or near the roofs of the buildings. Suppose a tank used at a hotel has radius 10 ft and the diameter of the outlet is 2.5 inches. An engineer has to determine placement of the tank so that the water pressure will be at least 2160 lb/ft^2 for a period of 10 minutes. One reason for this condition is in case of a fire. The electrical system could fail and it may take up to 10 minutes for the emergency generator and fire pump to be activated. What height should the engineer specify for the tank in order to guarantee this water pressure? Use the fact that the water pressure at a depth of d feet is $P = 62.5d$.

4. Not all water tanks are shaped like cylinders. Suppose a tank has cross-sectional area $A(h)$ at height h. Then the volume of water up to height h is $V = \int_0^h A(u)\,du$. By the Fundamental Theorem of Calculus: $\dfrac{dV}{dh} = A(h)$. Using the Chain Rule

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = A(h)\frac{dh}{dt}$$

and using Torricellis's Law

$$A(h)\frac{dh}{dt} = -a\sqrt{2gh}$$

(a) Suppose the tank has the shape of a sphere with radius 2 m and is initially half full of water. If the radius of the circular hole is 1 cm and we let $g = 10 \text{ m/s}^2$, show that h satisfies the differential equation

$$(4h - h^2)\frac{dh}{dt} = -0.0001\sqrt{20h}$$

(b) How long will it take for the water to drain completely?

Applied Project | Which Is Faster, Going Up or Coming Down?

Suppose you throw a ball into the air. Does it take longer to reach its maximum height or to fall back to Earth from its maximum height? We will answer that question in this project, but you might think about this situation and use your intuition to guess the answer before starting.

There are several possible models for the force due to air resistance, depending on the physical characteristics and speed of the ball. In this project, we use a linear model, $-pv$, but a quadratic model ($-pv^2$ on the way up and pv^2 on the way down) is another possibility, often used in cases with high speeds. For a golf ball, experiments have shown that a good model for the force due to air resistance is $-pv^{1.3}$ going up and $p|v|^{1.3}$ coming down. However, for any force function $-f(v)$, where $f(v) > 0$ for $v > 0$ and $f(v) < 0$ for $v < 0$, the answer to the project question remains the same. See F. Bauer, "What Goes Up Must Come Down, Eventually," *American Mathematical Monthly* 108 (2001) pp. 437–440.

1. A ball with mass m is projected vertically upward from Earth's surface with a positive initial velocity v_0. We assume the forces acting on the ball are the force of gravity, and an impeding force due to air resistance with direction opposite to the direction of motion and with magnitude $p|v(t)|$, where p is a positive constant and $v(t)$ is the velocity of the ball at time t.

In both the ascent and the descent, the total force acting on the ball is $-pv - mg$. During ascent, $v(t)$ is positive and the resistance acts downward; during descent, $v(t)$ is negative and resistance acts upward. By Newton's Second Law, the equation of motion is

$$mv' = -pv - mg$$

Solve this differential equation to show that the velocity is

$$v(t) = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}$$

2. Show that the height of the ball, until it hits the ground, is

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\left(1 - e^{-pt/m}\right) - \frac{mgt}{p}$$

3. Let t_1 be the time that the ball takes to reach its maximum height. Show that

$$t_1 = \frac{m}{p}\ln\left(\frac{mg + pv_0}{mg}\right)$$

Find this time for a ball with mass 1 kg and initial velocity 20 m/s. Assume the air resistance is $\frac{1}{10}$ of the speed.

4. Let t_2 be the time at which the ball falls back to Earth. For the particular ball in Problem 3, estimate t_2 by using a graph of the height function $y(t)$. Which is faster, going up or coming down?

5. In general, it's not easy to find t_2 because it's impossible to solve the equation $y(t) = 0$ explicitly. However, we can use an indirect method to determine whether ascent or descent is faster: consider the sign of $y(2t_1)$ (whether this value is positive or negative).

Show that

$$y(2t_1) = \frac{m^2 g}{p^2}\left(x - \frac{1}{x} - 2\ln x\right)$$

where $x = e^{pt_1/m}$. Then show that $x > 1$ and the function

$$f(x) = x - \frac{1}{x} - 2\ln x$$

is increasing for $x > 1$. Use this result to decide whether $y(2t_1)$ is positive or negative. Now draw a conclusion: Is it faster going up or going down?

7.4 Exponential Growth and Decay

One model for population growth considered earlier was based on the assumption that the population grows at a rate proportional to the size of the population. This assumption is expressed mathematically by using the differential equation

$$\frac{dP}{dt} = kP$$

This is a separable equation and easy to solve, but let's consider the assumption more carefully. Suppose we have a population (of bacteria, for example) with initial size $P = 1000$ and at a certain time it is growing at a rate of $P' = 300$ bacteria per hour. Now let's take another 1000 bacteria of the same type and put them with the first population. Each half of the combined population was previously growing at a rate of 300 bacteria per hour. We would expect the total population of 2000 to increase at a rate of 600 bacteria per hour initially (provided that there is enough room and nutrition). So, if we double the size, we double the growth rate. It seems reasonable that the growth rate should be proportional to the size.

The same assumption applies in other situations as well. In nuclear physics, the mass of a radioactive substance decays at a rate proportional to the mass. In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance. In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

In general, if $y(t)$ is the value of a quantity, or amount of a quantity present, at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

$$\frac{dy}{dt} = ky \qquad (1)$$

where k is a constant. Equation 1 is often called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$).

Equation 1 is a separable differential equation. Use the method of separation of variables to solve the initial-value problem with $y(0) = y_0 > 0$.

$$\int \frac{dy}{y} = \int k\,dt \qquad \text{Separate variables; integrate both sides.}$$

$$\ln|y| = kt + C \qquad \text{Antiderivatives.}$$

$$\ln|y_0| = \ln y_0 = k \cdot 0 + C \quad \Rightarrow \quad C = \ln y_0 \qquad \text{Use the initial value.}$$

$$\ln y = kt + \ln y_0 \qquad \text{Because } y(0) = y_0 > 0,\ y > 0 \text{ and } |y| = y.$$

$$y = e^{kt + \ln y_0} = y_0 e^{kt} \qquad \text{Solve for } y.$$

The solution of the initial-value problem

$$\frac{dy}{dt} = ky \quad y(0) = y_0$$

is

$$y(t) = y_0 e^{kt} \qquad (2)$$

Population Growth

Let's consider the significance of the proportionality constant k. In the context of population growth, we can write

$$\frac{dP}{dt} = kP \quad \text{or} \quad \frac{1}{P}\frac{dP}{dt} = k \tag{3}$$

The expression

$$\frac{1}{P}\frac{dP}{dt}$$

can be interpreted to mean the **relative growth rate** (the growth rate divided by the population size) is constant. Then Equation 2 says that a population with constant relative growth rate must grow exponentially.

Notice that the relative growth rate k appears as the coefficient of t in the exponential function $y_0 e^{kt}$. For example, if

$$\frac{dP}{dt} = 0.02P$$

and t is measured in years, then the relative growth rate is $k = 0.02$ and the population grows at a relative rate of 2% per year. If the population at time 0 is P_0, then the expression for the population at any time t is

$$P(t) = P_0 e^{0.02t}$$

Example 1 A Model for World Population

Assume that the population of the world grows at a rate proportional to its size. Use the data in Table 7.3 to model the population of the world from 1900 to 2020. What is the relative growth rate? How well does the model fit the data?

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870
2020	7755

Table 7.3
Table of world population.

Solution

Let time t be measured in years and let $t = 0$ correspond to the year 1900.
The population at time t, $P(t)$, is measured in millions of people.

The initial condition is $P(0) = 1650$.
Since we are assuming that the growth rate is proportional to population size, then the initial-value problem is

$$\frac{dP}{dt} = kP \qquad P(0) = 1650.$$

From Equation 2, the solution of this differential equation is

$P(t) = 1650e^{kt}$.

One way to estimate the relative growth rate k is to use another data point in Table 7.3. For example, in 1960, the population was 3040 million.

Therefore, $P(60) = 1650e^{k(60)} = 3040$.

Solve this equation for k.

$$e^{60k} = \frac{3040}{1650} \quad \Rightarrow \quad k = \frac{1}{60}\ln\frac{3040}{1650} \approx 0.0101847$$

Therefore, the relative growth rate is about 1.01% per year and the model becomes

$P(t) = 1650e^{0.0101847t}$.

Table 7.4 and Figure 7.32 present a numerical and graphical comparison of the actual population data and the predicted values using this model. Notice that the predictions are very inaccurate after about 60 years. And recall that in Section 1.4 we modeled the same data with an exponential function, but there we used the method of least squares.

Year	Model	Population
1900	1650	1650
1910	1827	1750
1920	2023	1860
1930	2240	2070
1940	2480	2300
1950	2746	2560
1960	3040	3040
1970	3366	3710
1980	3726	4450
1990	4126	5280
2000	4569	6080
2010	5059	6870
2020	5601	7755

Table 7.4
Numerical comparison of actual and predicted values of world population.

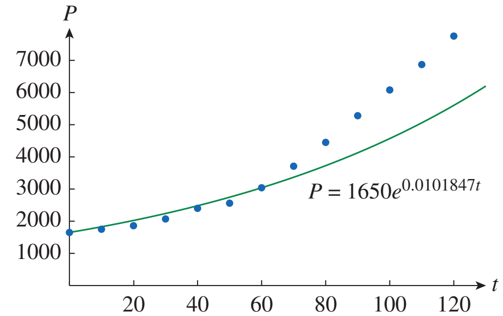


Figure 7.32
Graphical comparison of actual and predicted values of world population.

From Figure 7.32, it appears that we could produce a better model by using the given population in 1980, instead of 1960, to estimate k.

$$P(80) = 1650e^{80k} = 4450 \quad \Rightarrow \quad k = \frac{1}{80}\ln\frac{4450}{1650} \approx 0.0124016$$

The estimate for the relative growth rate is now 1.24% per year and the model is $P(t) = 1650e^{0.0124016t}$.

Figure 7.33 illustrates the second model. This exponential model is more accurate after 1980 but less accurate before 1960. ■

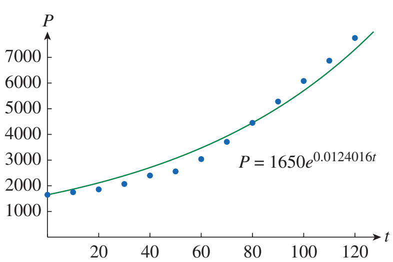


Figure 7.33
Another model for world population growth.

Example 2 Estimation and Prediction Using an Exponential Growth Model

Use the data in Table 7.3 to model the population of the world from 1950. Use the model to estimate the population in 1995 and to predict the population in 2025.

Solution

Here we let $t = 0$ correspond to the year 1950. The initial-value problem is

$$\frac{dP}{dt} = kP \quad P(0) = 2560.$$

The solution is $P(t) = 2560e^{kt}$.

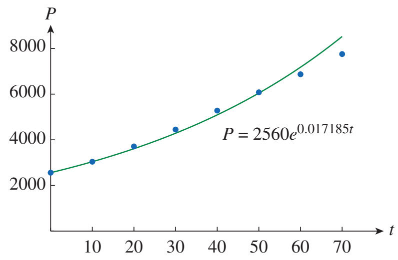


Figure 7.34
A model for world population from 1950.

Let's estimate k by using the population in 1960.

$$P(1) = 2560e^{10k} = 3040 \quad \Rightarrow \quad k = \frac{1}{10}\ln\frac{3040}{2560} \approx 0.017185$$

The relative growth rate is about 1.7% per year and the model is $P(t) = 2560e^{0.017185t}$. An estimate for the world population in 1995 is

$$P(45) = 2560e^{0.017185(45)} \approx 5547 \text{ million.}$$

Similarly, use the model to predict the population in 2025.

$$P(75) = 2560e^{0.017185(75)} \approx 9289 \text{ million.}$$

The graph in Figure 7.34 shows that the model is fairly accurate to the turn of the century, so the estimate for 1995 is probably reliable. But we have less confidence in the prediction for 2025. ■

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t, then the relative decay rate

$$-\frac{1}{m}\frac{dm}{dt}$$

has been found experimentally to be constant. Since dm/dt is negative, the relative decay rate is positive. It follows that

$$\frac{dm}{dt} = km$$

where k is a negative constant. This is interpreted as, radioactive substances decay at a rate proportional to the remaining mass. Therefore, we can use Equation 2 to show that the mass decays exponentially:

$$m(t) = m_0e^{kt}$$

Physicists often express the rate of decay in terms of **half-life**, the time required for half of any given quantity to decay.

Example 3 Use a Model for Exponential Decay

The half-life of radium-226 is 1590 years.

(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of radium-226 that remains after t years.
(b) Find the mass after 1000 years.
(c) When will the mass be reduced to 30 mg?

Solution

(a) Let $m(t)$ be the mass of radium-226 (in milligrams) that remains after t years.

Then $\dfrac{dm}{dt} = km$ and $m(0) = 100$.

Using Equation 2: $m(t) = m(0)e^{kt} = 100\,e^{kt}$.

To determine the value of k, use the fact that $m(1590) = \frac{1}{2}(100) = 50$.

$$m(1590) = 100e^{1590k} = 50 \quad \Rightarrow \quad e^{1590k} = \frac{1}{2}$$

$$1590k = \ln\frac{1}{2} = -\ln 2 \quad \Rightarrow \quad k = -\frac{\ln 2}{1590}$$

Therefore, $m(t) = 100e^{-(\ln 2)t/1590}$.

If we use the fact that $e^{\ln 2} = 2$, then an alternative expression for $m(t)$ is $m(t) = 100 \times 2^{-t/1590}$.

(b) The mass after 1000 years is

$$m(1000) = 100e^{-(\ln 2)1000/1590} \approx 64.67 \text{ mg}$$

(c) We want to find a value of t such that $m(t) = 30$.

$$m(t) = 100e^{-(\ln 2)t/1590} = 30 \quad \Rightarrow \quad e^{-(\ln 2)t/1590} = 0.3$$

Take the natural logarithm of both sides and solve for t.

$$-\frac{\ln 2}{1590}t = \ln 0.3 \quad \Rightarrow \quad t = -1590\frac{\ln 0.3}{\ln 2} \approx 2762 \text{ years}$$ ■

Newton's Law of Cooling

Note that this law also applies to an object warming.

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature for the surroundings, then we can translate Newton's Law of Cooling into a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant. We could solve this differential equation using the method of separation of variables, but an easier method is to first make a change of variables: let $y(t) = T(t) - T_s$. Because T_s is constant, then $y'(t) = T'(t)$ and the equation in terms of y and t is

$$\frac{dy}{dt} = ky$$

We can now use Equation 2 to find an expression for y, and then an expression for T.

Example 4 Use Newton's Law of Cooling to Predict Temperatures

A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After half an hour, the soda pop has cooled to 61°F.

(a) What is the temperature of the soda pop after another half hour?

(b) How long does it take for the soda pop to cool to 50°F?

Solution

(a) Let $T(t)$ be the temperature of the soda after t minutes.

The surrounding temperature is $T_s = 44$°F. Newton's Law of Cooling states

$$\frac{dT}{dt} = k(T - 44).$$

Let $y(t) = T(t) - 44 \quad \Rightarrow \quad y(0) = T(0) - 44 = 72 - 44 = 28.$

So, y is a solution of the initial-value problem

$$\frac{dy}{dt} = ky \qquad y(0) = 28.$$

Use Equation 2: $y(t) = y(0)e^{kt} = 28e^{kt}$.

Given $T(30) = 61 \Rightarrow y(30) = 61 - 44 = 17 \Rightarrow 28e^{30k} = 17 \Rightarrow e^{30k} = \frac{17}{28}.$

Take the natural logarithm of both sides to solve for k.

$$k = \frac{\ln(17/28)}{30} \approx -0.1663$$

Use the expression for y to solve for T.

$$y(t) = 28e^{-0.01663t} = T(t) - 44 \Rightarrow T(t) = 44 + 28e^{-0.01663t}$$

Find the temperature of the soda at time $t = 60$ minutes.

$$T(60) = 44 + 28e^{-0.01663(60)} \approx 54.3$$

After another half hour, the soda pop has cooled to about 54°F.

(b) Find the time at which $T(t) = 50$.

$$44 + 28e^{-0.01663t} = 50$$

$$e^{-0.01663t} = \frac{6}{28}$$

$$t = \frac{\ln(6/28)}{-0.01663} \approx 92.6$$

The soda pop cools to 50° after about 1 hour and 33 minutes.

Notice the following limit.

$$\lim_{t\to\infty} T(t) = \lim_{t\to\infty}\left(44 + 28e^{-0.01663t}\right) = 44 + 28 \cdot 0 = 44$$

We expect the soda pop to cool towards the temperature of the surroundings. The graph of the temperature function is shown in Figure 7.35. ■

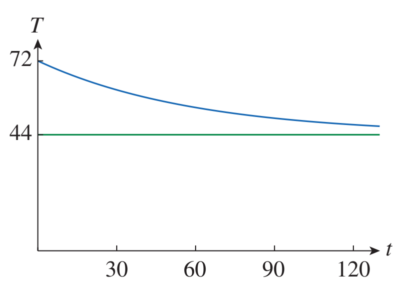


Figure 7.35
Graph of the temperature function.

Continuously Compounded Interest

Example 5 Value of an Investment

If \$1000 is invested at 6% interest, compounded annually, then after 1 year the investment is worth $\$1000(1.06) = \$1060$, after 2 years it's worth $\$[1000(1.06)]1.06 = \$1123.60$, and after t years it's worth $\$1000(1.06)^t$. In general, if an amount A_0 is invested at an interest rate r ($r = 0.06$ in this example), then after t years it's worth $A_0(1 + r)^t$.

Usually, however, interest is compounded more frequently, say, n times a year. Then in each compounding period, the interest rate is r/n and there are nt compounding periods in t years. So, the value of the investment is

$$A_0\left(1 + \frac{r}{n}\right)^{nt}$$

For instance, after 3 years at 6% interest, a \$1000 investment will be worth

$$\$1000(1.06)^3 = \$1191.02 \quad \text{with annual compounding}$$

$$\$1000(1.03)^6 = \$1194.05 \quad \text{with semiannual compounding}$$

$$\$1000(1.015)^{12} = \$1195.62 \quad \text{with quarterly compounding}$$

$$\$1000(1.005)^{36} = \$1196.68 \quad \text{with monthly compounding}$$

$$\$1000\left(1 + \frac{0.06}{365}\right)^{365 \cdot 3} = \$1197.20 \quad \text{with daily compounding}$$

These calculations demonstrate that interest paid increases as the number of compounding periods (n) increases. If we let $n \to \infty$, then interest is compounded **continuously** and the value of the investment will be

$$A(t) = \lim_{n\to\infty} A_0\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Value of investment as } n \text{ increases without bound.}$$

$$= \lim_{n\to\infty} A_0\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} \qquad \text{Multiply by 1 in a convenient form, } r/r.$$

$$= A_0\left[\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} \qquad \text{Property of limits.}$$

$$= A_0\left[\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^{m}\right]^{rt} \qquad \text{Change variables; let } m = n/r.$$

The limit in this expression is equal to the number e (see Equation 3.7.6). Therefore, if interest is compounded continuously at interest rate r, the amount after t years is

$A(t) = A_0e^{rt}$.

If we differentiate this expression: $\dfrac{dA}{dt} = rA_0e^{rt} = rA(t)$.

This is interpreted as, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Consider the example of \$1000 invested for 3 years at 6% interest.

If interest is compounded continuously, the value of the investment will be

$A(3) = 1000e^{(0.06)3} = 1000e^{0.18} = 1197.22$ dollars.

Notice how close this is to the amount of the investment for daily compounding, \$1197.20. ■

7.4 Exercises

1. A population of protozoa develops with a constant relative growth rate of 0.7944 per member per day. On day zero, the population consists of two members. Find the population size after 6 days.

2. The rate of change of bacteria in a population is proportional to the number present. There are initially 2000 bacteria in the culture, and at $t = 3$ hours, there are 2500. What is the bacteria population when $t = 5$ hours?

3. A radioactive substance initially measures 3 grams and has a half-life of 250 years. Write an expression for the amount of substance remaining after t years.

4. A certain bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 60 cells.
(a) Find the relative growth rate.
(b) Find an expression for the number of cells after t hours.
(c) Find the number of cells after 8 hours.
(d) Find the rate of growth after 8 hours.
(e) When will the population reach 20,000 cells?

5. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour, the population has increased to 420.
(a) Find an expression for the number of bacteria after t hours.
(b) Find the number of bacteria in the culture after 3 hours.
(c) Find the rate of growth after 3 hours.
(d) Find the time at which the population of bacteria will reach 10,000.

6. A bacteria culture grows with constant relative growth rate. The bacteria count was 400 after 2 hours and 25,600 after 6 hours.
(a) What is the relative growth rate? Express your answer as a percentage.
(b) What was the initial size of the culture?
(c) Find an expression for the number of bacteria after t hours.
(d) Find the number of cells after 4.5 hours.
(e) Find the rate of growth after 4.5 hours.
(f) Find the time at which the population reaches 50,000.

7. The table shows estimates of the population of Florida, in millions, from 1950 to 2020.

Year	Population	Year	Population
1950	2.810	1990	13.018
1960	5.004	2000	16.048
1970	6.791	2010	18.846
1980	9.840	2020	21.733

(a) Use the data for 1950 and 1990 to construct an exponential model for the population of Florida.
(b) Use your model to predict the population of Florida in 1970 and 2020. Compare with the actual figures.

8. The table shows the population of India, in millions, for selected years.

Year	Population	Year	Population
1950	376	1990	873
1960	451	2000	1057
1970	555	2010	1234
1980	699	2020	1380

(a) Use the data for 1950 and 1990 to construct an exponential model for the population of India. Use your model to predict the population in 2020.
(b) Use the data for 1950 and 2010 to construct an exponential model for the population of India. Use your model to predict the population in 2020.
(c) Graph both exponential functions in parts (a) and (b) together in a scatter plot of the actual population data. Do either of these models for population seem reasonable? Explain your reasoning.

9. For time $t \geq 0$, in hours, the rate of growth of a population of bacteria is given by $\dfrac{dP}{dt} = \dfrac{1}{4}P$. Initially, there are 6000 bacteria in the population.
(a) Write an expression for $P(t)$, the number of bacteria in the population at time t. Use this model to predict the number of bacteria in the population at time $t = 5$ hours.
(b) Find the average rate of bacteria growth for $0 \leq t \leq 8$.
(c) Use technology to find the average number of bacteria in the population for $0 \leq t \leq 8$.

10. Experiments show that if the chemical reaction

$$N_2O_5 \rightarrow 2NO_2 + \frac{1}{2}O_2$$

takes place at 45°C, the rate of the reaction of dinitrogen pentoxide is proportional to its concentration as follows:

$$-\frac{d[N_2O_5]}{dt} = 0.0005[N_2O_5]$$

(See Example 4 in Section 3.8.)
(a) Find an expression for the concentration $[N_2O_5]$ after t seconds if the initial concentration is C.
(b) How long will the reaction take to reduce the concentration of N_2O_5 to 90% of its original value?

11. Strontium-90 has a half-life of 28 days.
(a) A sample has a mass of 50 mg initially. Find a formula for the mass remaining after t days.
(b) Find the mass remaining after 40 days.
(c) How long does it take the sample to decay to a mass of 2 mg?
(d) Sketch a graph of the mass function.

12. The half-life of cesium-137 is 30 years. Suppose a sample has a mass of 100 mg initially.
(a) Find the mass that remains after t years.
(b) Find the amount of the sample remaining after 100 years.
(c) Find the time at which only 1 mg of the sample will remain.

13. A sample of tritium-3 decayed to 94.5% of its original amount after 1 year.
(a) Find the half-life of tritium-3.
(b) Find the time it takes for the sample to decay to 20% of its original amount.

14. Scientists can determine the age of ancient objects by the method of *radiocarbon dating*. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore, the level of radioactivity must also decay exponentially.

Suppose a parchment fragment was discovered that had about 74% as much ^{14}C radioactivity as does plant material on Earth today. Estimate the age of the parchment.

15. A curve passes through the point (0, 5) and has the property that the slope of the curve at every point P is twice the y-coordinate of P. Find the equation of the curve.

16. A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

(a) If the temperature of the turkey is 150°F after half an hour, find the temperature after 45 minutes.

(b) Find the time at which the temperature of the turkey will have cooled to 100°F.

17. When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room, its temperature has increased to 10°C.

(a) Find the temperature of the drink after 50 minutes.

(b) Find the time at which the temperature of the drink will be 15°C.

18. A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When the temperature of the coffee is 70°C, it is cooling at a rate of 1°C per minute. Find the time at which this occurs.

19. The rate of change of atmospheric pressure P with respect to altitude is proportional to P, provided that the temperature is constant. At 15°C, the pressure is 101.3 kPa at sea level and 87.14 kPa at $h = 1000$ m.

(a) Find the pressure at an altitude of 3000 m.

(b) Find the pressure at the top of Mount McKinley, at an altitude of 6187 m.

20. (a) Suppose \$1000 is invested at 8% interest. Find the value of the investment at the end of 3 years if the interest is compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) weekly, (v) daily, (vi) hourly, and (vii) continuously.

(b) Suppose \$1000 is invested and the interest is compounded continuously. Let $A(t)$ be the value of the investment after t years, where $0 \le t \le 3$. Sketch the graph of A for each of the interest rates 6%, 8%, and 10% on the same coordinate axes.

21. (a) Suppose \$3000 is invested at 5% interest. Find the value of the investment at the end of 5 years if the interest is compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) weekly, (v) daily, (vi) hourly, and (vii) continuously.

(b) Let $A(t)$ be the amount of the investment at time t for the case of continuous compounding. Write a differential equation and an initial condition satisfied by $A(t)$.

22. (a) How long will it take for an investment to double in value if the interest rate is 6% compounded continuously?

(b) Find the annual interest rate for an investment to double in the same amount of time as in part (a).

23. Consider a population $P = P(t)$ with constant relative birth and death rates α and β, respectively, and a constant emigration rate m, where α, β, and m are positive constants. Assume that $\alpha > \beta$. Then the rate of change of the population at time t is modeled by the differential equation

$$\frac{dP}{dt} = kP - m \quad \text{where} \quad k = \alpha - \beta$$

(a) Find the solution of this equation that satisfies the initial condition $P(0) = P_0$.

(b) What condition on m will lead to an exponential expansion of the population?

(c) What condition on m will result in a constant population? A population decline?

(d) In 1847, the population of Ireland was about 8 million, and the difference between the relative birth and death rates was 1.6% of the population. Because of the potato famine in the 1840s and 1850s, about 210,000 inhabitants per year emigrated from Ireland. Was the population expanding or declining at that time? Justify your answer.

24. Let c be a positive number. A differential equation of the form

$$\frac{dy}{dt} = ky^{1+c}$$

where k is a positive constant, is called a *doomsday equation* because the exponent in the expression ky^{1+c} is larger than exponent 1, for natural growth.

(a) Determine the solution that satisfies the initial condition $y(0) = y_0$.

(b) Show that there is a finite time $t = T$ (doomsday) such that $\lim_{t \to T^-} y(t) = \infty$.

(c) An especially prolific breed of rabbits has the growth term $ky^{1.01}$. If 2 such rabbits breed initially and the warren has 16 rabbits after 3 months, when is doomsday?

Applied Project | Calculus and Baseball

The purpose of this project is to explore two applications of calculus to baseball. The physical interactions associated with a game, especially the collision of the ball and the bat, are quite complex and related mathematical models are presented in a book by Robert Adair, *The Physics of Baseball*, 3rd Edition, New York, 2002.

1. When a batter hits a baseball thrown by a pitcher, the collision between the ball and the bat lasts only about a thousandth of a second. Let's try to calculate the average force on the bat during this collision by first computing the change in the ball's momentum.

The *momentum* p of an object is the product of its mass m and its velocity v, that is, $p = mv$. Suppose an object, moving along a straight line, is acted on by a force $F = F(t)$ that is a continuous function of time.

(a) Show that the change in momentum over a time interval $[t_0, t_1]$ is equal to the definite integral of F from t_0 to t_1; that is, show that

$$p(t_1) - p(t_0) = \int_{t_0}^{t_1} F(t)\,dt$$

This integral is called the *impulse* of the force over the time interval.

(b) A pitcher throws a 90-mi/h fastball to a batter, who hits a line drive directly back to the pitcher. The ball is in contact with the bat for 0.001 s and leaves the bat with velocity 110 mi/h. A baseball weighs 5 oz and, in U.S. Customary units, its mass is measured in slugs: $m = w/g$, where $g = 32\ \text{ft/s}^2$.

(i) Find the change in the ball's momentum.

(ii) Find the average force on the bat.

2. Now we will try to find the work required for a pitcher to throw a 90-mi/h fastball by first considering kinetic energy.

The *kinetic energy* K of an object of mass m and velocity v is given by $K = \frac{1}{2}mv^2$. Suppose an object of mass m, moving along a straight line, is acted on by a force $F = F(s)$ that depends on its position s. According to Newton's Second Law

$$F(s) = ma = m\,\frac{dv}{dt}$$

where a and v denote the acceleration and velocity of the object.

(a) Show that the work done in moving the object from a position s_0 to a position s_1 is equal to the change in the object's kinetic energy; that is, show that

$$W = \int_{s_0}^{s_1} F(s)\,ds = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2$$

where $v_0 = v(s_0)$ and $v_1 = v(s_1)$ are the velocities of the object at the positions s_0 and s_1.
Hint: By the Chain Rule,

$$m\,\frac{dv}{dt} = m\,\frac{dv}{ds}\frac{ds}{dt} = mv\,\frac{dv}{ds}$$

(b) How many foot-pounds of work does it take to throw a baseball at a speed of 90 mi/h?

3. (a) An outfielder catches a baseball 280 ft away from home plate and throws it directly to the catcher with an initial velocity of 100 ft/s. Assume that the velocity $v(t)$ of the ball after t seconds satisfies the differential equation $\frac{dv}{dt} = -\frac{1}{10}v$ because of air resistance. How long does it take for the ball to reach home plate? (Ignore any vertical motion of the ball.)

(b) The manager of a team wonders whether the ball will reach home plate sooner if it is relayed by an infielder. The shortstop can position himself directly between the outfielder and home plate, catch the ball thrown by the outfielder, turn, and throw the ball to the catcher with an initial velocity of 105 ft/s. The manager found the relay time of the shortstop (catching, turning, throwing) to be half a second. How far from home plate should the shortstop position himself to minimize the total time for the ball to reach home plate? Should the manager encourage a direct throw or a relayed throw? What if the shortstop can throw at 115 ft/s?

(c) For what throwing velocity of the shortstop does a relayed throw take the same time as a direct throw?

7.5 The Logistic Equation

In this section, we discuss in detail a model for population growth, the logistic model, that is more sophisticated than exponential growth. We will use many of the techniques we have learned; slope fields and Euler's method from Section 7.2 and the method of separation of variables for solving a differential equation from Section 7.3. Other possible models for population growth are introduced in the Exercises, some of which take into account harvesting and seasonal growth.

The Logistic Model

As we discussed in Section 7.1, the size of a population often increases exponentially in its early stages but levels off eventually and approaches its *carrying capacity* because of limited resources. If $P(t)$ is the size of the population at time t, we assume that

$$\frac{dP}{dt} \approx kP \quad \text{if } P \text{ is small}$$

This means that the growth rate is initially close to being proportional to the population's size. That is, the relative growth rate is almost constant when the population is small. But we also want to include in the model the fact that the relative growth rate decreases as the population P increases and becomes negative if P ever exceeds its **carrying capacity** M, the maximum population that the environment is capable of sustaining in the long run. The simplest expression for the relative growth rate that incorporates these assumptions is

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$$

If we multiply both sides by P, we obtain the model for population growth known as the **logistic differential equation**.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \tag{1}$$

A Closer Look

1. If P is small in comparison to M, then $\dfrac{P}{M}$ is close to 0, so $\dfrac{dP}{dt} \approx kP$.
2. As $P \to M$ (the population approaches its carrying capacity), then $\dfrac{P}{M} \to 1$, so $\dfrac{dP}{dt} \to 0$.
3. We can use Equation 1 to determine whether a particular solution of the logistic equation is increasing or decreasing to M.
   - **(a)** If the population P lies between 0 and M, then $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right) > 0$ and the population increases.
   - **(b)** If the population exceeds the carrying capacity $(P > M)$, then $1 - \dfrac{P}{M}$ is negative and $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right) < 0$, so the population decreases.
4. The logistic growth model also follows from the statement: the rate of change of a quantity (y) is jointly proportional to the size of the quantity and the difference between the quantity and the carrying capacity (M).

   In this case, the logistic differential equation is written as $\dfrac{dy}{dt} = ky(M - y)$, where k is the growth constant. ■

Slope Fields

Let's consider a more detailed analysis of the logistic differential equation by looking at a slope field.

Example 1 Logistic Differential Equation Slope Field

Draw a slope field for the logistic equation with $k = 0.08$ and carrying capacity $M = 1000$. What can you conclude about the solutions to this equation?

Solution

For these values of k and M, the differential equation is

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right).$$

A slope field for this equation is shown in Figure 7.36. Only the first quadrant is shown because neither negative populations nor negative time values are meaningful in this context.

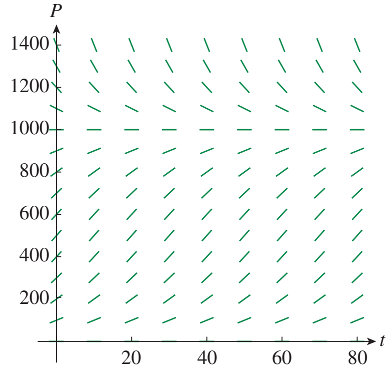


Figure 7.36
Slope field for the logistic differential equation.

The logistic equation is autonomous; $\dfrac{dP}{dt}$ depends only on P, not on t, so the slopes are the same along any horizontal line.

The slopes are positive for $0 < P < 1000$ and negative for $P > 1000$.

The slopes are small in magnitude when P is close to 0 or 1000 (the carrying capacity).

If the population is ever 0 or 1000, it remains that size. These two constant solutions are the equilibrium solutions. Notice that all other solutions move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = 1000$.

Solution curves with $P(0) = 100$, $P(0) = 400$, and $P(0) = 1300$ are shown in Figure 7.37.

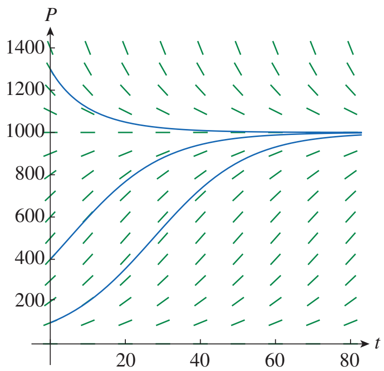


Figure 7.37
The slope field with several solution curves.

Notice that the solution curves that start below $P = 1000$ increase and those that start above $P = 1000$ decrease.

The slopes are greatest when $P \approx 500$, and therefore, the solution curves that start below $P = 1000$ have an inflection point when $P \approx 500$.

It can be shown that all solution curves that start below $P = 500$ have an inflection point when P is exactly 500 ($= M/2$). ■

Euler's Method

Next, let's use Euler's method to obtain numerical estimates for solutions of the logistic differential equation at specific times.

Example 2 Solution of a Logistic Equation Using Euler's Method

Use Euler's method with step sizes 20, 10, 5, 1, and 0.1 to estimate the population sizes $P(40)$ and $P(80)$, where P is the solution of the initial-value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

Solution

We have step size $h = 20$, $t_0 = 0$, $P_0 = 100$, and $F(t, P) = 0.08P\left(1 - \dfrac{P}{1000}\right)$.

Using the notation introduced in Section 7.2:

$t = 20$: $\quad P_1 = 100 + 20F(0, 100) = 244.$

$t = 40$: $\quad P_2 = 244 + 20F(20, 244) \approx 539.14.$

$t = 60$: $\quad P_3 = 539.14 + 20F(40, 539.14) \approx 936.69.$

$t = 80$: $\quad P_4 = 936.60 + 20F(60, 936.69) \approx 1031.57.$

Therefore, the estimates for the population sizes at times $t = 40$ and $t = 80$ are

$P(40) \approx 539 \quad$ and $\quad P(80) \approx 1032.$

Table 7.5 shows the approximations for smaller step sizes and Figure 7.38 shows a graph of the Euler approximations with step sizes $h = 10$ and $h = 1$.

Step size	Euler Estimate of $P(40)$	Euler Estimate of $P(80)$
20	539	1032
10	647	997
5	695	991
1	725	986
0.1	731	985

Table 7.5
Euler estimates for various step sizes.

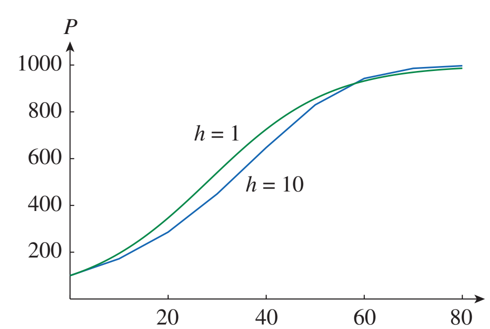


Figure 7.38
Graphs of Euler approximations of the solution curve.

Notice that the Euler approximation with $h = 1$ looks very similar to the lower solution curve sketched using a slope field in Figure 7.37. ■

The Analytic Solution

We can explicitly solve the logistic equation by using separation of variables.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$ The logistic equation.

$$\int \frac{dP}{P(1 - P/M)} = \int k\,dt$$ Separate variables; integrate both sides.

$$\frac{1}{P(1 - P/M)} = \frac{M}{P(M - P)} = \frac{1}{P} + \frac{1}{M - P}$$ Partial fraction decomposition on the left side.

$$\int\left(\frac{1}{P} + \frac{1}{M - P}\right)dP = \int k\,dt$$ Rewrite integrand on left side.

$$\ln|P| - \ln|M - P| = kt + C$$ Antiderivatives.

$$\ln\left|\frac{M - P}{P}\right| = -kt - C$$ Multiply both sides by -1; property of logarithms.

$$\left|\frac{M - P}{P}\right| = e^{-kt - C} = e^{-C}e^{-kt}$$ Exponentiate both sides.

$$\frac{M - P}{P} = Ae^{-kt}$$ Let $A = \pm e^{-C}$.

$$\frac{M}{P} - 1 = Ae^{-kt} \quad \Rightarrow \quad \frac{P}{M} = \frac{1}{1 + Ae^{-kt}}$$ Two separate fractions; add 1 to both sides, reciprocal.

$$P = \frac{M}{1 + Ae^{-kt}}$$ Multiply both sides by M.

Note that $P_0 > 0$.

If the initial condition is $P(0) = P_0$, then note that $P_0 > 0$.

$$P_0 = \frac{M}{1 + A \cdot e^0} \quad \Rightarrow \quad A = \frac{M - P_0}{P_0}$$

The solution of the logistic equation is

$$P(t) = \frac{M}{1 + Ae^{-kt}} \quad \text{where} \quad A = \frac{M - P_0}{P_0} \tag{2}$$

Consider the limit of $P(t)$ as t increases without bound.

$$\lim_{t\to\infty} \frac{M}{1 + Ae^{-kt}} = \frac{M}{1} = M$$

This mathematical result can be visualized by using the slope field (Figure 7.37), and it agrees with the model assumption that over time, the population moves toward the equilibrium solution, $P = M$.

Note: If we write the logistic equation in a more general form in which y represents a quantity, $\frac{dy}{dt} = ky(M - y)$, then the solution is

$$y(t) = \frac{M}{1 + Ae^{-kMt}} \quad \text{where} \quad A = \frac{M - y(0)}{y(0)}$$

Example 3 An Explicit Solution of the Logistic Equation

Consider the initial-value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

(a) Find the size of the population at time $t = 40$ and at time $t = 80$.
(b) At what time does the population reach 900?

Solution

(a) This is a logistic differential equation with $k = 0.08$, carrying capacity $M = 1000$, and initial population $P_0 = 100$.

Use Equation 2.

$$P(t) = \frac{1000}{1 + Ae^{-0.08t}} \text{ where } A = \frac{1000 - 100}{100} = 9 \quad \Rightarrow \quad P(t) = \frac{1000}{1 + 9e^{-0.08t}}$$

Compare these values with the Euler estimates from Example 2: $P(40) \approx 731$ and $P(80) \approx 985$.

The population sizes when $t = 40$ and when $t = 80$ are

$$P(40) = \frac{1000}{1 + 9e^{-0.08 \cdot 40}} = \frac{1000}{1 + 9e^{-3.2}} \approx 731.6.$$

$$P(80) = \frac{1000}{1 + 9e^{-0.08 \cdot 80}} = \frac{1000}{1 + 9e^{-6.4}} \approx 985.3.$$

(b) The population reaches 900 when $P(t) = 900$.

$$\frac{1000}{1 + 9e^{-0.08t}} = 900 \qquad \text{Set } P(t) = 900.$$

$$1 + 9e^{-0.08t} = \frac{10}{9} \qquad \text{Multiply by } 1 + 9e^{-0.08t}\text{; divide by 900.}$$

$$e^{-0.08t} = \frac{1}{81} \qquad \text{Subtract 1 from both sides; divide by 9.}$$

$$-0.08t = \ln\frac{1}{81} = -\ln 81 \qquad \text{Natural logarithm of both sides; property of logarithms.}$$

$$t = \frac{\ln 81}{0.08} \approx 54.9 \qquad \text{Divide by } -0.08.$$

The population reaches 900 when t is approximately 55. ■

Comparison of the Natural Growth and Logistic Models

In the 1930s, the biologist G. F. Gause conducted an experiment with the protozoan *Paramecium* and used a logistic equation to model his data. Table 7.6 gives his daily count of the population of protozoa. He estimated the initial relative growth rate to be 0.7944 protozoa/day and the carrying capacity to be 64.

t (days)	1	2	3	4	5	6	7	8	9	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57

Table 7.6
Daily count of protozoa.

Example 4 Models for the Protozoan Population

Find the exponential and logistic models for Gause's data. Compare the predicted values with the observed values for days 0–16 and comment on the accuracy of each model.

Solution

Given the relative growth rate $k = 0.7944$ and the initial population $P_0 = 2$, then the exponential model is $P(t) = P_0 e^{kt} = 2e^{0.7944t}$.

Gause used the same value for k for his logistic model. This is reasonable because $P_0 = 2$ is small in comparison with the carrying capacity, $M = 64$. The equation

$$\frac{1}{P_0}\left.\frac{dP}{dt}\right|_{t=0} = k\left(1 - \frac{2}{64}\right) \approx k$$

shows that the value of k for the logistic model is very close to the value for the exponential model.

The logistic equation is

$$P(t) = \frac{M}{1 + Ae^{-kt}} = \frac{64}{1 + Ae^{-0.7944t}}.$$

$$A = \frac{M - P_0}{P_0} = \frac{64 - 2}{2} = 31 \quad \Rightarrow \quad P(t) = \frac{64}{1 + 31e^{-0.7944t}}$$

Use these equations to calculate the predicted values (rounded to the nearest integer) and compare them in Table 7.7.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57
P (logistic model)	2	4	9	17	28	40	51	57	61	62	63	64	64	64	64	64	64
P (exponential model)	2	4	10	22	48	106	. . .										

Table 7.7
Predicted values of protozoa.

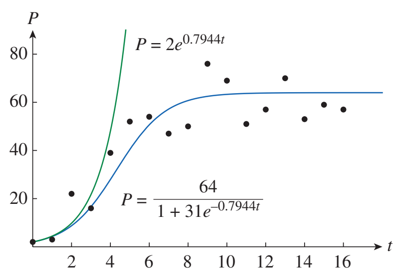


Figure 7.39
The exponential and logistic models for the *Paramecium* data.

Figure 7.39 shows a graph of the two models along with a scatter plot of the original data.

The table and the graph suggest that for the first 3 or 4 days, the exponential model provides good results. For $t \geq 5$, the exponential model is very inaccurate, but the logistic model fits the observations reasonably well for the entire time period. ■

Many cities and countries that once experienced exponential growth are now finding that their rates of population growth are declining and that the logistic model provides a better model. Figure 7.40 shows a scatter plot of the population of Los Angeles County, at time t from 1971 to 2021 and a graph of a shifted logistic function obtained using technology. This logistic model provides a very good fit.

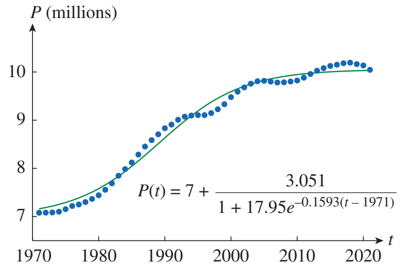


Figure 7.40
Logistic model for the population of Los Angeles County.

Other Models for Population Growth

The Law of Natural Growth and the logistic differential equation are not the only equations that have been proposed to model population growth. The exercises include examples of the Gompertz growth function and seasonal-growth models.

Two other models are modifications of the logistic model. The differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - c$$

has been used to model populations that are subject to harvesting of one sort or another. An example is a population of fish which are caught (harvested) at a constant rate.

For some species, there is a minimum population level m below which the species tends to become extinct. For example, adults may not be able to find suitable mates among a group that is too small. Such populations have been modeled by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

where the extra factor, $1 - \dfrac{m}{P}$, accounts for the consequences of a sparse population.

7.5 Exercises

A population grows according to the given logistic equation, where t is measured in weeks.

(a) What is the carrying capacity? What is the value of k?

(b) Write the solution of the equation.

(c) What is the population after 10 weeks?

1. $\dfrac{dP}{dt} = 0.04P\left(1 - \dfrac{P}{1200}\right), \quad P(0) = 60$

2. $\dfrac{dP}{dt} = 0.02P - 0.0004P^2, \quad P(0) = 40$

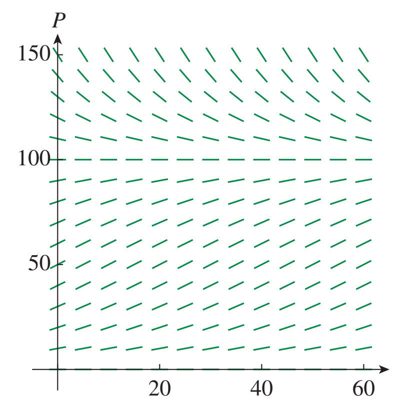
3. Suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 0.0005P^2$$

where t is measured in weeks.

(a) What is the carrying capacity? What is the value of k?

(b) A slope field is shown in the figure.



Where are the slopes close to 0? Where are they largest? Which solutions are increasing? Which solutions are decreasing?

(c) (i) Use the slope field to sketch solutions for initial populations of 20, 40, 60, 80, 120, and 140.

(ii) What do these solutions have in common? How do they differ?

(iii) Which solutions have inflection points? At what population levels do they occur?

(d) What are the equilibrium solutions? How are the other solutions related to these solutions?

4. Suppose that a population grows according to a logistic model with carrying capacity 6000 and $k = 0.0015$ per year.
(a) Write the logistic differential equation for this population growth.
(b) Draw a slope field for this differential equation. What does it suggest about the solution curves?
(c) Use the slope field to sketch the solution curves for initial populations of 1000, 2000, 4000, and 8000. What can you say about the concavity of the solution curves? What is the significance of the inflection points?
(d) Use Euler's method with step size $h = 1$ to estimate the population after 50 years if the initial population is 1000.
(e) If the initial population is 1000, write a formula for the population after t years. Use it to find the population after 50 years and compare with your estimate in part (d).
(f) Graph the solution curve in part (e) and compare with the solution curve sketched in part (c).

5. Suppose a population $P(t)$ satisfies the differential equation

$$\frac{dP}{dt} = 0.4P - 0.001P^2 \qquad P(0) = 50$$

where t is measured in years.
(a) What is the carrying capacity?
(b) Find the value of $P'(0)$.
(c) When will the population reach 50% of the carrying capacity?

6. Consider the logistic differential equation

$$\frac{dP}{dt} = -0.005P^2 + 2P$$

(a) Find $\lim_{t\to\infty} P(t)$. What does this value represent in the context of a logistic model?
(b) How large is the population when it is growing fastest?

7. Suppose a population grows according to a logistic model with initial population 1000 and carrying capacity 10,000. If the population grows to 2500 after 1 year, what will the population be after another 3 years?

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8. The rate at which oil is being produced from a petroleum reserve at time t satisfies the differential equation

$$\frac{dB}{dt} = 0.05B\left(1 - \frac{B}{12}\right)$$

where B is measured in millions of barrels of oil and t is measured in months. At time $t = 0$, one million barrels have already been produced. Find the value of B for which the production rate is increasing the fastest.

9. A population of chipmunks grows according to the logistic differential equation

$$\frac{dP}{dt} = -0.002P^2 + 6P$$

(a) Find $\lim_{t\to\infty} P(t)$. Explain the meaning of this value in the context of this problem.
(b) What is the population of chipmunks when it is growing fastest?

10. The number of antibodies y in a patient's bloodstream at time t is increasing according to a specific logistic growth model $y = B(t)$. Which of the following statements could be true?

(A) $\dfrac{dy}{dt} = 0.25y$

(B) $y = \dfrac{3000}{1 + e^{0.025t}}$

(C) $\dfrac{d^2y}{dt^2} > 0 \quad$ for all $t > 0$

(D) $\displaystyle\int \frac{dy}{y(3000 - y)} = \int 0.25\, dt$

11. For any time $t \geq 0$, where t is measured in weeks, the rate of spread of an epidemic is jointly proportional to the number of people who are infected and the number of people who are uninfected. The number of people $P(t)$ who are infected at time t satisfies the differential equation

$$\frac{dP}{dt} = 100P(1000 - P)$$

In a small, isolated town of 1000 inhabitants, 200 people have the disease after 1 week.
(a) Find $\lim_{t\to\infty} P(t)$. Explain the significance of this value in the context of this problem.
(b) How many people have been affected by the disease when it is spreading most rapidly?
(c) Is the disease spreading faster when 300 people have the disease or when 400 people have the disease? Explain your reasoning.
(d) Solve the differential equation and write the model for the number of people affected by the disease, $y = P(t)$, for any time $t \geq 0$.

12. For any time $t \geq 0$, where t is measured in years, a rabbit population is growing at a rate that is jointly proportional to the population and the difference between the rabbit population and its carrying capacity. Let the function $P(t)$ represent the rabbit population at time t, where P is measured in millions of rabbits. The function $P(t)$ satisfies the differential equation

$$\frac{dP}{dt} = 3P(1 - P)$$

(a) Find $\lim_{t\to\infty} P(t)$. Explain the significance of this value in the context of this problem.
(b) What is the rabbit population when it is growing most rapidly?
(c) Solve the differential equation using the initial condition $P(0) = 0.2$.
(d) Use the function $P(t)$ from part (c) to determine the time at which the rabbit population is growing most rapidly.

13. The Pacific halibut fishery has been modeled by the differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$$

where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7$ kg, and $k = 0.71$ per year.
(a) If $y(0) = 2 \times 10^7$ kg, find the biomass a year later.
(b) How long will it take for the biomass to reach 4×10^7 kg?

14. The table gives the number of yeast cells in a new laboratory culture at certain times.

Time (hours)	Yeast cells	Time (hours)	Yeast cells
0	18	10	509
2	39	12	597
4	80	14	640
6	171	16	664
8	336	18	672

(a) Plot the number of yeast cells over time and use the plot to estimate the carrying capacity for the yeast population.
(b) Use the data to estimate the initial relative growth rate.
(c) Use technology to find both an exponential model and a logistic model for these data.
(d) Compare the predicted values with the observed values, both in a table and with graphs. Comment on how well your models fit the data.
(e) Use your logistic model to estimate the number of yeast cells after 7 hours.

15. The population of the world was about 6.1 billion in 2000. The birth rate around this time was approximately 37.5 million per year and the death rate was approximately 17.5 million per year. Let's assume that the carrying capacity for the world population is 20 billion.
(a) Write the logistic differential equation for these data. Use an estimate of the initial relative growth rate for the value of k.
(b) Use the logistic model to estimate the world population in the year 2020 and compare with the actual population of 7.8 billion.
(c) Use the logistic model to predict the world population in the years 2100 and 2500.

16. (a) Make a guess as to the carrying capacity for the U.S. population. Use your guess and the fact that the population was 281.4 million in 2000 to formulate a logistic model for the U.S. population.
(b) Determine the value of k in your model by using the fact that the population in 2010 was 308.7 million.
(c) Use your model to predict the U.S. population in the years 2050 and 2100.
(d) Use your model to predict the year in which the U.S. population will exceed 375 million.

17. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor.
(a) Write a differential equation that incorporates this assumption and is satisfied by y.
(b) Solve the differential equation.
(c) A small town has 1000 inhabitants. At 8 AM, 80 people have heard a rumor. By noon, half the town has heard it. At what time will 90% of the population have heard the rumor?

18. Biologists stocked a lake with 400 fish and estimated the carrying capacity to be 10,000. The number of fish tripled in the first year.
(a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.
(b) How long will it take for the population to increase to 5000?

19. Suppose the function P satisfies the logistic differential equation.
(a) Show that

$$\frac{d^2P}{dt^2} = k^2P\left(1 - \frac{P}{M}\right)\left(1 - \frac{2P}{M}\right)$$

(b) Use the result in part (a) to show that a population grows fastest when it reaches half its carrying capacity.

20. For a fixed value of M, for example, $M = 10$, the family of logistic functions depends on the initial value P_0 and the proportionality constant k. Graph several members of this family. Explain how the graph changes when P_0 changes. How does the graph change when k varies?

21. The table gives the midyear population of Japan, in thousands, from 1960 to 2020.

Year	Population	Year	Population
1960	94,092	1995	125,327
1965	98,883	2000	126,776
1970	104,345	2005	127,715
1975	111,573	2010	127;579
1980	116,807	2015	126,920
1985	120,754	2020	126,476
1990	123,537		

Use technology to find both an exponential function and a logistic function to model these data. Graph the data points and both functions, and describe the accuracy of the models.

Hint: Subtract 94,000 from each of the population figures. Then, after obtaining a model using technology, add 94,000 to obtain your final model. It is also helpful to let $t = 0$ correspond to 1960.

22. The table gives the midyear population of Norway, in thousands, from 1960 to 2020.

Year	Population	Year	Population
1960	3581	1995	4359
1965	3723	2000	4492
1970	3877	2005	4625
1975	4007	2010	4891
1980	4086	2015	5208
1985	4152	2020	5368
1990	4242		

Use technology to find both an exponential function and a logistic function to model these data. Graph the data points and both functions, and describe the accuracy of the models.

Hint: Subtract 3500 from each of the population figures. Then, after obtaining a model using technology, add 3500 to obtain your final model. It is also helpful to let $t = 0$ correspond to 1960.

23. Consider the following modification of the logistic differential equation presented in Example 3 of this section.

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - 15$$

(a) Suppose $P(t)$ represents a fish population at time t, where t is measured in weeks. Explain the meaning of the final term in the equation (-15) in the context of this problem.
(b) Draw a slope field for this differential equation.
(c) What are the equilibrium solutions?
(d) Use the slope field to sketch several solution curves. Describe what happens to the fish population for various initial populations.
(e) Solve this differential equation explicitly, either by using partial fractions or with technology. Use the initial populations 200 and 300. Graph the solutions and compare with your sketches in part (d).

24. Consider the differential equation

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - c$$

as a model for a fish population, where t is measured in weeks and c is a constant.
(a) Use technology to sketch slope fields for various values of c.
(b) Use the slope fields from part (a) to determine the values of c for which there is at least one equilibrium solution. For what values of c does the fish population always die out?
(c) Use the differential equation to confirm your graphical observation in part (b).
(d) What would you recommend for a limit to the weekly catch of this fish population? Explain your reasoning.

25. There is considerable evidence to support the theory that for some species there is a minimum population m such that the species will become extinct if the size of the population falls below m. This condition can be incorporated into the logistic equation by introducing the factor $\left(1 - \frac{m}{P}\right)$. Thus, the modified logistic model is given by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

(a) Use the differential equation to show that any solution is increasing if $m < P < M$ and decreasing if $0 < P < m$.
(b) For the case in which $k = 0.08$, $M = 1000$, and $m = 200$, draw a slope field and use it to sketch several solution curves. Describe what happens to the population for various initial conditions. What are the equilibrium solutions?
(c) Solve the differential equation explicitly, by using either partial fractions or technology. Use P_0 as the initial population.
(d) Use the solution in part (c) to show that if $P_0 < m$, then the species will become extinct.
Hint: Show that the numerator in your expression for $P(t)$ is 0 for some value of t.

26. Another model for a growth function for a limited population is given by the **Gompertz function**, which is a solution of the differential equation

$$\frac{dP}{dt} = c \ln\left(\frac{M}{P}\right)P$$

where c is a constant and M is the carrying capacity.
(a) Solve this differential equation.
(b) Compute $\lim_{t\to\infty} P(t)$.
(c) Graph the Gompertz growth function for $M = 1000$, $P_0 = 100$, and $c = 0.05$, and compare it with the graph of the logistic function in Example 3. Explain similarities and/or differences.
(d) The logistic function grows fastest when $P = \frac{M}{2}$. Use the Gompertz differential equation to show that the Gompertz function grows fastest when $P = \frac{M}{e}$.

27. In a **seasonal-growth model**, a periodic function of time is introduced to account for seasonal variation in the rate of growth. Such variations could, for example, be caused by seasonal changes in the availability of food.
(a) Find the solution of the seasonal-growth model

$$\frac{dP}{dt} = kP\cos(rt - \phi) \qquad P(0) = P_0$$

where k, r, and ϕ are positive constants.
(b) Graph solution curves for several values of k, r, and ϕ. Explain how the values of k, r, and ϕ affect the solution. What can you conclude about $\lim_{t\to\infty} P(t)$?

28. Consider the following modification of the seasonal-growth model defined in Exercise 27.

$$\frac{dP}{dt} = kP\cos^2(rt - \phi) \qquad P(0) = P_0$$

(a) Solve this differential equation.
(b) Graph the solution for several values of k, r, and ϕ. Explain how the values of k, r, and ϕ affect the solution.
(c) What can you conclude about $\lim_{t\to\infty} P(t)$?

7.6 Predator-Prey Systems

We have looked at a variety of models for the growth of a single species that lives alone in an environment. In this section, we consider more realistic models that take into account the interaction of two species in the same habitat. We will see that these models take the form of a pair of linked differential equations.

We first consider the situation in which one species, called the *prey*, has an ample food supply and the second species, called the *predators*, feeds on the prey. Examples of prey and predators include rabbits and wolves in an isolated forest, food fish and sharks, aphids and ladybugs, and bacteria and amoebas. Our model will have two dependent variables and both are functions of time.

Suppose $R(t)$ is the number of prey (for example, R for rabbits) and $W(t)$ is the number of predators (with W for wolves) at time t. If there are no predators, an ample food supply would support exponential growth of the prey. Therefore,

$$\frac{dR}{dt} = kR \qquad \text{where } k \text{ is a positive constant}$$

If there were no prey, then we assume that the predator population would decline at a rate proportional to the number present. Therefore,

$$\frac{dW}{dt} = -rW \qquad \text{where } r \text{ is a positive constant}$$

If both species are present, we assume that the principal cause of death among the prey is caused by the predator, and the birth and survival rates of the predators depend on the available food supply, namely the prey. We also assume that the two species encounter each other at a rate that is proportional to both populations and is therefore proportional to the product RW. That is, the more there are of either population, the more encounters there are likely to be.

A system of two differential equations that incorporates these assumptions is

W represents the number of predators in the population, and *R* represents the number of prey.

$$\frac{dR}{dt} = kR - aRW \qquad \frac{dW}{dt} = -rW + bRW \tag{1}$$

where k, r, a, and b are positive constants. Notice that the term $-aRW$ decreases the natural growth rate of the prey and the term bRW increases the natural growth rate of the predators.

The Lotka-Volterra equations were proposed as a model to explain the variations in the shark and food-fish populations in the Adriatic Sea by the Italian mathematician Vito Volterra (1860–1940).

The equations in 1 are known as the **predator-prey equations**, or the **Lotka-Volterra equations**. A **solution** of this system of equations is a pair of functions $R(t)$ and $W(t)$ that describe the populations of prey and predator as functions of time. Because the system is coupled (R and W occur in both equations), we can't solve one equation and then the other; we have to solve them simultaneously. Unfortunately, it is usually impossible to find explicit formulas for R and W as functions of t. We can, however, use graphical methods to analyze the equations.

Example 1 Rabbits versus Wolves

Suppose that in a certain environment, populations of rabbits and wolves are described by the Lotka-Volterra equations with $k = 0.08$, $a = 0.001$, $r = 0.02$, and $b = 0.00002$. The time t is measured in months.

(a) Find the constant solutions (called the **equilibrium solutions**) and interpret these expressions.

(b) Use the system of differential equations to find an expression for $\dfrac{dW}{dR}$.

(c) Draw a slope field for the resulting differential equation in the RW-plane. Use the slope field to sketch several solution curves.

(d) Suppose that, at some point in time, there are 1000 rabbits and 40 wolves. Draw the corresponding solution curve and use it to describe the changes in both population levels.

(e) Use part (d) to sketch R and W as functions of t.

Solution

(a) Use the values given for k, a, r, and b. The Lotka-Volterra equations are

$$\frac{dR}{dt} = 0.08\,R - 0.001RW \text{ and } \qquad \frac{dW}{dr} = -0.02W + 0.00002RW.$$

The populations represented by R and W will be constant if both derivatives are 0.

$$\frac{dR}{dt} = R(0.08 - 0.001W) = 0 \quad \Rightarrow \quad R = 0,\ W = \frac{0.08}{0.001} = 80$$

$$\frac{dW}{dt} = W(-0.02 + 0.00002R) = 0 \quad \Rightarrow \quad W = 0,\ \ R = \frac{0.02}{0.00002} = 1000$$

One equilibrium solution is $R = 0$ and $W = 0$. This seems reasonable: if there are no rabbits or wolves, the populations are certainly not going to change.

The other constant solution is $R = 1000$ and $W = 80$. This means that 1000 rabbits are just enough to support a constant wolf population of 80. There are neither too many wolves, which would result in fewer rabbits, nor too few wolves, which would result in more rabbits.

(b) Use the Chain Rule to eliminate t.

$$\frac{dW}{dt} = \frac{dW}{dR}\frac{dR}{dt} \qquad \text{Solve for } \frac{dW}{dR}.$$

$$\frac{dW}{dR} = \frac{\frac{dW}{dt}}{\frac{dR}{dt}} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

(c) Think of W as a function of R. Use the differential equation

$\dfrac{dW}{dR} = \dfrac{-0.02W + 0.00002\,RW}{0.08R - 0.001\,RW}$ to sketch the slope field, as shown in Figure 7.41.

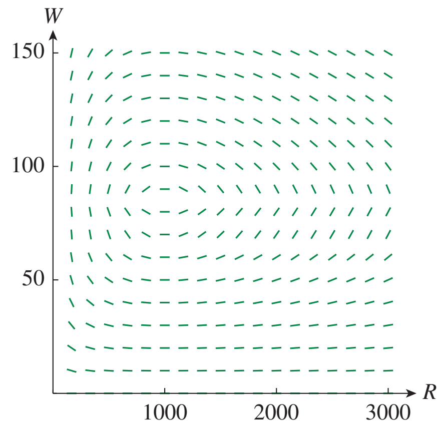


Figure 7.41
Slope field for the predator-prey (wolves-rabbits) system.

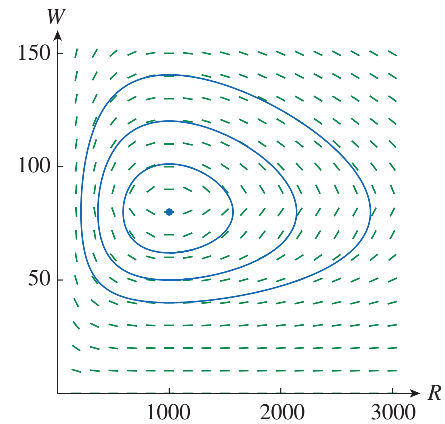


Figure 7.42
Phase portrait of the system.

Use this slope field to sketch several solution curves, as shown in Figure 7.42. These solution curves illustrate the relationship between R and W as time passes.

Notice that the curves appear to be *closed*. That is, as we trace along a curve, we always return to the same point.

Notice also that the point (1000, 80) is inside all of the solution curves. That point is called an *equilibrium point* because it corresponds to the equilibrium solution $R = 1000$, $W = 80$.

When we represent solutions of a system of differential equations as in Figure 7.42, the RW-plane is called the **phase plane**, and the solution curves are called **phase trajectories**.

Therefore, a phase trajectory is a path traced out by solutions (R, W) as times goes by.

A **phase portrait** consists of equilibrium points and typical phase trajectories, as shown in Figure 7.42.

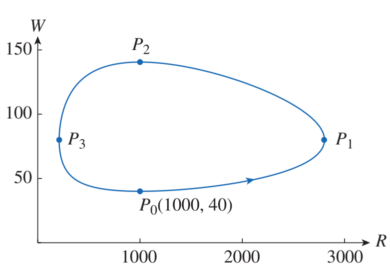


Figure 7.43
Phase trajectory through (1000, 40).

(d) Figure 7.43 shows the solution curve through the point $P_0(1000, 40)$, that is, starting with 1000 rabbits and 40 wolves. This phase trajectory does not include the slope field.

Starting at the point P_0 at time $t = 0$, we can determine the direction the populations move along the phase trajectory as t increases. Let $R = 1000$ and $W = 40$ in the first differential equation.

$$\frac{dR}{dt} = 0.08(1000) - 0.001(1000)(40) = 80 - 40 = 40$$

Since $\dfrac{dR}{dt} > 0$, R is increasing at P_0. Therefore, the populations are moving counterclockwise around the phase trajectory.

Here are some interpretations of the solution curve in Figure 7.43.

At P_0, there aren't enough wolves to maintain a balance between populations, so the rabbit population increases. This results in more wolves.

Eventually, there are so many wolves that the rabbits have a hard time avoiding them. The rabbit population reaches a maximum at P_1, and then begins to decline.

The number of wolves increases and reaches a maximum at P_2, until there aren't enough rabbits to support this many wolves. As the number of wolves begin to decline, the number of rabbits reaches a minimum at P_3.

At that point in time, the number of rabbits begin to increase until the populations return to their initial values, and the entire cycle begins again.

(e) Using the interpretation of the solution curve in part (d) of how the rabbit and wolf populations rise and fall, we can sketch the graphs of $R(t)$ and $W(t)$.

Suppose the points P_1, P_2, and P_3 in Figure 7.43 correspond to times t_1, t_2, and t_3. The graphs of R and W are shown in Figures 7.44 and 7.45.

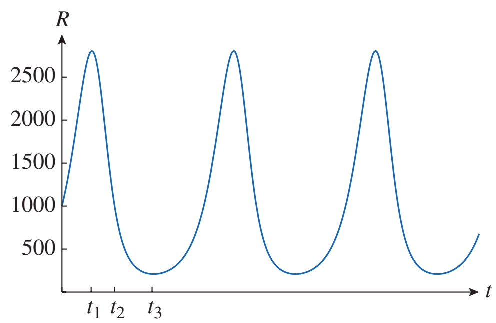


Figure 7.44
Graph of the rabbit population as a function of time.

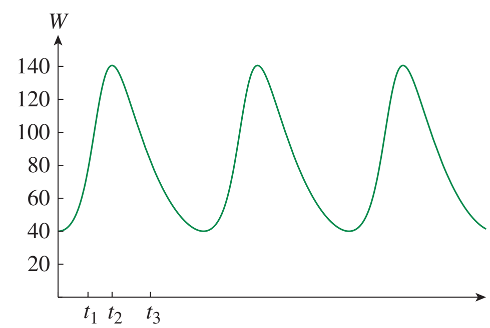


Figure 7.45
Graph of the wolf population as a function of time.

Another way to compare these two graphs is to combine them in one plot with the same horizontal axis but with different scales for R and W, the vertical axes, as in Figure 7.46. Notice that the rabbits reach their maximum populations about a quarter of a cycle before the wolves.

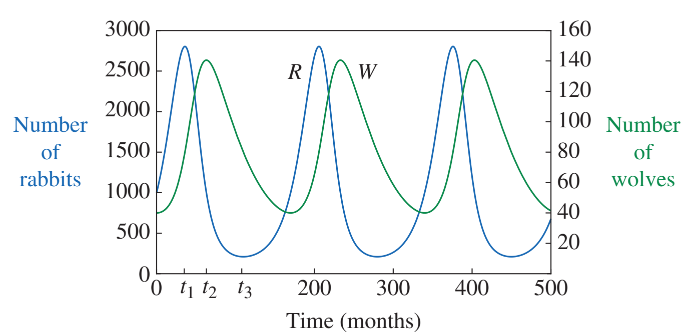


Figure 7.46
Comparison of the rabbit and wolf populations over time.

As discussed in Section 1.2, an important part of the modeling process is to interpret the mathematical conclusions as real-world predictions and to test the predictions against real data. The Hudson's Bay Company started trading in animal fur in Canada in 1670 and kept records that date back to the 1840s. Figure 7.47 shows graphs of the number of pelts of the snowshoe hare and its predator, the Canada lynx, traded by the company over a 90-year period. The coupled oscillations in the hare and lynx populations as predicted by the Lotka-Volterra model are evident in this graph. The period of these cycles is approximately 10 years.

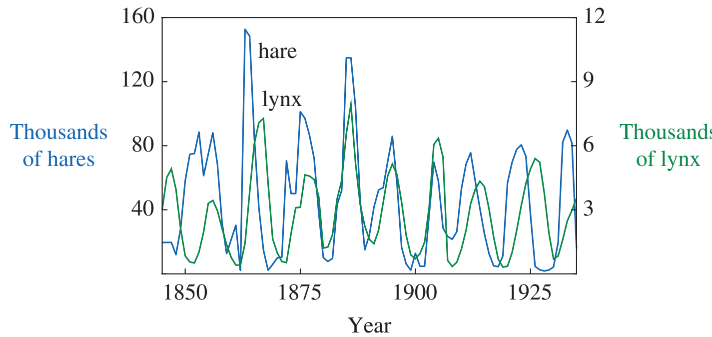


Figure 7.47
Relative abundance of hare and lynx from Hudson's Bay Company records.

Although the relatively simple Lotka-Volterra model has been used to explain and predict coupled populations, more sophisticated models have also been proposed. One way to modify the Lotka-Volterra equations is to assume that, in the absence of predators, the prey grow according to a logistic model with carrying capacity M. Then the Lotka-Volterra equations are replaced by the system of differential equations

$$\frac{dR}{dt} = kR\left(1 - \frac{R}{M}\right) - aRW \qquad \frac{dW}{dt} = -rW + bRW$$

There are some exercises in this section that involve this model.

Models have also been proposed to describe and predict population levels of two or more species that compete for the same resources or cooperate for mutual benefit. Some of these models are also explored in the Exercises.

7.6 Exercises

1. For each predator-prey system, determine which of the variables, x or y, represents the prey population and which represents the predator population. Is the growth of the prey restricted just by the predators or by other factors as well? Do the predators feed only on the prey or do they have additional food sources? Explain.

(a) $\dfrac{dx}{dt} = -0.05x + 0.0001xy$

$\dfrac{dy}{dt} = 0.1y - 0.005xy$

(b) $\dfrac{dx}{dt} = 0.2x - 0.0002x^2 - 0.006xy$

$\dfrac{dy}{dt} = -0.015y + 0.00008xy$

2. Each system of differential equations is a model for two species that either compete for the same resources or cooperate for mutual benefit (flowering plants and insect pollinators, for example). Decide whether each system describes competition or cooperation and explain why it is a reasonable model. (Think about the effect an increase in one species has on the growth rate of the other.)

(a) $\dfrac{dx}{dt} = 0.12x - 0.0006x^2 + 0.00001xy$

$\dfrac{dy}{dt} = 0.08x + 0.00004xy$

(b) $\dfrac{dx}{dt} = 0.15x - 0.0002x^2 - 0.0006xy$

$\dfrac{dy}{dt} = 0.2y - 0.00008y^2 - 0.0002xy$

3. The system of differential equations

$$\frac{dx}{dt} = 0.5x - 0.004x^2 - 0.001xy$$

$$\frac{dy}{dt} = 0.4y - 0.001y^2 - 0.002xy$$

is a model for the populations of two species.

(a) Does the model describe cooperation, or competition, or a predator-prey relationship? Explain your reasoning.

(b) Find the equilibrium solutions and explain their significance.

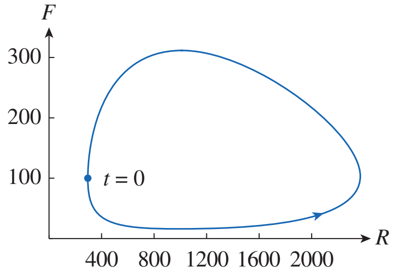
4. Flies, frogs, and crocodiles coexist in an environment. To survive, frogs need to eat flies and crocodiles need to eat frogs. In the absence of frogs, the fly population will grow exponentially and the crocodile population will decay exponentially. In the absence of crocodiles and flies, the frog population will decay exponentially. If $P(t)$, $Q(t)$, and $R(t)$ represent the populations of these three species at time t, write a system of differential equations as a model for their evolution. Suppose the constants in your equation are all positive, explain why you have used plus or minus signs in the system.

A phase trajectory is shown for populations of rabbits (R) and foxes (F).

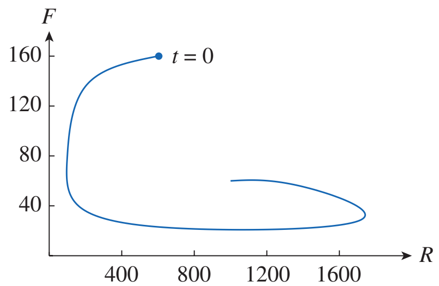
(a) Describe how each population changes as time goes by.

(b) Use your description to make a rough sketch of the graphs of R and F as functions of time.

5.

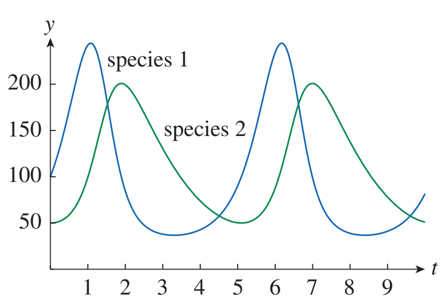


6.

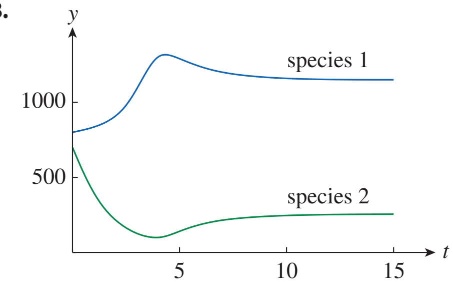


Graphs of populations of two species over time are shown. Use these graphs to sketch the corresponding phase trajectory.

7.



8.



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9. In Example 1(b), we showed that the rabbit and wolf populations satisfy the differential equation

$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.0001RW}$$

Solve this separable differential equation to show that

$$\frac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = C$$

where C is a constant.

It is impossible to solve this equation for W as an explicit function of R (or vice versa). However, a computer algebra system may be used to graph implicitly defined curves like this one. Use a CAS to draw the solution curve that passes through the point (1000, 40) and compare with Figure 7.43.

10. Populations of aphids and ladybugs are modeled by the equations

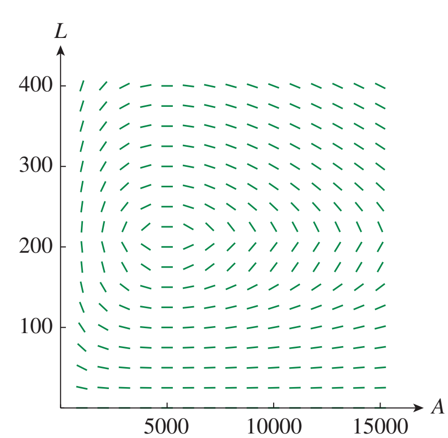
$$\frac{dA}{dt} = 2A - 0.01AL$$

$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

(a) Find the equilibrium solutions and explain their significance.

(b) Find an expression for $\dfrac{dL}{dA}$.

(c) The direction field for the differential equation in part (b) is shown. Use it to sketch a phase portrait. What do the phase trajectories have in common?



(d) Suppose that at time $t = 0$ there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it to describe how both populations change.

(e) Use part (d) to make rough sketches of the aphid and ladybug populations as functions of t. How are the graphs related to each other?

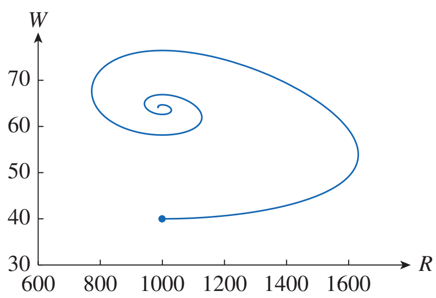
11. In Example 1, we used the Lotka-Volterra equations to model populations of rabbits and wolves. Consider these modified equations:

$$\frac{dR}{dt} = 0.08R(1 - 0.002R) - 0.001RW$$
$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

(a) According to these equations, what happens to the rabbit population in the absence of wolves?

(b) Find all the equilibrium solutions and explain their significance.

(c) The figure shows the phase trajectory that starts at the point (1000, 40). Describe what eventually happens to the rabbit and wolf populations.



(d) Sketch graphs of the rabbit and wolf populations as functions of time.

12. Consider the following modified Lotka-Volterra equations to model populations of aphids and ladybugs.

$$\frac{dA}{dt} = 2A(1 - 0.0001\,A) - 0.01AL$$
$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

(a) In the absence of ladybugs, what does the model predict about the aphids?

(b) Find the equilibrium solutions.

(c) Find an expression for $\dfrac{dL}{dA}$.

(d) Use a computer algebra system to draw a slope field for the differential equation in part (c). Then use the slope field to sketch a phase portrait. What do the phase trajectories have in common?

(e) Suppose that at time $t = 0$ there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it to describe how both populations change.

(f) Use part (e) to draw rough sketches of the aphid and ladybug populations as functions of t. How are the graphs related to each other?

7 Review

Concepts and Vocabulary

1. Define each expression in your own words.
   (a) Differential equation
   (b) The order of a differential equation
   (c) An initial condition
2. Explain the characteristics of the solutions of $y' = x^2 + y^2$ without solving the differential equation.
3. Explain the graph of a slope field for a differential equation of the form $y' = F(x, y)$.
4. Explain how to use Euler's method.
5. What is a separable differential equation? Explain how to solve a separable differential equation.
6. (a) Write a differential equation that expresses the law of natural growth. Interpret this differential equation in terms of relative growth rate.
   (b) Under what circumstances is this an appropriate model for population growth?
   (c) What are the solutions of this equation?
7. (a) Write the logistic differential equation.
   (b) Under what circumstances is this an appropriate model for population growth?
8. (a) Write the Lotka-Volterra equations to model populations of food fish (F) and sharks (S).
   (b) Explain the implications of these equations about each population in the absence of the other.

True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. All solutions of the differential equation $y' = -1 - y^4$ are decreasing functions.
2. The function $f(x) = \dfrac{\ln x}{x}$ is a solution of the differential equation $x^2y' + xy = 1$.
3. The differential equation $y' = x + y$ is separable.
4. The differential equation $y' = 3y - 2x + 6xy - 1$ is separable.
5. If y is the solution of the initial-value problem

$$\frac{dy}{dt} = 2y\left(1 - \frac{y}{5}\right) \qquad y(0) = 1$$

then $\lim_{t\to\infty} y = 5$.

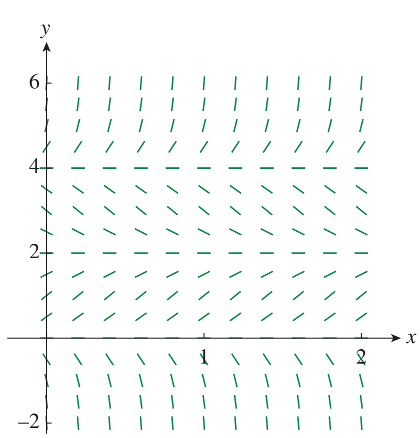
6. The solution of the initial-value problem

$$\frac{dy}{dt} = t(4 - 2y) \qquad y(0) = 2$$

is $y = 2$.

Exercises

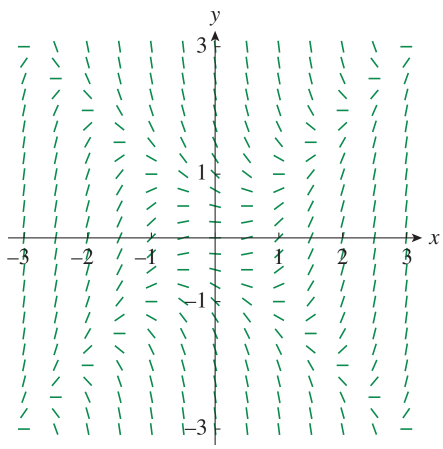
1. A slope field for the differential equation $y' = y(y - 2)(y - 4)$ is shown in the figure.



   (a) Sketch the graphs of the solutions that satisfy the given initial conditions.
      (i) $y(0) = -0.3$ (ii) $y(0) = 1$
      (iii) $y(0) = 3$ (iv) $y(0) = 4.3$
   (b) If the initial condition is $y(0) = c$, for what values of c is $\lim_{t\to\infty} y(t)$ finite? What are the equilibrium solutions?

2. Consider the differential equation $y' = \dfrac{x}{y}$.
   (a) Sketch the slope field and use it to sketch the four solutions that satisfy the initial conditions $y(0) = 1$, $y(0) = -1$, $y(2) = 1$, and $y(-2) = 1$.
   (b) Check your work in part (a) by solving the differential equation explicitly. What type of curve is the graph of each solution?

3. A slope field for the differential equation $y' = x^2 - y^2$ is shown in the figure.



(a) Sketch the solution of the initial-value problem

$$y' = x^2 - y^2 \qquad y(0) = 1$$

Use your graph to estimate the value of $y(0.3)$.

(b) Use Euler's method with step size 0.1 to estimate $y(0.3)$, where $y(x)$ is the solution of the initial-value problem in part (a). Compare with your estimate from part (a).

(c) On what lines are the centers of the horizontal line segments of the slope field in part (a) located? What happens when a solution curve crosses these lines?

4. Let $y(x)$ be the solution of the initial-value problem

$$y' = 2xy^2 \qquad y(0) = 1$$

(a) Use Euler's method with step size 0.2 to estimate $y(0.4)$.

(b) Repeat part (a) with step size 0.1.

(c) Find the exact solution of the differential equation and compare the value at 0.4 with the approximations in parts (a) and (b).

Solve each differential equation.

5. $\dfrac{dx}{dt} = 1 - t + x - tx$

6. $2ye^{y^2}y' = 2x + 3\sqrt{x}$

Solve the initial-value problem.

7. $\dfrac{dr}{dt} + 2tr = r, \quad r(0) = 5$

8. $(1 + \cos x)\,y' = (1 + e^{-y})\sin x, \qquad y(0) = 0$

9. $y' = xye^{-x}, \qquad y(0) = 1$

10. Solve the initial-value problem $y' = 3x^2e^y$, $y(0) = 1$, and graph the solution.

11. Suppose the function $y = f(x)$ is the solution of the differential equation $\dfrac{dy}{dx} = \dfrac{2x}{y}$ such that $f(4) = 6$.

(a) Write the equation of the tangent line to the graph of $y = f(x)$ at the point $(4, 6)$ and use it to estimate $f(4.8)$.

(b) Use Euler's method with two steps of equal size to estimate $f(4.8)$.

(c) Solve the differential equation for the initial value given and use your solution to evaluate $f(4.8)$.

Find the orthogonal trajectories of the family of curves.

12. $y = ke^x$

13. $y = e^{kx}$

14. Suppose a population $P(t)$ satisfies the initial-value problem

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{2000}\right) \qquad P(0) = 100$$

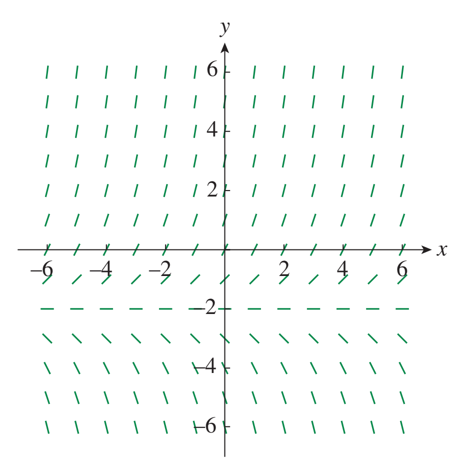
(a) Find a solution and use it to calculate the population when $t = 20$.

(b) When does the population reach 1200?

15. Suppose the function $y = f(x)$ is the solution of the differential equation $\dfrac{dy}{dx} = 2x^2y$ such that $f(0) = 2$. Use Euler's method with two steps of equal size to estimate $f(1)$.

16. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{-x}{y}$. Let $y = f(x)$ be a particular solution of this differential equation that satisfies the initial condition $f(0) = 3$. Use Euler's method with three steps of equal size to estimate $f(1.5)$.

17. Consider the slope field shown in the figure.



Which of the following differential equations could have this slope field?

(A) $\dfrac{dy}{dx} = y + 2$ (B) $\dfrac{dy}{dx} = y - 2$

(C) $\dfrac{dy}{dx} = x + 2$ (D) $\dfrac{dy}{dx} = x - 2$

18. A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour, the population has increased to 360 cells.
(a) Find the number of bacteria after t hours.
(b) Find the number of bacteria after 4 hours.
(c) Find the rate of growth after 4 hours.
(d) When will the population reach 10,000?

19. Cobalt-60 has a half-life of 5.24 years.
(a) Find the mass that remains from a 100 mg sample after 20 years.
(b) How long would it take for the mass to decay to 1 mg?

20. Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at that time. Therefore, $C'(t) = -kC(t)$, where k is a positive number called the *elimination constant* of the drug.
(a) If C_0 is the concentration at time $t = 0$, find the concentration at time t.
(b) If the body eliminates half the drug in 30 hours, how long does it take to eliminate 90% of the drug?

21. A cup of hot chocolate has temperature 80°C in a room kept at 20°C. After half an hour, the hot chocolate cools to 60°C.
(a) What is the temperature of the chocolate after another half hour?
(b) When will the chocolate have cooled to 40°C?

22. The population of the world was 6.1 billion in 2000 and 7.8 billion in 2020.
(a) Find an exponential model for these data and use the model to predict the world population in the year 2025.
(b) According to the model in part (a), when will the world population exceed 10 billion?
(c) Find a logistic model for the population. Assume a carrying capacity of 20 billion. Use the logistic model to predict the population in 2025. Compare with your prediction from part (a).
(d) According to the logistic model, when will the world population exceed 10 billion? Compare with your prediction in part (b).

23. The von Bertalanffy growth model is used to predict the length $L(t)$ of a fish over a period of time. If L_∞ is the largest length for a species, then the hypothesis is that the rate of growth in length is proportional to $L_\infty - L$, the length yet to be achieved.
(a) Write a differential equation to describe the rate of growth in length and solve it to find an expression for $L(t)$.
(b) For the North Sea haddock, it has been determined that $L_\infty = 53$ cm, $L(0) = 10$ cm, and the constant of proportionality is 0.2. Use this data to write the expression for $L(t)$.

24. One model for the spread of an epidemic is that the rate of spread is jointly proportional to the number of infected people and the number of uninfected people. In an isolated town of 5000 inhabitants, 160 people have a disease at the beginning of the week and 1200 have it at the end of the week. How long does it take for 80% of the population to become infected?

25. The Brentano-Stevens Law in psychology models the way that a subject reacts to a stimulus. It states that if R represents the reaction to an amount S of stimulus, then the relative rates of increase are proportional:

$$\frac{1}{R}\frac{dR}{dt} = \frac{k}{S}\frac{dS}{dt}$$

where k is a positive constant. Find R as a function of S.

26. A tank contains 100 L of pure water. Brine that contains 0.1 kg of salt per liter enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after 6 minutes?

27. The transport of a substance across a capillary wall in lung physiology has been modeled by the differential equation

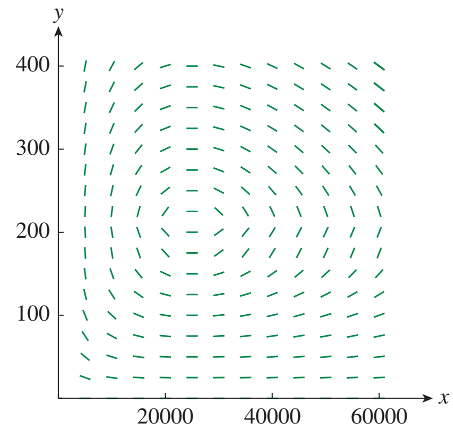
$$\frac{dh}{dt} = -\frac{R}{V}\left(\frac{h}{k + h}\right)$$

where h is the hormone concentration in the bloodstream, t is time, R is the maximum transport rate, V is the volume of the capillary, and k is a positive constant that measures the affinity between the hormones and the enzymes that assist the process. Solve this differential equation to find a relationship between h and t.

28. Populations of birds and insects are modeled by the equations

$$\frac{dx}{dt} = 0.4x - 0.002xy$$
$$\frac{dy}{dt} = -0.2y + 0.000008xy$$

(a) Which of the variables, x or y, represents the bird population and which represents the insect population? Explain your reasoning.
(b) Find the equilibrium solutions and explain their significance.
(c) Find an expression for $\dfrac{dy}{dx}$.
(d) The slope field for the differential equation in part (c) is shown in the figure. Use it to sketch the phase trajectory corresponding to initial populations of 100 birds and 40,000 insects. Then use the phase trajectory to describe how both populations change.



(e) Use part (d) to make rough sketches of the bird and insect populations as functions of time. How are these graphs related to each other?

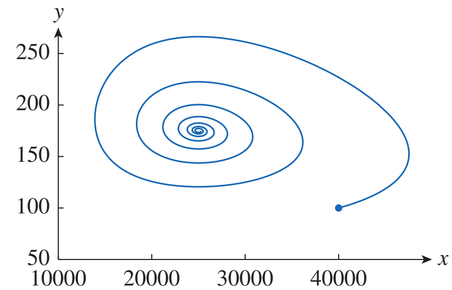
29. Suppose the model in Exercise 28 is replaced by the equations

$$\frac{dx}{dt} = 0.4x(1 - 0.000005x) - 0.02xy$$
$$\frac{dy}{dt} = -0.2y + 0.000008xy$$

(a) What do these equations imply about the insect population in the absence of birds?

(b) Find the equilibrium solutions and explain their significance.

(c) The figure shows the phase trajectory that starts with 100 birds and 40,000 insects. Use this graph to describe what eventually happens to the bird and insect populations.



(d) Sketch the graphs of the bird and insect populations as functions of time.

30. A person weighs 60 kg and is on a diet of 1600 calories per day, of which 850 are used automatically by basal metabolism. They spend about 15 cal/kg/day times their weight doing exercises. If 1 kg of fat contains 10,000 cal and we assume that the storage of calories in the form of fat is 100% efficient, formulate a differential equation and solve it to find their weight as a function of time. Does their weight ultimately approach an equilibrium weight?

Focus on Problem Solving

1. Find all functions f such that f' is continuous and

$$[f(x)]^2 = 100 + \int_0^x \left([f(t)]^2 + [f'(t)]^2\right) dt \quad \text{for all real } x$$

2. Suppose a student forgot the Product Rule for differentiation and made the mistake of thinking that $(fg)' = f'g'$. However, they were very lucky and got the correct answer anyway. The function f that they used was $f(x) = e^{x^2}$ and the domain of the problem was the interval $\left(\frac{1}{2}, \infty\right)$. Find the function g.

3. Let f be a function with the property that $f(0) = 1$, $f'(0) = 1$, and $f(a + b) = f(a)f(b)$ for all real numbers a and b. Show that $f'(x) = f(x)$ for all x and conclude that $f(x) = e^x$.

4. Find all functions f that satisfy the equation

$$\left(\int f(x)\, dx\right)\left(\int \frac{1}{f(x)}\, dx\right) = -1$$

5. Find the curve $y = f(x)$ such that $f(x) \geq 0$, $f(0) = 0$, $f(1) = 1$, and the area under the graph of f from 0 to x is proportional to the $(n + 1)$st power of $f(x)$.

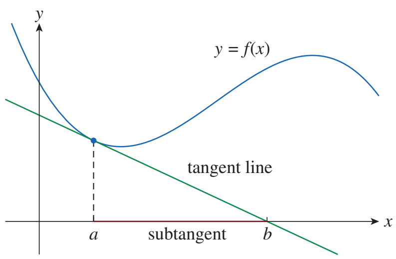


Figure 7.48
The subtangent is the portion of the x-axis between the points $(a, 0)$ and $(b, 0)$.

6. A *subtangent* is a portion of the x-axis that lies directly beneath the segment of a tangent line from the point of contact to the x-axis. Figure 7.48 shows an example of a subtangent. Find the curves that pass through the point $(c, 1)$ and whose subtangents all have length c.

7. An apple pie is removed from the oven at 5:00 PM. At that time, it is piping hot, 100°C. At 5:10 PM, its temperature is 80°C; at 5:20 PM, it is 65°C. What is the temperature of the room?

8. In Presque Isle, Maine, snow began to fall during the morning of February 2 and continued steadily into the afternoon. At noon, a snowplow began removing snow from a road at a constant rate. The plow traveled 6 km from noon to 1 PM but only 3 km from 1 PM to 2 PM. When did the snow begin to fall?

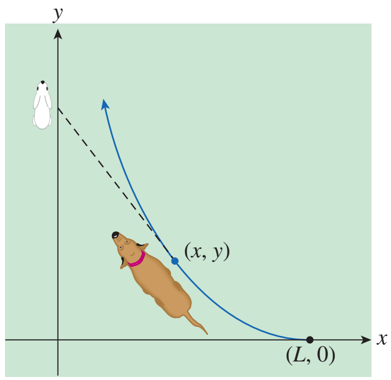
Hint: Let t be the time measured in hours after noon; let $x(t)$ be the distance traveled by the plow at time t; then the speed of the plow is $\dfrac{dx}{dt}$. Let b be the number of hours before noon that it began to snow. Find an expression for the height of the snow at time t. Then use the given information that the rate of removal R (in m^3/h) is constant.

9. A dog sees a rabbit running in a straight line across an open field and gives chase. In a rectangular coordinate system (as shown in the figure), assume:

(i) The rabbit is at the origin and the dog is at the point $(L, 0)$ at the instant the dog first sees the rabbit.

(ii) The rabbit runs along the y-axis and the dog always runs straight for the rabbit.

(iii) The dog runs at the same speed as the rabbit.



(a) Show that the dog's path is the graph of the function $y = f(x)$, where y satisfies the differential equation

$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(b) Determine the solution of the equation in part (a) that satisfies the initial conditions $y = y' = 0$ when $x = L$.

Hint: Let $z = \dfrac{dy}{dx}$ in the differential equation and solve the resulting first-order equation to find z; then integrate z to find y.

(c) Will the dog ever catch the rabbit? Justify your answer.

10. (a) Suppose the dog in Problem 9 runs twice as fast as the rabbit. Find a differential equation for the path of the dog. Then solve it to find the point where the dog catches the rabbit.

(b) Suppose the dog runs half as fast as the rabbit. How close does the dog get to the rabbit? What are their positions when they are closest?

11. A planning engineer for a new alum plant must present some estimates to their company regarding the capacity of a silo designed to contain bauxite ore until it is processed into alum. The ore resembles pink talcum powder and is poured from a conveyor at the top of the silo. The silo is a cylinder 100 ft high with a radius of 200 ft. The conveyor carries $60{,}000\pi$ ft^3/h and the ore maintains a conical shape whose radius is 1.5 times its height.

(a) If, at a certain time t, the pile is 60 ft high, how long will it take for the pile to reach the top of the silo?

(b) Management wants to know how much room will be left in the floor area of the silo when the pile is 60 ft high. How fast is the floor area of the pile growing at that height?

(c) Suppose a loader starts removing the ore at the rate of $20{,}000\pi$ ft^3/h when the height of the pile reaches 90 ft. Suppose, also, that the pile continues to maintain its shape. How long will it take for the pile to reach the top of the silo under these conditions?

12. Find the curve that passes through the point (3, 2) and has the property that if the tangent line is drawn at any point P on the curve, then the part of the tangent line that lies in the first quadrant is bisected at P.

13. Recall that the normal line to a curve at a point P on the curve is the line that passes through P and is perpendicular to the tangent line at P. Find the curve that passes through the point (3, 2) and has the property that if the normal line is drawn at any point on the curve, then the y-intercept of the normal line is always 6.

14. Find all curves with the property that if the normal line is drawn at any point P on the curve, then the part of the normal line between P and the x-axis is bisected by the y-axis.

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Large oil platforms can weigh over 200,000 tons. Many have special ice protection belts made from concrete, and can withstand waves of up to 18 meters high. Engineers use infinite series to examine the effect of waves and ice sheets when designing offshore structure like this.

8 Infinite Sequences and Series

Contents

Infinite sequences and series were used by Newton to represent functions as sums of infinite series. For example, in finding area, he often integrated a function by first expressing it as a series and then integrating term by term. We will investigate this approach in Section 8.7 in order to integrate functions like e^{-x^2}. Many of the functions that arise in mathematical physics and chemistry, such as Bessel functions, are defined as sums of series, so it is important to be familiar with the basic concepts of convergence of infinite sequences and series.

Physicists also use series in another way, as we will see in Section 8.8. In studying fields as diverse as optics, special relativity, and electromagnetism, they analyze phenomena by replacing a function with the first few terms in the series that represent it.

8.1 Sequences

Sequence Basics

A **sequence** is the primary element of a series and can be thought of as simply a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

Note that some sequences begin with $n = 0$. In this case the list of terms is $a_0, a_1, a_2, \ldots, a_n, \ldots$

The number a_1 is the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will study infinite sequences and so each term a_n will have a successor a_{n+1}.

Notice that for every positive integer n, there is a corresponding number a_n. Therefore, a sequence can be defined as a function whose domain is the set of positive integers. However, we usually write a_n rather than $a(n)$ or $f(n)$ for the value of the function at the number n.

Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

Example 1 Sequence Descriptions

Some sequences can be defined by giving a formula for the nth term. In the following examples, each sequence is described in three ways: by using the notation introduced above, by using a defining formula, and by writing out several terms of the sequence. Notice that n doesn't have to start at 1.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\quad a_n = \dfrac{n}{n+1}$ $\quad \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \ldots, \dfrac{n}{n+1}, \ldots\right\}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}$ $\quad a_n = \dfrac{(-1)^n(n+1)}{3^n}$ $\quad \left\{-\dfrac{2}{3}, \dfrac{3}{9}, -\dfrac{4}{27}, \dfrac{5}{81}, \ldots, \dfrac{(-1)^n(n+1)}{3^n}, \ldots\right\}$

(c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ $\quad a_n = \sqrt{n-3},\ n \geq 3$ $\quad \left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\right\}$

(d) $\left\{\cos\dfrac{n\pi}{6}\right\}_{n=0}^{\infty}$ $\quad a_n = \cos\dfrac{n\pi}{6},\ n \geq 0$ $\quad \left\{1, \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0, \ldots, \cos\dfrac{n\pi}{6}, \ldots\right\}$ ■

Example 2 Sequence Formula

Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

assuming that the pattern of the first few terms continues.

Solution

Write the first few terms of the sequence in greater detail in order to search for patterns.

$$a_1 = \frac{3}{5} \quad a_2 = -\frac{4}{25} \quad a_3 = \frac{5}{125} \quad a_4 = -\frac{6}{625} \quad a_5 = \frac{7}{3125}$$

The numerators of these fractions start with 3 and increase by 1 in each successive term. The second term has numerator 4, the third term has numerator 5; in general, the nth term will have numerator $n + 2$.

The denominators are the powers of 5, so a_n has denominator 5^n.

The signs of the terms alternate between positive and negative, so we need to multiply each term by a power of -1.

In Example 1(b), the factor $(-1)^n$ meant the sequence started with a negative term. Here, we want to start with a positive term, so we can use $(-1)^{n-1}$ or $(-1)^{n+1}$. Therefore,

$$a_n = (-1)^{n-1}\frac{n+2}{5^n}.$$

Example 3 A Sequence Without a Defining Equation

Here are some sequences that do not have a simple defining equation.

(a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n.

(b) Let a_n be the digit in the nth decimal place of the number e. Then $\{a_n\}$ is a well-defined sequence whose first few terms are

$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots\}$.

(c) **The Fibonacci sequence** $\{f_n\}$ is defined recursively by the conditions

$f_1 = 1 \quad f_2 = 1 \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3$

Each term is the sum of the two preceding terms. The first few terms are

$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$.

This sequence first appeared when the 13th-century Italian mathematician Fibonacci solved a problem concerning the breeding of pairs of rabbits (see Exercise 57).

A sequence like the one in Example 1(a), $a_n = \dfrac{n}{n+1}$, can be visualized either by plotting the terms of the sequence on a number line, as in Figure 8.1, or by plotting a graph, as in Figure 8.2. Since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated, discrete points in the plane with coordinates

$$(1, a_1), \quad (2, a_2), \quad (3, a_3), \quad \ldots, \quad (n, a_n), \quad \ldots$$

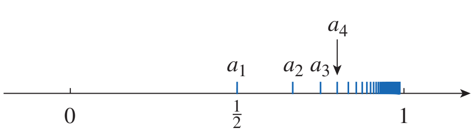


Figure 8.1
Plot of the sequence $a_n = \dfrac{n}{n+1}$ on a number line.

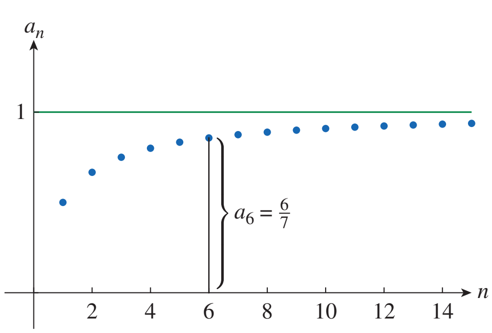


Figure 8.2
Graph of the sequence $a_n = \dfrac{n}{n+1}$ in the plane.

Limit of a Sequence

Figures 8.1 and 8.2 suggest that the terms of the sequence $a_n = \dfrac{n}{n+1}$ are approaching 1 as n becomes large. Consider the difference

$$1 - a_n = 1 - \frac{n}{n+1} = \frac{n+1-n}{n+1} = \frac{1}{n+1}$$

which can be made arbitrarily small by taking n sufficiently large. This sequence behavior is expressed symbolically as

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

In general, the notation

$$\lim_{n\to\infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large. This leads to the following definition of the limit of a sequence, which is very similar to the definition of a limit of a function at infinity in Section 2.5.

Definition • Limit of a Sequence

A sequence $\{a_n\}$ has the **limit** L, written as

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \quad \text{as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, then we say the sequence $\{a_n\}$ **converges**, or is **convergent**. Otherwise, we say the sequence **diverges**, or is **divergent**.

Figure 8.3 illustrates this definition by showing the graphs of three sequences that have the limit L. Notice that the terms of a sequence may *cross* the horizontal line $y = L$.

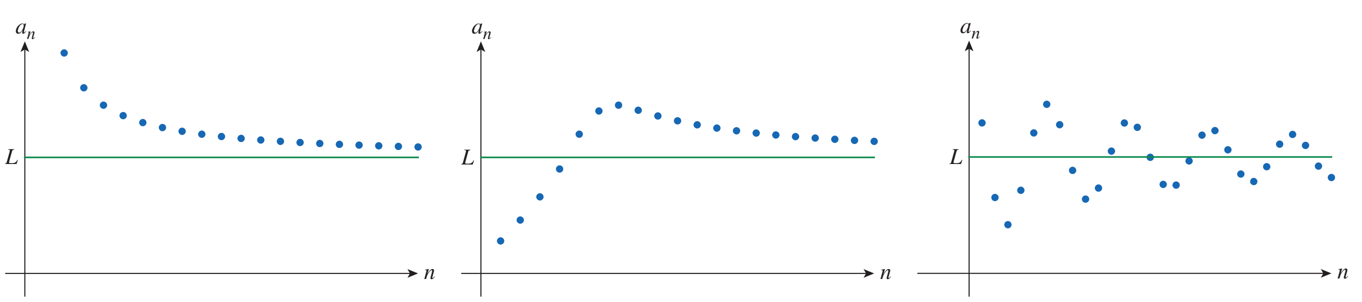


Figure 8.3
Graphs of sequences with $\lim_{n\to\infty} a_n = L$.

There are several reasons that a sequence might diverge. For example, the terms may oscillate, or they may increase (or decrease) without bound. Therefore, if a sequence diverges, the limit of the terms is not necessarily ∞.

The definitions for the limit of a sequence and the limit of a function are very similar. The only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n must be an integer. It seems reasonable that if a function converges to L, then so does the associated sequence. This concept is presented in the following theorem.

Theorem • Limit of a Function and Corresponding Sequence

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, where n is an integer, then $\lim_{n\to\infty} a_n = L$.

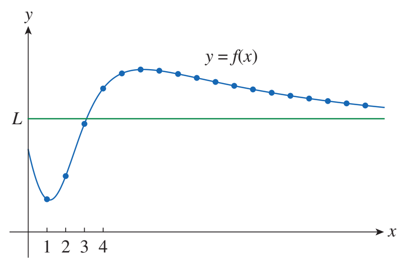


Figure 8.4
A graph of the function $y = f(x)$ and the sequence $a_n = f(n)$. If the function f converges to L, then so does the sequence.

Figure 8.4 illustrates this theorem.

Here is an example of this theorem in action. From Chapter 2, $\lim_{x\to\infty} \frac{1}{x^r} = 0$ when $r > 0$. Therefore,

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0 \tag{1}$$

If a_n becomes arbitrarily large as n increases without bound, we write $\lim_{n\to\infty} a_n = \infty$.

Limit Laws

The Limit Laws presented in Chapter 2 also hold for the limits of sequences and their proofs are similar. Here are the laws in terms of sequences.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

1. $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

2. $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$

3. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$

4. $\lim_{n\to\infty} c = c$

5. $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

6. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if} \quad \lim_{n\to\infty} b_n \neq 0$

7. $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if} \quad p > 0 \quad \text{and} \quad a_n > 0$

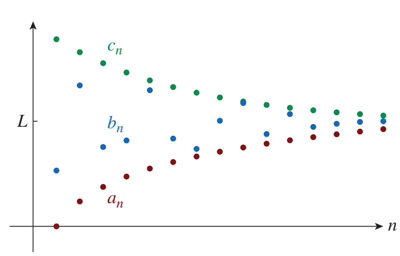


Figure 8.5
The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

The Squeeze Theorem can also be adapted for sequences.

Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Figure 8.5 illustrates this theorem.

Here is one more useful fact about limits of sequences. This result follows from the Squeeze Theorem because $-|a_n| \leq a_n \leq |a_n|$.

Theorem • Limit Implication

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example 4 Limit of a Sequence

Find $\lim_{n\to\infty} \dfrac{n}{n+1}$.

Solution

The method used here is similar to one used in Chapter 2: divide the numerator and the denominator by the highest power of n that occurs in the denominator. Then use the Limit Laws.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\dfrac{1}{n}} \qquad \text{Divide numerator and denominator by } n.$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \dfrac{1}{n}} \qquad \text{Limit Laws.}$$

$$= \frac{1}{1+0} = 1 \qquad \text{Limit Laws; Equation 3 with } r = 1.$$

Example 5 L'Hospital's Rule and a Related Function

Find $\lim_{n\to\infty} \dfrac{\ln n}{n}$.

Solution

Notice that $\lim_{n\to\infty} \ln n = \infty$ and $\lim_{n\to\infty} n = \infty$.

Both the numerator and the denominator increase without bound as $n \to \infty$.

We cannot apply l'Hospital's Rule directly because it applies to functions of a real variable, not to sequences.

However, we can apply l'Hospital's Rule to the related function $f(x) = \dfrac{\ln x}{x}$.

Note that $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x = \infty$. The limit $\lim_{x\to\infty} f(x)$ is in the indeterminate form $\frac{\infty}{\infty}$ and, therefore, we can use l'Hospital's Rule.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \qquad \text{L'Hospital's Rule; evaluate limit.}$$

Because the limit of the function is 0, the limit of the sequence is also 0, $\lim_{n\to\infty} \dfrac{\ln n}{n} = 0$.

Example 6 Oscillating Sequence

Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Solution

Write out the terms of this sequence: $\{-1, 1, -1, 1, -1, 1, -1, \ldots\}$.

Because the terms of the sequence oscillate between 1 and -1 infinitely often, a_n does not approach any number.

Therefore, $\lim_{n\to\infty} (-1)^n$ does not exist; the sequence $\{(-1)^n\}$ is divergent.

Figure 8.6 illustrates this result.

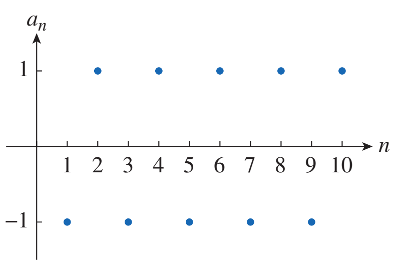


Figure 8.6
The graph of the sequence $\{(-1)^n\}$. This supports the conclusion that the sequence is divergent.

Example 7 Use the Absolute Value

Evaluate $\lim_{n\to\infty} \dfrac{(-1)^n}{n}$.

Solution

Consider the limit of the sequence of absolute values.

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0$$

Therefore, by the Limit Implication Theorem, $\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$. ■

Sequence-Function Links

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent. The proof is given in Appendix E.

Theorem • Limit of a Continuous Function Applied to a Sequence

If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

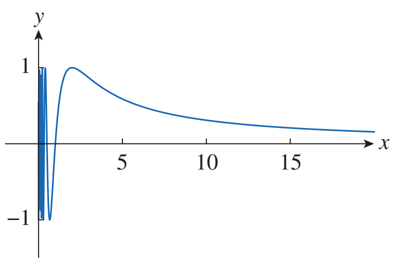


Figure 8.7
Graph of $y = \sin\left(\frac{\pi}{x}\right)$. Near 0 the graph oscillates back and forth between -1 and 1. As x increases without bound, $y \to 0$.

Example 8 Use a Continuous Function

Find $\lim_{n\to\infty}\sin\left(\frac{\pi}{n}\right)$.

Solution

Use Theorem 5 and the fact that the sine function is continuous at 0.

$$\lim_{n\to\infty}\sin\left(\frac{\pi}{n}\right) = \sin\left[\lim_{n\to\infty}\left(\frac{\pi}{n}\right)\right] = \sin 0 = 0$$

Figure 8.7 helps to illustrate and confirm this result. ■

Example 9 Use the Squeeze Theorem

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

Determine whether the sequence $a_n = \frac{n!}{n^n}$ is convergent or divergent.

Solution

Notice that both the numerator and denominator increase without bound as $n \to \infty$.

Therefore, a first attempt is to use l'Hospital's Rule.

However, there is no corresponding function for use with l'Hospital's Rule because $x!$ is undefined when x is not an integer.

Consider a few terms of the sequence analytically and graphically (see Figure 8.8) to get a sense of what happens to a_n as n gets large.

$$a_1 = 1 \qquad a_2 = \frac{1\cdot 2}{2\cdot 2}, \qquad a_3 = \frac{1\cdot 2\cdot 3}{3\cdot 3\cdot 3}, \qquad a_n = \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n\cdots n}$$

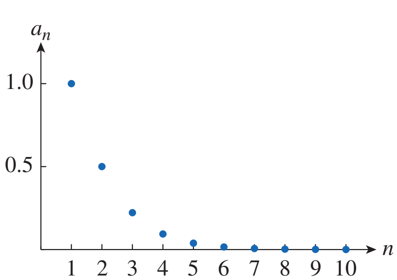


Figure 8.8
Graph of the sequence.

This exploration suggests that the terms of the sequence are decreasing and perhaps approaching 0.

To confirm this, rewrite the nth term of the sequence.

$$a_n = \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n\cdots n} = \frac{1}{n}\left(\frac{2\cdot 3\cdots n}{n\cdot n\cdots n}\right)$$

The expression in parentheses is at most 1 because the numerator is less than (or equal to) the denominator.

Therefore, $0 < a_n \leq \dfrac{1}{n}$.

As $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$, by the Squeeze Theorem, $\displaystyle\lim_{n\to\infty} a_n = 0$. ■

Example 10 Limit of a Geometric Sequence

For what values of r is the sequence $\{r^n\}$ convergent?

Solution

Recall from Chapter 2 and our study of exponential functions that

$$\lim_{x\to\infty} a^x = \infty \text{ for } a > 1 \quad \text{and} \quad \lim_{x\to\infty} a^x = 0 \text{ for } 0 < a < 1.$$

Let $a = r$ and use Theorem 2.

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

Note that $\displaystyle\lim_{n\to\infty} 1^n = 1$ and $\displaystyle\lim_{n\to\infty} 0^n = 0$.

If $-1 < r < 0$, then $0 < |r| < 1$, and therefore, $\displaystyle\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$ and $\displaystyle\lim_{n\to\infty} r^n = 0$.

If $r \leq 1$, the terms of the sequence increase in magnitude and alternate between positive and negative. Therefore, $\{r^n\}$ diverges in this case.

If $r = -1$, the sequence diverges, as shown in Example 6.

Figures 8.9 and 8.10 show the graphs of this sequence for various values of r.

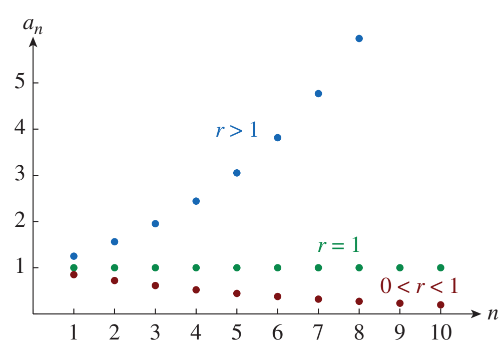


Figure 8.9
Graphs of the sequence $a_n = r^n$ for $r > 0$.

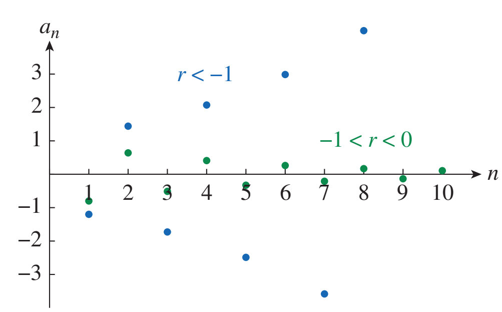


Figure 8.10
Graphs of the sequence $a_n = r^n$ for $r < 0$. ■

The sequence in Example 10 is extremely useful and important. The results are summarized as follows.

Definition • The Sequence $\{r^n\}$

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Sequence Characterizations

We can characterize sequences in ways similar to functions. Consider the following definition.

Definition • Increasing, Decreasing, and Monotonic Sequences

A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$.

A sequence $\{a_n\}$ is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.

A sequence is **monotonic** if it is either increasing or decreasing.

Example 11 Decreasing Sequence

The sequence $\left\{\dfrac{3}{n+5}\right\}$ is decreasing because

$$a_n = \frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6} = a_{n+1}.$$

Therefore, $a_n > a_{n+1}$ for all $n \geq 1$. ■

Example 12 Show That a Sequence Is Decreasing

Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

Solution 1

We need to show that $a_{n+1} < a_n$, that is

$$a_{n+1} = \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} = a_n.$$

This inequality is equivalent to the one we get by cross-multiplication.

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} \Leftrightarrow (n+1)(n^2+1) < n\left[(n+1)^2+1\right] \qquad \text{Cross-multiply.}$$

$$\Leftrightarrow n^3+n^2+n+1 < n^3+2n^2+2n \qquad \text{Multiply; collect terms.}$$

$$\Leftrightarrow 1 < n^2+n \qquad \text{Simplify.}$$

Because $n \geq 1$, the inequality $n^2+n > 1$ is true.

Therefore, $a_{n+1} < a_n$ and $\{a_n\}$ is decreasing.

Solution 2

Consider the function $f(x) = \dfrac{x}{x^2+1}$ and find the first derivative.

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} \qquad \text{Quotient Rule.}$$

$$= \frac{1-x^2}{(x^2+1)^2} < 0 \qquad \text{Simplify; inequality is true whenever } x^2 > 1.$$

This analysis shows that f is decreasing on $[1, \infty)$, so $f(n) > f(n+1)$.

Therefore, $\{a_n\}$ is decreasing. ■

The next definition applies to sequences that are bounded.

Definition • Bounded Sequence

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \le M \qquad \text{for all } n \ge 1$$

It is **bounded below** if there is a number m such that

$$m \le a_n \qquad \text{for all } n \ge 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

A Closer Look

1. If a sequence is bounded above, there are many values of M that satisfy the inequality $a_n \le M$. The smallest value of M is called the *least upper bound* or *supremum*.

   Similarly, if a sequence is bounded below, there are many values of m that satisfy the inequality $m \le a_n$. The largest value of m is called the *greatest lower bound* or *infimum*.

2. A sequence may be bounded above (below) but not bounded below (above). For example, the sequence $a_n = n$ is bounded below, $1 \le a_n$, but not above. The sequence $a_n = \dfrac{n}{n+1}$ is bounded because $0 \le a_n \le 1$. ■

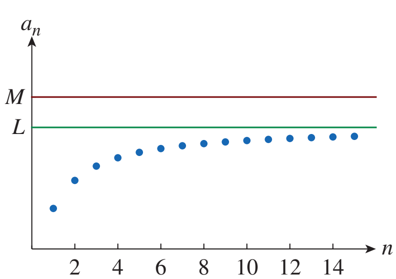


Figure 8.11
A bounded monotonic sequence.

It seems reasonable that there is a connection between sequences that are bounded and monotonic and convergence or divergence. We have already seen examples that indicate that not every bounded sequence is convergent. For example, consider the sequence $a_n = (-1)^n$. This sequence is bounded, $-1 \le a_n \le 1$, but divergent. In addition, not every monotonic sequence is convergent. Consider the sequence $a_n = n$, which is increasing but divergent.

However, if a sequence is both bounded *and* monotonic, then it must be convergent. This important observation is a theorem, presented below. Figure 8.11 provides some intuitive graphical understanding for this theorem. If $\{a_n\}$ is increasing and $a_n \le M$ for all n, then the terms are forced closer together (as n increases) and approach some number L.

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

Example 13 Recursively Defined Sequence

Investigate the sequence $\{a_n\}$ defined by the *recurrence relation*

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{2}(a_n + 6) \qquad \text{for } n = 1, 2, 3, \ldots$$

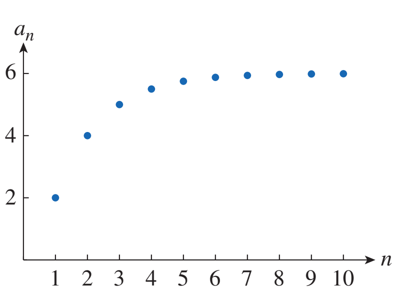


Figure 8.12
A graph of the first few terms of the sequence $\{a_n\}$.

Solution

Start by computing the first several terms. See Figure 8.12.

$a_1 = 2$ $\quad a_2 = \frac{1}{2}(2 + 6) = 4$ $\quad a_3 = \frac{1}{2}(4 + 6) = 5$

$a_4 = \frac{1}{2}(5 + 6) = 5.5$ $\quad a_5 = \frac{1}{2}(5.5 + 6) = 5.75$ $\quad a_6 = \frac{1}{2}(5.75 + 6) = 5.875$

$a_7 = 5.9375$ $\quad a_8 = 5.96875$ $\quad a_9 = 5.984375$

These initial terms suggest that the sequence is increasing and the terms are approaching 6.

Use mathematical induction to show that the sequence is increasing, that $a_{n+1} > a_n$ for all $n \geq 1$.

Show that this inequality is true for $n = 1$: $a_2 = 4 > 2 = a_1$.

Assume that it is true for $n = k$: $a_{k+1} > a_k$.

$$a_{k+1} + 6 > a_k + 6 \qquad \text{Add 6 to both sides.}$$

$$\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6) \qquad \text{Multiply both sides by } \tfrac{1}{2}.$$

$$a_{k+2} > a_{k+1} \qquad \text{True for } n = k + 1.$$

Therefore, the inequality is true for all n by induction.

Next, let's verify that the sequence $\{a_n\}$ is bounded by showing that $a_n < 6$ for all n. Because the sequence is increasing, we know it has a lower bound: $a_n \geq a_1 = 2$ for all n.

Use mathematical induction again.

For $n = 1$: $a_1 = 2 < 6$, so the assertion is true for $n = 1$.

Assume that it is true for $n = k$: $a_k < 6$.

$$a_k + 6 < 6 + 6 = 12 \qquad \text{Add 6 to both sides.}$$

$$\frac{1}{2}(a_k + 6) < \frac{1}{2}(12) = 6 \qquad \text{Multiply both sides by } \tfrac{1}{2}.$$

$$a_{k+1} < 6 \qquad \text{True for } n = k + 1.$$

Therefore by mathematical induction, $a_n < 6$ for all n.

Since the sequence $\{a_n\}$ is increasing and bounded, the Monotonic Sequence Theorem guarantees that it has a limit. However, it doesn't tell us the value of the limit.

Given that $L = \lim_{n \to \infty} a_n$ exists, use the given recurrence relation to find the value of L.

$$\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}\left(\lim_{n \to \infty} a_n + 6\right) = \frac{1}{2}(L + 6)$$

Because $a_n \to L$, it follows that $a_{n+1} \to L$ (as $n \to \infty$, $n + 1 \to \infty$ also).

Therefore, $L = \frac{1}{2}(L + 6) \quad \Rightarrow \quad 2L = L + 6 \quad \Rightarrow \quad L = 6.$

Finally, $\lim_{n \to \infty} a_n = 6$. ■

8.1 Exercises

1. (a) What is a sequence?
(b) What does it mean to say that $\lim_{n\to\infty} a_n = 8$?
(c) What does it mean to say that $\lim_{n\to\infty} a_n = \infty$?

2. (a) What is a convergent sequence? Give two examples.
(b) What is a divergent sequence? Give two examples.

3. List the first six terms of the sequence defined by

$$a_n = \frac{n}{2n+1}$$

Does the sequence appear to have a limit? If so, find it.

4. List the first nine terms of the sequence

$$a_n = \cos\left(\frac{n\pi}{3}\right)$$

Does the sequence appear to have a limit? If so, find it. If not, explain why.

Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

5. $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\right\}$

6. $\left\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots\right\}$

7. $\left\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \ldots\right\}$

8. $\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots\right\}$

9. $\{5, 8, 11, 14, 17, \ldots\}$

10. $\{5, 1, 5, 1, 5, 1, \ldots\}$

Determine whether the sequence converges or diverges. If it converges, find the limit.

11. $a_n = \dfrac{3+5n^2}{n+n^2}$

12. $a_n = \dfrac{3+5n^2}{1+n}$

13. $a_n = \dfrac{n^4}{n^3-2n}$

14. $a_n = 2 + (0.86)^n$

15. $a_n = 3^n 7^{-n}$

16. $a_n = \dfrac{3\sqrt{n}}{\sqrt{n}+2}$

17. $a_n = e^{-1/\sqrt{n}}$

18. $a_n = \dfrac{4^n}{1+9^n}$

19. $a_n = \sqrt{\dfrac{1+4n^2}{1+n^2}}$

20. $a_n = \tan\left(\dfrac{2n\pi}{1+8n}\right)$

21. $a_n = \dfrac{n^2}{\sqrt{n^3+4n}}$

22. $a_n = e^{2n/(n+2)}$

23. $a_n = \dfrac{(-1)^n}{2\sqrt{n}}$

24. $a_n = \dfrac{(-1)^{n+1}n}{n+\sqrt{n}}$

25. $\left\{\dfrac{e^n + e^{-n}}{e^{2n}-1}\right\}$

26. $a_n = \cos\left(\dfrac{2}{n}\right)$

27. $\{n^2 e^{-n}\}$

28. $\{\arctan 2n\}$

29. $a_n = \dfrac{\cos^2 n}{2^n}$

30. $\{n \cos n\pi\}$

31. $a_n = \left(1 + \dfrac{2}{n}\right)^n$

32. $a_n = \sqrt[n]{2^{1+3n}}$

33. $\left\{\dfrac{(2n-1)!}{(2n+1)!}\right\}$

34. $a_n = \dfrac{\sin 2n}{1+\sqrt{n}}$

35. $a_n = \sqrt[n]{n}$

36. $a_n = \ln(2n^2+1) - \ln(n^2+1)$

37. $a_n = \dfrac{(\ln n)^2}{n}$

38. $a_n = \arctan(\ln n)$

39. $a_n = n - \sqrt{n+1}\sqrt{n+3}$

40. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\}$

41. $\left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \ldots\right\}$

42. $a_n = \dfrac{n!}{2^n}$

43. $a_n = \dfrac{(-3)^n}{n!}$

Use a graph to conjecture whether the sequence is convergent or divergent. If the graph suggests that the sequence is convergent, guess the value of the limit from the graph, and then find the limit analytically.

44. $a_n = (-1)^n \dfrac{n}{n+1}$

45. $a_n = \dfrac{\sin n}{n}$

46. $a_n = \arctan\left(\dfrac{n^2}{n^2+4}\right)$

47. $a_n = \sqrt[n]{3^n + 5^n}$

48. $a_n = \dfrac{n^2 \cos n}{1+n^2}$

49. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$

50. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^n}$

51. If \$1000 is invested at 6% interest, compounded annually, then after n years, the investment is worth $a_n = 1000(1.06)^n$ dollars.
(a) Find the first five terms of the sequence $\{a_n\}$.
(b) Is the sequence convergent or divergent? Explain your reasoning.

52. If you deposit \$100 at the end of every month into an account that pays 3% interest compounded monthly, the amount of interest accumulated after n months is given by the sequence

$$I_n = 100\left(\frac{1.0025^n - 1}{0.0025} - n\right)$$

(a) Find the first six terms of the sequence.
(b) How much interest will you have earned after 2 years?

53. A fish farmer has 5000 catfish in his pond. The number of catfish increases by 8% per month, and the farmer harvests 300 catfish per month.
(a) Show that the catfish population P_n after n months is given recursively by

$$P_n = 1.08P_{n-1} - 300 \qquad P_0 = 5000$$

(b) How many catfish are in the pond after 6 months?

54. Find the first 40 terms of the sequence defined by

$$a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$$

and $a_1 = 11$. Do the same if $a_1 = 25$. Make a conjecture about this type of sequence.

55. Consider the sequence defined by

$$a_1 = 1 \qquad a_{n+1} = 4 - a_n \qquad \text{for } n \geq 1$$

(a) Determine whether the sequence is convergent or divergent.
(b) If $a_1 = 2$, determine whether the new sequence is convergent or divergent.

56. Suppose that $\lim_{n\to\infty} a_n = L$.
(a) Show that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$.
(b) A sequence $\{a_n\}$ is defined by

$$a_1 = 1 \qquad a_{n+1} = \frac{1}{1 + a_n} \qquad \text{for } n \geq 1$$

Find the first 10 terms of the sequence. Does it appear that the sequence is convergent? If so, estimate the value of the limit.
(c) Assuming that the sequence in part (b) has a limit, use the result in part (a) to find its exact value. Compare this limit with your estimate from part (b).

57. (a) Fibonacci posed the following problem: suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month? Show that the answer is f_n, where $\{f_n\}$ is the Fibonacci sequence defined in Example 3(c).
(b) Let $a_n = \dfrac{f_{n+1}}{f_n}$ and show that $a_{n-1} = 1 + \dfrac{1}{a_{n-2}}$.
(c) Assuming that $\{a_n\}$ is convergent, find its limit.

58. Find the limit of the sequence

$$\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\right\}$$

Determine whether the sequence is increasing, decreasing, or not monotonic, and if the sequence is bounded.

59. $a_n = \dfrac{1}{2n + 3}$

60. $a_n = \dfrac{2n - 3}{3n + 4}$

61. $a_n = n(-1)^n$

62. $a_n = n + \dfrac{1}{n}$

63. $a_n = \cos n$

64. $a_n = 3 - 2ne^n$

---

65. Suppose the sequence $\{a_n\}$ is decreasing and all of its terms lie between the numbers 5 and 8. Explain why the sequence has a limit. What can you say about the value of the limit?

66. Consider the sequence $\{a_n\}$ defined by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.
(a) Use induction to show that $\{a_n\}$ is increasing and bounded above by 3. Apply the Monotonic Sequence Theorem to show that $\lim_{n\to\infty} a_n$ exists.
(b) Find $\lim_{n\to\infty} a_n$.

67. Show that the sequence defined by

$$a_1 = 1 \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n. Show that $\{a_n\}$ is convergent and find its limit.

68. Show that the sequence defined by

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{3 - a_n}$$

satisfies the inequality $0 < a_n \leq 2$ and is decreasing. Show that the sequence is convergent and find its limit.

69. (a) Let $a_1 = a$, $a_2 = f(a)$, $a_3 = f(a_2) = f(f(a))$, $\ldots$, $a_{n+1} = f(a_n)$, where f is a continuous function. If $\lim_{n\to\infty} a_n = L$, show that $f(L) = L$.
(b) Illustrate part (a) by taking $f(x) = \cos x$, $a = 1$, and estimating the value of L.

70. Let $a_n = \left(1 + \dfrac{1}{n}\right)^n$.

(a) Show that if $0 \le a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n + 1)b^n$$

(b) Conclude that $b^n[(n + 1)a - nb] < a^{n+1}$.

(c) Use $a = 1 + \dfrac{1}{n + 1}$ and $b = 1 + \dfrac{1}{n}$ in part (b) to show that the sequence $\{a_n\}$ is increasing.

(d) Use $a = 1$ and $b = 1 + \dfrac{1}{2n}$ in part (b) to show that $a_{2n} < 4$.

(e) Use parts (c) and (d) to show that $a_n < 4$ for all n.

(f) Use the Monotonic Sequence Theorem to show that $\lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n$ exists. Do you recognize this limit?

71. Let a and b be positive numbers with $a > b$. Let a_1 be their arithmetic mean and b_1 their geometric mean:

$$a_1 = \frac{a + b}{2} \qquad b_1 = \sqrt{a \cdot b}$$

Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n \cdot b_n}$$

(a) Use mathematical induction to show that

$$a_n > a_{n+1} > b_{n+1} > b_n$$

(b) Conclude that both $\{a_n\}$ and $\{b_n\}$ are convergent.

(c) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Gauss called the common value of these limits the **arithmetic-geometric mean** of the numbers a and b.

72. The size of an undisturbed fish population has been modeled by the formula

$$p_{n+1} = \frac{bp_n}{a + p_n}$$

where p_n is the fish population after n years and a and b are positive constants that depend on the species and its environment. Suppose that the population in year 0 is $p_0 > 0$.

(a) Show that if $\{p_n\}$ is convergent, then the only possible values for its limit are 0 and $b - a$.

(b) Show that $p_{n+1} < \dfrac{b}{a} p_n$.

(c) Use part (b) to show that if $a > b$, then $\lim_{n\to\infty} p_n = 0$; that is, the population dies out.

(d) Now assume that $a < b$. Show that if $p_0 < b - a$, then $\{p_n\}$ is increasing and $0 < p_n < b - a$. Show also that if $p_0 > b - a$, then $\{p_n\}$ is decreasing and $p_n > b - a$. Conclude that if $a < b$, then $\lim_{n\to\infty} p_n = b - a$.

73. (a) Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

(b) If $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{1 + a_n}$$

find the first eight terms of the sequence $\{a_n\}$. Then use part (a) to show that $\lim_{n\to\infty} a_n = \sqrt{2}$. This gives the **continued fraction equation**

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}$$

Laboratory Project | Logistic Sequences

A sequence that arises in ecology as a model for population growth is defined by the **logistic difference equation**

$$p_{n+1} = kp_n(1 - p_n)$$

where p_n measures the size of the population of the nth generation of a single species. To keep the numbers manageable, p_n is a fraction of the maximal size of the population, so $0 \le p_n \le 1$. Notice that the form of this equation is similar to the logistic differential equation in Section 7.5. The discrete model, with sequences instead of continuous functions, is preferable for modeling insect populations, where mating and death occur in periodic fashion.

An ecologist is interested in predicting the size of the population as time goes on, and asks these questions: Will the population stabilize at a limiting value? Will the population change in a cyclical manner? Or will it exhibit random behavior?

Use technology to compute the first n terms of this sequence starting with an initial population p_0, where $0 < p_0 < 1$. Use this general procedure to answer the following.

1. Calculate 30 terms of the sequence for $p_0 = \frac{1}{2}$ and for two values of k such that $1 < k < 3$. Graph each sequence. Do the sequences appear to converge? Repeat this process for a different value of p_0 between 0 and 1. Does the limit depend on the choice of p_0? Does it depend on the choice of k?

2. Calculate several terms of the sequence for a value of k between 3 and 3.4 and plot this sequence. Describe the behavior of the terms of this sequence.

3. Experiment with values of k between 3.4 and 3.5. Explain how the terms of the sequence vary.

4. For values of k between 3.6 and 4, compute and plot at least 100 terms of the sequence and describe the behavior of the sequence. What happens if you change p_0 by 0.001? This type of behavior is called *chaotic* and is exhibited by insect populations under certain conditions.

8.2 Series

Series Basics

In this section, we consider methods to sum the terms of a sequence $\{a_n\}$. At first, it doesn't seem possible to sum an *infinite* sequence of terms. However, consider a number expressed as an infinite decimal, for example, the irrational number π:

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ \ldots$$

The principle behind our decimal notation is that any number can be written as an infinite sum. In this example, that means

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

This intuitive description of an infinite decimal representation certainly suggests a limit.

where the three dots $(\cdots)$ indicate that the sum continues forever, and the more terms we add, the closer we get to the actual value of π.

In general, we would like to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$. An expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \tag{1}$$

is called an **infinite series** (or just **series**) and is denoted by

$$\sum_{i=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

We need to make some mathematical sense out of a sum of infinitely many terms.

It is impossible to find a finite sum for the series

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

because if we start adding the terms, we get the cumulative sums: 1, 3, 6, 10, 15, 21, . . . After the nth term, the sum is $\dfrac{n(n+1)}{2}$, which can be made arbitrarily large as n increases without bound.

n	Sum of first n terms
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

Table 8.1
Partial sums of an infinite series.

However, suppose we add the terms of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots + \frac{1}{2^n} + \cdots$$

The cumulative sums are $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots, 1 - \frac{1}{2^n}, \ldots$ Table 8.1 suggests that as we add more and more terms, these cumulative sums, or *partial sums*, get closer and closer to 1. In fact, by adding sufficiently many terms of the series, we can make the partial sums as close as we like to 1. So, it seems reasonable to say that the sum of this infinite series is 1 and to write this conclusion as

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

We can extend this idea of finding cumulative sums to determine whether or not a general series, as in Equation 1, has a finite sum. Consider the **partial sums**

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \end{aligned}$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If $\lim_{n\to\infty} s_n = s$ exists (as a finite number), then, as in the preceding example, we call s the sum of the infinite series Σa_n.

Definition • Convergent Series

Given a series $\sum_{n=1}^{\infty} = a_1 + a_2 + a_3 + \cdots,$

let s_n denote its nth partial sum: $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series Σa_n is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

A Closer Look

1. This definition means that the sum of a series is the limit of the sequence of partial sums. When we write $\sum_{i=1}^{\infty} a_i = s$, the limit interpretation means that by adding sufficiently many terms of the series, we can get arbitrarily close to the number s.
2. Here is another way to write, and think about, this definition in terms of limits.

   $$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

   This is analogous to the definition of a Type 1 improper integral.

   $$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^{t} f(x)\,dx$$

   To evaluate this improper integral, we integrate first from 1 to t, and then determine the limit as $t \to \infty$. For a series, we first find the sum from 1 to n, and then evaluate the limit as $n \to \infty$.

Series Compilation

Example 1 Geometric Series

An important example of an infinite series is the **geometric series**.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r. At the beginning of this section, we considered the special case in which $a = \frac{1}{2}$ and $r = \frac{1}{2}$.

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$.

Since $\lim_{n\to\infty} s_n$ does not exist, the geometric series diverges in this case.

If $r \neq 1$, then

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} \qquad n\text{th partial sum.}$$

$$rs_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n \qquad \text{Multiply both sides by } r.$$

Subtract these equations: $s_n - rs_n = s_n(1 - r) = a - ar^n$.

Solve for s_n

$$s_n = \frac{a(1 - r^n)}{1 - r} \qquad (2)$$

If $-1 < r < 1$, we know that $r^n \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} \lim_{n\to\infty} r^n \qquad \text{Limit Properties.}$$

$$= \frac{a}{1 - r} - \frac{a}{1 - r} \cdot 0 = \frac{a}{1 - r} \qquad r^n \to 0 \text{ as } n \to \infty.$$

Therefore, when $|r| < 1$, the geometric series is convergent, and its sum is $\frac{a}{1 - r}$. If $r \leq -1$ or $r > 1$, the sequence $\{r_n\}$ is divergent, so by Equation 3, $\lim_{n\to\infty} s_n$ does not exist. Therefore, the geometric series diverges in those cases.

Figure 8.13 provides a geometric illustration of this result.

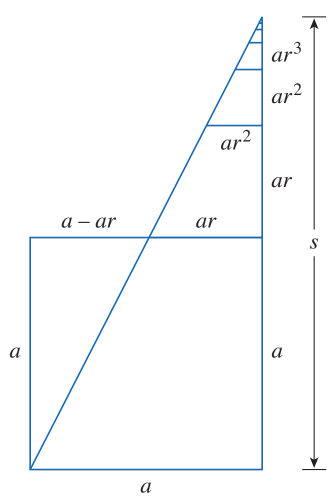


Figure 8.13
A geometric illustration of the result in Example 2. If the triangles are constructed as shown and s is the sum of the series, then by similar triangles $\frac{s}{a} = \frac{a}{a - ar}$, so $s = \frac{a}{1 - r}$.

Here is a summary of Example 1.

In words: The sum of a convergent geometric series is $\dfrac{\text{first term}}{1 - \text{common ratio}}$

The Geometric Series

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$, and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, then the geometric series is divergent.

Example 2 Sum of a Geometric Series

Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

Solution

Rewrite the series to identify a and r.

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = 5 + 5 \cdot \left(-\frac{2}{3}\right) + 5 \cdot \left(-\frac{2}{3}\right)^2 + 5 \cdot \left(-\frac{2}{3}\right)^3 + \cdots$$

The first term is $a = 5$ and the common ratio is $r = -\frac{2}{3}$.

Since $|r| = \frac{2}{3} < 1$, the series is convergent.

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{a}{1-r} = \frac{5}{1 - \left(-\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3$$

A Closer Look

It is important to understand what is really meant by saying the sum of the series in Example 2 is 3. We can't literally add an infinite number of terms, one by one. But according to definition, the total sum is the limit of the sequence of partial sums. That means by taking the sum of sufficiently many terms, we can get as close as we like to the number 3. Table 8.2 shows the first ten partial sums s_n (some numerical confirmation) and the graph in Figure 8.14 (graphical confirmation) shows how the sequence of partial sums approaches 3.

n	s_n
1	5.00000000
2	1.66666667
3	3.88888889
4	2.40740741
5	3.39506173
10	2.94797541
20	2.99909781
30	2.99998435
40	2.99999973

Table 8.2
The partial sums numerically.

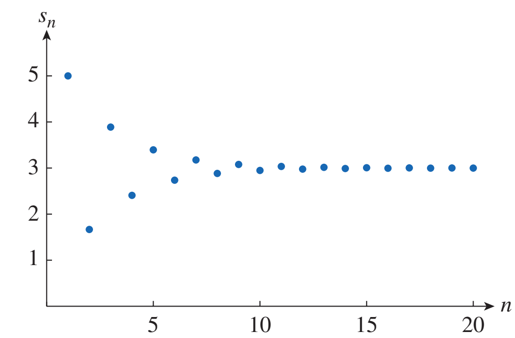


Figure 8.14
The partial sums graphically.

Example 3 Determine Whether a Series Is Convergent or Divergent

Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

Solution

Rewrite the nth term to identify a and r.

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

This is a geometric series with $a = 4$ and $r = \frac{4}{3}$.

Since $r > 1$, the series diverges.

Example 4 Express a Repeating Decimal as a Rational Number

Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

Solution

Rewrite the number as a series.

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term, we have a geometric series with $a = \frac{17}{10^3}$ and $r = \frac{1}{10^2}$.

Therefore,

$$2.3\overline{17} = 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}}$$ Sum of a geometric series; simplify.

$$= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495}.$$ Simplify entire expression.

Example 5 A Series with Variable Terms

Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

Solution

Write the terms of the series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometric series with $a = 1$ and $r = x$.

Since $|r| = |x| < 1$, the series converges, and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \tag{5}$$

■

Example 6 A Telescoping Sum

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Solution

This is not a geometric series, so consider the general definition of a convergent series. Compute the partial sums.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We can simplify this expression by using a partial fraction decomposition.

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Rewrite the nth partial sum.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \qquad \text{Use the PFD.}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \qquad \text{Write the terms of the series.}$$

$$= 1 - \frac{1}{n+1} \qquad \text{Cancel terms in pairs.}$$

Notice that the terms cancel in pairs. This is an example of a **telescoping sum**. Because of all the cancellations, the sum collapses into just two terms.

Consider the limit of the partial sums.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

Therefore, the series is convergent, and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Figure 8.15 illustrates this conclusion by showing the graphs of the sequence of terms $a_n = \frac{1}{n(n+1)}$ and the sequence $\{s_n\}$ of partial sums. This graph suggests that $a_n \to 0$ and confirms that $s_n \to 1$.

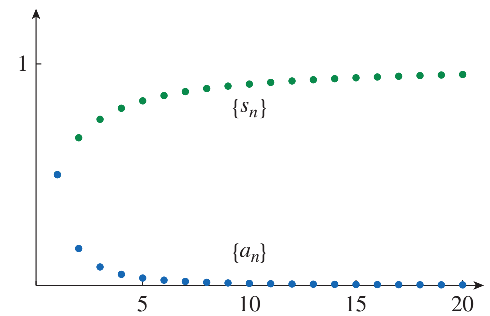


Figure 8.15
Graphs of $\{a_n\}$ and $\{s_n\}$. Where is a_1?

Example 7 Harmonic Series

Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Solution

To show that this series is divergent, it is convenient to consider the partial sums s_2, s_4, s_8, s_{16}, s_{32}, . . . and show that they become arbitrarily large.

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$\begin{aligned} s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2} \end{aligned}$$

$$\begin{aligned} s_{16} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2} \end{aligned}$$

Similarly, $s_{32} > 1 + \frac{5}{2}$, $s_{64} > 1 + \frac{6}{2}$, and in general

$$s_{2^n} > 1 + \frac{n}{2}.$$

That's the 2^nth partial sum.

This shows that as n increases, the partial sums can be made arbitrarily large.

That is, $s_{2^n} \to \infty$ as $n \to \infty$, and so $\{s_n\}$ is divergent.

Therefore, the harmonic series diverges. ■

This is a very important fact that will be used repeatedly in the study of series: the harmonic series diverges.

Convergent Series Behavior

The next theorem tells us about the behavior of the terms in a convergent series.

Theorem • Limit of the Terms of a Convergent Series

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Proof

Let $s_n = a_1 + a_2 + \cdots + a_n$ (the nth partial sum).

Then $a_n = s_n - s_{n-1}$.

Since Σa_n is convergent, the sequence of partial sums $\{s_n\}$ is convergent.

Let $\lim_{n\to\infty} s_n = s$. Since $n - 1 \to \infty$ as $n \to \infty$, then $\lim_{n\to\infty} s_{n-1} = s$.

Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$ ■

A Closer Look

1. There are two *sequences* associated with every *series*. Associated with the series Σa_n, there is the sequence of terms, $\{a_n\}$, and the sequence of partial sums, $\{s_n\}$.
   If Σa_n is convergent, then the limit of the sequence $\{s_n\}$ is s, the sum of the series, and, as the theorem above shows, the limit of the sequence $\{a_n\}$ is 0.
2. The converse of the theorem above is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that Σa_n is convergent.
   For example, for the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ the terms of the series converge to 0, $a_n = \frac{1}{n} \to 0$ as $n \to \infty$, but the series is divergent, as shown in Example 7. ■

The following test for divergence is the contrapositive of Theorem 6.

The Test for Divergence (or *n*th Term Test)

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 8 Use the Test for Divergence

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$ diverges.

Solution

Consider the limit of the terms of the series.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4}$$ Limit of the sequence $\{a_n\}$.

$$= \lim_{n\to\infty} \frac{1}{5+\dfrac{4}{n^2}} = \frac{1}{5} \neq 0$$ Divide numerator and denominator by n^2; evaluate the limit.

The series diverges by the Test for Divergence.

A Closer Look

If $\lim_{n\to\infty} a_n \neq 0$, then we know that Σa_n is divergent.

If $\lim_{n\to\infty} a_n = 0$, then we cannot conclude anything about the convergence or divergence of Σa_n. The series might converge, or it might diverge. We need to do something more in order to draw a conclusion.

Combining Series

The next theorem helps us to combine known convergent series into new convergent series.

Theorem • Properties of Convergent Series

If Σa_n and Σb_n are convergent series, then so are the series Σca_n (where c is a constant), $\Sigma(a_n + b_n)$, $\Sigma(a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

These properties of convergent series follow from the corresponding Limit Laws for Sequences. For example, here is a proof of Theorem 8, part (ii).

Write the sum of each series and the partial sums as

$$s_n = \sum_{i=1}^{n} a_i \qquad s = \sum_{n=1}^{\infty} a_n \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

The nth partial sum of the series $\Sigma(a_n + b_n)$ is $u_n = \displaystyle\sum_{i=1}^{n} (a_i + b_i)$.

Consider the limit of these partial sums.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sum_{i=1}^{n} (a_i + b_i) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\right)$$ Property of summations.

$$= \lim_{n\to\infty} \sum_{i=1}^{n} a_i + \lim_{n\to\infty} \sum_{i=1}^{n} b_i$$ Limit Law.

$$= \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = s + t$$ Σa_n and Σb_n are convergent.

Therefore, $\Sigma(a_n + b_n)$ is convergent, and its sum is

$$\sum_{n=1}^{\infty}(a_n + b_n) = s + t = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

Example 9 Combine Two Series

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

Solution

The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.

Therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$

In Example 6, we concluded that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1.$

Use the properties of convergent series to combine these results. The given series is convergent, and

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) &= 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)} + \sum_{n=1}^{\infty}\frac{1}{2^n}.\\ &= 3 \cdot 1 + 1 = 4\end{aligned}$$

A Closer Look

A finite number of terms does not affect the convergence or divergence of a series. For example, suppose that we know that the series $\displaystyle\sum_{n=4}^{\infty}\frac{n}{n^3+1}$ is convergent.

Since $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty}\frac{n}{n^3+1},$

then the entire series $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+1}$ is convergent (because all we did was add a finite number of terms at the beginning of a convergent series).

Similarly, if it is known that the series $\displaystyle\sum_{n=N+1}^{\infty} a_n$ converges, then the full, or extended, series $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$ is also convergent.

8.2 Exercises

1. (a) Explain the difference between a sequence and a series.
(b) Explain what is meant by a convergent series and a divergent series.

2. Explain what it means to say that $\displaystyle\sum_{n=1}^{\infty} a_n = 5$.

Calculate the first ten terms of the sequence of partial sums. Does it appear that the series is convergent or divergent? Explain your reasoning.

3. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4+n^2}$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$

5. $\displaystyle\sum_{n=1}^{\infty} \sin n$ **6.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$ **8.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{e^n}$

Find at least ten partial sums of the series. Graph both the sequence of terms and the sequence of partial sums in the same viewing rectangle. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{12}{(-5)^n}$ **10.** $\displaystyle\sum_{n=1}^{\infty} \cos n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+4}}$ **12.** $\displaystyle\sum_{n=1}^{\infty} \frac{7^{n+1}}{10^n}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ **14.** $\displaystyle\sum_{n=1}^{\infty} \left(\sin\frac{1}{n} - \sin\frac{1}{n+1}\right)$

15. Let $a_n = \dfrac{2n}{3n+1}$.

(a) Determine whether the sequence $\{a_n\}$ is convergent.

(b) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

16. (a) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{j=1}^{n} a_j$$

(b) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{i=1}^{n} a_j$$

Determine whether the geometric series is convergent or divergent. If it is convergent, find the sum.

17. $3 - 4 + \dfrac{16}{3} - \dfrac{64}{9} + \cdots$

18. $4 + 3 + \dfrac{9}{4} + \dfrac{27}{16} + \cdots$

19. $10 - 2 + 0.4 - 0.08 + \cdots$

20. $2 + 0.5 + 0.125 + 0.03125 + \cdots$

21. $3 - \dfrac{2}{3} + \dfrac{4}{27} - \dfrac{8}{243} + \cdots$

22. $\displaystyle\sum_{n=1}^{\infty} 12(0.73)^{n-1}$ **23.** $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\pi^n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$ **25.** $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$ **27.** $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$

Determine whether the series is convergent or divergent. If it is convergent, find its sum.

28. $\dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{9} + \dfrac{1}{12} + \dfrac{1}{15} + \cdots$

29. $\dfrac{1}{3} + \dfrac{2}{9} + \dfrac{1}{27} + \dfrac{2}{81} + \dfrac{1}{243} + \dfrac{2}{729} + \cdots$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$ **31.** $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^2-2k+5}$

32. $\displaystyle\sum_{n=1}^{\infty} 3^{n+1}4^{-n}$ **33.** $\displaystyle\sum_{n=1}^{\infty} [(-0.2)^n + (0.6)^{n-1}]$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$ **35.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n}$

36. $\displaystyle\sum_{k=1}^{\infty} (\sin 100)^k$ **37.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$

38. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ **39.** $\displaystyle\sum_{k=0}^{\infty} (\sqrt{2})^{-k}$

40. $\displaystyle\sum_{n=1}^{\infty} \arctan n$ **41.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$

42. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right)$ **43.** $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

Determine whether the series is convergent or divergent by expressing the nth partial sum s_n as a telescoping sum. If it is convergent, find its sum.

44. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ **45.** $\displaystyle\sum_{n=1}^{\infty} \ln\frac{n}{n+1}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ **47.** $\displaystyle\sum_{n=4}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

48. $\displaystyle\sum_{n=1}^{\infty} (e^{1/n} - e^{1/(n+1)})$ **49.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-n}$

50. Let $x = 0.99999\ldots$.

(a) Do you think that $x < 1$ or $x = 1$?

(b) Construct a geometric series for x and use it to find the value of x.

(c) How many decimal representations does the number 1 have?

(d) Which numbers have more than one decimal representation?

51. Suppose the sequence $\{a_n\}$ is defined by

$$a_1 = 1 \qquad a_n = (5-n)a_{n-1}$$

Calculate $\displaystyle\sum_{n=1}^{\infty} a_n$.

Express the number as a ratio of integers.

52. $0.\overline{8} = 0.888\ldots$

53. $0.\overline{46} = 0.46464646\ldots$

54. $0.5\overline{29} = 0.5292929\ldots$

55. $2.\overline{516} = 2.516516516\ldots$

56. $10.1\overline{35} = 10.135353535\ldots$

57. $1.234\overline{567}$

58. $5.\overline{71358}$

Find the values of x for which the series converges. Find the sum of the series for those values of x.

59. $\displaystyle\sum_{n=1}^{\infty} (-5)^n x^n$

60. $\displaystyle\sum_{n=1}^{\infty} (x+2)^n$

61. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$

62. $\displaystyle\sum_{n=0}^{\infty} (-4)^n (x-5)^n$

63. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{x^n}$

64. $\displaystyle\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n}$

65. $\displaystyle\sum_{n=0}^{\infty} e^{nx}$

66. $\displaystyle\sum_{n=4}^{\infty} (2 \ln x)^n$

67. The harmonic series is a divergent series whose terms approach 0. Show that the series

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$$

has the same properties.

Use technology to find an appropriate partial fraction decomposition and an expression for the partial sum. Use this expression to find the sum of the series. Check your answer using technology to sum the series directly.

68. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2 + 3n + 1}{(n^2 + n)^3}$

69. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^5 - 5n^3 + 4n}$

70. Suppose the nth partial sum of a series $\displaystyle\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n-1}{n+1}$$

Find a_n and $\displaystyle\sum_{n=1}^{\infty} a_n$.

71. Suppose the nth partial sum of a series $\displaystyle\sum_{n=1}^{\infty} a_n$ is $s_n = 3 - n2^{-n}$. Find a_n and $\displaystyle\sum_{n=1}^{\infty} a_n$.

72. A doctor prescribes a 100-mg antibiotic tablet for a patient to be taken every 8 hours. Just before each tablet is taken, 20% of the drug remains in the body.

(a) How much of the drug is in the body just after the second tablet is taken? After the third tablet?

(b) If Q_n is the quantity of the antibiotic in the body just after the nth tablet is taken, find an equation that expresses Q_{n+1} in terms of Q_n.

(c) What quantity of the antibiotic remains in the body in the long run?

73. To control the medfly (Mediterranean fruit fly and agricultural pest), N sterilized male flies are released into the general fly population every day. If s is the proportion of these sterilized flies that survive a given day, then Ns^k will survive for k days.

(a) How many sterile flies survive after n days? What happens to the number of sterile flies in the long run?

(b) If $s = 0.9$ and 10,000 sterilized males are needed to control the medfly population in a given area, how many should be released every day?

74. When money is spent on goods and services, those who receive the money also spend some of it. The people receiving some of the twice-spent money will spend some of that, and so on. Economists call this chain reaction the *multiplier effect*. In a hypothetical isolated community, the local government begins the process by spending D dollars. Suppose that each recipient of spent money spends $100c\%$ and saves $100s\%$ of the money that they receive. The values c and s are called the *marginal propensity to consume* and the *marginal propensity to save*, respectively, and, of course, $c + s = 1$.

(a) Let S_n be the total spending that has been generated after n transactions. Find an equation for S_n.

(b) Show that $\displaystyle\lim_{n\to\infty} S_n = kD$, where $k = \dfrac{1}{s}$. The number k is called the *multiplier*. What is the multiplier if the marginal propensity to consume is 80%?

Note: The federal government uses this principle to justify deficit spending. Banks use this principle to justify lending a large percentage of the money that they receive in deposits.

75. A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where $0 < r < 1$. Suppose that the ball is dropped from an initial height of H meters.

(a) Assuming that the ball continues to bounce indefinitely, find the total distance that it travels.

(b) Calculate the total time that the ball travels. Use the fact that the ball falls $\frac{1}{2}gt^2$ meters in t seconds.

(c) Suppose that each time the ball strikes the surface with velocity v, it rebounds with velocity $-kv$, where $0 < k < 1$. How long will it take for the ball to come to rest?

76. Find the value of c if

$$\sum_{n=2}^{\infty} (1 + c)^{-n} = 2$$

77. Find the value of c such that

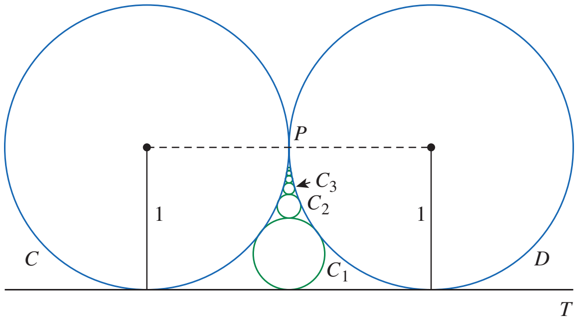
$$\sum_{n=0}^{\infty} e^{nc} = 10$$

78. Here is the outline of another method to show that the harmonic series is divergent, making use of the fact that $e^x > 1 + x$ for any $x > 0$.
If s_n is the nth partial sum of the harmonic series, show that $e^{s_n} > n + 1$. Why does this imply that the harmonic series is divergent?

79. Graph the functions defined by $y = x^n$, $0 \le x \le 1$, for $n = 0, 1, 2, 3, 4, 5, \ldots$ on the same coordinate axes. Find the area between successive curves and use this to give a geometric demonstration of the fact that

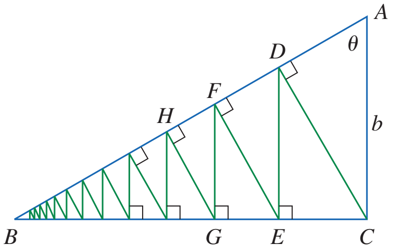
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

80. The figure shows two circles C and D of radius 1 that touch at P.



The line T is a common tangent line: C_1 is the circle that touches C, D, and T; C_2 is the circle that touches C, D, and C_1; C_3 is the circle that touches C, D, and C_2. This procedure can be continued indefinitely and produces an infinite sequence of circles $\{C_n\}$. Find an expression for the diameter of C_n and thus provide another geometric demonstration of Example 6.

81. A right triangle ABC is given with $\angle A = \theta$ and $|AC| = b$. CD is drawn perpendicular to AB, $DE \perp BC$, $EF \perp AB$, and this process is continued indefinitely, as shown in the figure.



Find the total length of all the perpendiculars

$$|CD| + |DE| + |EF| + |FG| + \cdots$$

in terms of b and θ.

82. Explain the flaw in this calculation.

$$\begin{aligned} 0 &= 0 + 0 + 0 + \cdots \\ &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\ &= 1 + (-1+1) + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + 0 + \cdots = 1 \end{aligned}$$

83. Suppose that $\sum_{n=1}^{\infty} a_n$ $(a_n \neq 0)$ is known to be a convergent series. Prove that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is a divergent series.

84. If Σa_n is divergent and $c \neq 0$, show that Σca_n is divergent.

85. If Σa_n is convergent and Σb_n is divergent, show that the series $\Sigma(a_n + b_n)$ is divergent. Hint: Argue by contradiction.

86. If Σa_n and Σb_n are both divergent, is $\Sigma(a_n + b_n)$ necessarily divergent?

87. Suppose that a series Σa_n has positive terms and its partial sums s_n satisfy the inequality $s_n \le 1000$ for all n. Explain why Σa_n must be convergent.

88. The Fibonacci sequence was defined in Section 8.1 by the equations

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}$$

Show that each of the following statements is true.

(a) $\dfrac{1}{f_{n-1}f_{n+1}} = \dfrac{1}{f_{n-1}f_n} - \dfrac{1}{f_n f_{n+1}}$

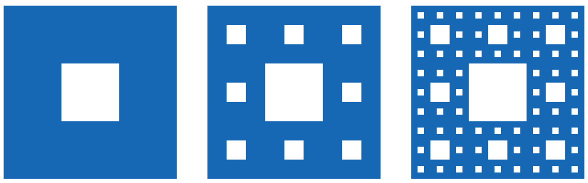
(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = 1$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = 2$

89. The **Cantor set**, named after the German mathematician Georg Cantor (1845–1918), is constructed as follows. Start with the closed interval $[0, 1]$ and remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. That leaves the two intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. Remove the open middle third of each. Four intervals remain and again we remove the open middle third of each of them. Continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the preceding step. The Cantor set consists of the numbers that remain in $[0, 1]$ after all those intervals have been removed.

(a) Show that the total length of all the intervals that are removed is 1. Despite that, the Cantor set contains infinitely many numbers. Give examples of some numbers in the Cantor set.

(b) The **Sierpiński carpet** is a two-dimensional counterpart of the Cantor set. It is constructed by removing the center one-ninth of a square of side 1, then removing the centers of the eight smaller remaining squares, and so on. The figure shows the first three steps of the construction.



Show that the sum of the areas of the removed squares is 1. This implies that the Sierpiński carpet has area 0.

90. (a) A sequence $\{a_n\}$ is defined recursively by the equation

$$a_n = \frac{1}{2}(a_{n-1} + a_{n-2}) \quad \text{for } n \geq 3$$

where a_1 and a_2 can be any real numbers. Consider several terms of this sequence for various values of a_1 and a_2. Use technology to guess the limit of this sequence.

(b) Find $\lim_{n\to\infty} a_n$ in terms of a_1 and a_2 by expressing $a_{n+1} - a_n$ in terms of $a_2 - a_1$ and summing a series.

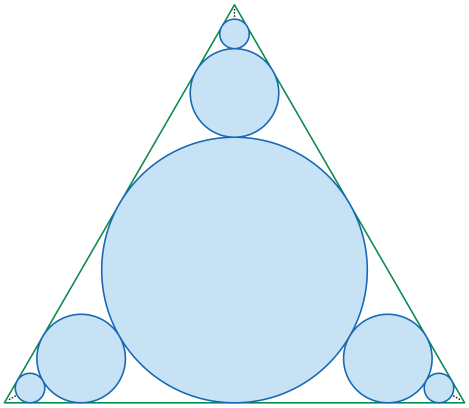
91. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

(a) Find the partial sums s_1, s_2, s_3 and s_4. Do you recognize the denominators? Use the pattern to guess a formula for s_n.

(b) Use mathematical induction to prove your guess.

(c) Show that the given infinite series is convergent and find its sum.

92. In the figure, there are infinitely many circles approaching the vertices of an equilateral triangle, each circle touching other circles and sides of the triangle.



If the triangle has sides of length 1, find the total area occupied by the circles.

8.3 The Integral and Comparison Tests; Estimating Sums

In general, it is difficult to find the exact sum of a series. We can find the sum of a geometric series and certain other series, for example $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, because in each case we can find a formula for the nth partial sum s_n. But it usually isn't easy to find such a formula. Therefore, in this section, we develop several tests that enable us to determine whether a series is convergent or divergent without actually finding its sum. In some cases, we will be able to find good estimates of the sum, but not the actual value.

In this section, we will consider only series with positive terms, so the partial sums are increasing. If we can show that the partial sums are bounded, then by the Monotonic Sequence Theorem, the series is convergent.

Testing with an Integral

Consider the series whose terms are the reciprocals of the squares of the positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.463611
10	1.549768
50	1.625133
100	1.634984
500	1.642936
1000	1.643935
5000	1.644734
10000	1.644834

Table 8.3
Table of partial sums.

There is no simple formula for the sum s_n of the first n terms. However, examine the approximate values for s_n in Table 8.3. This table suggests that the partial sums are approaching a number near 1.64 as $n \to \infty$. Therefore, it certainly looks as though the series is convergent.

We can confirm this conjecture with a geometric argument. Figure 8.16 shows the graph of the function $y = \frac{1}{x^2}$ and specific associated rectangles that lie below the curve.

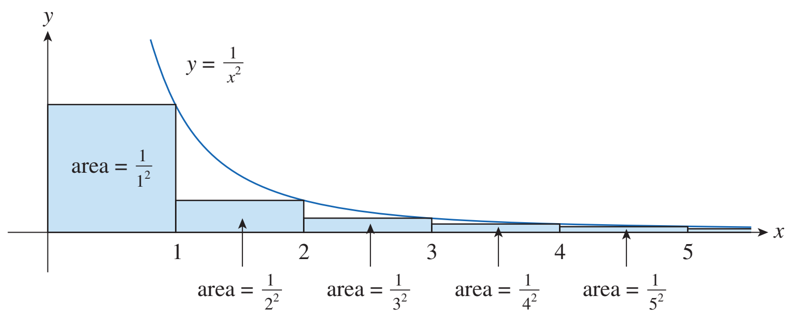


Figure 8.16
Graph of $y = \frac{1}{x^2}$. All of the rectangles lie below the graph.

The base of each rectangle is an interval of length 1; the height is equal to the value of the function $y = \frac{1}{x^2}$ at the right endpoint of the interval.

The sum of the areas of the rectangles is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the graph of $y = \frac{1}{x^2}$ for $x \geq 1$, which is the value of the integral $\int_1^{\infty} \frac{1}{x^2}\,dx$. In Section 5.10, we showed that this improper integral is convergent and has value 1. Therefore, the figure shows that all the partial sums are less than

$$\frac{1}{1^2} + \int_1^{\infty} \frac{1}{x^2}\,dx = 2$$

This means the partial sums of the series are bounded. We also know that the partial sums are increasing, because all the terms are positive. Therefore, the partial sums converge, by the Monotonic Sequence Theorem, and so the series is convergent.

We do not know the exact value of the sum of the series (the limit of the partial sums), but we do know now that this sum is less than 2.

The exact sum of this series was found by the Swiss mathematician Leonhard Euler (1707–1783) to be $\frac{\pi^2}{6}$ ($\approx$1.644934).

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots < 2$$

Now consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681
10000	198.5446

Table 8.4
Table of partial sums.

Table 8.4 shows values for s_n and suggests that the partial sums are not approaching a finite number but instead are increasing without bound. This observation suggests that

the series is divergent. Consider a graph for confirmation. Figure 8.17 shows the graph of $y = \dfrac{1}{\sqrt{x}}$, but this time with rectangles whose top portions lie *above* the curve.

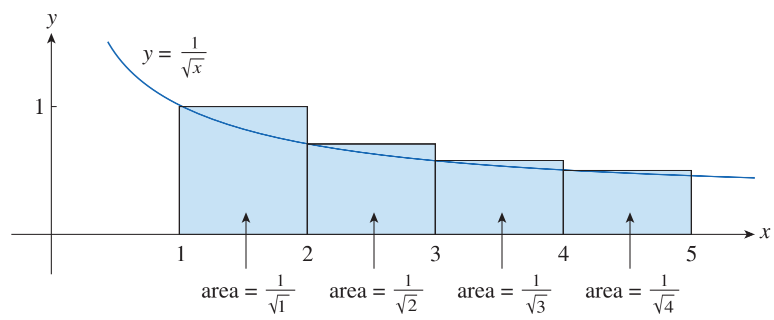


Figure 8.17

Graph of $y = \dfrac{1}{\sqrt{x}}$. A portion of each rectangle lies above the graph.

The base of each rectangle is an interval of length 1. The height is equal to the value of the function $y = \dfrac{1}{\sqrt{x}}$ at the *left* endpoint of the interval. So, the sum of the areas of all the rectangles is

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

The total area of all the rectangles is greater than the area under the graph of $y = \dfrac{1}{\sqrt{x}}$ for $x \geq 1$, which is equal to the integral $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$. In Section 5.10, we showed that this improper integral is divergent. Geometrically, this is interpreted as the area under the graph is unbounded, or infinite. So, the sum of the series must be infinite also, that is, the series is divergent.

The geometric arguments used in these two series examples can be used to prove the following test for series convergence.

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(a) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

A Closer Look

1. When using the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For example, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \quad \text{we use} \quad \int_4^{\infty} \frac{1}{(x-3)^2}\,dx$$

2. In addition, it is not necessary for f to be decreasing over the entire interval. It is important that f be *ultimately* decreasing, that is, decreasing for x larger than some number N. Then, if $\sum_{n=N}^{\infty} a_n$ is convergent, so is $\sum_{n=1}^{\infty} a_n$. ■

Example 1 Use the Integral Test

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution

The associated function $f(x) = \frac{\ln x}{x}$ is positive and continuous for $x > 1$ (it is a quotient of continuous functions).

To check whether f is decreasing, first find its derivative.

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot (1)}{x^2} = \frac{1 - \ln x}{x^2} \qquad \text{Quotient Rule; simplify.}$$

There is only one critical point: $x = e$, and $f'(x) < 0$ when $x > e$. Therefore, f is *ultimately* decreasing when $x \geq e$ and we can use the Integral Test.

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2}\right]_1^t \qquad \text{Improper integral; use the substitution } u = \ln x.$$

$$= \lim_{t\to\infty} \left[\frac{(\ln t)^2}{2} - \frac{(\ln 1)^2}{2}\right] = \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty \qquad \text{FTC2; evaluate the limit.}$$

Since the improper integral is divergent, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is also divergent by the Integral Test. ■

Example 2 Analysis of the p-series

For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Solution

If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$.

If $p = 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$.

In both cases, since the limit of the terms is nonzero, the series diverges by the Test for Divergence.

If $p > 0$, then the associated function $f(x) = \dfrac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$.

In Section 5.10, we learned that $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $p \le 1$.

Therefore, using the Integral Test, we can conclude the following:

the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Note: For $p = 1$, this series is the harmonic series, which diverges, as shown in Section 8.2. ■

The series in Example 2 is called a ***p*-series**. It is an extremely valuable reference tool. Here is a summary of the result from Example 2.

p-Series

A p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p-series with $p = 3 > 1$. But the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p-series with $p = \dfrac{1}{3} < 1$.

Testing by Comparing

The concept of using comparison tests is straightforward: compare a given series with a series that is known to converge or diverge. But you need good problem-solving skills in order to select an appropriate series for comparison. For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} \tag{1}$$

is similar to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is a geometric series with $a = \dfrac{1}{2}$ and $r = \dfrac{1}{2}$, and is, therefore, convergent. Because the series in Equation 2 is so similar to a convergent series, this suggests that it must also be convergent. Indeed, it is. The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

shows that the series in Equation 2 has smaller terms than those of the geometric series, and therefore, all its partial sums are also smaller than 1, the sum of the geometric series. This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series.

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1$$

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent. The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

The Comparison Test

Suppose Σa_n and Σb_n are series with positive terms.

(a) If Σb_n is convergent and $a_n \leq b_n$ for all n, then Σa_n is also convergent.

(b) If Σb_n is divergent and $a_n \geq b_n$ for all n, then Σa_n is also divergent.

Common Series for Use with the Comparison Test

When using the Comparison Test we must, of course, have some known series Σb_n for the purpose of comparison. In a lot of cases, we use one of these series.

(1) A p-series:

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

(2) A geometric series:

$\displaystyle\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. ■

Example 3 Use the Comparison Test and a *p*-Series

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

Solution

For large n, the dominant term in the denominator is $2n^2$.

Therefore, it seems reasonable to compare the given series with $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2}$.

In Comparison Test terminology, a_n is the left side and b_n is the right side.

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

The left side has a larger denominator.

Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$.

This is convergent because it is a p-series with $p = 2 > 1$.

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ is convergent by the Comparison Test. ■

A Closer Look

Although the condition $a_n \le b_n$ or $a_n \ge b_n$ in the Comparison Test is given for all n, we need to verify only that the inequality is true for $n \ge N$, where N is some fixed integer. Remember that the convergence of a series is not affected by a finite number of initial terms.

Example 4 Use the Comparison Test

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution

Previously, we used the Integral Test to show that this series diverges. But we can also use the Comparison Test.

Observe that if $n \ge 3$, then $\ln n > 1$ and $\frac{\ln n}{n} > \frac{1}{n}$.

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (it is a p-series with $p = 1$; the harmonic series).

Therefore, the given series is divergent by the Comparison Test.

To use the Comparison Test, we must have

(i) $a_n \le b_n$ and Σb_n convergent or

(ii) $a_n \ge b_n$ and Σb_n divergent.

If

(i) $a_n \ge b_n$ and Σb_n is convergent, or

(ii) $a_n \le b_n$ and Σb_n is divergent,

then we cannot draw a conclusion; the Comparison Test does not apply.

For example, consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$.

The inequality

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

is true but not helpful. The Comparison Test cannot be used because

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

is convergent and $a_n > b_n$.

But we still have a feeling that $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is convergent; it is very similar to the convergent geometric series, Σb_n, and it seems as though the 1 in the denominator should not affect convergence. In cases like this, consider the following test.

The Limit Comparison Test

Suppose that Σa_n and Σb_n are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Although we will not present a proof of the Limit Comparison Test, the result seems reasonable because for large n, $a_n \approx cb_n$.

Example 5 Use the Limit Comparison Test

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges or diverges.

Solution

Use the Limit Comparison Test with $a_n = \dfrac{1}{2^n - 1}$ and $b_n = \dfrac{1}{2^n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{1}{2^n - 1}}{\dfrac{1}{2^n}}$$

Write the limit expression in the Limit Comparison Test.

$$= \lim_{n\to\infty} \frac{2^n}{2^n - 1}$$

Simplify the quotient.

$$= \lim_{n\to\infty} \frac{1}{1 - \dfrac{1}{2^n}} = 1 > 0$$

Divide the numerator and denominator by 2^n; evaluate the limit.

Since this limit exists and is greater than 0, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series, the given series converges by the Limit Comparison Test. ■

Estimating the Sum of a Series

Suppose we use the Integral Test to show that a series Σa_n is convergent. It would be nice if we could also find an approximation to the sum s of the series. We know that *any* partial sum s_n is an approximation to s because $\displaystyle\lim_{n\to\infty} s_n = s$. But we would really like some measure of the accuracy of the approximation. To measure accuracy, we need to estimate the magnitude of the remainder,

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder R_n is the error made when s_n, the sum of the first n terms, is used as an approximation to the total sum.

Assume that the associated function f is decreasing on $[n, \infty)$ and use the same notation and concepts as in the Integral Test. Compare the areas of the rectangles with the area under the graph of $y = f(x)$ for $x > n$ in Figure 8.18. This suggests that

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \le \int_n^{\infty} f(x)\,dx$$

Similarly, Figure 8.19 suggests that

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^{\infty} f(x)\,dx$$

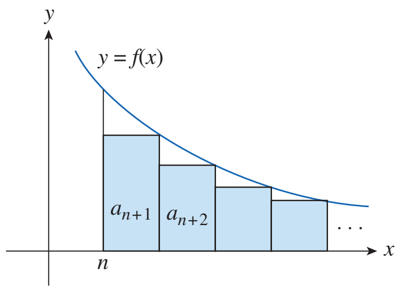


Figure 8.18

$R_n \leq \int_n^{\infty} f(x)\,dx.$

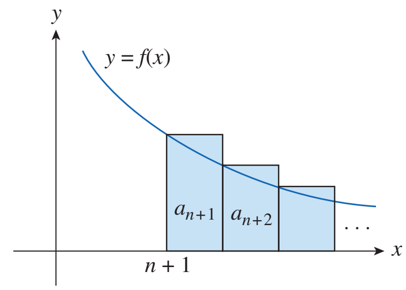


Figure 8.19

$R_n \geq \int_{n+1}^{\infty} f(x)\,dx.$

This geometric argument leads to the following error estimate.

Remainder Estimate for the Integral Test

Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and Σa_n is convergent with sum s and sequence of partial sums $\{s_n\}$. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$$

Example 6 Estimate the Sum of a Series

(a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first ten terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Solution

The associated function is $f(x) = \frac{1}{x^3}$, which is positive and decreasing (on $(0, \infty)$), and therefore, satisfies the conditions of the Integral Test.

In both parts (a) and (b), we will need to know $\int_n^{\infty} f(x)\,dx$.

$$\int_n^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left[-\frac{1}{2t^2} + \frac{1}{2n^2}\right] = \frac{1}{2n^2}$$

(a) Find the tenth partial sum.

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

Use the remainder estimate to determine the accuracy of this estimate.

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2(10)^2} = \frac{1}{200} = 0.005$$

The magnitude of the error is at most 0.005.

(b) To ensure that the estimate is within 0.0005 means that we have to find a value of n such that $R_n \leq 0.0005$.

Use the remainder estimate.

$$R_n \leq \int_{n}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2} < 0.0005$$

Solve this inequality for n.

$$n^2 > \frac{1}{0.001} = 1000$$ Multiply both sides by 2; take the reciprocal of both sides.

$$n > \sqrt{1000} \approx 31.623$$ Square root of both sides.

We always round up to the next integer to guarantee the remainder term is less than or equal to the desired value.

Using technology, $s_{32} = 1.20158$.t

Therefore, we need 32 terms to ensure accuracy to within 0.0005. ■

To find a lower and upper bound for the sum of a series, add S_n to the inequality in the remainder estimate. Remember that $s_n + R_n = s$.

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_{n}^{\infty} f(x)dx \qquad (4)$$

This inequality provides an interval for the sum of a series and conveys more accuracy than a single partial sum.

Example 7 An Improved Estimate

Although Euler was able to calculate the exact sum of the p-series for $p = 2$, no one has yet been able to find the exact sum for $p = 3$.

Use Equation 4 with $n = 10$ to estimate the sum of the series. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Solution

Use the inequalities in Equation 4.

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^3}dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^3}dx$$

From Example 6, we have:

$$\int_{n}^{\infty} \frac{1}{x^3}dx = \frac{1}{2n^2}.$$

Use this result in the inequality statement.

$$s_{10} + \frac{1}{2(11)^2} \leq s \leq s_{10} + \frac{1}{2(10)^2}$$

Using technology, we get $1.201664 \leq s \leq 1.202532$.

It seems reasonable to approximate s by the midpoint of this interval. Then the error is at most half the length of the interval.

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n^3} \approx \frac{1.201664 + 1.202532}{2} = 1.2021 \quad \text{with error} < 0.0005.$$

If we compare Examples 6 and 7, we see that the interval estimate can be much better than a single estimate s_n. To ensure that the error is smaller than 0.0005, we had to use 32 terms in Example 6 but only 10 terms in Example 7.

If we have already used the Comparison Test to show that a series Σa_n converges by comparison with a series Σb_n, then we may be able to estimate the sum Σa_n by comparing remainders. Consider the following example.

Example 8 The First 100 Terms

Use the sum of the first 100 terms to approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3+1}$.

Estimate the error involved in this approximation.

Solution

Since $\dfrac{1}{n^3+1} < \dfrac{1}{n^3}$, the given series is convergent by the Comparison Test.

The remainder T_n for the comparison series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ was estimated in Example 6, using the remainder estimate for the Integral Test.

Therefore, $T_n \le \displaystyle\int_n^{\infty}\frac{1}{x^3}\,dx = \frac{1}{2n^2}$.

The remainder R_n for the given series satisfies $R_n \le T_n \le \dfrac{1}{2n^2}$.

For $n = 100$, $R_n \le \dfrac{1}{2(100)^2} = 0.00005$.

Using technology, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3+1} \approx \sum_{n=1}^{100}\frac{1}{n^3+1} = 0.6864538$.

The error in this estimation is less than 0.00005.

8.3 Exercises

1. Draw a picture to show that

$$\sum_{n=2}^{\infty}\frac{1}{n^{1.3}} < \int_1^{\infty}\frac{1}{x^{1.3}}\,dx$$

What can you conclude about the series?

2. Suppose f is a continuous positive decreasing function for $x \ge 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order.

$$\int_1^6 f(x)\,dx \qquad \sum_{i=1}^{5} a_i \qquad \sum_{i=2}^{6} a_i$$

3. Suppose Σa_n and Σb_n are series with positive terms and Σb_n is known to be convergent.
(a) If $a_n > b_n$ for all n, what, if anything, can you say about Σa_n? Explain.
(b) If $a_n < b_n$ for all n, what, if anything, can you say about Σa_n? Explain.

4. Suppose Σa_n and Σb_n are series with positive terms and Σb_n is known to be divergent.
(a) If $a_n > b_n$ for all n, what, if anything, can you say about Σa_n? Explain.
(b) If $a_n < b_n$ for all n, what, if anything, can you say about Σa_n? Explain.

5. It is important to distinguish between

$$\sum_{n=1}^{\infty} n^b \quad \text{and} \quad \sum_{n=1}^{\infty} b^n$$

What is the first series called? The second? For what values of b does the first series converge? For what values of b does the second series converge?

Use the Integral Test to determine whether the series is convergent or divergent.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5n-1}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

11. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Use the Comparison Test to determine whether the series is convergent or divergent.

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$

13. $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{n \sin^2 n}{1+n^3}$

Determine whether the series is convergent or divergent.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$

17. $\displaystyle\sum_{n=1}^{\infty} n^{-0.9999}$

18. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$

19. $\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \cdots$

20. $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \cdots$

21. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$

22. $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$

23. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

26. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

27. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{e^n}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{n+3^n}{n+2^n}$

37. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n\sqrt{n}}$

39. $\displaystyle\sum_{n=0}^{\infty} \frac{1+\sin n}{10^n}$

40. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

Explain why the Integral Test cannot be used to determine whether the series is convergent.

44. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{1+n^2}$

Find the values of p for which the series is convergent.

46. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

48. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.
(a) Find the tenth partial sum, s_{10}. Estimate the error in using s_{10} as an approximation to the sum of the series.
(b) Use Equation 4 to obtain an improved estimate of the sum.
(c) Find a value of n so that s_n is within 0.00001 of the sum.

49. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.
(a) Find the tenth partial sum, s_{10}. Estimate the error in using s_{10} as an approximation to the sum of the series.
(b) Use Equation 4 to obtain an improved estimate of the sum.
(c) Find a value of n so that s_n is within 0.001 of the sum.

50. Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ correct to three decimal places.

51. Estimate $\sum_{n=1}^{\infty} (2n+1)^{-6}$ correct to five decimal places.

52. How many terms of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ are required to ensure that the sum is accurate to within 0.01?

Use the sum of the first 10 terms to approximate the sum of the series. Estimate the error.

53. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+1}}$

54. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$

55. (a) Use a graph of $y = 1/x$ to show that if s_n is the nth partial sum of the harmonic series, then

$$s_n \leq 1 + \ln n$$

(b) The harmonic series diverges, but very slowly. Use part (a) to show than the sum of the first million terms is less than 15 and the sum of the first billion terms is less than 22.

56. Show that if we want to approximate the sum of the series $\sum_{n=1}^{\infty} n^{-1.001}$ so that the error is less than 5 in the ninth decimal place, then we need to add more than $10^{11,301}$ terms!

57. The meaning of the decimal representation of a number $0.d_1d_2d_3\ldots$, where d_i is one of the numbers 0, 1, 2, . . . , 9, is that

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots$$

Show that this series always converges.

58. Show that if $a_n > 0$ and Σa_n is convergent, then $\Sigma \ln(1 + a_n)$ is also convergent.

59. If Σa_n is a convergent series with positive terms, is it true that $\Sigma \sin(a_n)$ is also convergent? Explain your reasoning.

60. Find all positive values of b for which the series $\sum_{n=1}^{\infty} b^{\ln n}$ converges.

61. Show that if $a_n > 0$ and $\lim_{n\to\infty} na_n \neq 0$, then Σa_n is divergent.

62. Find all values of c for which the following series converges.

$$\sum_{n=1}^{\infty}\left(\frac{c}{n} - \frac{1}{n+1}\right)$$

63. Show that if $a_n \geq 0$ and Σa_n converges, then Σa_n^2 also converges.

64. If Σa_n and Σb_n are both convergent series with positive terms, is it true that $\Sigma a_n b_n$ is also convergent? Explain your reasoning.

8.4 Other Convergence Tests

The tests for convergence (or divergence) that we have considered so far apply only to series with positive terms. In this section, we learn how to deal with series whose terms are not necessarily positive. These concepts involve *alternating series*, whose terms alternate in sign.

Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

These examples suggest that the nth term of an alternating series is of the form

$$a_n = (-1)^{n-1}b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where b_n is a positive number. In fact, $b_n = |a_n|$.

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

The Alternating Series Test (AST)

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

(i) $b_{n+1} \leq b_n$, for all n, and

(ii) $\lim\limits_{n\to\infty} b_n = 0$,

then the series is convergent.

We won't present a formal proof of this test, but Figure 8.20 illustrates the idea behind the proof.

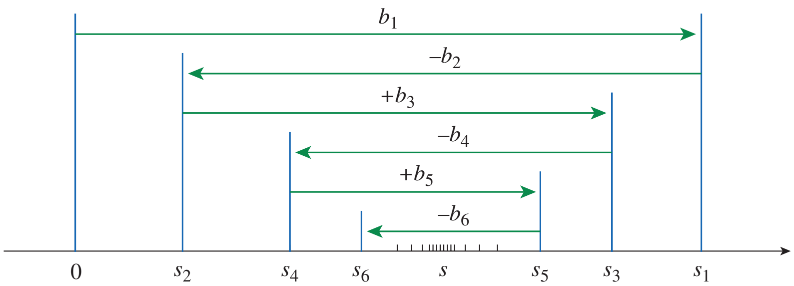


Figure 8.20
A geometric illustration of a convergent alternating series.

Compute and plot the partial sums. Plot $s_1 = b_1$ on a number line. To find s_2, subtract b_2, so s_2 is to the left of s_1. To find s_3, add b_3, so s_3 is to the right of s_2. But since $b_3 < b_2$, s_3 is to the left of s_1. Continuing in this manner, we confirm that the partial sums oscillate back and forth.

Since $b_{n+1} \leq b_n$ and $b_n \to 0$, the successive steps become smaller and smaller. The even partial sums $s_2, s_4, s_6, \ldots$ are increasing, and the odd partial sums $s_1, s_3, s_5, \ldots$ are decreasing. So it seems reasonable that both are converging to some number s, which is the sum of the series. Therefore, the proof involves the even and odd partial sums separately.

Example 1 The Alternating Harmonic Series

Consider the alternating harmonic series.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} \qquad \text{where } b_n = \frac{1}{n}$$

Examine the terms.

(i) $b_{n+1} < b_n$ because $\dfrac{1}{n+1} < \dfrac{1}{n}$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Therefore, the series is convergent by the Alternating Series Test.

Figure 8.21 illustrates this result by showing graphs of the terms $a_n = \frac{(-1)^{n-1}}{n}$ and the partial sums s_n.

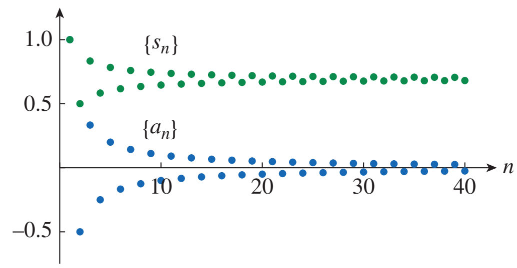


Figure 8.21
The graphs of the terms and the partial sums of the alternating harmonic series.

Notice how the values of s_n seesaw across the limiting value, which appears to be about 0.7. It can be shown that the exact sum of the series is $\ln 2 \approx 0.693$. ■

Example 2 Case in Which the AST Fails

Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$.

Common Error

An alternating series fails to satisfy one of the conditions of the AST and therefore diverges.

Correct Method

The AST can never be used to argue that a given series diverges, only that it converges.

Check the conditions of the AST.

(i) Let $f(x) = \frac{3x}{4x-1} \quad \Rightarrow \quad f'(x) = -\frac{3}{(1-4x)^2} < 0 \quad \text{for all } x \neq \frac{1}{4}.$

Therefore, $b_{n+1} < b_n$.

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4 - \frac{1}{n}} = \frac{3}{4}$

Condition (ii) is not satisfied. This limit *suggests* that the series is divergent.

Consider the limit of the nth term of the series.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$

This limit does not exist because the terms oscillate back and forth between positive and negative values (close to -0.75 and 0.75) and never approach any one single value.

Therefore, the series diverges by the Test for Divergence. ■

Example 3 Use the AST

Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ converges or diverges.

Solution

The given series is alternating, so check conditions (i) and (ii) of the AST.

(i) It isn't obvious that the sequence given by $b_n = \dfrac{n^2}{n^3 + 1}$ is decreasing.

Consider the related function $f(x) = \dfrac{x^2}{x^3 + 1} \Rightarrow f'(x) = \dfrac{x(2 - x^3)}{(x^3 + 1)^2}$

We are concerned only with positive values for x.

$f'(x) < 0$ if $2 - x^3 < 0 \Rightarrow x > \sqrt[3]{2}$.

Therefore, f is decreasing on the interval $[\sqrt[3]{2}, \infty)$.

This shows that $f(n+1) < f(n)$ and $b_{n+1} < b_n$ when $n \geq 2$.

(ii) $\displaystyle \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3 + 1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^3}} = 0$

Both conditions are satisfied; the series converges by the Alternating Series Test. ■

Estimating Sums

A partial sum s_n of any convergent series can be used as an approximation to the total sum s. But this approximation isn't very helpful without an indication of accuracy. We know that the error involved in using $s_n \approx s$ is the remainder $R_n = s - s_n$.

The next theorem provides a bound on the error for a series that satisfies the Alternating Series Test. The conclusion is easy to prove and visualize geometrically (see Figure 8.20). The bound on the error is b_{n+1}, the absolute value of the next term.

In Figure 8.20, notice that $s - s_4 < b_5$, $|s - s_5| < b_6$, and so on. The sum of the series, s, lies between any two consecutive partial sums.

Alternating Series Error Bound Theorem

If $s = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

(i) $b_{n+1} \leq b_n$, for all n, and

(ii) $\displaystyle\lim_{n\to\infty} b_n = 0$,

then the Alternating Series Error Bound is $|R_n| = |s - s_n| \leq b_{n+1}$.

Example 4 Find an Error Bound for an Alternating Series

Remember, by definition, $0! = 1$.

Estimate the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

Solution

Check to make sure the series is convergent.

(i) $b_{n+1} = \dfrac{1}{(n+1)!} = \dfrac{1}{n!(n+1)} < \dfrac{1}{n!} = b_n$

(ii) $0 < \dfrac{1}{n!} < \dfrac{1}{n} \to 0 \quad$ so $\quad b_n = \dfrac{1}{n!} \to 0$ as $n \to \infty$

The series converges by the Alternating Series Test.

To gain some insight into how many terms we need to use in the approximation, let's write out the first few terms of the series.

Common Error

The error bound in estimating any series using the nth partial sum s_n is b_{n+1}.

Correct Method

The Alternating Series Error Bound applies only to series that satisfy the conditions in the theorem. This error bound does not apply to other types of series.

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots$$
$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \cdots$$

Notice that

$$b_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002$$

and

$$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = 0.368056.$$

By the Alternating Series Error Bound Theorem, we know that $|s - s_6| \le b_7 < 0.0002$.

This error of less than 0.0002 does not affect the third decimal place, so we have $s \approx 0.368$ correct to three decimal places.

Note: Soon we will be able to show that this sum is exactly $\frac{1}{e}$. ■

Absolute Convergence

Given any series Σa_n, consider the associated series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

We have convergence tests for series with positive terms and for alternating series. But what if the terms alternate in signs in no discernible pattern? The concept of absolute convergence may help in these cases.

Definition • Absolutely Convergent

A series Σa_n is called **absolutely convergent** if the series of absolute values $\Sigma |a_n|$ is convergent.

Notice that if Σa_n is a series of positive terms, then $|a_n| = a_n$ and absolute convergence is the same as convergence in this case.

Example 5 Determine Absolute Convergence

Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \frac{1}{5^2} + \cdots.$$

This series is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series ($p = 2$). ■

Example 6 Convergent but Not Absolutely Convergent

Consider the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

This series is convergent by the AST but not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots,$$

which is the harmonic series (p-series with $p = 1$) and is, therefore, divergent. ■

The properties of the series in Example 6 lead to this definition.

Definition • Conditionally Convergent

A series Σa_n is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 6 shows that the alternating harmonic series is conditionally convergent. Therefore, it is possible for a series to be convergent but not absolutely convergent. However, absolute convergence is a *stronger* result. The next theorem shows that absolute convergence implies convergence.

Theorem • Absolute Convergence Implies Convergence

If a series Σa_n is absolutely convergent, then it is convergent.

To understand why this theorem is true, consider the inequality

$$0 \le a_n + |a_n| \le 2|a_n|$$

This is true because $|a_n|$ is either a_n or $-a_n$. If Σa_n is absolutely convergent, then $\Sigma |a_n|$ is convergent (by definition). Therefore, $\Sigma 2|a_n|$ is convergent. And $\Sigma(a_n + |a_n|)$ is convergent by the Comparison Test. Then

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is the difference of two convergent series and is, therefore, convergent. ■

Example 7 Series Involving a Trigonometric Function

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^3} + \cdots$$

is convergent or divergent.

Solution

This series has both positive and negative terms, but it is not alternating. The first term is positive, the next three are negative, and the following three are positive: the signs change irregularly.

Consider the series of absolute values

$$\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}.$$

Since $|\cos n| \le 1$ for all n, $\dfrac{|\cos n|}{n^2} \le \dfrac{1}{n^2}$.

The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (it is a p-series with $p = 2$).

Therefore, the series $\displaystyle\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$ is convergent by the Comparison Test.

This shows that the given series, $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$, is absolutely convergent and, therefore, convergent.

Figure 8.22 shows the graphs of the terms a_n and the partial sums s_n for the series. Notice that the series is not alternating but does indeed have positive and negative terms. These graphs also suggest that the terms $a_n \to 0$ and the partial sums are converging.

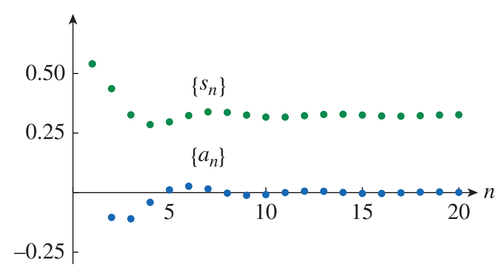


Figure 8.22
The series is not alternating but has positive and negative terms.

The Ratio Test

The following test is useful in determining whether a given series is absolutely convergent.

The Ratio Test

(i) If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent and therefore convergent.

(ii) If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty} a_n$.

The ratio test can be proved by comparing the given series to a geometric series. It seems reasonable that geometric series are involved because, for those series, the ratio r of consecutive terms is constant and the series converges if $|r| < 1$. In part (i) of the

Ratio Test, the ratio of consecutive terms isn't constant but $|a_{n+1}/a_n| \to L$ so, for large n, $|a_{n+1}/a_n|$ is almost constant and the series converges if $L < 1$.

A Closer Look

Part (iii) of the Ratio Test says that if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the test gives no information.

Here are two examples that demonstrate this.

1. Consider the *convergent* series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^2} = 1$$

2. Now consider the *divergent* series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

The Ratio Test is usually helpful and conclusive if the nth term of the series contains an exponential or factorial.

Therefore, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the series $\sum_{n=1}^{\infty} a_n$ might converge or it might diverge. In this case the Ratio Test fails, and we need to use some other test to draw a conclusion about convergence or divergence. ■

Example 8 Use the Ratio Test

Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

Solution

Use the Ratio Test with $a_n = (-1)^n \frac{n^3}{3^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1$$

So, by the Ratio Test, the given series is absolutely convergent, and therefore convergent. ■

Example 9 Series Involving Factorials

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Solution

The general term of the series a_n contains both an exponential and a factorial. Use the Ratio Test.

Since the terms $a_n = \dfrac{n^n}{n!}$ are positive, we do not need the absolute value symbols of the ratio.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Since $e > 1$, the given series is divergent by the Ratio Test. ■

Note: Although the Ratio Test works in Example 9, another method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \geq n$$

it follows that a_n does not approach 0 as $n \to \infty$. Therefore, the given series is divergent by the Test for Divergence.

8.4 Exercises

1. (a) What is an alternating series?
(b) Under what conditions does an alternating series converge?
(c) If these conditions are satisfied, what can you say about the remainder after n terms?

2. What can you conclude about the series Σa_n in each case?

(a) $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 8$

(b) $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0.8$

(c) $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

Determine whether the series is convergent or divergent.

3. $\dfrac{2}{3} - \dfrac{2}{5} + \dfrac{2}{7} - \dfrac{2}{9} + \dfrac{2}{11} - \cdots$

4. $-\dfrac{2}{5} + \dfrac{4}{6} - \dfrac{6}{7} + \dfrac{8}{8} - \dfrac{10}{9} + \cdots$

5. $\dfrac{1}{\ln 3} - \dfrac{1}{\ln 4} + \dfrac{1}{\ln 5} - \dfrac{1}{\ln 6} + \dfrac{1}{\ln 7} - \cdots$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$

10. $\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{-n}$

11. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$

12. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$

13. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$

14. Is the 50th partial sum s_{50} of the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ an overestimate or an underestimate of the total sum? Explain your reasoning.

15. Calculate the first 10 partial sums of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$$

and graph both the sequence of terms and the sequence of partial sums on the same coordinate axes. Estimate the error in using the 10th partial sum to approximate the total sum.

In each case, determine the values of p for which the given series converges.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+p}$

18. $\displaystyle\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$

Show that the series is convergent. How many terms of the series are necessary in order to estimate the sum to the indicated accuracy?

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n5^n}$ $\quad |\text{error}| < 0.0001$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ $\quad |\text{error}| < 0.00005$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 2^n}$ $\quad |\text{error}| < 0.0005$

22. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n e^{-n}$ $\quad |\text{error}| < 0.01$

Graph both the sequence of terms and the sequence of partial sums in the same viewing window. Use the graph to make a rough estimate of the sum of the series. Then use the Alternating Series Error Bound to estimate the sum correct to four decimal places.

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!}$

24. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{8^n}$

Approximate the sum of the series correct to four decimal places.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!}$

Determine whether the series is absolutely convergent.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{n!}$

33. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$

34. $\displaystyle\sum_{n=1}^{\infty} n e^{-n}$

35. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n}\right)^n$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^n}{n 10^{n+1}}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n n!}{(2n)!}$

39. $\displaystyle 1 - \frac{2!}{1\cdot 3} + \frac{3!}{1\cdot 3\cdot 5} - \frac{4!}{1\cdot 3\cdot 5\cdot 7} + \cdots + (-1)^{n-1} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)} + \cdot$

40. $\displaystyle \frac{2}{3} + \frac{2\cdot 5}{3\cdot 5} + \frac{2\cdot 5\cdot 8}{3\cdot 5\cdot 7} + \frac{2\cdot 5\cdot 8\cdot 11}{3\cdot 5\cdot 7\cdot 9} + \cdots$

41. The terms of a series are defined recursively by the equations

$$a_1 = 2 \qquad a_{n+1} = \frac{5n+1}{4n+3} a_n$$

Determine whether Σa_n converges or diverges.

42. The terms of a series are defined recursively by the equations

$$a_1 = 1 \qquad a_{n+1} = \frac{2+\cos n}{\sqrt{n}} a_n$$

Determine whether Σa_n converges or diverges.

43. For which of the following series is the Ratio Test inconclusive (that is, it fails to provide a definite answer)?

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ $\qquad$ (B) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

(C) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$ $\qquad$ (D) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$

Let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

The **Root Test** says the following:

(i) If $L < 1$, then Σa_n is absolute convergent.
(ii) If $L > 1$ or $L = \infty$, then Σa_n is divergent.
(iii) If $L = 1$, then the Root Test is inconclusive.

Similar to the Ratio Test, the Root Test is proved by comparison with a geometric series. Use the Root Test to determine whether the series is convergent or divergent.

44. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

46. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(\ln n)^n}$

47. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

48. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

49. $\displaystyle\sum_{n=0}^{\infty} (\arctan n)^n$

50. For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

51. (a) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x.

(b) Deduce that $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x.

52. Around 1910, the Indian mathematician Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}}$$

William Gosper used this series in 1985 to compute the first 17 million digits of π.

(a) Verify that the series is convergent.
(b) How many correct decimal places of π do you get if you use just the first term of the series? What if you use two terms?

8.5 Power Series

Up to this point we have considered only series of constants. In this section, we begin a discussion of series that represent functions.

Power Series Basics

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \tag{1}$$

where x is a variable and the c_n's are constants called the **coefficients** of the series. For each fixed x, the series in Equation 1 is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f looks like a polynomial. The difference here is that f has infinitely many terms.

Suppose we let $c_n = 1$ for all n; then the power series becomes a geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \tag{2}$$

is called a **power series in $(x - a)$** or a **power series centered at a** or a **power series about a**. Notice that in writing the term corresponding to $n = 0$ in Equations 1 and 2, we use the convention that $(x-a)^0 = 1$ even when $x = a$. In addition, when $x = a$, all of the terms are 0 for $n \geq 1$, so the power series in Equation 2 always converges when $x = a$.

Trigonometric Series
A power series is a series in which each term is a power function. A **trigonometric series**

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is a series whose terms are trigonometric functions.

Example 1 A Power Series That Converges Only at Its Center

For what values of x is the series $\displaystyle\sum_{n=0}^{\infty} n!x^n$ convergent?

Solution

Let $a_n = n!x^n$ denote the nth term of the series, and use the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| && \text{Ratio Test limit.}\\
&= \lim_{n\to\infty}\left|\frac{(n+1)n!x^{n+1}}{n!x^n}\right| && \text{Definition of factorial.}\\
&= \lim_{n\to\infty}(n+1)|x| = \infty && \text{Simplify; evaluate limit } (x \neq 0).
\end{aligned}$$

By the Ratio Test, the series diverges for $x \neq 0$.

The given series converges only when $x = 0$. ■

Example 2 Use the Ratio Test to Determine Where a Power Series Converges

For what values of x is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

Solution

Let $a_n = \dfrac{(x-3)^n}{n}$.

Use the Ratio Test; first consider the relevant quotient.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| \qquad \text{Use definition for } a_n\text{; division of fractions.}$$

$$= \left|\frac{n}{n+1}(x-3)\right| \qquad \text{Simplify.}$$

$$= \left|\frac{n}{n+1}\right| |x-3| = \frac{n}{n+1}|x-3| \qquad \text{Property of absolute value; quotient is positive.}$$

Now consider the limit as $n \to \infty$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+1}|x-3|$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}|x-3| \qquad \text{Divide numerator and denominator by } n.$$

$$= (1)|x-3| = |x-3| \qquad \text{Evaluate the limit; remember, } x \text{ is constant as } n \to \infty.$$

By the Ratio Test, when $|x-3| < 1$, the given series is absolutely convergent and therefore convergent. When $|x-3| > 1$, the series is divergent.

In order to explicitly determine the values for which the series converges, solve the following inequality for x.

$$|x-3| < 1 \iff -1 < x-3 < 1 \iff 2 < x < 4$$

Therefore the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.

The Ratio Test is inconclusive when $|x-3| = 1$.
So we need to consider the cases $x = 2$ and $x = 4$ separately.

$$x = 4: \sum_{n=1}^{\infty}\frac{(x-3)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n} \qquad \text{This is the harmonic series, which diverges.}$$

$$x = 2: \sum_{n=1}^{\infty}\frac{(x-3)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \qquad \text{This is the alternating harmonic series, which converges.}$$

Therefore, the given power series converges for $2 \le x < 4$, and the *interval of convergence* is $[2, 4)$. ■

Power series provide a way to represent some of the most important functions that arise in mathematics, physics, and chemistry. For example, the **Bessel function**, named after the German astronomer Friedrich Bessel (1784–1846), is defined as a power series. This kind of function first arose when Bessel solved Kepler's equation for describing planetary motion. Since that time, these functions have been applied in many different physical situations, including the temperature distribution in a circular plate and the shape of a vibrating drumhead. See Figure 8.23.

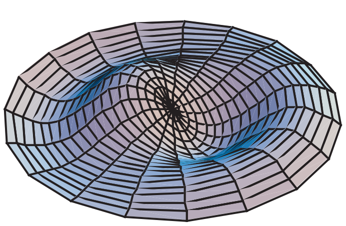


Figure 8.23
This computer-generated model of a vibrating drumhead involves Bessel functions and cosine functions.

Example 3 A Power Series That Converges for All Values of x

Find the domain of the function defined by $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution

Let $a_n = \frac{x^n}{n!}$.

Use the Ratio Test; consider the relevant quotient.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

Definition of nth term; division of fractions.

$$= \left|\frac{1}{n+1}x\right| = \frac{1}{n+1}|x|$$

Simplify.

Consider the limit associated with the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1}|x| = 0 \cdot |x| = 0 < 1 \quad \text{for all } x$$

Therefore, by the Ratio Test, the given series converges for all values of x. The domain for the function f is all real numbers, $\mathbb{R} = (-\infty, \infty)$. ■

Recall that the sum of a series is equal to the limit of the sequence of partial sums. So, in Example 3, for the function defined as a sum of a series, for every real number x,

$$f(x) = \lim_{n\to\infty} s_n(x) \quad \text{where} \quad s_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!}$$

The first few partial sums are

$s_0(x) = 1,$

$s_1(x) = 1 + x,$

$s_2(x) = 1 + x + \frac{x^2}{2},$

$s_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6},$ and

$s_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$

Figure 8.24 shows the graphs of these partial sums, which are polynomials. Each polynomial is an approximation to the function f.

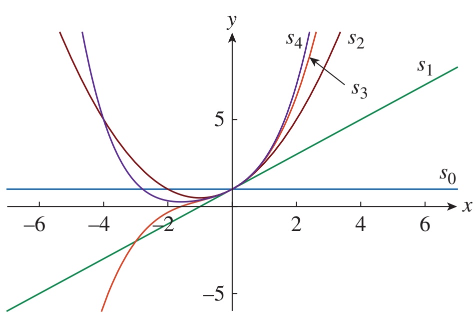


Figure 8.24
Graphs of the partial sums.

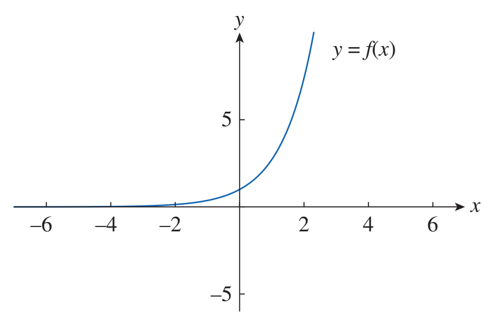


Figure 8.25
Graph of the function f.

The approximations seem to become *better* as more terms are included; that is, the graphs appear to be more *moderate* and *stable* as n increases. We will learn later that these graphs approach the graph of the natural exponential function e^x (see Figure 8.25) and that the series in Example 3 represents e^x.

Power Series Convergence

For the power series that we have considered so far, the set of values of x for which the series is convergent turned out to be an interval: for the geometric series in Example 2, a finite interval; in Example 3, the infinite interval $(-\infty, \infty)$; and in Example 1, a collapsed interval $[0, 0] = \{0\}$. The next theorem, which we will not prove, says that this is true in general.

Theorem • Power Series Convergence

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities (associated with convergence).

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii).

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

In case (i), the interval consists of a single value a.

In case (ii), the interval is $(-\infty, \infty)$.

This means that we *always* have to check each endpoint of the interval separately for convergence.

In case (iii), the inequality corresponding to $|x-a| < R$ is $a - R < x < a + R$. When x is an *endpoint* of this interval, that is, $x = a \pm R$, anything can happen; the series might converge at one or both endpoints, or it might diverge at both endpoints.

Therefore, in case (iii), there are four possibilities for the interval of convergence.

$$(a-R,\ a+R) \quad (a-R,\ a+R] \quad [a-R,\ a+R) \quad [a-R,\ a+R]$$

This situation, case (iii), is illustrated in Figure 8.26.

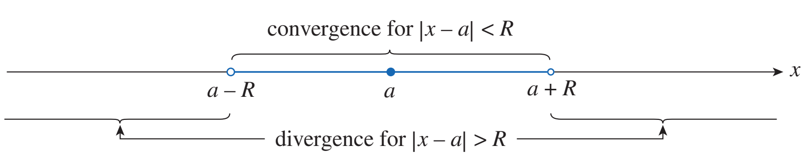


Figure 8.26
Graphical illustration of the interval of convergence (case (iii)).

Table 8.5 provides a summary of the radius and interval of convergence for each of the examples considered in this section.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$R = \infty$	$(-\infty, \infty)$

Table 8.5
Examples of radius and interval of convergence.

In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R. Remember, the Ratio and Root Tests are always inconclusive when x is an endpoint of the interval of convergence. The endpoints must be checked with some other test.

Example 4 Radius and Interval of Convergence

Find the radius of convergence and the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Note that this power series is centered at $a = 0$.

Solution

Let $a_n = \dfrac{(-3)^n x^n}{\sqrt{n+1}}$ and consider the quotient for the Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right|$$ Definition of nth term; division of fractions.

$$= \left|-3x\sqrt{\frac{n+1}{n+2}}\right| = 3\sqrt{\frac{n+1}{n+2}}|x|$$ Simplify; property of absolute value.

Evaluate the limit associated with the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 3\sqrt{\frac{n+1}{n+2}}|x|$$ Use simplified expression.

$$= \lim_{n\to\infty} 3\sqrt{\frac{1+(1/n)}{1+(2/n)}}|x| = 3|x|$$ Divide numerator and denominator by n^2; evaluate the limit.

By the Ratio Test, the given series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Therefore, it converges if $|x| < \dfrac{1}{3}$ and diverges if $|x| > \dfrac{1}{3}$.

It might be easier to determine the radius of convergence if we use $a = 0$ in the inequalities. We can write: the series converges if $|x - 0| < \dfrac{1}{3}$ and diverges if $|x - 0| > \dfrac{1}{3}$.

This means that the radius of convergence is $R = \dfrac{1}{3}$.

The series converges for x in the interval

$$\left(a - \frac{1}{3}, a + \frac{1}{3}\right) = \left(0 - \frac{1}{3}, 0 + \frac{1}{3}\right) = \left(-\frac{1}{3}, \frac{1}{3}\right).$$

Now, test the endpoints of this interval.

$$x = -\frac{1}{3}: \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

When $x = -\frac{1}{3}$ the series diverges; it is a p series with $p = \frac{1}{2} < 1$.

$$x = \frac{1}{3}: \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

When $x = \frac{1}{3}$, the series converges by the Alternating Series Test.

The given power series converges when $-\frac{1}{3} < x \le \frac{1}{3}$.

The interval of convergence of the series Σa_n is $\left(-\frac{1}{3}, \frac{1}{3}\right]$. ■

Example 5 Radius and Interval of Convergence

Find the radius of convergence and the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

Note that this is a power series centered at $a = -2$.

Solution

Let $a_n = \dfrac{n(x+2)^n}{3^{n+1}}$ and consider the quotient for the Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right|$$

Definition of nth term; division of fractions.

$$= \left|\frac{n+1}{n} \cdot \frac{x+2}{3}\right| = \frac{n+1}{n} \cdot \frac{|x+2|}{3}$$

Simplify; property of absolute value.

Evaluate the limit associated with the Ratio Test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{|x+2|}{3}$$

Use simplified expression.

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \frac{|x+2|}{3} = \frac{|x+2|}{3}$$

Divide numerator and denominator by n; evaluate the limit.

Since $a = -2$, the series converges if $|x - (-2)| < 3$. The series diverges if $|x - (-2)| > 3$.

The series converges if $\dfrac{|x+2|}{3} < 1 \iff |x+2| < 3$.

The series diverges if $\dfrac{|x+2|}{3} > 1 \iff |x+2| > 3$.

The radius of convergence is $R = 3$.

Rewrite the inequality: $|x+2| < 3 \iff -5 < x < 1$.

Test the endpoints of this interval.

$$x = -5: \quad \sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n n$$

When $x = -5$, the series diverges by the nth Term Test: $\lim_{n\to\infty} (-1)^n n \neq 0$.

$$x = 1: \quad \sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n$$

When $x = 1$, the series also diverges by the nth Term Test.

The interval of convergence of the series Σa_n is $(-5, 1)$. ■

8.5 Exercises

1. What is the general form of a power series?

2. (a) What is the radius of convergence of a power series? Explain how you would find it.
(b) What is the interval of convergence of a power series? Explain how you would find it.

Find the radius of convergence and interval of convergence of the series.

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^n n x^n$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\sqrt[3]{n}}$

5. $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{2n-1}$

6. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n^2}$

7. $\displaystyle\sum_{n=1}^{\infty} n^n x^n$

8. $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^4 4^n}$

9. $\displaystyle\sum_{n=1}^{\infty} 2^n n^2 x^n$

10. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 4^n}{\sqrt{n}} x^n$

11. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n 5^n} x^n$

12. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2^n(n^2+1)} x^n$

13. $\displaystyle\sum_{n=1}^{\infty}\frac{x^{2n}}{n!}$

14. $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{n^2+1}$

15. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)2^n}(x-1)^n$

16. $\displaystyle\sum_{n=2}^{\infty}\frac{(x+2)^n}{2^n \ln n}$

17. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{8^n}(x+6)^n$

18. $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^n}$

19. $\displaystyle\sum_{n=1}^{\infty}\frac{(2x-1)^n}{5^n\sqrt{n}}$

20. $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$

21. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{b^n}(x-a)^n, \quad b>0$

22. $\displaystyle\sum_{n=2}^{\infty}\frac{b^n}{\ln n}(x-a)^n, \quad b>0$

23. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2 x^n}{2\cdot 4\cdot 6\cdots(2n)}$

24. $\displaystyle\sum_{n=1}^{\infty}\frac{(5x-4)^n}{n^3}$

25. $\displaystyle\sum_{n=2}^{\infty}\frac{x^{2n}}{n(\ln n)^2}$

26. $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

27. $\displaystyle\sum_{n=1}^{\infty}\frac{n!x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

28. If $\displaystyle\sum_{n=0}^{\infty} c_n 4^n$ is convergent, can you conclude that the series is convergent?

(a) $\displaystyle\sum_{n=0}^{\infty} c_n(-2)^n$ (b) $\displaystyle\sum_{n=0}^{\infty} c_n(-4)^n$

29. Suppose that $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ converges when $x=-4$ and diverges when $x=6$. What can you conclude about the convergence or divergence of the series?

(a) $\displaystyle\sum_{n=0}^{\infty} c_n$ (b) $\displaystyle\sum_{n=0}^{\infty} c_n 8^n$

(c) $\displaystyle\sum_{n=0}^{\infty} c_n(-3)^n$ (d) $\displaystyle\sum_{n=0}^{\infty}(-1)^n c_n 9^n$

30. If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty}\frac{(n!)^k}{(kn)!}x^n$$

31. Graph the first several partial sums $s_n(x)$ of the series $\displaystyle\sum_{n=0}^{\infty} x^n$ together with the function $f(x) = \dfrac{1}{1-x}$ in the same viewing rectangle. On what interval do these partial sums appear to be converging to $f(x)$?

32. The function J_1 defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

is called the *Bessel function of order 1*.

(a) Find the domain of J_1.

(b) Graph the first several partial sums in the same viewing rectangle.

(c) If the technology you are using has built-in Bessel functions, graph J_1 in the same viewing rectangle as the partial sums in part (b) and observe how the partial sums approximate J_1.

33. The function A defined by

$$A(x) = 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \cdots$$

is called an *Airy function* after the English mathematician and astronomer Sir George Airy (1801–1892).

(a) Find the domain of the Airy function.

(b) Graph the first several partial sums in the same viewing rectangle.

(c) If the technology you are using has a built-in Airy function, graph A in the same viewing rectangle as the partial sums in part (b) and observe how the partial sums approximate A.

34. A function f is defined by

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$$

that is, its coefficients are $c_{2n} = 1$ and $c_{2n+1} = 2$ for all $n \geq 0$. Find the interval of convergence of the series and find an explicit formula for $f(x)$.

35. If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_{n+4} = c_n$ for all $n \geq 0$, find the interval of convergence of the series and a formula for $f(x)$.

36. Suppose the series $\Sigma c_n x^n$ has radius of convergence 2 and the series $\Sigma d_n x^n$ has radius of convergence 3. What is the radius of convergence of the series $\Sigma (c_n + d_n) x^n$?

37. Suppose the radius of convergence of the power series $\Sigma c_n x^n$ is R. What is the radius of convergence of the power series $\Sigma c_n x^{2n}$?

38. Is it possible to find a power series whose interval of convergence is $[0, \infty)$? Explain your reasoning.

39. Let p and q be real numbers with $p < q$. Find series whose interval of convergence is

(a) (p, q) (b) $(p, q]$ (c) $[p, q)$ (d) $[p, q]$

40. Suppose that the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ satisfies $c_n \neq 0$ for all n. Show that if $\lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right|$ exists, then it is equal to the radius of convergence of the power series.

8.6 Representations of Functions as Power Series

In this section, we will learn how to represent certain types of functions as sums of power series. We'll do this by working with geometric series, or by differentiating or integrating an appropriate series. Although it may seem odd to represent a known function as a sum of infinitely many terms, it is actually a useful strategy for integrating functions that do not have elementary antiderivatives, for solving differential equations, and for approximating functions by polynomials.

For example, consider the function $f(x) = e^{x^2}$. We can't find a closed form antiderivative for f. However, if we could write $f(x)$ as an infinite polynomial, then it seems reasonable that we can antidifferentiate term by term. This would give us an infinite polynomial that represents the antiderivative. The partial sums could be used to approximate the antiderivative.

Using Geometric Series

Recall the following equation.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1 \tag{1}$$

This equation was derived from the geometric series with $a = 1$ and $r = x$. But our focus here is different. Consider Equation 1 as expressing the function $f(x) = \dfrac{1}{1-x}$ as a sum of a power series.

Figure 8.27 is a geometric illustration of Equation 1.

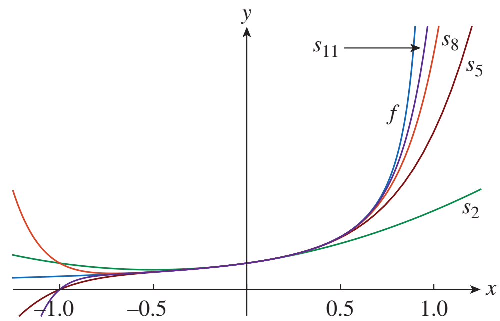


Figure 8.27
As n increases, the partial sum $s_n(x)$ becomes a better approximation to $f(x)$.

Because the sum of a series is the limit of the sequence of partial sums,

$$\frac{1}{1-x} = \lim_{n\to\infty} s_n(x) \quad \text{where} \quad s_n(x) = 1 + x + x^2 + \cdots + x^n$$

Figure 8.27 suggests that as n increases, the nth partial sum, $s_n(x)$, becomes a better approximation to $f(x)$ for $-1 < x < 1$.

Example 1 Find a New Power Series from an Old One

Express $\dfrac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Solution

Replace x by $-x^2$ in Equation 1.

$$\frac{1}{1+x^2} = \frac{1}{1-\left(-x^2\right)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n \qquad \text{Use Equation 1.}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \qquad \text{Rewrite the } n\text{th term; expand.}$$

Because this is a geometric series, it converges when

$$\left|-x^2\right| < 1 \iff x^2 < 1 \iff |x| < 1.$$

Therefore, the interval of convergence is $(-1, 1)$.

Note that we could also determine the radius of convergence by using the Ratio Test. ■

Example 2 Factor and Rewrite to Find a New Power Series Representation

Find the power series representation for $\dfrac{1}{x+2}$.

Solution

Work algebraically to put the given function in the form of Equation 1.

$$\frac{1}{2+x} = \frac{1}{2\left(1+\dfrac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\dfrac{x}{2}\right)\right]}$$

Factor out 2; rewrite as in Equation 1.

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

Rewrite as a power series.

The series converges when $\left|-\dfrac{x}{2}\right| < 1 \iff |x| < 2$.

The interval of convergence is $(-2, 2)$. ■

Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence of the series. To differentiate or integrate the function f, the following theorem says (without a proof here) that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial. This is called **term-by-term differentiation and integration**.

Theorem • Differentiation and Integration of a Power Series

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\displaystyle\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

A Closer Look

1. In part (ii), $\int c_0\,dx = c_0x + C_1$ is written as $c_0(x-a) + C$, where $C = c_1 + ac_0$, so all the terms of the series have the same form.
2. Equations (i) and (ii) in this theorem can be written in the form

(iii) $\displaystyle\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\frac{d}{dx}\left[c_n(x-a)^n\right]$

(iv) $\displaystyle\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\int c_n(x-a)^n\,dx$

For finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. Equations (iii) and (iv) extend these concepts to infinite sums, provided that we are dealing with *power series.*

For other types of series of functions, the result is not as simple; see, for example, Exercise 42.

3. This theorem says that the *radius of convergence* remains the same when a power series is differentiated or integrated. However, this does not mean that the *interval of convergence* remains the same.

A differentiated power series may lose one or both endpoints of the interval of convergence if the original power series was convergent there. An integrated power series may gain one or both endpoints of the interval of convergence if the original power series was not convergent there. See, for example, Exercise 43. ■

Example 3 Differentiate a Power Series

In Section 8.5, we mentioned the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

which is a power series that is defined for all x.

Therefore, using the previous theorem, J_0 is differentiable for all x and its derivative is found by term-by-term differentiation as follows:

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$$ ■

Example 4 A New Power Series by Differentiating an Old One

Express $\dfrac{1}{(1-x)^2}$ as a power series by differentiating Equation 1.

Solution

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$ Equation 1.

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$$ Differentiate each side of the equation.

Note that often it is more convenient for the series to start at $n = 0$. In this case, if we replace n by $n + 1$, we get

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

According to the previous theorem, the radius of convergence of the differentiated series is the same as the radius of convergence of the original series, that is, $R = 1$. ■

Example 5 A New Power Series by Integrating an Old One

Find a power series representation for $\ln(1 + x)$ and its radius of convergence.

Solution

Search for a calculus relationship between $\ln(1 + x)$ and a function with known power series.

Notice that $\dfrac{d}{dx}[\ln(1 + x)] = \dfrac{1}{1 + x}$.

Using Equation 1, we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots \quad |x| < 1.$$

Integrate both sides of this equation.

$$\ln(1 + x) = \int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx$$ Use power series representation.

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C$$ Integrate term by term.

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} + C \quad |x| < 1$$ Write using summation notation.

To determine the value of C, let $x = 0$ in this equation.

$\ln(1 + 0) = \ln 1 = 0 = C$

Therefore,

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} \quad |x| < 1.$$

The radius of convergence is the same as for the original series: $R = 1$.

Note that this series does converge at the right endpoint: by the AST when $x = 1$, it becomes the alternating harmonic series. This means that the alternating series converges to $\ln(1 + 1) = \ln 2$, which is an amazing result. ■

Example 6 Power Series Representation for an Inverse Trigonometric Function

Find a power series representation for $f(x) = \tan^{-1} x$.

Solution

Notice that $f'(x) = \dfrac{1}{1 + x^2}$.

You can see that finding some of these power series requires good problem-solving skills.

Therefore, we can find the series we need by integrating the power series for $\dfrac{1}{1+x^2}$ from Example 1.

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + \cdots)\,dx$$ Use series from Example 1.

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$ Integrate term by term.

Let $x = 0 \Rightarrow C = \tan^{-1} 0 = 0$.

Therefore,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$ Use $C = 0$.

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$ Write using summation notation.

Note that if $x = 1$, the series converges to $\tan^{-1} 1 = \dfrac{\pi}{4}$. This result is known as the Leibniz formula for π.

The radius of convergence of the power series for $\dfrac{1}{1+x^2}$ is $R = 1$.

So, the radius of convergence of this series for $\tan^{-1}x$ is also 1. ■

Example 7 Power Series and Estimation

(a) Write $\displaystyle\int \frac{1}{1+x^7}\,dx$ as a power series.

(b) Use part (a) to approximate $\displaystyle\int_0^{0.5} \frac{1}{1+x^7}\,dx$ correct to within 10^{-7}.

Solution

(a) First, express $\dfrac{1}{1+x^7}$ as a power series.

Start with Equation 1 and replace x by $-x^7$.

$$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty}(-x^7)^n$$
$$= \sum_{n=0}^{\infty}(-1)^n x^{7n} = 1 - x^7 + x^{14} - x^{21} + \cdots$$

Integrate term by term.

$$\int \frac{1}{1+x^7}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{7n}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{7n+1}}{7n+1}$$
$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots$$

This series converges for $|-x^7| < 1 \quad \Leftrightarrow \quad |x| < 1$.

(b) In applying the Fundamental Theorem of Calculus, it doesn't matter which antiderivative we use, so let's use the antiderivative from part (a) with $C = 0$.

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx = \left[x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots\right]_0^{0.5}$$
$$= \frac{1}{2} - \frac{1}{8\cdot 2^8} + \frac{1}{15\cdot 2^{15}} - \frac{1}{22\cdot 2^{22}} + \cdots + \frac{(-1)^n}{(7n+1)2^{7n+1}} + \cdots$$

This infinite series is the exact value of the definite integral, but since it is an alternating series, we can approximate the sum using the Alternating Series Error Bound.

If we stop adding after the term with $n = 3$, the error is smaller than the term with $n = 4$.

$$\text{Error} < \frac{1}{29\cdot 2^{29}} \approx 6.4 \times 10^{-11}$$

and the definite integral is approximately

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx \approx \frac{1}{2} - \frac{1}{8\cdot 2^8} + \frac{1}{15\cdot 2^{15}} - \frac{1}{22\cdot 2^{22}} \approx 0.49951374.$$ ■

8.6 Exercises

1. If the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is 10, what is the radius of convergence of the series $\sum_{n=1}^{\infty} n\, c_n x^{n-1}$? Why?

2. Suppose we know that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$. What can you say about convergence of the following series? Why?

$$\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$$

3. Explain why the functions $f(x) = \dfrac{1}{1-x}$ and $S(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$ are not identical.

Find a power series representation for the function and determine the interval of convergence.

4. $f(x) = \dfrac{1}{1+x}$

5. $f(x) = \dfrac{5}{1-4x^2}$

6. $f(x) = \dfrac{2}{3-x}$

7. $f(x) = \dfrac{4}{2x+3}$

8. $f(x) = \dfrac{x^2}{x^4+16}$

9. $f(x) = \dfrac{x}{2x^2+1}$

10. $f(x) = \dfrac{x-1}{x+2}$

11. $f(x) = \dfrac{x+a}{x^2+a^2}, \quad a > 0$

12. (a) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2}$$

(b) Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}$$

(c) Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}$$

13. (a) Use Equation 1 to find a power series representation for $f(x) = \ln(1-x)$. What is the radius of convergence?
(b) Use part (a) to find a power series for $f(x) = x\ln(1-x)$.
(c) Let $x = \frac{1}{2}$ in your result from part (a) and express $\ln 2$ as the sum of an infinite series.

Find a power series representation for the function and determine the radius of convergence.

14. $f(x) = \ln(5-x)$

15. $f(x) = x^2 \tan^{-1}(x^3)$

16. $f(x) = \dfrac{x}{(1+4x)^2}$

17. $f(x) = \left(\dfrac{x}{2-x}\right)^3$

18. $f(x) = \dfrac{1+x}{(1-x)^2}$

19. $f(x) = \dfrac{x^2+x}{(1-x)^3}$

Find a power series representation for f, and graph f and several partial sums $s_n(x)$ in the same viewing rectangle. Describe what happens to the graphs of the partial sums as n increases.

20. $f(x) = \dfrac{x^2}{x^2+1}$

21. $f(x) = \ln(1+x^4)$

22. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

23. $f(x) = \tan^{-1}(2x)$

Evaluate the indefinite integral as a power series. Find the radius of convergence.

24. $\displaystyle\int \frac{x}{1-x^8}\,dx$

25. $\displaystyle\int \frac{x}{1+x^3}\,dx$

26. $\displaystyle\int x^2 \ln(1+x)\,dx$

27. $\displaystyle\int \frac{\tan^{-1} x}{x}\,dx$

28. $\displaystyle\int \frac{\ln(1-x)}{x}\,dx$

29. $\displaystyle\int \frac{x - \tan^{-1} x}{x^3}\,dx$

Use a power series to approximate the definite integral to six decimal places.

30. $\displaystyle\int_0^{0.3} \frac{x}{1+x^3}\,dx$

31. $\displaystyle\int_0^{1/2} \arctan\left(\frac{x}{2}\right)dx$

32. $\displaystyle\int_0^{0.2} x\ln(1+x^2)\,dx$

33. $\displaystyle\int_0^{0.3} \frac{x^2}{1+x^4}\,dx$

34. Consider the power series

$$S(x) = 6x + 6x^2 + 6x^3 + 6x^4 + \cdots + 6x^n + \cdots$$

(a) Does $S(5)$ exist? Explain your reasoning.
(b) Does $S(0.5)$ exist? Explain your reasoning.
(c) Find the domain of the function S.
(d) Find a rational function f that is identical to S over the domain found in part (c).
(e) Find $S(0.1)$ and $f(0.1)$, where f is the function found in part (d). What does the value $|f(0.1) - S(0.1)|$ represent?

35. Consider the rational function $f(x) = \dfrac{4x^2}{1 - 2x^2}$.

(a) Find a power series $s(x)$ that converges to f over some interval of convergence.
(b) Find the interval of convergence of $s(x)$.

36. Suppose $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{(x-3)^n}{n \cdot 3^n}$ for some interval of convergence.

(a) Find the interval of convergence of this power series.
(b) Find the power series for $f'(x)$.
(c) Find a rational function for $f'(x)$ over its interval of convergence.
(d) Find the value of k if $f(x) = \ln x + k$.

---

37. Use the result of Example 6 to compute arctan 0.2 correct to five decimal places.

38. Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is a solution of the differential equation

$$f''(x) + f(x) = 0$$

39. The Bessel function of order 0 is defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

(a) Show that J_0 satisfies the differential equation

$$x^2 J_0''(x) + x\, J_0'(x) + x^2 J_0(x) = 0$$

(b) Evaluate $\int_0^1 J_0(x)\, dx$ correct to three decimal places.

40. The Bessel function of order 1 is defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

(a) Show that J_1 satisfies the differential equation

$$x^2 J_1''(x) + x J_1'(x) + (x^2 - 1) J_1(x) = 0$$

(b) Show that $J_0'(x) = -J_1(x)$.

41. (a) Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a solution of the differential equation $f'(x) = f(x)$.

(b) Show that $f(x) = e^x$.

42. Let $f_n(x) = \dfrac{\sin nx}{n^2}$.

(a) Show that the series $\sum_{n=0}^{\infty} f_n(x)$ converges for all values of x but the series of derivatives $\sum_{n=0}^{\infty} f_n'(x)$ diverges when $x = 2n\pi$, where n is an integer.

(b) For what values of x does $\sum_{n=0}^{\infty} f_n''(x)$ converge?

43. Let $f(x) = \sum_{n=1}^{\infty} \dfrac{x^n}{n^2}$.

Find the intervals of convergence for f, f', and f''.

44. Consider the geometric series $\sum_{n=0}^{\infty} x^n$.

(a) Find the sum of the series $\sum_{n=1}^{\infty} nx^{n-1}$, $|x| < 1$.

(b) Find the sum of each of the following series.

(i) $\sum_{n=1}^{\infty} nx^n$, $|x| < 1$ (ii) $\sum_{n=1}^{\infty} \dfrac{n}{2^n}$

(c) Find the sum of each of the following series.

(i) $\sum_{n=2}^{\infty} n(n-1)x^n$, $|x| < 1$

(ii) $\sum_{n=2}^{\infty} \dfrac{n^2 - n}{2^n}$

(iii) $\sum_{n=1}^{\infty} \dfrac{n^2}{2^n}$

45. Use the power series for $\tan^{-1} x$ to show the following expression for π as the sum of an infinite series.

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

46. (a) By completing the square, show that

$$\int_0^{1/2} \frac{dx}{x^2 - x + 1} = \frac{\pi}{3\sqrt{3}}$$

(b) By factoring $x^3 + 1$ as a sum of cubes, rewrite the integral in part (a). Then express $\dfrac{1}{x^3 + 1}$ as the sum of a power series and use it to show the following formula for π.

$$\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2} \right)$$

8.7 Taylor and Maclaurin Series

In Section 8.6, we were able to find power series representations for certain types of functions using clever algebraic manipulations, differentiation, and integration. In this section, we will consider the more general problems: Which functions have power series representations? How can we find these representations?

Introduction

To determine whether a function has a power series representation, start by working backward. Suppose f is any function that can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \qquad |x-a| < R \tag{1}$$

Let's try to determine what the coefficients c_n must be in terms of f. First notice that if we let $x = a$ in Equation 1, then all terms after the first one are 0, and we get

$$f(a) = c_0$$

Using Theorem 8.6.2, we can differentiate the series in Equation 1 term by term.

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \qquad |x-a| < R \tag{2}$$

Let $x = a$ in Equation 2.

$$f'(a) = c_1$$

Differentiate both sides of Equation 2.

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \cdots \qquad |x-a| < R \tag{3}$$

Again, let $x = a$ in Equation 3.

$$f''(a) = 2c_2$$

Let's try this procedure one more time. Differentiate both sides of Equation 3.

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \cdots \qquad |x-a| < R \tag{4}$$

Let $x = a$ in Equation 4.

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

By now you can recognize a pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot nc_n = n!c_n$$

Solve this equation for the nth coefficient:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula is valid even for $n = 0$ if we use the conventions that $0! = 1$ and $f^{(0)} = f$. This derivation proves the following theorem.

Theorem • Coefficients of a Power Series

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substitute this formula for c_n back into the series. Then we can say that if f has a power series expansion at a, it must be of the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned} \tag{5}$$

The series in Equation 6 is called the **Taylor series of the function f at a** (or **about a** or **centered at a**). For the special case $a = 0$, the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \tag{6}$$

This special case is called a **Maclaurin series**.

A Closer Look

The argument above shows that if f can be represented as a power series about a, then f is equal to the sum of its Taylor series. But there exist functions that are not equal to the sum of their Taylor series. An example of such a function is given in Exercise 87. ■

Example 1 Maclaurin Series for the Exponential Function

Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Solution

If $f(x) = e^x \;\Rightarrow\; f^{(n)}(x) = e^x \;\Rightarrow\; f^{(n)}(0) = 1$ for all n.

Therefore, the Taylor series for f about 0, that is, the Maclaurin series, is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

To find the radius of convergence, let $a_n = \dfrac{x^n}{n!}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \qquad \text{Ratio Test quotient.}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1 \qquad \text{Evaluate the limit.}$$

By the Ratio Test, the series converges for all x, and the radius of convergence is $R = \infty$. ■

The conclusion we can draw from Theorem 5 and Example 1 is that if e^x has a power series expansion at 0, then we found it:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Now, we need to determine whether e^x really *does* have a power series representation.

Functions Equal to Their Taylor Series

Here's the more general question: Under what conditions is a function equal to the sum of its Taylor series? In other words, if f has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums. In the case of Taylor series, the partial sums are

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

The expression for $T_n(x)$ is a polynomial of degree n called the ***n*th-degree Taylor polynomial of *f* at *a***. For example, for the exponential function $f(x) = e^x$, the result in Example 1 shows that the Taylor polynomials at 0 (Maclaurin polynomials) with $n = 1$, 2, and 3 are

$$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!} \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The graphs of the exponential function and these three Taylor polynomials are shown in Figure 8.28.

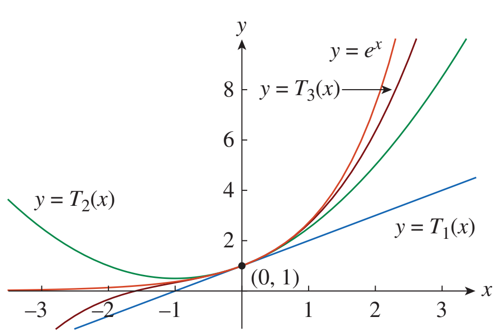


Figure 8.28
As n increases, the graph of $T_n(x)$ appears to approach the graph of e^x. This suggests that e^x is equal to the sum of its Taylor series.

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n \to \infty} T_n(x)$$

To evaluate this limit and, ultimately, to determine when a function is equal to the sum of its Taylor series, we consider the remainder.

Let

$$R_n(x) = f(x) - T_n(x) \quad \text{so that} \quad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is called the **remainder** of the Taylor series. If we can somehow show that $\lim_{n\to\infty} R_n(x) = 0$, then

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} [f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

This argument is a proof for the following theorem.

Theorem • A Function Equal to the Sum of its Taylor Series

If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

To show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f, we often use the following fact.

Taylor's Inequality

If $|f^{(n+1)}(x)| \le M$ for $|x - a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x - a|^{n+1} \quad \text{for } |x - a| \le d$$

To understand why this result is true, let's consider the case for $n = 1$. Assume that $|f''(x)| \le M$.

If $|f''(x)| \le M$, then $f''(x) \le M$ also, and for $a \le x \le a + d$,

$$\int_a^x f''(t)\,dt \le \int_a^x M\,dt$$

An antiderivative of f'' is f', so by Part 2 of the Fundamental Theorem of Calculus,

$$f'(x) - f'(a) \le M(x - a) \quad \text{or} \quad f'(x) \le f'(a) + M(x - a)$$

Integrate both sides of this equation.

$$\int_a^x f'(t)\,dt \le \int_a^x [f'(a) + M(t - a)]\,dt$$

$$f(x) - f(a) \le f'(a)(x - a) + M\frac{(x-a)^2}{2}$$ f is an antiderivative of f'; FTC2.

$$f(x) - f(a) - f'(a)(x - a) \le \frac{M}{2}(x - a)^2$$ Rearrange terms.

$$R_1(x) \le \frac{M}{2}(x - a)^2$$ $R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x - a)$.

A similar argument, using $f''(x) \geq -M$, shows that

$$R_1(x) \geq -\frac{M}{2}(x-a)^2$$

Therefore,

$$|R_1(x)| \leq \frac{M}{2}|x-a|^2$$

We assumed that $x > a$. However, similar calculations show that this inequality is also true for $x < a$.

This argument proves Taylor's Inequality for the case in which $n = 1$. The result for $n > 1$ is proved in a similar way by integrating $n + 1$ times.

A Closer Look

1. There are alternatives to Taylor's Inequality, that is, other formulas involving the remainder term. For example, if $f^{(n+1)}$ is continuous on an interval I and $x \in I$, then

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

This is called the *integral form of the remainder term.*

2. Another formula, called *Lagrange's form of the remainder term*, states that there exists a number z between x and a such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

This expression for the remainder looks very similar to the $(n+1)$st term in a Taylor series except that we evaluate $f^{(n+1)}$ at z, not a. Usually, we cannot find the value z (between x and a), and therefore, the exact remainder. However, we can often bound the value of $|f^{(n+1)}(z)|$.

Suppose that the maximum value of $|f^{(n+1)}(z)|$ is M for z between x and a. Then the **Lagrange Error Bound** is

$$|R_n(x)| \leq \max_{z \text{ between } x \text{ and } a} |f^{(n+1)}(z)| \cdot \frac{|x-a|^{n+1}}{(n+1)!} = M \cdot \frac{|x-a|^{n+1}}{(n+1)!}$$

Note that finding the value of M could involve a complicated optimization problem. Therefore, we generally are satisfied with a reasonable bound on M.

3. We will use Taylor's Inequality in approximating functions. For now, focus on the use of this inequality to show that the remainder term goes to 0 and, therefore, that a function is equal to its Taylor series. ■

In applying the previous two theorems, it is often useful to make use of the following limit.

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \text{for every real number } x \tag{7}$$

This is true because the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x, so the nth term must approach 0.

Example 2 The Maclaurin Series for e^x

Prove that e^x is equal to the sum of its Maclaurin series.

Solution

If $f(x) = e^x \;\Rightarrow\; f^{(n+1)}(x) = e^x$ for all n.

If d is any positive number and $|x| \le d$, then $|f^{(n+1)}(x)| = e^x \le e^d$.

Use Taylor's Inequality with $a = 0$ and $M = e^d$.

$$|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1} \quad \text{for } |x| \le d$$

Notice that the same constant $M = e^d$ works for every value of n.

Using Equation 10, we have

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

By the Squeeze Theorem, $\lim_{n\to\infty} |R_n(x)| = 0$, and therefore, $\lim_{n\to\infty} R_n(x) = 0$ for all values of x.

Therefore, e^x is equal to the sum of its Maclaurin series, that is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x \tag{8}$$

■

In particular, if we let $x = 1$ in Equation 8, we obtain the following expression for the number e as a sum of an infinite series.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \tag{9}$$

Example 3 A Taylor Series for e^x

Find the Taylor series for $f(x) = e^x$ centered at $a = 2$.

Solution

Since $f^{(n)}(x) = e^x \Rightarrow f^{(n)}(2) = e^2$.

Let $a = 2$ in the definition of a Taylor series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

Again, it can be verified, as in Example 1, that the radius of convergence is $R = \infty$.

As in Example 2, we can also verify that $\lim_{n\to\infty} R_n(x) = 0$.

Therefore,

$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n \quad \text{for all } x \tag{10}$$

■

We now have two power series expansions for e^x: the Maclaurin series in Equation 8 and the Taylor series in Equation 10. The first is better to use if we are interested in values of x near 0, and the second is better if x is near 2.

Example 4 The Maclaurin Series for sin *x*

Find the Maclaurin series for sin x and prove that it represents sin x for all x.

Solution

Start by finding derivatives, and evaluate each at 0.

$$\begin{aligned} f(x) &= \sin x & f(0) &= \sin 0 = 0 \\ f'(x) &= \cos x & f'(0) &= \cos 0 = 1 \\ f''(x) &= -\sin x & f''(0) &= -\sin 0 = 0 \\ f'''(x) &= -\cos x & f'''(0) &= -\cos 0 = -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= \sin 0 = 0 \end{aligned}$$

The derivatives repeat in a cycle of four.
Therefore, we can write the Maclaurin series as:

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Since $f^{(n+1)}(x)$ is $\pm \sin x$ or $\pm \cos x$, then $|f^{(n+1)}(x)| \le 1$ for all x.

So, we can let $M = 1$ in Taylor's Inequality:

$$|R_n(x)| \le \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!} \tag{11}$$

By Equation 10, $\lim\limits_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)!} = 0$.

Therefore, $\lim\limits_{n\to\infty} |R_n(x)| = 0$ by the Squeeze Theorem, and $\lim\limits_{n\to\infty} R_n(x) = 0$.

So, sin x is equal to the sum of its Maclaurin series.

Figure 8.29 shows the graph of sin x together with its Taylor (Maclaurin) polynomial.

$$T_1(x) = x \qquad T_3(x) = x - \frac{x^3}{3!} \qquad T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Notice that as n increases, the graph of $T_n(x)$ approaches the graph of sin x.

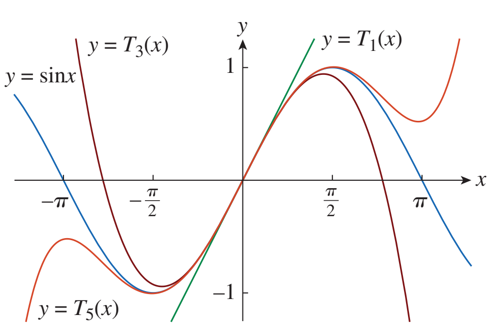


Figure 8.29
The graph of sin x and three of its Maclaurin polynomials.

■

The Maclaurin series for sin x is an important result. Here it is again for future reference.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x \tag{12}$$

Example 5 Find a Maclaurin Series by Differentiating a Known Series

Find the Maclaurin series for cos x.

Solution

The Maclaurin series for e^x, sin x, and cos x that we have found were discovered, using different methods, by Newton. These equations are absolutely remarkable because they say that we know everything about each of these functions if we know all its derivatives at the single number 0.

We could solve this problem as in Example 4. Find derivatives and evaluate each at 0.

However, it's easier to differentiate the Maclaurin series for sin x.

$$\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots \qquad \text{Differentiate term by term.}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{Simplify.}$$

Since the Maclaurin series for sin x converges for all x, the differentiated series for cos x also converges for all x. Therefore,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x \tag{13}$$

■

Example 6 A Shortcut for Obtaining a Maclaurin Series

Find the Maclaurin series for the function $f(x) = x \cos x$.

Solution

We could again find derivatives and evaluate each at 0.

However, it's easier to multiply the series for cos x by x.

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$ ■

The power series that we obtained by indirect methods in Examples 5 and 6 and in Section 8.6 are indeed the Taylor or Maclaurin series of the given functions. The theorem about the coefficients of a power series asserts that, no matter how a power series representation $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is obtained, it is always true that $c_n = \dfrac{f^{(n)}(a)}{n!}$. In other words, the coefficients are uniquely determined.

Example 7 A Taylor Series for sin *x*

Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\dfrac{\pi}{3}$.

Solution

Find derivatives and evaluate each at $\dfrac{\pi}{3}$.

$$
\begin{aligned}
f(x) &= \sin x & f\left(\frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} \\
f'(x) &= \cos x & f'\left(\frac{\pi}{3}\right) &= \frac{1}{2} \\
f''(x) &= -\sin x & f''\left(\frac{\pi}{3}\right) &= -\frac{\sqrt{3}}{2} \\
f'''(x) &= -\cos x & f'''\left(\frac{\pi}{3}\right) &= -\frac{1}{2}
\end{aligned}
$$

This pattern repeats indefinitely. Therefore, the Taylor series is

$$
\begin{aligned}
f\left(\frac{\pi}{3}\right) &+ \frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(x-\frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3 + \cdots \\
&= \frac{\sqrt{3}}{2} + \frac{1}{2\cdot 1!}\left(x-\frac{\pi}{3}\right) - \frac{\sqrt{3}}{2\cdot 2!}\left(x-\frac{\pi}{3}\right)^2 - \frac{1}{2\cdot 3!}\left(x-\frac{\pi}{3}\right)^3 + \cdots
\end{aligned}
$$

This series does indeed represent $\sin x$ for all x, and the proof is similar to that in Example 4; just replace x by $x - \dfrac{\pi}{3}$.

We can write this series using summation notation if we separate the terms that contain $\sqrt{3}$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{3}}{2(2n)!}\left(x-\frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!}\left(x-\frac{\pi}{3}\right)^{2n+1}$$ ■

A Closer Look

We now have two different series representations for $\sin x$: the Maclaurin series in Example 4 and the Taylor series in Example 7. To approximate values for $\sin x$, it is best to use the Maclaurin series for values of x near 0 and the Taylor series for values of x near $\dfrac{\pi}{3}$. Figure 8.30 shows graphically that the third-degree Taylor polynomial T_3 is a good approximation to $\sin x$ near $\dfrac{\pi}{3}$, but not so good near 0.

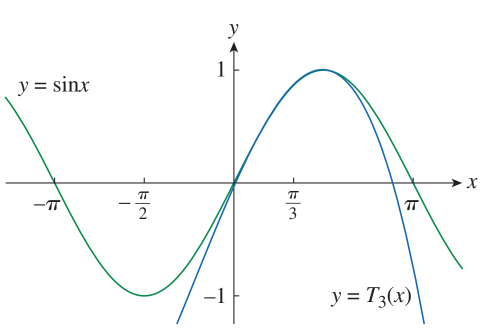


Figure 8.30
The graph of $y = \sin x$ and $y = T_3(x)$, the third-degree Taylor polynomial centered at $\dfrac{\pi}{3}$.

Compare this with the third-degree Maclaurin polynomial in Figure 8.29. This polynomial is a good approximation to $\sin x$ near 0, but not as good near $\dfrac{\pi}{3}$. ■

The Binomial Series

Example 8 Use the Direct Method to Find a Maclaurin Series

Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

Solution

Find the derivatives and evaluate each at 0.

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1\\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k\\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1)\\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2)\\
&\vdots & &\vdots\\
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
\end{aligned}$$

Therefore, the Maclaurin series of $f(x) = (1 + x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n.$$

This series is called the **binomial series**. Notice that if k is a nonnegative integer, then the terms are eventually 0 and so the series is finite.

For other values of k, none of the terms is 0 and so we can try the Ratio Test to test for convergence. If the nth term is a_n, then consider the appropriate ratio:

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right|\\
&= \frac{|k-n|}{n+1}|x|
\end{aligned}$$

Find the appropriate limit.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|k-n|}{n+1}|x| = \lim_{n\to\infty}\frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x| = |x|$$

By the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$. ■

In probability and statistics, $\binom{k}{n}$ is **combination** and is read as "k choose n" or "k items taken n at a time."

The traditional notation for the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

and these numbers are called the **binomial coefficients**.

The following theorem states that $(1 + x)^k$ is indeed equal to the sum of its Maclaurin series. It is possible to prove this by showing that the remainder term $R_n(x)$ approaches 0. However, the proof outlined in Exercise 89 is easier.

The Binomial Series

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

A Closer Look

1. The binomial series always converges when $|x| < 1$.
2. Convergence at the endpoints, ± 1, depends on the value of k.

   The series converges at 1 if $-1 < k \leq 0$.

   The series converges at both endpoints if $k \geq 0$.
3. If k is a positive integer and $n > k$, then the expression $\binom{k}{n}$ contains a factor $(k-k)$, so $\binom{k}{n} = 0$ for $n > k$.

   This means the series terminates and reduces to the ordinary Binomial Theorem when k is a positive integer. ■

Example 9 Use a Binomial Series to Obtain a Maclaurin Series

Find the Maclaurin series for the function $f(x) = \dfrac{1}{\sqrt{4-x}}$ and its radius of convergence.

Solution

Rewrite $f(x)$ in a form in which we can use the binomial series.

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

Use the binomial series with $k = -\dfrac{1}{2}$ and x replaced by $-\dfrac{x}{4}$.

$$\begin{aligned}
\frac{1}{\sqrt{4-x}} &= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n \\
&= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x}{4}\right)^3 \right. \\
&\qquad \left. + \cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-\frac{x}{4}\right)^n + \cdots\right] \\
&= \frac{1}{2}\left[1 + \frac{1}{8}x + \frac{1\cdot 3}{2!8^2}x^2 + \frac{1\cdot 3\cdot 5}{3!8^3}x^3 + \cdots + \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!8^n}x^n + \cdots\right] \\
&= \frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2 + \frac{5}{2048}x^3 + \cdots + \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2n!8^n}x^n + \cdots
\end{aligned}$$

This series converges when $\left|-\frac{x}{4}\right| < 1 \Rightarrow |x| < 4.$

Therefore, the radius of convergence is $R = 4$. ■

Series Summary

Table 8.6 presents a summary of some important Maclaurin series that we have derived in this section and the preceding one.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Table 8.6
Important Maclaurin series and their radii of convergence.

Example 10 Sum of a Series

Find the sum of the series $\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \cdots$

Solution

Write the given series using summation notation.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n \cdot 2^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n}$$

Using Table 8.6, this series matches the entry for $\ln(1 + x)$ with $x = \frac{1}{2}$.

Therefore,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n \cdot 2^n} = \ln\left(1 + \frac{1}{2}\right) = \ln\frac{3}{2}.$$ ■

One reason that Taylor series are important is that they enable us to integrate functions that we previously realized have no closed-form antiderivative. For example, the function $f(x) = e^{-x^2}$ cannot be integrated by any technique discussed so far because its antiderivative is not an elementary function. In Example 11, we will use the idea of integrating a function by first expressing it as a power series and then integrating term by term.

Example 11 Use a Series to Evaluate an Integral

(a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

(b) Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

Solution

(a) Find the Maclaurin series for $f(x) = e^{-x^2}$.

We could use the direct method, find derivatives, and evaluate each at 0. However, it's easier if we replace x by $-x^2$ in the series for e^x. Use Table 8.6.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

Integrate term by term.

$$\int e^{-x^2}\,dx = \int\left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx$$

$$= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

This series converges for all x because the original series for e^{-x^2} converges for all x.

(b) Use the Fundamental Theorem of Calculus with $C = 0$.

$$\int_0^1 e^{-x^2}\,dx = \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \cdots\right]_0^1$$

Antiderivative with $C = 0$.

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots$$

FTC2; every term in the antiderivative is 0 when $x = 0$.

$$\approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} = 0.7475$$

Use the first five terms as an approximation.

The Alternating Series Error Bound associated with this approximation is

$$|\text{error}| \le \frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001.$$

Another use of Taylor series is associated with limits. There are other ways to find the limit in Example 12, for example, l'Hospital's Rule. But we will illustrate the use of series.

Example 12 Use a Series to Evaluate a Limit

Evaluate $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

Solution

Consider the limit of the numerator and the limit of the denominator.

$$\lim_{x \to 0} (e^x - 1 - x) = 0 \qquad \lim_{x \to 0} x^2 = 0$$

Therefore, this limit is a good candidate for l'Hospital's Rule.

However, using the Maclaurin series for e^x, we have

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}=\lim_{x\to 0}\frac{\left(1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots\right)-1-x}{x^2}$$

Use the Maclaurin series for e^x.

$$=\lim_{x\to 0}\frac{\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\cdots}{x^2}$$

Simplify the numerator.

$$=\lim_{x\to 0}\left(\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\frac{x^3}{5!}+\cdots\right)=\frac{1}{2}$$

Divide each term in the numerator by x^2; evaluate the limit.

Just a reminder: we can evaluate this limit term by term because power series are continuous functions. ■

Multiplication and Division of Power Series

We have already learned that we can add or subtract power series just as though they were polynomials. The following example suggests that power series can also be multiplied and divided, just like polynomials. In these cases, we usually find only the first few terms because the calculations are tedious and time consuming. And we know that the initial terms are really the most important ones for approximations.

Example 13 Find a Maclaurin Series by Multiplication and Division

Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

Solution

(a) Use the Maclaurin series for e^x and $\sin x$ from Table 8.6.

$$e^x \sin x=\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)$$

Use known Maclaurin series.

$$=\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots\right)+\left(x^2-\frac{x^4}{6}+\frac{x^6}{120}-\cdots\right)$$
$$+\left(\frac{x^3}{2}-\frac{x^5}{12}+\frac{x^7}{240}-\cdots\right)+\cdots$$

Systematic multiplication.

$$=x+x^2+\frac{x^3}{3}-\frac{x^5}{30}+\cdots$$

Simplify to obtain the first few terms in the product.

(b) Use the Maclaurin series for $\sin x$ and $\cos x$ from Table 8.6.

$$\tan x=\frac{\sin x}{\cos x}=\frac{x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\cdots}{1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\cdots}$$

Use a procedure like long division.

$$
\begin{array}{r@{}l}
 & x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots \\
\cline{2-2}
1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4 - \cdots \big) & x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \cdots \\
 & x - \tfrac{1}{2}x^3 + \tfrac{1}{24}x^5 - \cdots \\
\cline{2-2}
 & \tfrac{1}{3}x^3 - \tfrac{1}{30}x^5 + \cdots \\
 & \tfrac{1}{3}x^3 - \tfrac{1}{6}x^5 + \cdots \\
\cline{2-2}
 & \tfrac{2}{15}x^5 + \cdots
\end{array}
$$

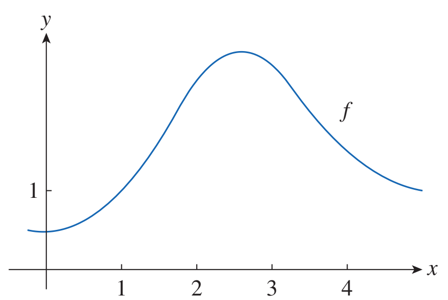
Therefore, $\tan x = x + \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5 + \cdots$ ■

Although there is no justification for the formal manipulations used in Example 13, they are indeed legitimate. There is a theorem that states that if both $f(x) = \Sigma c_n x^n$ and $g(x) = \Sigma b_n x^n$ converge for $|x| < R$ and the series are multiplied as if they were polynomials, then the resulting series also converges for $|x| < R$ and represents $f(x)g(x)$. For division, we require $b_0 \neq 0$, in which case the resulting series will have derivatives of all orders.

8.7 Exercises

1. If $f(x) = \sum_{n=0}^{\infty} b_n(x-5)^n$ for all x, write a formula for b_8.

2. The graph of a function f is shown in the figure.



(a) Explain why the series

$$1.6 - 0.8(x-1) + 0.4(x-1)^2 - 0.1(x-1)^3 + \cdots$$

is *not* the Taylor series for f centered at 1.

(b) Explain why the series

$$2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$$

is *not* the Taylor series for f centered at 2.

3. If $f^{(n)}(0) = (n+1)!$ for $n = 0, 1, 2, \ldots$, find the Maclaurin series for f and its radius of convergence.

4. Find the Taylor series for f centered at 4 if

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$$

What is the radius of convergence of the Taylor series?

Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. Assume that f has a power series expansion, but you do not need to show that $R_n(x) \to 0$. Find the associated radius of convergence.

5. $f(x) = (1-x)^{-2}$

6. $f(x) = \ln(1+x)$

7. $f(x) = \sin \pi x$

8. $f(x) = \cos 3x$

9. $f(x) = e^{-2x}$

10. $f(x) = xe^x$

11. $f(x) = 2^x$

12. $f(x) = x \cos x$

Find the Taylor series for $f(x)$ centered at the given value of a. Assume that f has a power series expansion, but you do not need to show that $R_n(x) \to 0$. Find the associated radius of convergence.

13. $f(x) = x^5 + 2x^3 + x, \quad a = 2$

14. $f(x) = x^6 - x^4 + 2, \quad a = -2$

15. $f(x) = \ln x, \quad a = 2$

16. $f(x) = \dfrac{1}{x}, \quad a = -3$

17. $f(x) = e^{2x}, \quad a = 3$

18. $f(x) = \cos x, \quad a = \dfrac{\pi}{2}$

19. $f(x) = \sin x, \quad a = \pi$

20. $f(x) = \sqrt{x}, \quad a = 16$

21. $f(x) = \dfrac{1}{\sqrt{x}}, \quad a = 9$

22. $f(x) = \dfrac{1}{x^2}, \quad a = 1$

23. Prove that the series obtained in Exercise 7 represents $\sin \pi x$ for all x.

24. Prove that the series obtained in Exercise 19 represents $\sin x$ for all x.

Use the binomial series to expand the function as a power series. Find the radius of convergence.

25. $\sqrt[4]{1-x}$

26. $\sqrt[3]{8+x}$

27. $\dfrac{1}{(2+x)^3}$

28. $(1+x)^{3/4}$

Use the Maclaurin series in Table 8.6 to obtain the Maclaurin series for the function.

29. $f(x) = \arctan(x^2)$

30. $f(x) = \sin\left(\dfrac{\pi x}{4}\right)$

31. $f(x) = x \cos 2x$

32. $f(x) = e^{3x} - e^{2x}$

33. $f(x) = x \cos\left(\dfrac{x^2}{2}\right)$

34. $f(x) = x^2 \ln(1 + x^3)$

35. $f(x) = \dfrac{x}{\sqrt{4 + x^2}}$

36. $f(x) = \dfrac{x^2}{\sqrt{2 + x}}$

37. $f(x) = \sin^2 x$ Hint: Use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

38. $f(x) = \begin{cases} \dfrac{x - \sin x}{x^3} & \text{if } x \neq 0 \\[2mm] \dfrac{1}{6} & \text{if } x = 0 \end{cases}$

Find the Maclaurin series of f (by any method) and its radius of convergence. Graph f and its first few Taylor polynomials in the same viewing rectangle. Describe the relationship between these graphs of the polynomials and the graph of f.

39. $f(x) = \cos(x^2)$

40. $f(x) = \ln(1 + x^2)$

41. $f(x) = xe^{-x}$

42. $f(x) = \tan^{-1}(x^3)$

43. Use the Maclaurin series for $\cos x$ to compute $\cos \dfrac{\pi}{36}$ correct to five decimal places.

44. Use the Maclaurin series for e^x to calculate $\dfrac{1}{\sqrt[10]{e}}$ correct to five decimal places.

45. (a) Use the binomial series to expand $\dfrac{1}{\sqrt{1 - x^2}}$.

(b) Use part (a) to find the Maclaurin series for $\sin^{-1} x$.

46. (a) Expand $\dfrac{1}{\sqrt[4]{1+x}}$ as a power series.

(b) Use part (a) to estimate $\dfrac{1}{\sqrt[4]{1.1}}$ correct to three decimal places.

Evaluate the indefinite integral as an infinite series.

47. $\displaystyle\int \sqrt{1 + x^3}\, dx$

48. $\displaystyle\int x^2 \sin(x^2)\, dx$

49. $\displaystyle\int \frac{\cos x - 1}{x}\, dx$

50. $\displaystyle\int \arctan(x^2)\, dx$

51. $\displaystyle\int x \cos(x^3)\, dx$

52. $\displaystyle\int \frac{e^x - 1}{x}\, dx$

Use series to approximate the definite integral to within the indicated accuracy.

53. $\displaystyle\int_0^{1/2} x^3 \arctan x\, dx$ (four decimal places)

54. $\displaystyle\int_0^1 \sin x^4\, dx$ (four decimal places)

55. $\displaystyle\int_0^{0.4} \sqrt{1 + x^4}\, dx$ $(|\text{error}| < 5 \times 10^{-6})$

56. $\displaystyle\int_0^{0.5} x^2 e^{-x^2}\, dx$ $(|\text{error}| < 0.001)$

Use series to evaluate the limit.

57. $\displaystyle\lim_{x \to 0} \frac{x - \ln(1+x)}{x^2}$

58. $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x}$

59. $\displaystyle\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$

60. $\displaystyle\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2}$

61. $\displaystyle\lim_{x \to 0} \frac{x^3 - 3x + 3\tan^{-1} x}{x^5}$

62. Use the series in Example 13(b) to evaluate

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

We found this limit earlier using l'Hospital's Rule three times. Which method do you prefer and why?

Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function.

63. $y = e^{-x^2} \cos x$

64. $y = \sec x$

65. $y = \dfrac{x}{\sin x}$

66. $y = e^x \ln(1 + x)$

67. $y = (\arctan x)^2$

68. $y = e^x \sin^2 x$

Find the sum of the series.

69. $\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{n!}$

70. $\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

71. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{n5^n}$

72. $\sum_{n=0}^{\infty}\frac{3^n}{5^n n!}$

73. $\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$

74. $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \cdots$

75. $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots$

76. $\frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \cdots$

77. Suppose f is a function such that $f(1) = 2$, $f'(1) = 3$, $f''(1) = 4$, $f'''(1) = 5$, and $|f^{(4)}(x)| < 3$ for all values of x, $1 \le x \le 2$.

(a) Find the third-degree Taylor polynomial, $T_3(x)$, for f centered at $a = 1$.

(b) Use $T_3(x)$ to approximate $f(1.5)$.

(c) Show that $|T_3(1.5) - f(1.5)| < 0.01$.

78. The third-degree Taylor polynomial centered at $a = 1$ for a function f is given by

$$T_3(x) = 5 + 12(x-1) - 3(x-1)^2 + 18(x-1)^3.$$

(a) Use the first degree Taylor polynomial for f centered at $a = 1$ to approximate $f(2)$. Explain why this approximation is the same as a tangent line approximation.

(b) Suppose that f'' does not change signs for $x \ge 1$. Determine, if possible, whether the actual value of $f(2)$ is greater than 14 or less than 14. Explain your reasoning.

(c) The fourth derivative of f evaluated at $x = 1$ is $f^{(4)}(1) = 312$. Find the fourth-degree Taylor polynomial for f centered at $a = 1$ and use it to approximate $f(2)$.

79. The Taylor series $\sum_{n=0}^{\infty} a_n(x-5)^n$ converges to a function f for all values in its interval of convergence. For this function, it is known that the nth derivative for f evaluated at 5 is given by $f^{(n)}(5) = \frac{(-1)^n n!}{2^n(n+2)}$. Find the value of a_3.

80. Let $f(x) = e^{2x} - \cos(x^2)$. Find the coefficient of x^4 in the Maclaurin series for f.

81. Suppose $f'(x) = \sum_{n=0}^{\infty}(-1)^n \frac{(3x)^{2n+1}}{(2n+1)!}$ and $f(0) = 2$. Find $f\left(\frac{\pi}{3}\right)$.

82. Let g be a function that has derivatives of all orders for all real numbers x. It is known that $g(2) = 5$, $g'(2) = 6$, $g'(2.5) = 10$, $g''(2) = 28$, and $g'''(2) = 48$. For $2 \le x \le 3$, $|g^{(4)}(x)| \le 80$.

(a) Find an equation of the line tangent to the graph of g at the point where $x = 2$. Use this tangent line to approximate $g(3)$.

(b) Use a left Riemann sum with two subintervals of equal length and the values given above to approximate

$$\int_2^3 g'(x)\,dx.$$

Use the approximation for $\int_2^3 g'(x)\,dx$ to estimate $g(3)$. Show the computations that lead to your answer.

(c) Find $T_3(x)$, the third-degree Taylor polynomial for g centered at $a = 2$. Use $T_3(x)$ to find an approximation for $g(3)$.

(d) Use the Lagrange error bound to show that $g(3) \neq 40$.

83. The function f is twice differentiable, $f(5) = 3$, and $f''(5) = 20$. The table gives values of f' at selected values of x.

x	5.0	5.1	5.2	5.3	5.4
$f'(x)$	4	5	7	8	10.5

On the interval $5 \le x \le 5.4$, $|f^{(3)}(x)| \le 7$.

(a) Write the second-degree Taylor polynomial $T_2(x)$ for f centered at $a = 5$.

(b) Use the Taylor polynomial to approximate $f(5.4)$.

(c) Use the Lagrange error bound to show that $|f(5.4) - T_2(5.4)| < 0.6$. Show the computations that lead to your conclusion.

84. The power series given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ is the Taylor series centered at $a = 1$ for a certain function f.

(a) Find the interval of convergence for the power series.

(b) For $n \ge 1$, find an expression for $f^{(n)}(1)$ in terms of n.

(c) Show that the Taylor series for $f\left(\frac{1}{2}x + \frac{1}{2}\right)$ centered at $a = 1$ is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$.

(d) Suppose $f\left(\frac{3}{4}\right) = -\ln 2$. Use the first three terms of the Taylor series to approximate $-\ln 2$.

(e) Find the first three terms and the general term of the Taylor series for $f'(x)$.

(f) There is a function g such that $g'(x) = f(x)$ and $g(1) = 2$. Find the fourth-degree Taylor polynomial for g centered at $a = 1$.

85. Let f be a function having derivatives of all orders for all real numbers x. The third-degree Taylor polynomial for f centered at $a = 1$ is given by

$$T_3(x) = 4 + a_1(x - 1) + 11(x - 1)^2 + 15(x - 1)^3$$

where a_1 is a constant whose value is to be determined.

(a) The linear approximation for $f(2)$ given by the line tangent to the graph of f at the point where $x = 1$ is $f(2) \approx 14$. Find the value of a_1.

(b) Find the value of $f''(1)$. Given that f'' does not change signs for $x \geq 1$, determine whether the actual value of $f(2)$ is greater than 14 or less than 14. Give a reason for your answer.

(c) The fourth derivative of f evaluated at $x = 1$ is $f^{(4)}(1) = 312$. Find the fourth-degree Taylor polynomial for f centered at $a = 1$ and use it to approximate $f(2)$.

(d) Suppose that $x = 4$ is in the interval of convergence of the Taylor series for f centered at $a = 1$. Is there enough information given to determine whether or not $x = 3$ is in the interval of convergence? Is there enough information given to determine whether or not $x = -2$ is in the interval of convergence? Explain your reasoning.

(e) Let $g(x) = \int_1^x f(t)\, dt$. Find the second-degree Taylor polynomial for g centered at $a = 1$.

---

86. Show that if p is an nth-degree polynomial, then

$$p(x + 1) = \sum_{i=0}^{n} \frac{p^{(i)}(x)}{i!}$$

87. (a) Show that the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not equal to its Maclaurin series.

(b) Graph the function in part (a) and describe its behavior near the origin.

88. Prove Taylor's Inequality for $n = 2$, that is, prove that if $|f'''(x)| \leq M$ for $|x - a| \leq d$, then

$$|R_2(x)| \leq \frac{M}{6}|x - a|^3 \quad \text{for } |x - a| \leq d$$

89. Use the following steps to prove the Binomial Series.

(a) Let $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$. Differentiate this series to show that

$$g'(x) = \frac{kg(x)}{1 + x} \qquad -1 < x < 1$$

(b) Let $h(x) = (1 + x)^{-k} g(x)$ and show that $h'(x) = 0$.

(c) Deduce that $g(x) = (1 + x)^k$.

90. In Exercise 42 in Section 6.4, the length of the ellipse described by $x = a \sin \theta$, $y = b \cos \theta$, where $a > b > 0$, was shown to be

$$L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta$$

where $e = \sqrt{a^2 - b^2}/a$ is the eccentricity of the ellipse. Expand the integrand as a binomial series and use the result of Exercise 54 in Section 5.6 to express L as a series in powers of the eccentricity up to the term in e^6.

Laboratory Project | An Elusive Limit

This project involves the function f defined by

$$f(x) = \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}$$

1. Use technology to evaluate $f(x)$ for $x = 1, 0.1, 0.05, 0.025, 0.01$, and 0.0075. Do these values suggest that f has a limit as $x \to 0$? Why or why not?

2. Use technology to graph f near $x = 0$. Does this graph suggest that f has a limit as $x \to 0$? Why or why not?

3. Try to evaluate $\lim_{x \to 0} f(x)$ using l'Hospital's Rule. You may need to use technology (a CAS) to find the derivatives of the numerator and the denominator. Explain your results. How many applications of l'Hospital's Rule are necessary in order to find the limit?

4. Use technology (a CAS) to find the Maclaurin series of the numerator and the denominator. Evaluate $\lim_{x \to 0} f(x)$ using these series.

5. Use technology (a CAS) to find $\lim_{x \to 0} f(x)$ directly.

6. How do you resolve the answers to problems 4 and 5 with the results from problems 1 and 2?

Writing Project | How Newton Discovered the Binomial Series

The Binomial Theorem, which gives the expansion of $(a + b)^k$, was known to Chinese mathematicians many centuries before the time of Newton for the case where the exponent k is a positive integer. In 1665, when he was 22, Newton was the first to discover the infinite series expansion of $(a + b)^k$ when k is a fractional exponent (positive or negative). He didn't publish his discovery, but he stated it and gave examples of how to use it in a letter (now called the *epistola prior*) dated June 13, 1676, that he sent to Henry Oldeburg, secretary of the Royal Society of London, to transmit to Leibniz.

When Leibniz replied, he asked how Newton had discovered the binomial series. Newton wrote a second letter, the *epistola posterior* of October 24, 1676, in which he explained in great detail how he arrived at his discovery by a very indirect route. He was investigating the areas under the curves $y = (1 - x^2)^{n/2}$ from 0 to x for $n = 0, 1, 2, 3, 4, \ldots$. These are easy to calculate if n is even. By observing patterns and interpolating, Newton was able to guess the answers for odd values of n. Then he realized he could get the same answers by expressing $(1 - x^2)^{n/2}$ as an infinite series.

Write an essay about Newton's discovery of the binomial series. Start by giving the statement of the binomial series in Newton's notation (see the *epistola prior* on page 285 of [4] or page 402 of [2]). Explain why Newton's version is equivalent to Theorem 17. Then read Newton's *epistola posterior* (page 287 in [4] or page 404 in [2]) and explain the patterns that Newton discovered in the areas under the curves $y = (1 - x^2)^{n/2}$. Show how he was able to guess the areas under the remaining curves and how he verified his answers. Finally, explain how these discoveries led to the binomial series. The books by Edwards [1] and Katz [3] contain commentaries on Newton's letters.

1. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 178–187.

2. John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader* (London: MacMillan Press, 1987).

3. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), pp. 463–466.

4. D. J. Struik, ed., *A Sourcebook in Mathematics*, *1200–1800* (Princeton, NJ: Princeton University Press, 1969).

8.8 Applications of Taylor Polynomials

In this section, we consider two types of applications of Taylor polynomials. First we will see how they are used to approximate functions; computer scientists use these types of approximations because polynomials are the simplest of functions. Then we will investigate how physicists and engineers use Taylor polynomials to study special relativity, optics, blackbody radiation, electric dipoles, and even the construction of highways across a desert.

Approximating Functions by Polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor Series centered at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

In Section 8.7, we defined the expression for $T_n(x)$; a polynomial of degree n called the nth-degree Taylor polynomial of f at a. Therefore,

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Since f is the sum of its Taylor series, we know that $T_n(x) \to f(x)$ as $n \to \infty$. Therefore, T_n can be used as an approximation to f: $f(x) \approx T_n(x)$.

Notice that the first-degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x-a)$$

is the same as the linearization of f at a discussed in Section 3.9. Notice also that $T_1(a) = f(a)$ and $T'_1(a) = f'(a)$. In general, it can be shown that the derivatives of T_n at a agree with those of f up to and including derivatives of order n.

To illustrate these ideas, let's take another look at the graphs of $y = e^x$ and its first few Taylor polynomials (centered at 0), as shown in Figure 8.31. The graph of T_1 is the tangent line to the graph of $y = e^x$ at (0, 1); this tangent line is the best linear approximation to e^x near (0, 1). The graph of T_2 is the parabola $y = 1 + x + \frac{x^2}{2}$, and the graph of T_3 is the cubic curve $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$, which is a better fit to the exponential curve $y = e^x$ than T_2. The next Taylor polynomial T_4 would be an even better approximation, and so on.

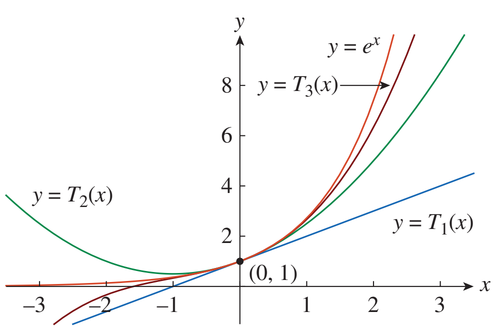


Figure 8.31
The graphs of the exponential function and three Taylor polynomials.

Expression	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

Table 8.7
Numerical approximations using Taylor polynomials.

The values in Table 8.7 provide a numerical illustration of the convergence of the Taylor polynomials $T_n(x)$ to the function $y = e^x$. The table shows that when $x = 0.2$, the convergence is very fast, but when $x = 3$, it is a little slower. In fact, the farther x is from 0, the more slowly $T_n(x)$ converges to e^x.

If we use a Taylor polynomial T_n to approximate a function f, then we need to think about a few questions: How accurate is an approximation? How large should n be in

order to achieve a desired accuracy? To answer these questions, we need to consider the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

There are three possible methods for estimating the size of the error:

1. If technology is used, then we can graph $|R_n|$ and estimate the error.

2. If the series is an alternating series, then we can use the Alternating Series Error Bound.

3. In all cases we can use Taylor's Inequality (Theorem 8.7.9), which says that if $|f^{(n+1)}(x)| \le M$, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x - a|^{n+1}$$

Example 1 Approximate a Root Function by a Quadratic Function

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 centered at $a = 8$.
(b) How accurate is this approximation when $7 \le x \le 9$?

Solution

(a) Find the derivatives and evaluate each at 8.

$$\begin{aligned} f(x) &= \sqrt[3]{x} = x^{1/3} & f(8) &= 2 \\ f'(x) &= \frac{1}{3}x^{-2/3} & f'(8) &= \frac{1}{12} \\ f''(x) &= -\frac{2}{9}x^{-5/3} & f''(8) &= -\frac{1}{144} \\ f'''(x) &= \frac{10}{27}x^{-8/3} & f'''(8) &= \frac{5}{3456} \end{aligned}$$

The second-degree Taylor polynomial is

$$\begin{aligned} T_2(x) &= f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2. \end{aligned}$$

The approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2.$$

(b) The Taylor series is not alternating when $x < 8$, so we cannot use the Alternating Series Error Bound in this case.

But we can use Taylor's Inequality with $n = 2$ and $a = 8$.

$$|R_2(x)| \le \frac{M}{3!}|x-8|^3 \quad \text{where} \quad |f'''(x)| \le M$$

$f'''(x) = \dfrac{10}{27}x^{-8/3}$ is positive and decreasing on $(0, \infty)$ and therefore on $[7, 9]$.

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \leq \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021 \;\Rightarrow\; \text{let } M = 0.0021$$

For x in the interval $[7, 9]$:

$$7 \leq x \leq 9 \;\Leftrightarrow\; -1 \leq x - 8 \leq 1 \;\Leftrightarrow\; |x - 8| \leq 1,$$

Taylor's Inequality becomes $|R_2(x)| \leq \dfrac{0.0021}{3!} \cdot 1^3 = \dfrac{0.0021}{6} < 0.0004.$

Therefore, if $7 \leq x \leq 9$, the approximation in part (a) is accurate to within 0.0004. Figure 8.32 shows the graphs of $y = \sqrt[3]{x}$ and $y = T_2(x)$ are very close together when x is near 8.

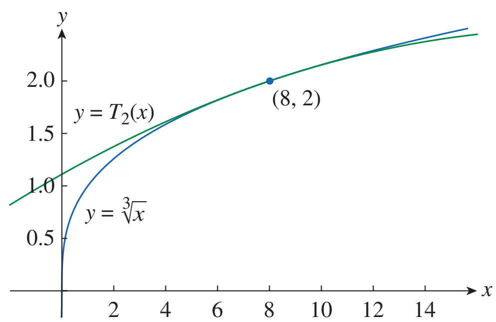


Figure 8.32
Graphs of $y = \sqrt[3]{x}$ and $y = T_2(x)$.

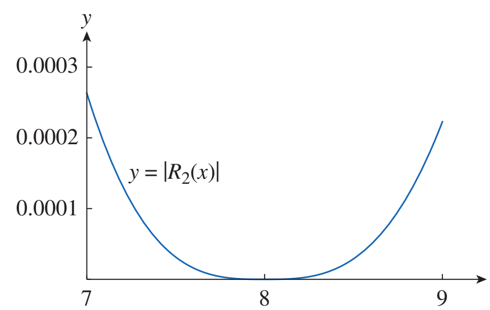


Figure 8.33
The graph of $y = |R_2(x)|$.

Figure 8.33 shows the graph of $y = |R_2(x)|$ defined by the expression $|R_2(x)| = |\sqrt[3]{x} - T_2(x)|$.

This graph suggests that $|R_2(x)| < 0.0003$ when $7 \leq x \leq 9$. This graphical error estimate is slightly better (smaller) than the analytical approximation using Taylor's Inequality in this case. ■

Example 2 Approximate sin *x* by a Fifth-Degree Taylor Polynomial

Consider the fifth-degree Taylor (Maclaurin) polynomial for $\sin x$ centered at 0.

$$\sin x \approx T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

(a) What is the maximum error possible in using this approximation when $-0.3 \leq x \leq 0.3$? Use $T_5(x)$ to approximate $\sin\left(\dfrac{\pi}{15}\right)$.

(b) For what values of x is this approximation accurate to within 0.00005?

Solution

(a) The Maclaurin series for $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

is alternating for all nonzero values of x, and successive terms decrease in size because $|x| < 1$. Therefore, we can use the Alternating Series Error Bound.

If $-0.3 \le x \le 0.3$, then $|x| \le 0.3$. So, the error is smaller than

$\dfrac{(0.3)^7}{5040} \approx 4.3 \times 10^{-8}$. The next term in the series is $\dfrac{x^7}{7!}$.

The approximation for $\sin\left(\dfrac{\pi}{15}\right)$ is

$$\sin\left(\frac{\pi}{15}\right) \approx \frac{\pi}{15} - \left(\frac{\pi}{15}\right)^3 \cdot \frac{1}{3!} + \left(\frac{\pi}{15}\right)^5 \cdot \frac{1}{5!} \approx 0.20791169.$$

(b) The error in the approximation will be smaller than 0.00005 if $\dfrac{|x|^7}{5040} < 0.00005$.

Solve this inequality for x.

$|x|^7 < 0.252 \;\Leftrightarrow\; |x| < (0.252)^{1/7} \approx 0.821$

$T_5(x)$ is accurate to within 0.00005 when $|x| < 0.821$.

Suppose we use Taylor's Inequality to solve Example 2. Since $f^{(7)}(x) = -\cos x$, then $|f^{(7)}(x)| \le 1$ and therefore,

$$|R_6(x)| \le \frac{1}{7!}|x|^7$$

This leads to the same estimates as with the Alternating Series Error Bound.

Let's consider using graphical methods to estimate the error of estimation. Figure 8.34 shows the graph of

$$|R_6(x)| = \left|\sin x - \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\right|$$

which suggests that $|R_6(x)| < 4.3 \times 10^{-8}$ when $|x| \le 0.3$. This is the same estimate we obtained in Example 2. For part (b), we want $|R_6(x)| < 0.00005$. Figure 8.35 shows a graph of $y = |R_6(x)|$ and $y = 0.00005$. If we use technology to find the intersection point on the right, we find that the inequality is satisfied when $|x| < 0.822$. Again, this is very close to the estimate that we obtained in the solution to Example 2.

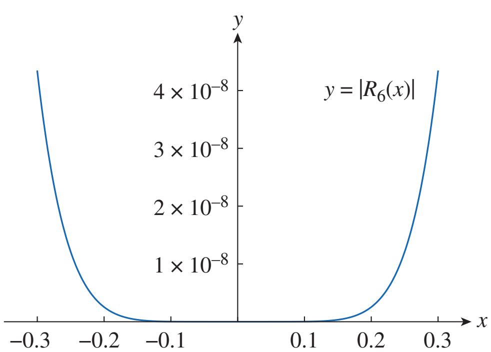


Figure 8.34
The graph suggests that $|R_6(x)| < 4.3 \times 10^{-8}$ when $|x| \le 0.3$.

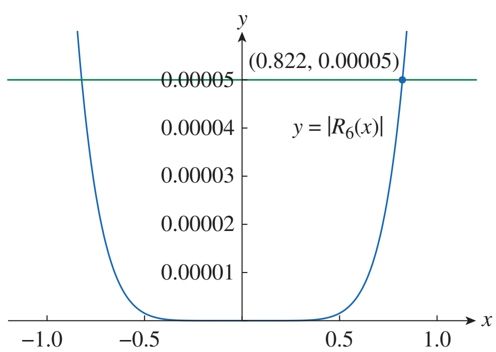


Figure 8.35
Graphs of $y = |R_6(x)|$ and $y = 0.00005$.

Suppose we had been asked to approximate $\sin\left(\frac{2\pi}{5}\right)$ instead of $\sin\left(\frac{\pi}{15}\right)$ in Example 2. Then it would be prudent to use the Taylor polynomials centered at $a = \frac{\pi}{3}$ (instead of $a = 0$) because they are better approximations to $\sin x$ for values close to $\frac{\pi}{3}$. The angle $\frac{2\pi}{5}$ is close to $\frac{\pi}{3}$ and the derivatives of $\sin x$ are easy to compute at $\frac{\pi}{3}$.

Figure 8.36 shows the graphs of the Maclaurin polynomial approximations

$$T_1(x) = x \qquad T_3(x) = x - \frac{x^3}{3!}$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

to the sine function.

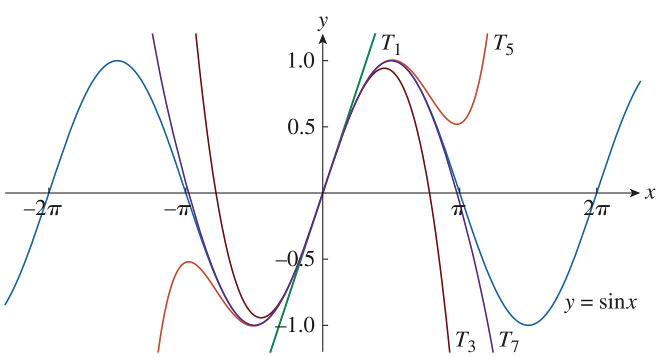


Figure 8.36
Graphs of Maclaurin polynomial approximations to the sine function.

These graphs suggest that as n increases, $T_n(x)$ is a good approximation to $\sin x$ on a larger and larger interval.

One application of the approximations presented in Examples 1 and 2 occurs in technology calculations. For example, if we use the sin or e^x key on a calculator, or if we use a computer subroutine to compute a trigonometric, exponential, or Bessel function, in many cases a polynomial approximation is used. The polynomial is often a Taylor polynomial that has been modified so that the error is spread more evenly throughout an interval.

Application to Physics

Taylor polynomials are also used frequently in physics. In order to gain insight into a complicated equation, a physicist often simplifies a function by considering only the first two or three terms in its Taylor series. That is, the physicist uses a Taylor polynomial as an approximation to the function. Taylor's inequality can then be used to gauge the accuracy of the approximation. The next example shows one way in which this idea is used in special relativity.

Example 3 Use Taylor to Compare Einstein and Newton

In Einstein's theory of special relativity, the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2$$

(a) Show that when v is very small compared with c, the expression for K agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.

(b) Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| \le 100$ m/s.

Solution

(a) Use the expression for m to rewrite K.

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2 = m_0c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

If $x = -\frac{v^2}{c^2}$, then the Maclaurin series for $(1 + x)^{-1/2}$ is easiest to compute as a binomial series with $k = -\frac{1}{2}$. Notice that $|x| < 1$ because $v < c$.

Therefore, the binomial series is

$$(1 + x)^{-1/2} = 1 - \frac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \cdots$$
$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$$

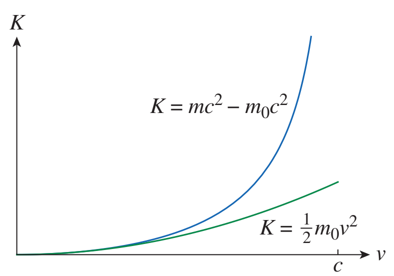


Figure 8.37
The upper curve is the expression for the kinetic energy K of an object with velocity v in special relativity. The lower curve shows the function used for K in classical Newtonian physics. When v is smaller than the speed of light, the curves are very similar.

Use this binomial series with $x = -v^2/c^2$ in the expression for K.

$$K = m_0c^2\left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) - 1\right]$$
$$= m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right)$$

If v is much smaller than c, then all terms after the first are very small when compared with the first term. Therefore, if we omit the terms after the first, then we get

$$K \approx m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2}\right) = \frac{1}{2}m_0v^2.$$

Figure 8.37 shows a graphical comparison of the two expressions for K.

(b) If $x = -\frac{v^2}{c^2}$, $f(x) = m_0c^2[(1 + x)^{-1/2} - 1]$, and M is a number such that $|f''(x)| \le M$, then we can use Taylor's Inequality to write

$$|R_1(x)| \le \frac{M}{2!}x^2.$$

We have $f''(x) = \frac{3}{4}m_0c^2(1 + x)^{-5/2}$ and we are given that $|v| \le 100$ m/s. Therefore,

$$|f''(x)| = \frac{3m_0c^2}{4(1 - v^2/c^2)^{5/2}} \leq \frac{3m_0c^2}{4(1 - 100^2/c^2)^{5/2}} \ (= M).$$

Use $c = 3 \times 10^8$ m/s:

$$|R_1(x)| \leq \frac{1}{2} \cdot \frac{3m_0c^2}{4(1 - 100^2/c^2)^{5/2}} \cdot \frac{100^4}{c^4} < (4.17 \times 10^{-10})m_0$$

Therefore, when $|v| \leq 100$ m/s, the magnitude of the error in using the Newtonian expression for kinetic energy is at most $(4.2 \times 10^{-10})m_0$. ■

Another application to physics occurs in optics. Figure 8.38 is adapted from *Optics*, 4th ed., by Eugene Hecht (San Francisco, 2002), page 153. It depicts a wave from the point source S meeting a spherical interface of radius R centered at C. The ray SA is refracted toward P.

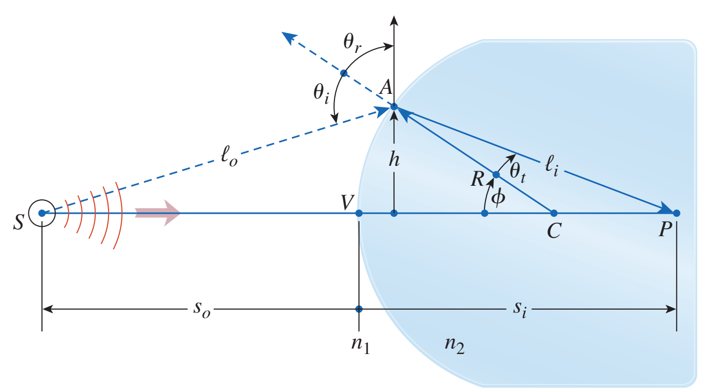


Figure 8.38
Refraction at a spherical interface.

Using Fermat's principle that light travels so as to minimize the time taken, Hecht derives the equation

$$\frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = \frac{1}{R}\left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o}\right) \tag{1}$$

Use the identity $\cos(\pi - \phi) = \cos \phi$.

where n_1 and n_2 are indexes of refraction and ℓ_o, ℓ_i, s_o, and s_i are the distances indicated in Figure 8.38. By the Law of Cosines, applied to triangles ACS and ACP, we have,

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\phi} \tag{2}$$
$$\ell_i = \sqrt{R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\phi}$$

Because Equation 1 is difficult to work with, Gauss, in 1841, simplified it by using the linear approximation $\cos\phi \approx 1$ for small values of ϕ. This corresponds to using the Taylor polynomial of degree 1. Then Equation 1 can be written in the following simpler manner:

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} \tag{3}$$

The resulting optical theory is known as *Gaussian optics*, or *first-order optics*, and has become the basic theoretical tool used to design lenses.

A more accurate theory is obtained by using the Taylor polynomial of degree 3 (which is the same as the Taylor polynomial of degree 2) to approximate $\cos\phi$. This accounts for rays in which ϕ is not so small, that is, rays that strike the surface at greater distances h above the axis. In Exercise 34(b), you are asked to use this approximation to derive the more accurate equation

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} + h^2\left[\frac{n_1}{2s_o}\left(\frac{1}{s_o} + \frac{1}{R}\right)^2 + \frac{n_2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)^2\right] \tag{4}$$

The resulting optical theory is known as *third-order optics.*

Other applications of Taylor polynomials to physics and engineering are explored in Exercises 35–38 and in the Applied Project on page 769.

8.8 Exercises

1. (a) Find the Taylor polynomials up to degree 5 for $f(x) = \sin x$ centered at $a = 0$. Graph f and these polynomials in the same viewing rectangle.

(b) Evaluate f and these polynomials at $x = \frac{\pi}{4}, \frac{\pi}{2}$, and π.

(c) Explain graphically how the Taylor polynomials converge to $f(x)$.

2. (a) Find the Taylor polynomials up to degree 3 for $f(x) = \tan x$ centered at $a = 0$. Graph f and these polynomials in the same viewing rectangle.

(b) Evaluate f and these polynomials at $x = \frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{3}$.

(c) Explain graphically how the Taylor polynomials converge to $f(x)$.

3. (a) Find the Taylor polynomials up to degree 3 for $f(x) = \frac{1}{x}$ centered at $a = 1$. Graph f and these polynomials in the same viewing rectangle.

(b) Evaluate f and these polynomials at $x = 0.9$ and 1.3.

(c) Explain graphically how the Taylor polynomials converge to $f(x)$.

Find the Taylor polynomial $T_3(x)$ for the function f centered at the number a. Graph f and T_3 in the same viewing rectangle.

4. $f(x) = e^x, \quad a = 1$

5. $f(x) = \sin x, \quad a = \frac{\pi}{6}$

6. $f(x) = \cos x, \quad a = \frac{\pi}{2}$

7. $f(x) = e^{-x}\sin x, \quad a = 0$

8. $f(x) = x + e^{-x}, \quad a = 0$

9. $f(x) = \ln x, \quad a = 1$

10. $f(x) = \frac{\ln x}{x}, \quad a = 1$

11. $f(x) = x\cos x, \quad a = 0$

12. $f(x) = xe^{-2x}, \quad a = 0$

13. $f(x) = \tan^{-1} x, \quad a = 1$

Use technology to find the Taylor polynomial T_n centered at a for $n = 2, 3, 4, 5$. Then graph these polynomials and f in the same viewing rectangle.

14. $f(x) = \cot x, \quad a = \frac{\pi}{4}$

15. $f(x) = \sqrt[3]{1 + x^2}, \quad a = 0$

For the function f:

(a) Approximate f by a Taylor polynomial with degree n at the number a.

(b) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

(c) Check your result in part (b) by graphing $y = |R_n(x)|$.

16. $f(x) = \frac{1}{x}, \quad a = 1, \quad n = 2, \quad 0.7 \le x \le 1.3$

17. $f(x) = x^{-1/2}, \quad a = 4, \quad n = 2, \quad 3.5 \le x \le 4.5$

18. $f(x) = x^{2/3}, \quad a = 1, \quad n = 3, \quad 0.8 \le x \le 1.2$

19. $f(x) = \sin x, \quad a = \frac{\pi}{6}, \quad n = 4, \quad 0 \le x \le \frac{\pi}{3}$

20. $f(x) = \sec x, \quad a = 0, \quad n = 2, \quad -0.2 \le x \le 0.2$

21. $f(x) = \ln(1 + 2x), \quad a = 1, \quad n = 3, \quad 0.5 \le x \le 1.5$

22. $f(x) = e^{x^2}, \quad a = 0, \quad n = 3, \quad 0 \le x \le 0.1$

23. $f(x) = x\ln x, \quad a = 1, \quad n = 3, \quad 0.5 \le x \le 1.5$

24. $f(x) = x\sin x, \quad a = 0, \quad n = 4, \quad -1 \le x \le 1$

25. Use the information from Exercise 6 to estimate $\cos\left(\frac{4\pi}{9}\right)$ correct to five decimal places.

26. Use the information from Exercise 19 to estimate $\sin\left(\frac{19\pi}{90}\right)$ correct to five decimal places.

27. Use Taylor's Inequality to determine the number of terms of the Maclaurin series for e^x that should be used to estimate $e^{0.1}$ to within 0.00001.

28. How many terms of the Maclaurin series for $\ln(1+x)$ are needed to estimate ln1.4 to within 0.001?

Use the Alternating Series Error Bound or Taylor's Inequality to estimate the range of values of x for which the given approximation is accurate to within the stated error. Check your answer graphically.

29. $\sin x \approx x - \frac{x^3}{6} \quad (|\text{error}| < 0.01)$

30. $\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24} \quad (|\text{error}| < 0.005)$

31. $\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5} \quad (|\text{error}| < 0.05)$

32. Suppose that

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$$

and the Taylor series of f centered at 4 converges to $f(x)$ for all x in the interval of convergence. Show that the fifth-degree Taylor polynomial approximates $f(5)$ with error less than 0.0002.

33. A car is moving with speed 20 m/s and acceleration 2 m/s^2 at a given instant. Using a second-degree Taylor polynomial, estimate how far the car moves in the next second. Would it be reasonable to use this polynomial to estimate the distance traveled during the next minute? Explain your reasoning.

34. (a) Derive Equation 3 for Gaussian optics from Equation 1 by approximating $\cos\ \phi$ in Equation 2 by its first-degree Taylor polynomial.

(b) Show that if $\cos\ \phi$ is replaced by its third-degree Taylor polynomial in Equation 2, then Equation 1 becomes Equation 4 for third-order optics.
Hint: Use the first two terms in the binomial series for ℓ_o^{-1} and ℓ_i^{-1}, and the fact that $\phi \approx \sin\ \phi$.

35. An electric dipole consists of two electric charges of equal magnitude and opposite sign. If the charges are q and $-q$ and are located at a distance d from each other, then the electric field E at the point P in the figure is

$$E = \frac{q}{D^2} - \frac{q}{(D+d)^2}$$

By expanding the repression for E as a series in powers of d/D, show that E is approximately proportional to $1/D^3$ where P is far away from the dipole.

36. The resistivity ρ of a conducting wire is the reciprocal of the conductivity and is measured in units of ohm-meters (Ω-m). The resistivity of a given metal depends on the temperature according to the equation

$$\rho(t) = \rho_{20} e^{\alpha(t-20)}$$

where t is the temperature in °C. There are tables that list the values of α (called the temperature coefficient) and ρ_{20} (the resistivity at 20°C) for various metals. Except at very low temperatures, the resistivity varies almost linearly with temperature and so it is common to approximate the expression for $\rho(t)$ by its first- or second-degree Taylor polynomial at $t = 20$.

(a) Find expressions for these linear and quadratic approximations.

(b) For copper, the table values are $\alpha = 0.0039/°\text{C}$ and $\rho_{20} = 1.7 \times 10^{-8}\ \Omega$-m. Graph the resistivity of copper and the linear and quadratic approximations for $-250°\text{C} \le t \le 1000°\text{C}$.

(c) For what values of t does the linear approximation agree with the exponential expression to within one percent?

37. If a surveyor measures differences in elevation when making plans for a highway across a desert, corrections must be made for the curvature of Earth.

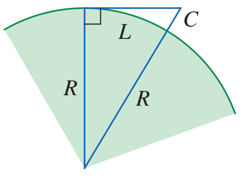
(a) If R is the radius of Earth and L is the length of the highway, show that the correction is

$$C = R\sec\left(\frac{L}{R}\right) - R$$

(b) Use a Taylor polynomial to show that

$$C \approx \frac{L^2}{2R} + \frac{5L^4}{24R^3}$$

(c) Compare the corrections given by the formulas in parts (a) and (b) for a highway that is 100 km long. (Use 6370 km as the radius of the earth.)



38. The period of a pendulum with length L that makes a maximum angle θ_0 with the vertical is

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity. (In Exercise 34, Section 5.9, we approximated this integral using Simpson's Rule.)

(a) Expand the integrand as a binomial series and use the result of Exercise 54, Section 5.6 to show that

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1^2}{2^2}k^2 + \frac{1^2 3^2}{2^2 4^2}k^4 + \frac{1^2 3^2 5^2}{2^2 4^2 6^2}k^6 + \cdots\right]$$

If θ_0 is not too large, the approximation $T \approx 2\pi\sqrt{L/g}$, obtained by using only the first term in the series, is often used. A better approximation is obtained by using two terms:

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{4}k^2\right)$$

(b) Notice that all the terms in the series after the first one have coefficients that are at most $1/4$. Use this fact to compare this series with a geometric series and show that

$$2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{4}k^2\right) \le T \le 2\pi\sqrt{\frac{L}{g}}\left(\frac{4 - 3k^2}{4 - 4k^2}\right)$$

(c) Use the inequalities in part (b) to estimate the period of a pendulum with $L = 1$ meter and $\theta_0 = 10°$. How does it compare the estimate $T \approx 2\pi\sqrt{L/g}$? What if $\theta_0 = 42°$?

39. In Section 4.7, we considered Newton's method for approximating a root r of the equation $f(x) = 0$, and from an initial approximation x_1 we obtained successive approximations $x_2, x_3, \ldots$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Use Taylor's Inequality with $n = 1$, $a = x_n$, and $x = r$ to show that if $f''(x)$ exists on an interval I containing r, x_n, and x_{n+1}, and $|f''(x)| \le M$, $|f'(x)| \ge K$ for all $x \in I$, then

$$|x_{n+1} - r| \le \frac{M}{2K}|x_n - r|^2$$

This means that if x_n is accurate to d decimal places, then x_{n+1} is accurate to about $2d$ decimal places. More precisely, if the error at stage n is at most 10^{-m}, then the error at stage $n + 1$ is at most $(M/2K)10^{-2m}$.

Applied Project | Radiation from the Stars

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Any object emits radiation when heated. A *blackbody* is a system that absorbs all the radiation that falls on it. For instance, a matte black surface or a large cavity with a small hole in its wall (like a blast furnace) is a blackbody and emits blackbody radiation. Even the radiation from the sun is close to being blackbody radiation.

Proposed in the late 19th century, the Rayleigh–Jeans Law expresses the energy density of blackbody radiation of wavelength λ as

$$f(\lambda) = \frac{8\pi kT}{\lambda^4}$$

where λ is measured in meters, T is the temperature in kelvins (K), and k is Boltzmann's constant. The Rayleigh–Jeans Law agrees with experimental measurements for long wavelengths but disagrees drastically for short wavelengths. The law predicts that $f(\lambda) \to \infty$ as $\lambda \to 0^+$ but experiments have shown that $f(\lambda) \to 0$. This fact is called the *ultraviolet catastrophe.*

In 1900, Max Planck found a better model (known now as Planck's Law) for blackbody radiation:

$$f(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1}$$

where λ is measured in meters, T is the temperature (in kelvins), and

h = Planck's constant $= 6.6262 \times 10^{-34}$ J · s

c = speed of light $= 2.997925 \times 10^8$ m/s

k = Boltzmann's constant $= 1.387 \times 10^{-23}$ J/K

1. Use l'Hospital's Rule to show that

$$\lim_{\lambda \to 0^+} f(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda \to \infty} f(\lambda) = 0$$

for Planck's Law. These results suggest that this law models blackbody radiation better than the Rayleigh–Jeans Law for short wavelengths.

2. Use a Taylor polynomial to show that, for large wavelengths, Planck's Law gives approximately the same values as the Rayleigh–Jeans Law.

3. Use technology to graph f as given by both laws in the same viewing rectangle. Comment on the similarities and differences in the graphs. Use $T = 5700$ K (the temperature of the sun). It may be practical to change from meters to the more convenient unit of micrometers: 1 μm $= 10^{-6}$m.

4. Use your graph in Problem 3 to estimate the value of λ for which $f(\lambda)$ is a maximum under Planck's Law.

5. Investigate how the graph of f changes as T varies. (Use Planck's Law.) In particular, graph f for the stars Betelgeuse ($T = 3400$ K), Procyon ($T = 6400$ K), and Sirius ($T = 9200$ K) as well as the sun. How does the total radiation emitted (the area under the curve) vary with T? Use the graph to comment on why Sirius is known as a blue star and Betelgeuse as a red star.

8 Review

Concepts and Vocabulary

1. (a) What is a convergent sequence?
(b) What is a convergent series?
(c) What does $\lim_{n\to\infty} a_n = 3$ mean?
(d) What does $\sum_{n=1}^{\infty} a_n = 3$ mean?

2. (a) What is a bounded sequence?
(b) What is a monotonic sequence?
(c) What can you conclude about a bounded monotonic sequence?

3. (a) What is a geometric series? Under what circumstances is it convergent? What is its sum?
(b) What is a p-series? Under what circumstances is it convergent?

4. Suppose $\Sigma a_n = 3$ and s_n is the nth partial sum of the series. What is $\lim_{n\to\infty} a_n$? What is $\lim_{n\to\infty} s_n$?

5. State the following.
(a) The Test for Divergence
(b) The Integral Test
(c) The Comparison Test
(d) The Limit Comparison Test
(e) The Alternating Series Test
(f) The Ratio Test

6. (a) What is an absolutely convergent series?
(b) What can you conclude about an absolutely convergent series?

7. (a) If a series is convergent by the Integral Test, how do you estimate its sum?
(b) If a series is convergent by the Comparison Test, how do you estimate its sum?
(c) If a series is convergent by the Alternating Series Test, how do you estimate its sum?

8. (a) Write the general form of a power series.
(b) What is the radius of convergence of a power series?
(c) What is the interval of convergence of a power series?

9. Suppose $f(x)$ is the sum of a power series with radius of convergence R.
(a) How do you differentiate f? What is the radius of convergence of the series for f'?
(b) How do you integrate f? What is the radius of convergence of the series for $\int f(x)\,dx$?

10. (a) Write an expression for the nth-degree Taylor polynomial of f centered at a.
(b) Write an expression for the Taylor series of f centered at a.
(c) Write an expression for the Maclaurin series of f.
(d) How do you show that $f(x)$ is equal to the sum of its Taylor series?
(e) State Taylor's Inequality.

11. Write the Maclaurin series and the interval of convergence for each of the following functions.
(a) $\dfrac{1}{1-x}$ (b) e^x (c) $\sin x$
(d) $\cos x$ (e) $\tan^{-1} x$ (f) $\ln(1+x)$

12. Write the binomial series expansion of $(1+x)^k$. What is the radius of convergence of this series?

True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. If $\lim_{n\to\infty} a_n = 0$, then Σa_n is convergent.

2. The series $\sum_{n=1}^{\infty} n^{-\sin 1}$ is convergent.

3. If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{2n+1} = L$.

4. If $\Sigma c_n 6^n$ is convergent, then $\Sigma c_n(-2)^n$ is convergent.

5. If $\Sigma c_n 6^n$ is convergent, then $\Sigma c_n(-6)^n$ is convergent.

6. If $\Sigma c_n x^n$ diverges when $x = 6$, then it diverges when $x = 10$.

7. The Ratio Test can be used to determine whether $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$ converges.

8. The Ratio Test can be used to determine whether $\sum_{n=1}^{\infty} \dfrac{1}{n!}$ converges.

9. If $0 \le a_n \le b_n$ and Σb_n diverges, then Σa_n diverges.

10. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$

11. If $-1 < \alpha < 1$, then $\lim_{n\to\infty} \alpha^n = 0$.

12. If Σa_n is divergent, then $\Sigma |a_n|$ is divergent.

13. If $f(x) = 2x - x^2 + \frac{1}{3}x^3 - \cdots$ converges for all x, then $f'''(0) = 2$.

14. If $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n + b_n\}$ is divergent.

15. If $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n b_n\}$ is divergent.

16. If $\{a_n\}$ is decreasing and $a_n > 0$ for all n, then $\{a_n\}$ is convergent.

17. If $a_n > 0$ and Σa_n converges, then $\Sigma(-1)^n a_n$ converges.

18. If $a_n > 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\lim_{n\to\infty} a_n = 0$.

19. $0.99999\ldots = 1$.

20. If $\lim_{n\to\infty} a_n = 2$, then $\lim_{n\to\infty} (a_{n+3} - a_n) = 0$.

21. If a finite number of terms are added to a convergent series, then the new series is still convergent.

22. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then $\sum_{n=1}^{\infty} a_n b_n = AB$.

Exercises

Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

1. $a_n = \dfrac{2 + n^3}{1 + 2n^3}$

2. $a_n = \dfrac{9^{n+1}}{10^n}$

3. $a_n = \dfrac{n^3}{1 + n^2}$

4. $a_n = \cos(n\pi/2)$

5. $a_n = \dfrac{n \sin n}{n^2 + 1}$

6. $a_n = \dfrac{\ln n}{\sqrt{n}}$

7. $\left\{\left(1 + \dfrac{3}{n}\right)^{4n}\right\}$

8. $\left\{\dfrac{(-10)^n}{n!}\right\}$

9. A sequence is defined recursively by the equations

$$a_1 = 1 \qquad a_{n+1} = \frac{1}{3}(a_n + 4)$$

Show that $\{a_n\}$ is increasing and $a_n < 2$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

Determine whether the series is convergent or divergent.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n + 1}}$

14. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

15. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n + 1}\right)$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos 3n}{1 + (1.2)^n}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1 + 2n^2)^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{5^n n!}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$

20. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n + 1}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n + 1} - \sqrt{n - 1}}{n}$

Find the sum of the series.

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n + 3)}$

24. $\displaystyle\sum_{n=1}^{\infty} [\tan^{-1}(n + 1) - \tan^{-1} n]$

25. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{3^{2n}(2n)!}$

26. $1 - e + \dfrac{e^2}{2!} - \dfrac{e^3}{3!} + \dfrac{e^4}{4!} - \cdots$

27. Express the repeating decimal $4.17326326326\ldots$ as a fraction.

28. For what values of x does the series $\sum_{n=1}^{\infty} (\ln x)^n$ converge?

29. Find the sum of the series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^5}$ correct to four decimal places.

30. Consider the series $\sum_{n=1}^{\infty} \dfrac{1}{n^6}$.

(a) Find the partial sum s_5 of the series and estimate the error in using it as an approximation to the sum of the series.
(b) Find the sum of this series correct to five decimal places.

31. Use the sum of the first eight terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} (2 + 5^n)^{-1}$$

Estimate the error involved in this approximation.

32. (a) Show that the series $\sum_{n=1}^{\infty} \dfrac{n^n}{(2n)!}$ is convergent.

(b) Deduce that $\lim_{n\to\infty} \dfrac{n^n}{(2n)!} = 0$.

33. Prove that if the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then the series

$$\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)a_n$$

is also absolutely convergent.

Find the radius of convergence and interval of convergence of the series.

34. $\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n^2 5^n}$

35. $\sum_{n=1}^{\infty}\frac{(x+2)^n}{n4^n}$

36. $\sum_{n=1}^{\infty}\frac{2^n(x-2)^n}{(n+2)!}$

37. $\sum_{n=0}^{\infty}\frac{2^n(x-3)^n}{\sqrt{n+3}}$

---

38. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}x^n$$

39. Find the Taylor series for $f(x) = \sin x$ centered at $a = \frac{\pi}{6}$.

40. Find the Taylor series for $f(x) = \cos x$ centered at $a = \frac{\pi}{3}$.

Find the Maclaurin series for f and its radius of convergence. Use either the direct method (definition of a Maclaurin series) or a known series such as geometric series, binomial series, or the Maclaurin series for e^x, $\sin x$, $\tan^{-1} x$, and $\ln(1 + x)$.

41. $f(x) = \dfrac{x^2}{1+x}$

42. $f(x) = \tan^{-1}(x^2)$

43. $f(x) = \ln(4 - x)$

44. $f(x) = xe^{2x}$

45. $f(x) = \sin(x^4)$

46. $f(x) = 10^x$

47. $f(x) = \dfrac{1}{\sqrt[4]{16-x}}$

48. $f(x) = (1 - 3x)^{-5}$

---

49. Evaluate $\int \frac{e^x}{x}\,dx$ as an infinite series.

50. Use series to approximate $\int_0^1 \sqrt{1+x^4}\,dx$ correct to two decimal places.

(a) Approximate f by a Taylor polynomial with degree n at the number a.

(b) Graph f and T_n in the same viewing rectangle.

(c) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

(d) Check your result in part (c) by graphing $y = |R_n(x)|$.

51. $f(x) = \sqrt{x}, \quad a = 1, \quad n = 3, \quad 0.9 \le x \le 1.1$

52. $f(x) = \sec x, \quad a = 0, \quad n = 2, \quad 0 \le x \le \frac{\pi}{6}$

---

53. The function f satisfies the inequality $|f^{(n)}(x)| \le 4$. Suppose the third-degree Taylor polynomial for $f(x)$ centered at $a = 2$ is used to approximate $f(3)$. Use Taylor's Inequality to estimate the accuracy of the approximation.

54. Let $T_3(x)$ be the third-degree Taylor polynomial for $\dfrac{x}{1+x}$ centered at $a = 1$. Find the coefficient on the term $(x - 1)^3$.

55. Use series to evaluate the limit

$$\lim_{x\to 0}\frac{\sin x - x}{x^3}$$

56. Suppose that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for all x.

(a) If f is an odd function, show that

$$c_0 = c_2 = c_4 = \cdots = 0$$

(b) If f is an even function, show that

$$c_1 = c_3 = c_5 = \cdots = 0$$

57. If $f(x) = e^{x^2}$, show that $f^{(2n)}(0) = \dfrac{(2n)!}{n!}$.

58. The force due to gravity on an object with mass m at a height h above the surface of Earth is

$$F = \frac{mgR^2}{(R+h)^2}$$

where R is the radius of Earth and g is the acceleration due to gravity.

(a) Express F as a series in powers of $\frac{h}{R}$.

(b) Observe that if we approximate F by the first term in the series, we get the expression $F \approx mg$ that is usually used when h is much smaller than R. Use the Alternating Series Error Bound to estimate the range of values of h for which the approximation $F \approx mg$ is accurate to within one percent. Use $R = 6400$ km.

Focus on Problem Solving

Example Pattern Recognition

Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!}$.

Solution

One of the most important problem-solving skills is *recognizing something familiar.* This series doesn't look exactly like a series we already know, but it does have some characteristics in common with the Maclaurin series for the exponential function.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Replace x by $x + 2$ in this series.

$$e^{x+2} = \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} = 1 + (x+2) + \frac{(x+2)^2}{2!} + \frac{(x+2)^3}{3!} + \frac{(x+2)^4}{4!} + \cdots$$

This is beginning to look a lot like the given series. However, the exponent in the numerator matches the number factorial in the denominator (unlike the given series).

Hold this series for e^{x+2} aside.

In the given series, multiply by 1 in a convenient form.

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} = \frac{(x+2)^3}{(x+2)^3} \cdot \sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!}$$

Multiply by 1 in a convenient form.

$$= \frac{1}{(x+2)^3} \sum_{n=0}^{\infty} \frac{(x+2)^{n+3}}{(n+3)!}$$

$(x + 2)^3$ passes freely through the summation symbol.

$$= (x+2)^{-3}\left[\frac{(x+2)^3}{3!} + \frac{(x+2)^4}{4!} + \cdots\right]$$

Consider the first few terms of the series.

The series in the brackets is the series for e^{x+2} with the first three terms missing. So,

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} = (x+2)^{-3}\left[e^{x+2} - 1 - (x+2) - \frac{(x+2)^2}{2!}\right].$$ ■

Problems

1. If $f(x) = \sin(x^3)$, find $f^{(15)}(0)$.

2. The function f is defined by

$$f(x) = \lim_{n \to \infty} \frac{x^{2n} - 1}{x^{2n} + 1}$$

Where is f continuous?

3. (a) Show that $\tan \frac{1}{2}x = \cot \frac{1}{2}x - 2 \cot x$.

(b) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n}$$

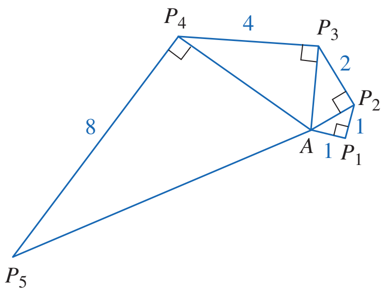


Figure 8.39
Sequence of points as described in Problem 4.

4. Let $\{P_n\}$ be a sequence of points determined as in Figure 8.39. Therefore, $|AP_1| = 1, |P_n P_{n+1}| = 2^{n-1}$, and angle $AP_n P_{n+1}$ is a right angle. Find $\lim_{n\to\infty} \angle P_n A P_{n+1}$.

5. To construct the **snowflake curve**, start with an equilateral triangle with sides of length 1. Step 1 in the construction is to divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part (see Figure 8.40). Step 2 is to repeat step 1 for each side of the resulting polygon. This process is repeated at each succeeding step. The snowflake curve is the curve that results from repeating this process indefinitely.

(a) Let s_n, l_n, and p_n represent the number of sides, the length of a side, and the total length of the nth approximating curve (the curve obtained after step n of the construction), respectively. Find formulas for s_n, l_n, and p_n.

(b) Show that $p_n \to \infty$ as $n \to \infty$.

(c) Sum an infinite series to find the area enclosed by the snowflake curve.

Note: Parts (b) and (c) show that the snowflake curve is infinitely long but encloses a finite area.

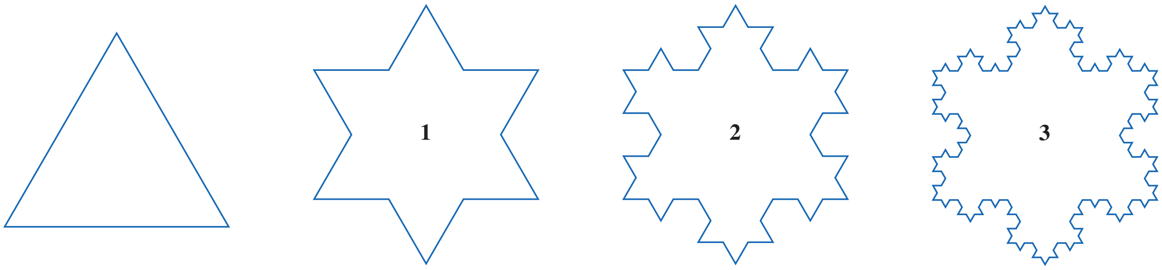


Figure 8.40
The first few steps in the construction of the snowflake curve.

6. Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

7. (a) Show that for $xy \neq 1$,

$$\arctan x - \arctan y = \arctan \frac{x - y}{1 + xy}$$

if the left side lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

(b) Show that $\arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4}$.

(c) Deduce the following formula of John Machin (1680–1751):

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

(d) Use the Maclaurin series for arctan to show that

$$0.1973955597 < \arctan \frac{1}{5} < 0.1973955616$$

(e) Show that

$$0.004184075 < \arctan \frac{1}{239} < 0.004184077$$

(f) Deduce that, correct to seven decimal places, $\pi \approx 3.1415927$.

Machin used this method in 1706 to find π correct to 100 decimal places. Recently, with the aid of computers, the value of π has been computed to increasingly greater accuracy. In 2019, Emma Haruka Iwao, a Google employee, computed the value of π to over 31 trillion digits!

8. (a) Prove a formula similar to the one in Problem 7(a) that involves the arccot instead of arctan.

(b) Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \operatorname{arccot}(n^2 + n + 1)$.

9. Use the result of Problem 7(a) to find the sum of the series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{2}{n^2}\right)$$

10. If $a_0 + a_1 + a_2 + \cdots + a_k = 0$, show that

$$\lim_{n\to\infty} (a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}) = 0$$

Hint: Try the special cases $k = 1$ and $k = 2$ first. Look for a pattern, and then try to prove the more general case.

11. Find the interval of convergence of

$$\sum_{n=1}^{\infty} n^3 x^n$$

and find its sum.

12. Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$.

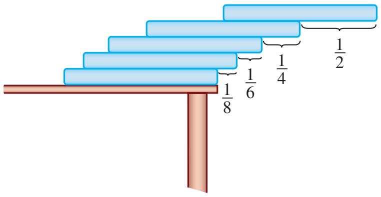


Figure 8.41
An illustration of the method of stacking.

13. Suppose you have a large supply of calculus books, all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the one beneath it. Show that it is possible to do this so that the top book extends entirely beyond the table. In fact, show that the top book can extend any distance at all beyond the edge of the table if the stack is high enough.

Use the following method of stacking: The top book extends half its length beyond the second book. The second book extends a quarter of its length beyond the third. The third extends one-sixth of its length beyond the fourth, and so on. See Figure 8.41. You might try this stacking technique yourself with a deck of cards. Consider centers of mass.

14. Let

$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots$$
$$v = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots$$
$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots$$

Show that $u^3 + v^3 + w^3 - 3uvw = 1$.

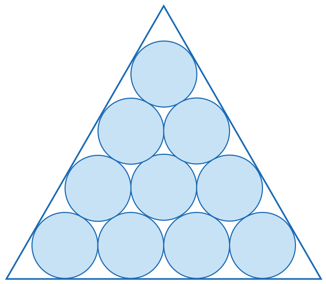


Figure 8.42
Circles packed in an equilateral triangle.

15. If $p > 1$, evaluate the expression

$$\frac{1 + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \dfrac{1}{4^p} + \cdots}{1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \cdots}$$

16. Suppose that circles of equal diameter are packed tightly in n rows inside an equilateral triangle. For example, Figure 8.42 illustrates the case $n = 4$. If A is the area of the triangle and A_n is the total area occupied by the n rows of circles, show that

$$\lim_{n\to\infty} \frac{A_n}{A} = \frac{\pi}{2\sqrt{3}}$$

17. A sequence $\{a_n\}$ is defined recursively by the equations

$$a_0 = a_1 = 1 \qquad n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}$$

Find the sum of the series $\sum_{n=0}^{\infty} a_n$.

18. If the region bounded above by the graph of $y = e^{-x/10}\sin x$, below by the x-axis, for $x \geq 0$ is rotated about the x-axis, the resulting solid looks like an infinite decreasing string of beads. See Figure 8.43.

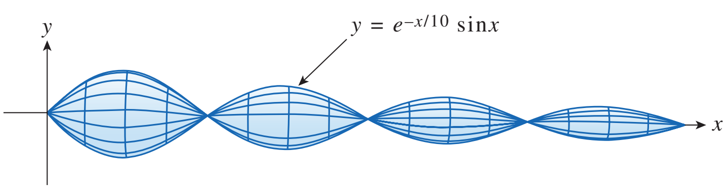


Figure 8.43
The resulting solid of revolution.

(a) Find the exact volume of the nth bead.
(b) Find the total volume of the beads, that is, the total volume of the solid.

19. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$.

20. Starting with the vertices $P_1(0, 1)$, $P_2(1, 1)$, $P_3(1, 0)$, $P_4(0, 0)$ of a square, construct additional points as shown in Figure 8.44: P_5 is the midpoint of $P_1 P_2$, P_6 is the midpoint of $P_2 P_3$, P_7 is the midpoint of $P_3 P_4$, and so on. The polygonal spiral path $P_1 P_2 P_3 P_4 P_5 P_6 P_7 \ldots$ approaches a point P inside the square.

(a) If the coordinates of P_n are $(x_n\ y_n)$, show that

$$\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$$

and find a similar equation for the y-coordinates.

(b) Find the coordinates of P.

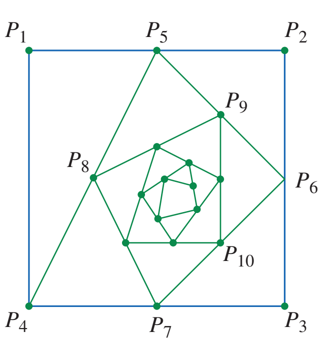


Figure 8.44
The polygonal spiral path.

21. Follow the steps outlined to show that

$$\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} + \cdots = \ln 2$$

(a) Use the formula for the sum of a finite geometric series to write an expression for

$$1 - x + x^2 - x^3 + \cdots + x^{2n-2} - x^{2n-1}$$

(b) Integrate the result obtained in part (a) from 0 to 1 to find an expression for

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

as an integral.

(c) Use part (b) to show

$$\left| \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2n-1)(2n)} - \int_0^1 \frac{dx}{1+x} \right| < \int_0^1 x^{2n} dx$$

(d) Finally, use part (c) to show that the sum of the given series is ln 2.

22. Find all the solutions of the equation

$$1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots = 0$$

Hint: Consider the cases $x \geq 0$ and $x < 0$ separately.

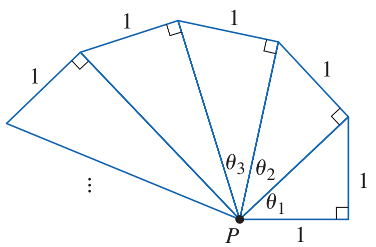


Figure 8.45
Right-angled triangles construction.

23. Right-angled triangles are constructed as in Figure 8.45. Each triangle has height 1 and its base is the hypotenuse of the preceding triangle. Show that this sequence of triangles makes indefinitely many turns around P by showing that $\Sigma\theta_n$ is a divergent series.

24. Consider the series whose terms are the reciprocals of the positive integers that can be written in base 10 notation without using the digit 0. Show that this series is convergent and the sum is less than 90.

25. (a) Show that the Maclaurin series of the function

$$f(x) = \frac{x}{1 - x - x^2} \quad \text{is} \quad \sum_{n=1}^{\infty} f_n x^n$$

where f_n is the nth Fibonacci number, that is, $f_1 = 1, f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.

Hint: Write

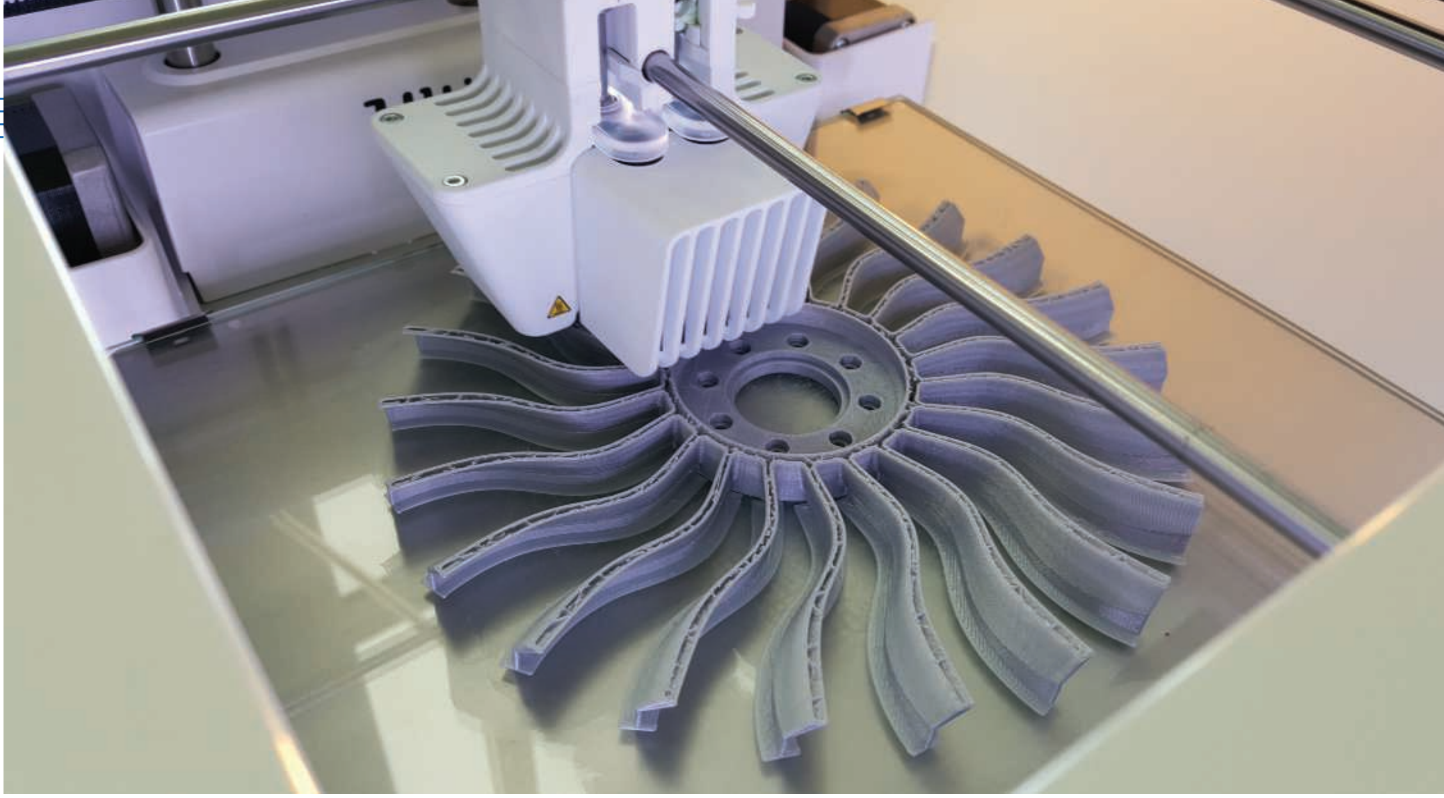
$$\frac{x}{1 - x - x^2} = c_0 + c_1 x + c_2 x^2 + \cdots$$

and multiply both sides of this equation by $1 - x - x^2$.

(b) Write $f(x)$ as a sum of partial fractions and obtain the Maclaurin series in a different way. Use this expression to find an explicit formula for the nth Fibonacci number.

26. Prove that if $n > 1$, the nth partial sum of the harmonic series is not an integer.

Hint: Let 2^k be the largest power of 2 that is less than or equal to n and let M be the product of all odd integers that are less than or equal to n. Suppose that $s_n = m$, an integer. Then $M2^k s_n = M2^k m$. The right side of this equation is even. Prove that the left side is odd by showing that each of its terms is an even integer, except for the last one.



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3D printing has become very popular recently as the technology has improved and materials have become less expensive. 3D printing applications are used in manufacturing, especially in the quick development and evaluation of prototypes, medicine where bioprinting involves 3D printing of artificial organs, and construction for extrusion, power bonding, and additive welding.

The Cartesian 3D printer is the most common design, usually with one electrical motor for each axis. A Polar 3D printer uses one linear dimension and two angles to correctly position an object. Each printer is based on a different method for representing a point in space.

9 Vectors and the Geometry of Space

Contents

In this chapter, we will introduce vectors and coordinate systems for three-dimensional space. This is the background necessary for the study of functions of two variables because the graph of such a function is a surface in space. Vectors provide alternate, simpler descriptions of lines and planes in space as well as the velocity and acceleration of an object as it moves in space.

9.1 Three-Dimensional Coordinate Systems

Location in Space

To locate a point in a plane, two numbers are necessary. We know that any point in the plane can be represented as an ordered pair (a, b) of real numbers, where a is the x-coordinate and b is the y-coordinate. For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required. We represent any point in space by an ordered triple (a, b, c) of real numbers.

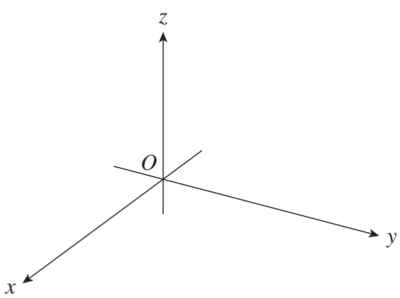


Figure 9.1
The coordinate axes.

In order to represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the **coordinate axes**, and labeled as the x-axis, y-axis, and the z-axis. Usually, we identify the x- and y-axes as being horizontal and the z-axis as being vertical, and we typically draw the orientation of the axes as in Figure 9.1.

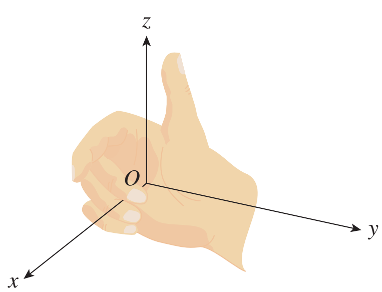


Figure 9.2
Illustration of the right-hand rule.

The direction of the z-axis is determined by the **right-hand rule** as illustrated in Figure 9.2: if a person curls the fingers of their right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then their thumb points in the positive direction of the z-axis.

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 9.3. The xy-plane is the plane that contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes. These three coordinate planes divide space into eight parts, called **octants**. The **first octant**, in the foreground, is determined by the positive axes.

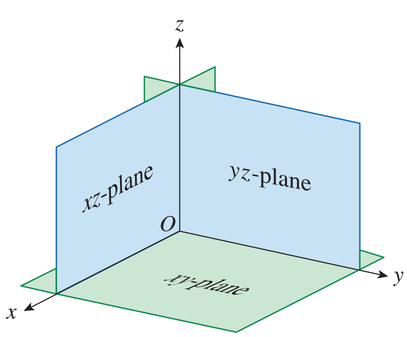


Figure 9.3
The coordinate planes.

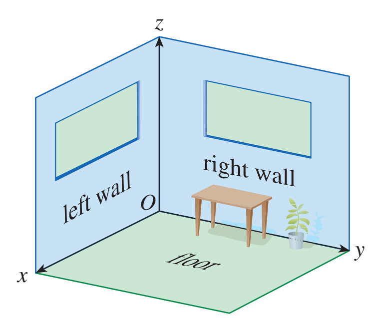


Figure 9.4
Room-wall analogy for coordinate planes.

It may be helpful to visualize the coordinate planes and three-dimensional figures in general by using the following analogy, as illustrated in Figure 9.4. Look at any bottom corner of a room and let this corner be the origin. The wall on your left is in the xz-plane, the wall on your right is in the yz-plane, and the floor is in the xy-plane. The x-axis runs along the intersection of the floor and the left wall. The y-axis runs along the intersection of the floor and the right wall. The z-axis runs up from the floor toward the ceiling along the intersection of the two walls. You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point O.

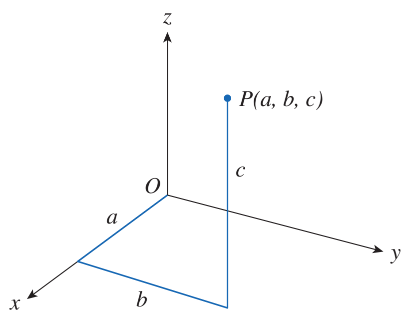


Figure 9.5
Location of the point $P(a, b, c)$ in space.

Now, consider a point P in space. Let a be the (directed) distance from the yz-plane to P, let b be the distance from the xz-plane to P, and let c be the distance from the xy-plane to P. We represent the point P by the ordered triple (a, b, c) of real numbers and the numbers a, b, and c are the **coordinates** of P; a is the x-coordinate, b is the y-coordinate, and c is the z-coordinate. Therefore, to locate the point (a, b, c), we can start at the origin O and move a units along the x-axis, then b units parallel to the y-axis, and then c units parallel to the z-axis, as illustrated in Figure 9.5.

The point $P(a, b, c)$ determines a rectangular box as shown in Figure 9.6. If we draw a perpendicular line from P to the xy-plane, we obtain a point Q with coordinates $(a, b, 0)$ called the **projection** of P on the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P on the yz-plane and xz-plane, respectively.

Figures 9.7 and 9.8 illustrate the method for plotting points in space.

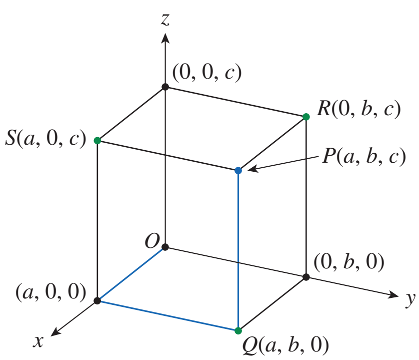


Figure 9.6
The projections of P onto the coordinate planes.

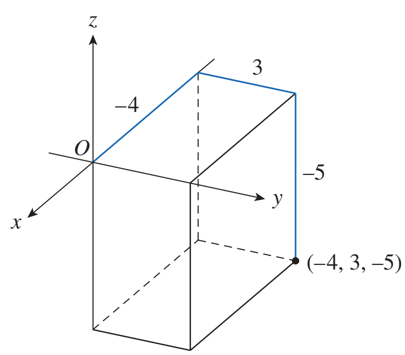


Figure 9.7
The point $(-4, 3, -5)$ plotted in space.

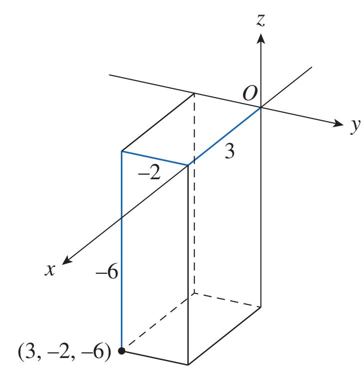


Figure 9.8
The point $(3, -2, -6)$ plotted in space.

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. There is a one-to-one correspondence between points P in space and ordered triples (a, b, c) in $\mathbb{R}^3$. This is called a **three-dimensional rectangular coordinate system**. Note that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

Graphs in Space

In two-dimensional analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In three-dimensional analytic geometry, an equation in x, y, and z represents a *surface* in $\mathbb{R}^3$.

Example 1 Graphing Equations

Graph and describe the surfaces in $\mathbb{R}^3$ represented by the following equations.
(a) $z = 3$ (b) $y = 5$

Solution

(a) The equation $z = 3$ represents the set $\{(x, y, z) \mid z = 3\}$.

This is the set of all points in $\mathbb{R}^3$ whose z-coordinate is 3.

Therefore, this equation represents a horizontal plane that is parallel to the xy-plane and three units above it, as illustrated in Figure 9.9.

(b) The equation $y = 5$ represents the set of all points in $\mathbb{R}^3$ whose y-coordinate is 5.

This is a vertical plane that is parallel to the xz-plane and five units to the right of it, as illustrated in Figure 9.10.

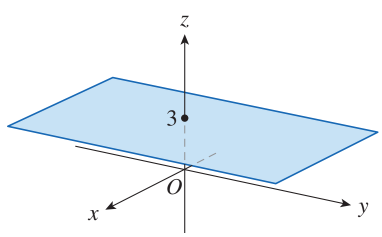


Figure 9.9
Graph of the plane $z = 3$ in $\mathbb{R}^3$.

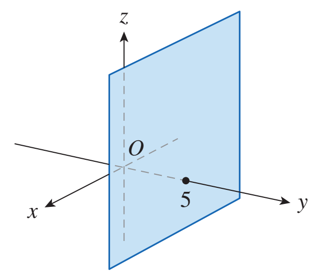


Figure 9.10
Graph of the plane $y = 5$ in $\mathbb{R}^3$.

A Closer Look

1. The context in which an equation is given determines whether it represents a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$. In Example 1, $y = 5$ represents a plane in $\mathbb{R}^3$, but $y = 5$ can also represent a line in $\mathbb{R}^2$ if the context is two-dimensional analytic geometry.
2. In general, if k is a constant, then $x = k$ represents a plane parallel to the yz-plane, $y = k$ is a plane parallel to the xz-plane, and $z = k$ is a plane parallel to the xy-plane. For example, in Figure 9.6, the faces of the rectangular box are formed by the three coordinate planes $x = 0$ (the yz-plane), $y = 0$ (the xz-plane), and $z = 0$ (the xy-plane), and the planes $x = a$, $y = b$, and $z = c$.

Example 2 Describe Regions Represented by Equations

(a) Describe the points (x, y, z) that simultaneously satisfy the equations

$$x^2 + y^2 = 1 \qquad \text{and} \qquad z = 3$$

(b) Describe the surface in $\mathbb{R}^3$ represented by the equation $x^2 + y^2 = 1$.

Solution

(a) The equation $z = 3$ represents a horizontal plane.

Because $x^2 + y^2 = 1$, the points lie on a circle with radius 1 and center on the z-axis.

Therefore, the points that satisfy both equations lie on a circle with radius 1 in the horizontal plane $z = 3$. See Figure 9.11.

(b) The equation $x^2 + y^2 = 1$ has no restrictions on z, so the point (x, y, z) could lie on a circle in any horizontal plane $z = k$.

Therefore, the surface $x^2 + y^2 = 1$ in $\mathbb{R}^3$ consists of all possible horizontal circles $x^2 + y^2 = 1, z = k$.

So, the surface is an infinite circular cylinder with radius 1 whose axis is the z-axis. See Figure 9.12.

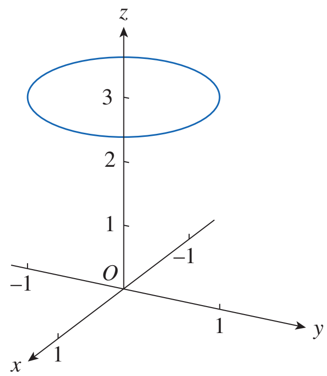


Figure 9.11
The circle $x^2 + y^2 = 1$, $z = 3$.

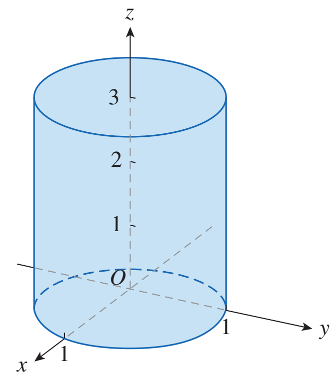


Figure 9.12
The cylinder $x^2 + y^2 = 1$, which extends infinitely far both up and down.

Example 3 A Plane Region

Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $y = x$.

Solution

The equation represents the set of all points in $\mathbb{R}^3$ whose x- and y-coordinates are equal, that is, $\{(x, x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$.

This is a vertical plane that intersects the xy-plane in the line $y = x$, $z = 0$.

The portion of this plane that lies in the first octant is shown in Figure 9.13.

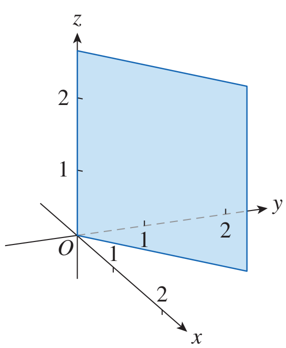


Figure 9.13
Graph of the plane $y = x$.

Distance in Space

The familiar formula for the distance between two points in a plane can be extended to the following three-dimensional formula.

Distance Formula in Three Dimensions

The distance $|P_1 P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

To understand why this formula is true, consider a rectangular box as illustrated in Figure 9.14. The points P_1 and P_2 are opposite vertices and the faces of the box are parallel to the coordinate planes.

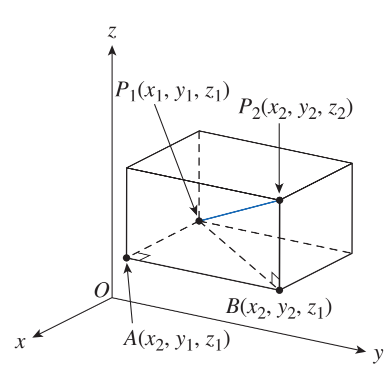


Figure 9.14
Rectangular box to illustrate the distance formula.

If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are the vertices of the box as indicated in the figure, then

$$|P_1A| = |x_2 - x_1| \qquad |AB| = |y_2 - y_1| \qquad |BP_2| = |z_2 - z_1|$$

The triangles P_1BP_2 and P_1AB are right triangles. Use the Pythagorean Theorem twice:

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2 \quad \text{and} \quad |P_1B|^2 = |P_1A|^2 + |AB|^2$$

Combine these two equations.

$$\begin{aligned}|P_1P_2|^2 &= (|P_1A|^2 + |AB|^2) + |BP_2|^2\\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2\\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\end{aligned}$$

Therefore, $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Example 4 Distance Between Two Points

The distance between the point $P(2, -1, 7)$ and the point $Q(1, -3, 5)$ is

$$|PQ| = \sqrt{(1-2)^2 + (-3+1)^2 + (5-7)^2} = \sqrt{1 + 4 + 4} = 3.$$ ■

Example 5 Use Distance to Define a Sphere

Find an equation of a sphere with radius r and center $C(h, k, l)$.

Solution

By definition, a sphere is the set of all points $P(x, y, z)$ whose distance from C is r. See Figure 9.15.

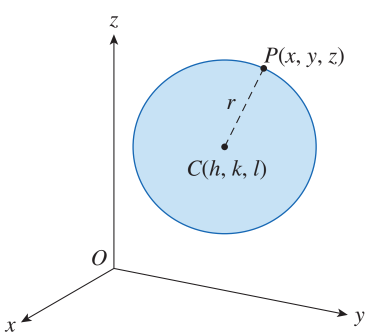


Figure 9.15
Sphere centered at $C(h, k, l)$ with radius r.

Therefore, P is on the sphere if and only if $|PC| = r$.

Write out the expression for the distance from P to C, and then square both sides of this equation.

$|PC| = r$

$\sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2} = r$ Distance from P to C.

$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$ Square both sides.

■

The equation in Example 5 is used often and is highlighted here.

Equation of a Sphere

An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

In particular, if the center of the sphere is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

Example 6 Find the Center and Radius of a Sphere

Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

Solution

Rewrite the given equation by completing the square associated with each variable.

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) = -6 + 4 + 9 + 1$$ Complete the square for each variable.

$$(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8$$ Write each expression as a square.

Compare this equation with the standard form of the equation of a sphere.

The resulting equation represents a sphere with center $(-2, 3, -1)$ and radius $\sqrt{8} = 2\sqrt{2}$. ■

Example 7 Identify a Region

Describe the region in $\mathbb{R}^3$ represented by the following inequalities.

$$1 \le x^2 + y^2 + z^2 \le 4 \qquad z \le 0$$

Solution

Rewrite the equation $1 \le x^2 + y^2 + z^2 \le 4$.

$$1 \le \sqrt{x^2 + y^2 + z^2} \le 2$$ Take the square root of each part of the inequality.

Therefore, points (x, y, z) that satisfy this inequality have distance from the origin of at least 1 and at most 2.

The additional inequality, $z \le 0$, means that the points must lie on or below the xy-plane.

Therefore, the given inequalities represent the region that lies between (or on) the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and beneath (or on) the xy-plane. Figure 9.16 shows this region. ■

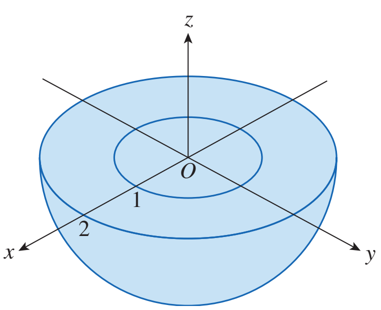


Figure 9.16
The region in $\mathbb{R}^3$ described by the inequalities $1 \le x^2 + y^2 + z^2 \le 4, z \le 0$.

9.1 Exercises

1. Find the coordinates of a point that is obtained by starting at the origin, moving along the x-axis 4 units in the positive direction, and then moving downward 3 units.

2. Plot the points $(0, 5, 2)$, $(4, 0, -1)$, $(2, 4, 6)$, and $(1, -1, 2)$ on the same set of coordinate axes.

3. Which of the points $P(6, 2, 3)$, $Q(-5, -1, 4)$, and $R(0, 3, 8)$ is closest to the xz-plane? Which point lies in the yz-plane?

4. Find the projections of the point $(2, 3, 5)$ on the xy-, yz-, and xz-planes. Draw a rectangular box with the origin and $(2, 3, 5)$ as opposite vertices and with its faces parallel to the coordinate planes. Label all vertices of the box. Find the length of the diagonal of the box.

5. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $x + y = 2$.

6. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $x^2 + z^2 = 9$.

7. (a) Describe the curve in $\mathbb{R}^2$ represented by the equation $x = 4$. Describe the surface in $\mathbb{R}^3$ represented by the equation $x = 4$. Sketch both the curve and the surface.
(b) Describe the surface in $\mathbb{R}^3$ represented by $y = 3$. Describe the surface in $\mathbb{R}^3$ represented by $z = 5$. Describe the set of points (x, y, z) such that $y = 3$ and $z = 5$, that is, the set of points where the two surfaces intersect. Sketch the set of intersection points.

8. Find the lengths of the sides of the triangle PQR. Determine if PQR is a right triangle, an isosceles triangle, or neither.
(a) $P(3, -2, -3)$, $Q(7, 0, 1)$, $R(1, 2, 1)$
(b) $P(2, -1, 0)$, $Q(4, 1, 1)$, $R(4, -5, 4)$

9. Find the distance from $(3, 7, -5)$ to each of the following.
(a) The xy-plane (b) The yz-plane
(c) The xz-plane (d) The x-axis
(e) The y-axis (f) The z-axis

10. Determine whether the points lie on a straight line.
(a) $A(2, 4, 2)$, $B(3, 7, -2)$, $C(1, 3, 3)$
(b) $D(0, -5, 5)$, $E(1, -2, 4)$, $F(3, 4, 2)$

11. Find an equation of the sphere with center $(2, -6, 4)$ and radius 5. Describe its intersection with each of the coordinate planes.

12. Find an equation of the sphere that passes through the point $(4, 3, -1)$ and has center $(3, 8, 1)$.

13. Find an equation of the sphere that passes through the origin and whose center is $(1, 2, 3)$.

Show that the equation represents a sphere, and find its center and radius.

14. $x^2 + y^2 + z^2 - 6x + 4y - 2z = 11$

15. $x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0$

16. $2x^2 + 2y^2 + 2z^2 = 8x - 24z + 1$

17. $3x^2 + 3y^2 + 3z^2 = 10 + 6y + 12z$

18. $x^2 - 10x + y^2 - 4y + z^2 - 2\sqrt{3}z + 32 = 0$

19. $x^2 + x + y^2 + \dfrac{22y}{7} + z^2 + \dfrac{337}{196} = 0$

20. (a) Prove that the midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

(b) Find the lengths of the medians of the triangle with vertices $A(1, 2, 3)$, $B(-2, 0, 5)$, and $C(4, 1, 5)$.

21. Find an equation of a sphere if one of its diameters has endpoints $(2, 1, 4)$ and $(4, 3, 10)$.

22. Find equations of the spheres with center $(2, -3, 6)$ that touch (a) the xy-plane, (b) the yz-plane, (c) the xz-plane.

23. Find an equation of the largest sphere with center $(5, 4, 9)$ that is contained in the first octant.

Describe the region in $\mathbb{R}^3$ represented by the equations or inequalities.

24. $x = 5$

25. $y = -2$

26. $y < 8$

27. $x \geq -3$

28. $0 \leq z \leq 6$

29. $z^2 = 1$

30. $x^2 + y^2 = 4, z = -1$

31. $y^2 + z^2 = 16$

32. $x^2 + y^2 + z^2 \leq 3$

33. $x = z$

34. $x^2 + z^2 \leq 9$

35. $x^2 + y^2 + z^2 > 2z$

36. $y = \sqrt{4 - x^2}$

37. $z = x^2$

Write inequalities to describe the region.

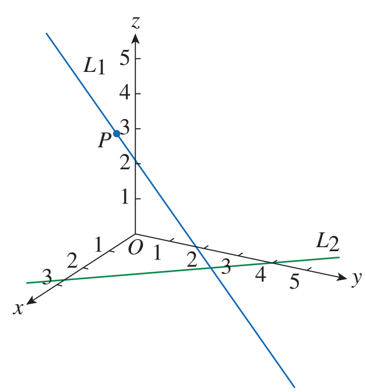
38. The region between the yz-plane and the vertical plane $x = 5$.

39. The solid cylinder that lies on or below the plane $z = 8$ and on or above the disk in the xy-plane with center at the origin and radius 2.

40. The region consisting of all points between (but not on) the spheres of radius r and R centered at the origin, where $r < R$.

41. The solid upper hemisphere of the sphere of radius 2 centered at the origin.

42. The figure shows a line L_1 in space and a second line L_2, which is the projection of L_1 on the xy-plane. That is, the points on L_2 are directly beneath or above, the points on L_1.



(a) Use the graph to find the coordinates of the point P on the line L_1.
(b) Locate on the diagram the points A, B, and C, where the line L_1 intersects the xy-plane, the yz-plane, and the xz-plane, respectively.

43. Consider the points P such that the distance from P to $A(-1, 5, 3)$ is twice the distance from P to $B(6, 2, -2)$. Show that the set of all such points is a sphere, and find its center and radius.

44. Find an equation of the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$. Describe the set.

45. Find the volume of the solid that lies inside both spheres

$$x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$$

and

$$x^2 + y^2 + z^2 = 4$$

46. Find the distance between the spheres represented by $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 4x + 4y + 4z - 11$.

47. Describe and sketch a solid with the following properties. When illuminated by rays parallel to the z-axis, its shadow is a circular disk. If the rays are parallel to the y-axis, its shadow is a square. If the rays are parallel to the x-axis, its shadow is an isosceles triangle.

9.2 Vectors

Definitions and Notation

The term **vector** is used to indicate a quantity that has both magnitude and direction, for example, displacement, velocity, or force. A vector is often graphically represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector, and the arrow points in the direction of the vector. We usually denote a vector by using a boldface letter, for example, **v**, or by placing an arrow above the letter, for example, $\vec{v}$.

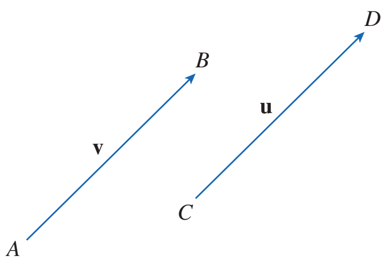


Figure 9.17
The vectors **u** and **v** are equivalent.

Suppose a particle moves along a line segment from point A to point B. The corresponding **displacement vector v**, shown in Figure 9.17, has **initial point** A (the tail) and **terminal point** B (the tip). We indicate these designations by writing $\mathbf{v} = \overrightarrow{AB}$.

In Figure 9.17, the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as v even though it is in a different position. We say that **u** and **v** are **equivalent** (or **equal**) and we write $\mathbf{u} = \mathbf{v}$. The **zero vector**, denoted **0**, has length 0. It is the only vector with no specific direction.

Combining Vectors

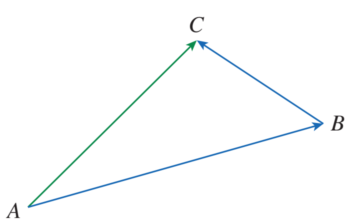


Figure 9.18
The vector $\overrightarrow{AC}$ is the sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

Suppose a particle moves from A to B, so its displacement vector is $\overrightarrow{AB}$. Then the particle changes direction and moves from B to C, with displacement vector $\overrightarrow{BC}$ as illustrated in Figure 9.18. The combined effect of these displacements is that the particle has moved from A to C. The resulting displacement vector $\overrightarrow{AC}$ is called the *sum* of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

In general, to define the sum of vectors **u** and **v**, graphically, move **v** so that its tail coincides with the tip of **u**. Here is the formal definition.

> **Definition • Vector Addition**
>
> If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u + v** is the vector from the initial point of **u** to the terminal point of **v**.

The definition of vector addition is illustrated in Figure 9.19. This figure shows why this definition is sometimes called the **Triangle Law**.

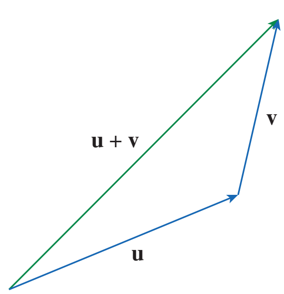


Figure 9.19
Illustration of vector addition: the Triangle Law.

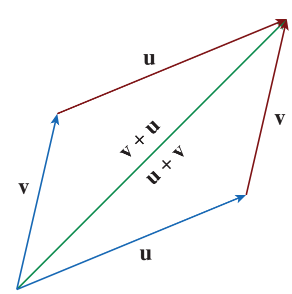


Figure 9.20
Another way to construct the sum: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$; the Parallelogram Law.

In Figure 9.20, suppose we start with the same vectors **u** and **v** as in Figure 9.19. Draw another copy of **v** with the same initial point as **u**. Complete the parallelogram; this shows graphically that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. This also suggests another way to construct the sum: place **u** and **v** so they start at the same point. Then $\mathbf{u} + \mathbf{v}$ lies along the diagonal of the parallelogram with **u** and **v** as sides. This is called the **Parallelogram Law**.

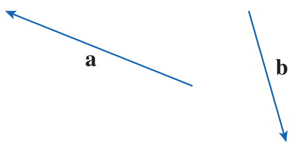


Figure 9.21
Vectors in Example 1.

Example 1 Sketch a Vector Sum

Draw the sum of the vectors **a** and **b** shown in Figure 9.21.

Solution

Translate, or move, **b** and place its tail at the tip of **a**. This copy of **b** must have the same length and direction.

Draw the vector $\mathbf{a} + \mathbf{b}$ as in Figure 9.22, starting at the initial point of **a** and ending at the terminal point of the copy of **b**.

Alternatively, we could use the Parallelogram Law.

Place **b** so that it starts where **a** starts and then $\mathbf{a} + \mathbf{b}$ is the diagonal of the parallelogram, as shown in Figure 9.23.

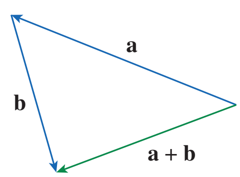


Figure 9.22
The sum $\mathbf{a} + \mathbf{b}$ using the Triangle Law.

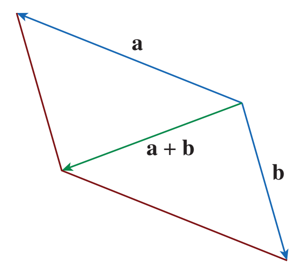


Figure 9.23
The sum $\mathbf{a} + \mathbf{b}$ using the Parallelogram Law.

We can also multiply a vector by a real number c, and in this context c is called a **scalar** to distinguish it from a vector. For example, it seems reasonable that $2\mathbf{v}$ should be the same as $\mathbf{v} + \mathbf{v}$, which has the same direction as **v** but is twice as long. Here is the general definition for scalar multiplication.

Definition • Scalar Multiplication

If c is a scalar and **v** is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite to **v** if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

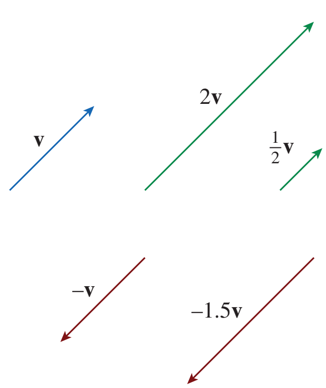


Figure 9.24
Examples of scalar multiples of **v**.

This definition is illustrated in Figure 9.24. In this case, real numbers work like scaling factors on vectors; that's why we call them scalars. Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another. In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$ has the same length as **v** but points in the opposite direction. This vector is called the **negative** of **v**.

The **difference** $\mathbf{u} - \mathbf{v}$ of two vectors can be defined using addition and scalar multiplication.

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

Here's how we can visualize the difference $\mathbf{u} - \mathbf{v}$. Draw the negative of $\mathbf{v}$, $-\mathbf{v}$, and then add it to $\mathbf{u}$ by the Parallelogram Law as illustrated in Figure 9.25. Another way to picture this difference is to write $\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$. This means the vector $\mathbf{u} - \mathbf{v}$, when added to $\mathbf{v}$, results in the vector $\mathbf{u}$. So, we could construct $\mathbf{u} - \mathbf{v}$ using the Triangle Law, as illustrated in Figure 9.26.

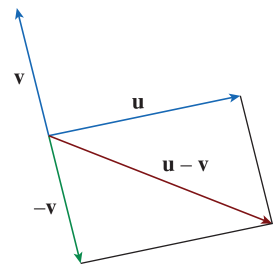


Figure 9.25
The difference $\mathbf{u} - \mathbf{v}$ using the Parallelogram Law.

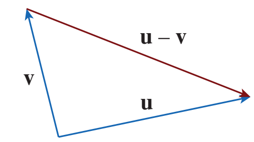


Figure 9.26
The difference $\mathbf{u} - \mathbf{v}$ using the Triangle Law.

Figure 9.27
Vectors in Example 2.

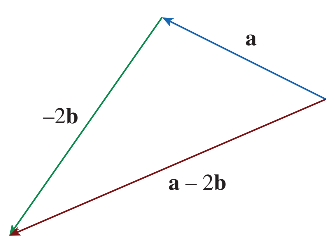


Figure 9.28
The vector $\mathbf{a} - 2\mathbf{b}$.

Example 2 Sketch a Vector Difference

Draw the vector $\mathbf{a} - 2\mathbf{b}$ if $\mathbf{a}$ and $\mathbf{b}$ are the vectors shown in Figure 9.27.

Solution

Draw the vector $-2\mathbf{b}$ pointing in the direction opposite to $\mathbf{b}$ and twice as long.

Place the tail of the vector $-2\mathbf{b}$ at the tip of $\mathbf{a}$ and then use the Triangle Law to draw $\mathbf{a} + (-2\mathbf{b})$ as illustrated in Figure 9.28. ■

■ Vector Components

For many applications, it is best to introduce a coordinate system and treat vectors algebraically. If we place the initial point of a vector $\mathbf{a}$ at the origin of a rectangular coordinate system, then the terminal point of $\mathbf{a}$ has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether the relevant coordinate system is two- or three-dimensional. These coordinates are called the **components** of $\mathbf{a}$ and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation $\langle a_1, a_2 \rangle$ for the ordered pair that refers to a vector, or directed line segment, so that it is not confused with the ordered pair (a_1, a_2), which refers to a point in the plane. See Figures 9.29 and 9.30.

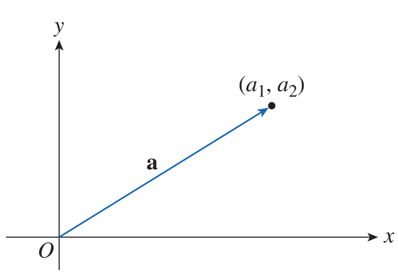


Figure 9.29
A visualization of the vector $\mathbf{a} = \langle a_1, a_2 \rangle$ in a two-dimensional rectangular coordinate system.

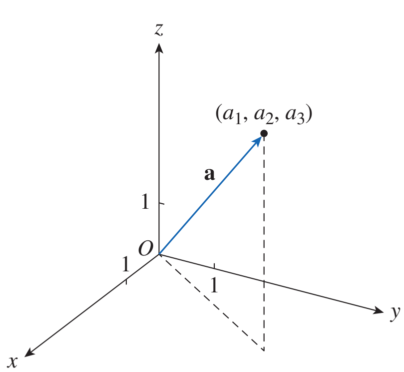


Figure 9.30
A visualization of the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ in a three-dimensional rectangular coordinate system.

For example, the vectors shown in Figure 9.31 are all equivalent to the vector $\overrightarrow{OP} = \langle 3, 2 \rangle$, whose terminal point is $P(3, 2)$. For each of these vectors, the terminal point is reached from the initial point by a displacement of three units to the right and two units upward. We can think of these geometric vectors as **representations** of the algebraic vector $\mathbf{a} = \langle 3, 2 \rangle$. The particular representation $\overrightarrow{OP}$ from the origin to the point $P(3, 2)$ is called the **position vector** of the point P.

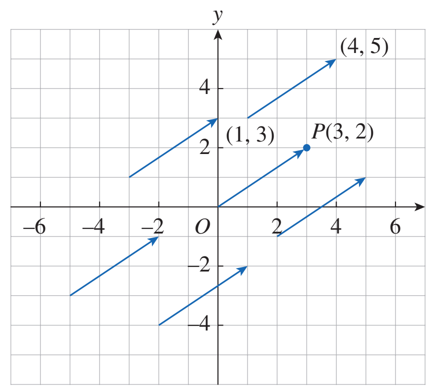


Figure 9.31
Various representations of the vector $\mathbf{a} = \langle 3, 2 \rangle$.

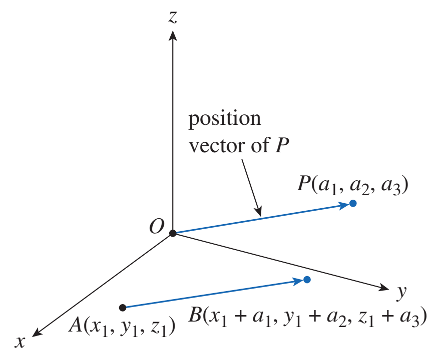


Figure 9.32
Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$.

In three dimensions, the vector $\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$. If we consider any other representation $\overrightarrow{AB}$ of $\mathbf{a}$, where the initial point is $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$, then the following equations must be true:

$$x_1 + a_1 = x_2, \qquad y_1 + a_2 = y_2, \qquad z_1 + a_3 = z_2$$

and therefore,

$$a_1 = x_2 - x_1, \qquad a_2 = y_2 - y_1, \qquad a_3 = z_2 - z_1$$

See Figure 9.32.

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \tag{1}$$

Example 3 Represent a Displacement Vector

Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

Solution

Use Equation 3. The vector corresponding to $\overrightarrow{AB}$ is

$\mathbf{a} = \langle -2 - 2, 1 - (-3), 1 - 4 \rangle = \langle -4, 4, -3 \rangle.$ ■

The **magnitude** or **length** of a vector $\mathbf{v}$ is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$. By using the distance formula to compute the length of a segment OP, we obtain the following formulas.

Magnitude of a Vector

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Operations and Properties

We have seen how to add and subtract vectors graphically. However, we need to be able to perform these and other operations on vectors algebraically. Figure 9.33 shows that if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then the sum is $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, at least in this case where the components are positive. In other words, *to add vectors algebraically*, *we add their components*. Similarly, *to subtract vectors*, *we subtract components*. Using similar triangles in Figure 9.34, the components of the vector $c\mathbf{a}$ are ca_1 and ca_2. Therefore, *to multiply a vector by a scalar*, *we multiply each component by that scalar*.

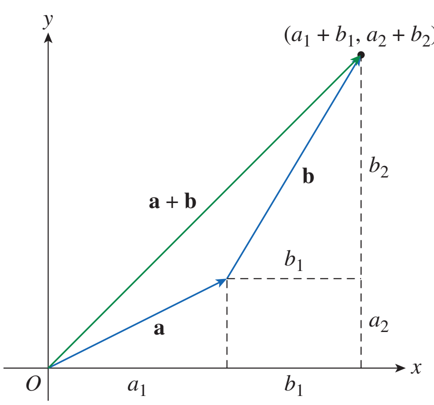


Figure 9.33
Graphical justification for the algebraic addition of two vectors.

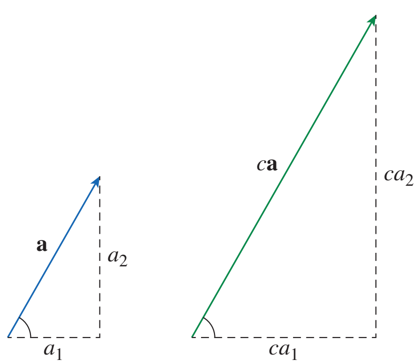


Figure 9.34
Graphical justification for the algebraic scalar multiple of a vector.

Here is a summary of these results.

Operations on Vectors Algebraically

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle \qquad c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

Example 4 Operations on Vectors

If $\mathbf{a} = \langle 4, 0, 3\rangle$ and $\mathbf{b} = \langle -2, 1, 5\rangle$, find $|\mathbf{a}|$, and the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{b}$, and $2\mathbf{a} + 5\mathbf{b}$.

Solution

Use the formula for magnitude of a vector and the formulas for operations on vectors algebraically.

$$|\mathbf{a}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5 \qquad \text{Formula for magnitude.}$$

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= \langle 4, 0, 3\rangle + \langle -2, 1, 5\rangle \\
&= \langle 4 + (-2), 0 + 1, 3 + 5\rangle = \langle 2, 1, 8\rangle && \text{Add components.} \\
\mathbf{a} - \mathbf{b} &= \langle 4, 0, 3\rangle - \langle -2, 1, 5\rangle \\
&= \langle 4 - (-2), 0 - 1, 3 - 5\rangle = \langle 6, -1, -2\rangle && \text{Subtract components.} \\
3\mathbf{b} &= 3\langle -2, 1, 5\rangle \\
&= \langle 3(-2), 3(1), 3(5)\rangle = \langle -6, 3, 15\rangle && \text{Multiply each component by 3.} \\
2\mathbf{a} + 5\mathbf{b} &= 2\langle 4, 0, 3\rangle + 5\langle -2, 1, 5\rangle \\
&= \langle 8, 0, 6\rangle + \langle -10, 5, 25\rangle = \langle -2, 5, 31\rangle && \text{Multiply by appropriate scalar; add components.}
\end{aligned}$$

■

Vectors in n dimensions are used to list various quantities in an organized way. For example, the components of a six-dimensional vector

$$\mathbf{p} = \langle p_1, p_2, p_3, p_4, p_5, p_6\rangle$$

might represent the prices of six different ingredients required to make a particular product, or the doses of six prescription medications taken by a patient in the hospital.

Four-dimensional vectors $\langle x, y, z, t\rangle$ are used in relativity theory, where the first three components specify a position in space and the fourth represents time.

We denote the set of all two-dimensional vectors by V_2 and the set of all three dimensional vectors by V_3. More generally, we denote the set of all n-dimensional vectors by V_n. An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n\rangle$$

where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of $\mathbf{a}$. Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

Properties of Vectors

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$	4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$	6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$	8. $1\mathbf{a} = \mathbf{a}$

These eight properties of vectors can be verified either geometrically or algebraically. For example, Property 1 can be seen from Figure 9.20; it is equivalent to the Parallelogram Law. Or it can be shown algebraically; here is the proof for the case $n = 2$:

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= \langle a_1, a_2\rangle + \langle b_1, b_2\rangle = \langle a_1 + b_1, a_2 + b_2\rangle && \text{Addition of vectors algebraically.} \\
&= \langle b_1 + a_1, b_2 + a_2\rangle && \text{Addition is commutative.} \\
&= \langle b_1, b_2\rangle + \langle a_1, a_2\rangle = \mathbf{b} + \mathbf{a} && \text{Addition of vectors algebraically.}
\end{aligned}$$

Property 2, the associative law, is illustrated in Figure 9.35. We can prove this property by applying the Triangle Law several times: the vector $\overrightarrow{PQ}$ is obtained either by first constructing $\mathbf{a} + \mathbf{b}$ and then adding $\mathbf{c}$, or by adding $\mathbf{a}$ to the vector $\mathbf{b} + \mathbf{c}$.

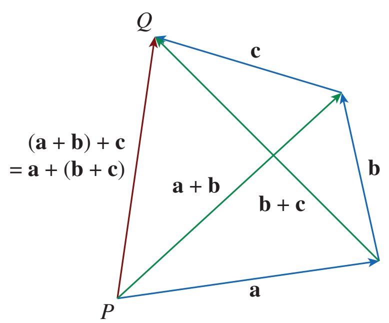


Figure 9.35
The geometry to justify Property 2, the associative law.

Basis Vectors

Certain vectors in two, three, and even n dimensions play a special role. In three dimensions, consider the **standard basis vectors**

$$\mathbf{i} = \langle 1, 0, 0\rangle \qquad \mathbf{j} = \langle 0, 1, 0\rangle \qquad \mathbf{k} = \langle 0, 0, 1\rangle$$

The vectors **i**, **j**, and **k** have length 1 and point in the directions of the positive x-, y-, and z-axes. Similarly, the analogous standard basis vectors in two dimensions are $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$. Figures 9.36 and 9.37 illustrate these vectors in two and three dimensions.

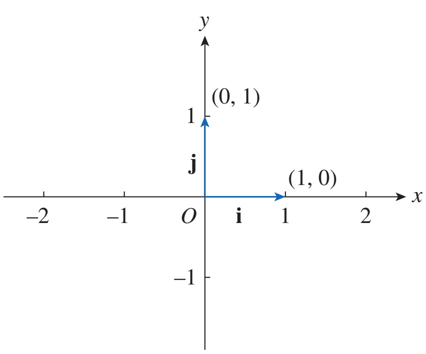


Figure 9.36
Standard basis vectors in two dimensions.

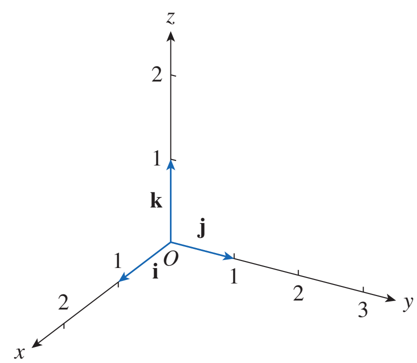


Figure 9.37
Standard basis vectors in three dimensions.

Suppose $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, then we can write **a** as

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, a_3\rangle = \langle a_1, 0, 0\rangle + \langle 0, a_2, 0\rangle + \langle 0, 0, a_3\rangle \\ &= a_1\langle 1, 0, 0\rangle + a_2\langle 0, 1, 0\rangle + a_3\langle 0, 0, 1\rangle \\ \mathbf{a} &= a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k} \end{aligned} \tag{2}$$

Therefore, any vector in V_3 can be expressed in terms of the standard basis vectors **i**, **j**, and **k**. For example,

$$\langle 1, -2, 6\rangle = \mathbf{i} - 2\,\mathbf{j} + 6\,\mathbf{k}$$

Similarly, in two dimensions, we can write any vector **a** as

$$\mathbf{a} = \langle a_1, a_2\rangle = a_1\,\mathbf{i} + a_2\,\mathbf{j} \tag{3}$$

Figures 9.39 and 9.38 show geometric interpretations of Equations 3 and 2.

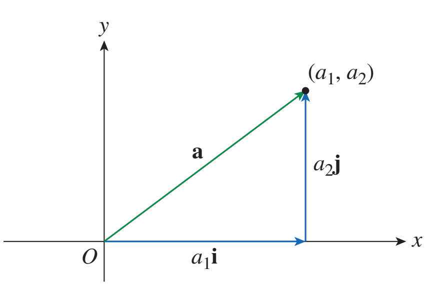


Figure 9.38
In two dimensions: $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j}$.

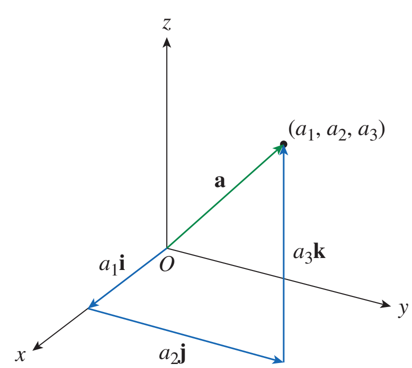


Figure 9.39
In three dimensions: $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$.

Example 5 Basis Vectors Expression

If $\mathbf{a} = \mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$ and $\mathbf{b} = 4\,\mathbf{i} + 7\,\mathbf{k}$, express $2\mathbf{a} + 3\mathbf{b}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Solution

Use Properties 1, 2, 5, 6, and 7 of vectors.

$$\begin{aligned} 2\mathbf{a} + 3\mathbf{b} &= 2(\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}) + 3(4\,\mathbf{i} + 7\,\mathbf{k}) \\ &= 2\,\mathbf{i} + 4\,\mathbf{j} - 6\,\mathbf{k} + 12\,\mathbf{i} + 21\,\mathbf{k} = 14\,\mathbf{i} + 4\,\mathbf{j} + 15\,\mathbf{k} \end{aligned}$$

A **unit vector** is a vector whose length is 1. For example, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as $\mathbf{a}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|} \tag{4}$$

In order to verify this, let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and because c is a positive scalar, $\mathbf{u}$ has the same direction as $\mathbf{a}$. In addition,

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1$$

Example 6 Unit Vector

Find the unit vector in the direction of the vector $2\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}$.

Solution

The given vector has length

$$|2\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3.$$

Use Equation 6. The unit vector with the same direction is

$$\frac{1}{3}(2\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}) = \frac{2}{3}\,\mathbf{i} - \frac{1}{3}\,\mathbf{j} - \frac{2}{3}\,\mathbf{k}.$$

Applications

Vectors are used in many aspects of physics and engineering. In Chapter 10, we will use vectors to describe the velocity and acceleration of objects moving in space. For now, we will consider an example involving force.

A force can be represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

Example 7 A Weight Hung from Two Wires

A 100-lb weight hangs from two wires as shown in Figure 9.40. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and their magnitudes.

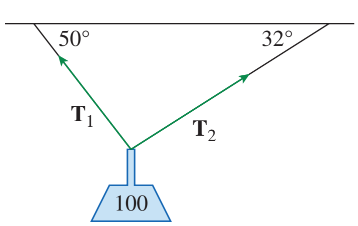


Figure 9.40
The weight, two wires, and forces.

Solution

Express $\mathbf{T}_1$ and $\mathbf{T}_2$ in terms of their horizontal and vertical components.
Use Figure 9.41.

$$\mathbf{T}_1 = -|\mathbf{T}_1|\cos 50°\,\mathbf{i} + |\mathbf{T}_1|\sin 50°\,\mathbf{j}$$

$$\mathbf{T}_2 = |\mathbf{T}_2|\cos 32°\,\mathbf{i} + |\mathbf{T}_2|\sin 32°\,\mathbf{j}$$

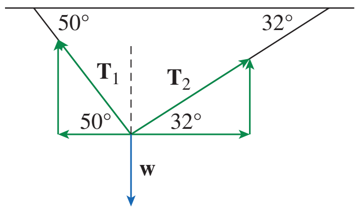


Figure 9.41
Horizontal and vertical components of $\mathbf{T}_1$ and $\mathbf{T}_2$

The resultant vector of the tensions $\mathbf{T}_1 + \mathbf{T}_2$ counterbalances the weight $\mathbf{w}$. Therefore, $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100\,\mathbf{j}$.

Use the component expressions for $\mathbf{T}_1$ and $\mathbf{T}_2$.

$$(-|\mathbf{T}_1|\cos 50° + |\mathbf{T}_2|\cos 32°)\,\mathbf{i} + (|\mathbf{T}_1|\sin 50° + |\mathbf{T}_2|\sin 32°)\,\mathbf{j} = 100\,\mathbf{j}$$

Equate components.

$$-|\mathbf{T}_1|\cos 50° + |\mathbf{T}_2|\cos 32° = 0$$

$$|\mathbf{T}_1|\sin 50° + |\mathbf{T}_2|\sin 32° = 100$$

Solve the first equation for $|\mathbf{T}_2|$ and substitute into the second.

$$|\mathbf{T}_1|\sin 50° + \frac{|\mathbf{T}_1|\cos 50°}{\cos 32°}\sin 32° = 100$$

The magnitudes of the tensions are:

$$|\mathbf{T}_1| = \frac{100}{\sin 50° + \tan 32° \cos 50°} \approx 85.64 \text{ lb} \quad \text{and}$$

$$|\mathbf{T}_2| = \frac{|\mathbf{T}_1|\cos 50°}{\cos 32°} \approx 64.91 \text{ lb.}$$

Substitute these values into the original component expression for $\mathbf{T}_1$ and $\mathbf{T}_2$.

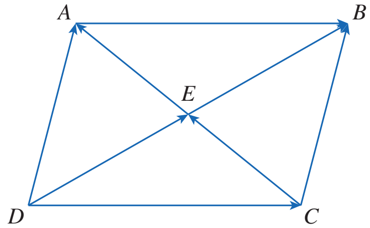
$$\mathbf{T}_1 \approx -55.05\,\mathbf{i} + 65.60\,\mathbf{j} \qquad \mathbf{T}_2 \approx 55.05\,\mathbf{i} + 34.40\,\mathbf{j}$$

9.2 Exercises

1. Determine whether the quantity is a vector or scalar. Explain your reasoning.
(a) The cost of a theater ticket
(b) The current in a river
(c) The initial speed and flight path from Denver to Omaha
(d) The population of the world
(e) The amount of ice cream in one scoop
(f) The rate of drift and trajectory of an iceberg

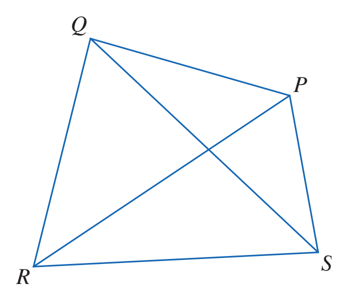
2. Explain the relationship between the point (4, 7) and the vector $\langle 4, 7\rangle$. Illustrate your answer with a sketch.

3. Identify all the vectors in the parallelogram.



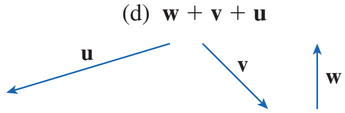
4. Use the figure to write each combination of vectors as a single vector.

(a) $\overrightarrow{PQ} + \overrightarrow{QR}$ (b) $\overrightarrow{RP} + \overrightarrow{PS}$
(c) $\overrightarrow{QS} - \overrightarrow{PS}$ (d) $\overrightarrow{RS} + \overrightarrow{SP} + \overrightarrow{PQ}$



5. Copy the vectors in the figure and use them to draw the following vectors.

(a) $\mathbf{u} + \mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$
(c) $\mathbf{v} + \mathbf{w}$ (d) $\mathbf{w} + \mathbf{v} + \mathbf{u}$



6. Copy the vectors in the figure and use them to draw the following vectors.

(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$
(c) $\frac{1}{2}\mathbf{a}$ (d) $-3\mathbf{b}$
(e) $\mathbf{a} + 2\mathbf{b}$ (f) $2\mathbf{b} - \mathbf{a}$

Find a vector **a** with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

7. $A(-1, 3)$, $B(2, 2)$ **8.** $A(2, 1)$, $B(0, 6)$

9. $A(-2, -5)$, $B(3, 4)$ **10.** $A(0, 3, 1)$, $B(2, 3, -1)$

11. $A(4, 0, -2)$, $B(4, 2, 1)$ **12.** $A(-2, -3, -1)$, $B(3, -1, 6)$

Find the sum of the given vectors and illustrate geometrically.

13. $\langle -1, 4 \rangle$, $\langle 6, -2 \rangle$ **14.** $\langle -2, -1 \rangle$, $\langle 5, 7 \rangle$

15. $\langle 3, -5 \rangle$, $\langle 0, 4 \rangle$ **16.** $\langle 0, 1, 2 \rangle$, $\langle 0, 0, -3 \rangle$

17. $\langle -1, 0, 2 \rangle$, $\langle 0, 4, 0 \rangle$ **18.** $\langle 1, 2, 0 \rangle$, $\langle -1, 3, 3 \rangle$

Find $\mathbf{a} + \mathbf{b}$, $2\mathbf{a} + 3\mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{a} - \mathbf{b}|$.

19. $\mathbf{a} = \langle 5, -12 \rangle$, $\mathbf{b} = \langle -3, -6 \rangle$

20. $\mathbf{a} = 4\,\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - 2\,\mathbf{j}$

21. $\mathbf{a} = \mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$, $\mathbf{b} = -2\,\mathbf{i} - \mathbf{j} + 5\,\mathbf{k}$

22. $\mathbf{a} = 2\,\mathbf{i} - 4\,\mathbf{j} + 4\,\mathbf{k}$, $\mathbf{b} = 2\,\mathbf{j} - \mathbf{k}$

23. $\mathbf{a} = -2\,\mathbf{i} + 3\,\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 3\,\mathbf{i} - 3\,\mathbf{j} + 2\,\mathbf{k}$

Find a unit vector that has the same direction as the given vector.

24. $-3\,\mathbf{i} + 7\,\mathbf{j}$ **25.** $\langle 3, -3 \rangle$

26. $\langle -4, 2, 4 \rangle$ **27.** $8\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$

28. Find a vector that has the same direction as $\langle -2, 4, 2 \rangle$ with length 6.

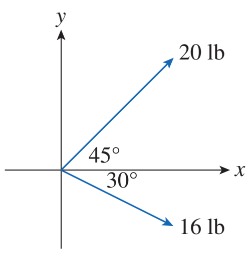
29. Suppose **v** lies in the first quadrant, forms an angle of $\pi/3$ with the positive x-axis, and $|\mathbf{v}| = 4$. Find **v** in component form.

30. Suppose a child pulls a sled through the snow on a level path with a force of 50 N exerted at an angle of 38° above the horizontal. Find the horizontal and vertical components of the force.

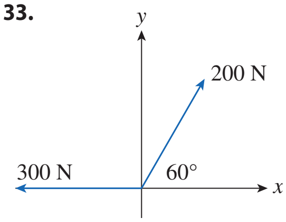
31. A quarterback throws a football with angle of elevation 40° and speed 60 ft/s. Find the horizontal and vertical components of the velocity vector.

Find the magnitude of the resultant force and the angle it makes with the positive x-axis.

32.



33.

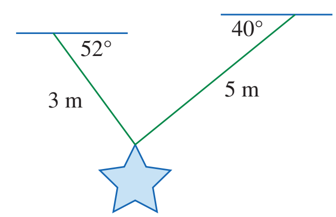


34. The magnitude of a velocity vector is called *speed*. Suppose that a wind is blowing from the direction N45°W at a speed of 50 km/h. This means that the direction from which the wind blows is 45° west of the northerly direction. A pilot is steering a plane in the direction N60°E at an airspeed (speed in still air) of 250 km/h.

The *true course*, or *track*, of the plane is the direction of the resultant of the velocity vectors of the plane and the wind. The *ground speed* of the plane is the magnitude of the resultant. Find the true course and the ground speed of the plane.

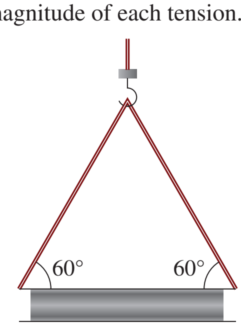
35. A person walks due west on the deck of a ship at 3 mi/h. The ship is moving north at a speed of 22 mi/h. Find the speed and direction of the person relative to the surface of the water.

36. Ropes 3 m and 5 m in length are fastened to a holiday decoration that is suspended over a town square. The decoration has a mass of 5 kg. The ropes, fastened at different heights, make angles of 52° and 40° with the horizontal. Find the tension in each wire and the magnitude of each tension.

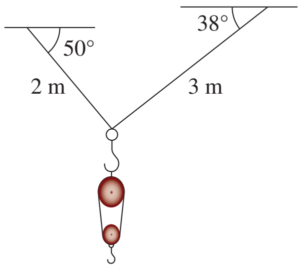


37. A clothesline is tied between two poles, 8 m apart. The line is quite taut and has negligible sag. When a wet shirt with a mass of 0.8 kg is hung at the middle of the line, the midpoint is pulled down 8 cm. Find the tension in each half of the clothesline.

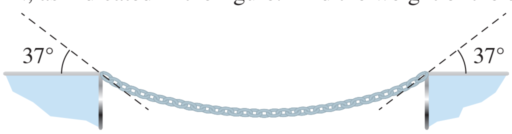
38. A crane suspends a 500-lb steel beam horizontally by support cables (with negligible weight) attached from a hook to each end of the beam. The support cables each make an angle of 60° with the beam. Find the tension vector in each support cable and the magnitude of each tension.



39. A block-and-tackle pulley hoist is suspended in a warehouse by ropes of lengths 2 m and 3 m. The hoist weights 350 N. The ropes, fastened at different heights, make angles of 50° and 38° with the horizontal. Find the tension in each rope and the magnitude of each tension.



40. The tension vector T at each end of a chain has magnitude 25 N, as indicated in the figure. Find the weight of the chain.



41. A boater wants to cross a canal that is 3 km wide and wants to land at a point 2 km upstream from their starting point. The current in the canal flows at 3.5 km/h and the speed of their boat is 13 km/h.

(a) In what direction should they steer?

(b) How long will the trip take?

42. Three forces act on an object. Two of the forces are at an angle of 100° to each other and have magnitudes 25 N and 12 N. The third is perpendicular to the plane of these two forces and has magnitude 4 N. Calculate the magnitude of the force that would exactly counterbalance these three forces.

43. Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.

44. (a) Find the unit vectors that are parallel to the tangent line to the graph of $y = 2 \sin x$ at the point $(\pi/6, 1)$.

(b) Find the unit vectors that are perpendicular to the tangent line.

(c) Sketch the graph of $y = 2 \sin x$. Add to the graph the vectors in parts (a) and (b) such that their initial points are $(\pi/6, 1)$.

---

45. If A, B, and C are the vertices of a triangle, find

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$$

46. Let C be the point on the line segment AB that is twice as far from B as it is from A. If $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$, show that $\mathbf{c} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.

47. (a) Draw the vectors $\mathbf{a} = \langle 3, 2 \rangle$, $\mathbf{b} = \langle 2, -1 \rangle$, and $\mathbf{c} = \langle 7, 1 \rangle$.

(b) Use a sketch to show that there are scalars s and t such that $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$.

(c) Use your sketch to estimate the values of s and t .

(d) Find the exact values of s and t.

48. Suppose that **a** and **b** are nonzero vectors that are not parallel and **c** is any vector in the plane determined by **a** and **b**. Give a geometric argument to show that **c** can be written as $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ for appropriate scalars s and t. Then write an algebraic argument using vector components.

49. Suppose **a** is a three-dimensional unit vector in the first octant that starts at the origin and makes an angles of 60° and 72° with the positive x- and y-axes, respectively. Express **a** in terms of its components.

50. Suppose a vector **a** makes angles α, β, and γ with the positive x-, y-, and z-axes, respectively. Find the components of **a** and show that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

The numbers $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are called the *direction cosines* of **a**.

51. If $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, describe the set of all points (x, y, z) such that $|\mathbf{r} - \mathbf{r}_0| = 1$.

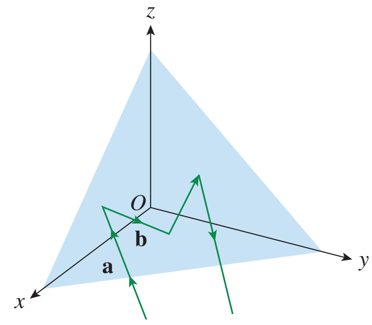
52. If $\mathbf{r} = \langle x, y \rangle$, $\mathbf{r}_1 = \langle x_1, y_1 \rangle$, and $\mathbf{r}_2 = \langle x_2, y_2 \rangle$, describe the set of all points (x, y) such that $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2| = k$, where $k > |\mathbf{r}_1 - \mathbf{r}_2|$?

53. Figure 9.35 provides a geometric illustration of Property 2 of vectors. Use components to give an algebraic proof of this property for the case $n = 2$.

54. Prove Property 5 of vectors algebraically for the case $n = 3$. Then use similar triangles to provide a geometric proof.

55. Use vectors to show that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

56. Suppose the three coordinate planes are all mirrored and a light ray given by the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ first strikes the xz-plane, as shown in the figure. Use the fact that the angle of incidence equals the angle of reflection to show that the direction of the reflected ray is given by $\mathbf{b} = \langle a_1, -a_2, a_3 \rangle$. Show that, after being reflected by all three mutually perpendicular mirrors, the resulting ray is parallel to the initial ray. (American space scientists used this principle, together with laser beams and an array of corner mirrors on the moon, to calculate very precisely the distance from Earth to the moon.)



9.3 The Dot Product

In our study of vectors, we have learned how to add two vectors and how to multiply a vector by a scalar. It seems reasonable to ask if it is possible to multiply two vectors so that their product is a useful quantity. The dot product is a form of vector multiplication, and will be considered in this section. Another form of vector multiplication is the cross product, which is presented in the next section.

Work and the Dot Product

An example of a situation in physics and engineering where we need to combine two vectors occurs in calculating the work done by a force. In Section 6.5, we defined the work done by a constant force F in moving an object through a distance d as $W = Fd$. However, this equation applies only when the force is directed along the line of motion of the object.

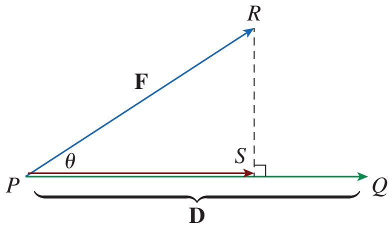


Figure 9.42
A constant force as a vector **F**.

Suppose, more generally, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as illustrated in Figure 9.42. If the force moves the object from P to Q, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. So, in this case, we have two vectors: the force **F** and the displacement **D**.

The **work** done by **F** is defined as the magnitude of the displacement, $|\mathbf{D}|$, multiplied by the magnitude of the applied force in the direction of the motion, which, from Figure 9.42, is

$$|\overrightarrow{PS}| = |\mathbf{F}|\cos\theta$$

So, the work done by **F** is defined to be

$$W = |\mathbf{D}|(|\mathbf{F}|\cos\theta) = |\mathbf{F}||\mathbf{D}|\cos\theta \tag{1}$$

Notice that work is a scalar quantity; it has no direction. But its value depends on the angle θ between the force and displacement vectors.

The expression in Equation 1 can be used to define the dot product of two vectors even when they do not represent force or displacement.

Definition • Dot Product

The **dot product** of two nonzero vectors **a** and **b** is the number

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle between **a** and **b**, $0 \le \theta \le \pi$. (So, θ is the smaller angle between the vectors when they are drawn with the same initial point.) If either **a** or **b** is **0**, we define $\mathbf{a} \cdot \mathbf{b} = 0$.

This product is called the **dot product** because of the dot in the notation $\mathbf{a} \cdot \mathbf{b}$. The result of computing $\mathbf{a} \cdot \mathbf{b}$ is a real number, that is, a scalar; it is not a vector. For this reason, the dot product is sometimes called the **scalar product**.

In the example of finding the work done by a force **F** in moving an object through a displacement $\mathbf{D} = \overrightarrow{PQ}$ by calculating $\mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos\theta$, it does not make sense for

the angle θ between $\mathbf{F}$ and $\mathbf{D}$ to be $\pi/2$ or larger. In this case, movement from P to Q could not take place. Notice that there is no such restriction in the general definition of $\mathbf{a} \cdot \mathbf{b}$; θ can be any angle from 0 to π.

Example 1 Compute a Dot Product from Lengths and the Contained Angle

If the vectors $\mathbf{a}$ and $\mathbf{b}$ have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

Solution

Use the definition of the dot product.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\left(\frac{\pi}{3}\right) = 4 \cdot 6 \cdot \frac{1}{2} = 12$$

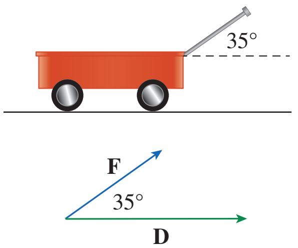


Figure 9.43
The force and displacement vectors.

Example 2 Find the Work Done by a Force

A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle on the wagon is held at an angle 35° above the horizontal. Find the work done by the force.

Solution

If $\mathbf{F}$ and $\mathbf{D}$ are the force and displacement vectors, as illustrated in Figure 9.43, then the work done is

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos 35^\circ$$
$$= (70)(100)\cos 35^\circ \approx 5734.06 \text{ N} \cdot \text{m} = 5734.06 \text{ J}.$$

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \frac{\pi}{2}$. Therefore, if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular vectors, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\left(\frac{\pi}{2}\right) = 0$$

Conversely, if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos\theta = 0$, so $\theta = \pi/2$. The zero vector $\mathbf{0}$ is considered to be perpendicular to all vectors.

$$\text{Two vectors } \mathbf{a} \text{ and } \mathbf{b} \text{ are orthogonal if and only if } \mathbf{a} \cdot \mathbf{b} = 0. \tag{2}$$

If $0 \le \theta < \frac{\pi}{2}$, then $\cos\theta > 0$ and if $\frac{\pi}{2} < \theta \le \pi$, then $\cos\theta < 0$. Therefore, $\mathbf{a} \cdot \mathbf{b}$ is positive for $\theta < \frac{\pi}{2}$ and negative for $\theta > \frac{\pi}{2}$. We can use these results to think of $\mathbf{a} \cdot \mathbf{b}$ as a measure of the extent to which $\mathbf{a}$ and $\mathbf{b}$ point in the same direction.

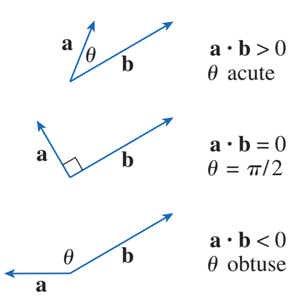


Figure 9.44
The dot product as a measure of the extent to which **a** and **b** point in the same direction.

The dot product $\mathbf{a} \cdot \mathbf{b}$ is positive if $\mathbf{a}$ and $\mathbf{b}$ point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions. See Figure 9.44. In the extreme case where $\mathbf{a}$ and $\mathbf{b}$ point in exactly the same direction, we have $\theta = 0$, so $\cos\theta = 1$ and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$$

If $\mathbf{a}$ and $\mathbf{b}$ point in exactly opposite directions, then we have $\theta = \pi$, $\cos\theta = -1$, and $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.

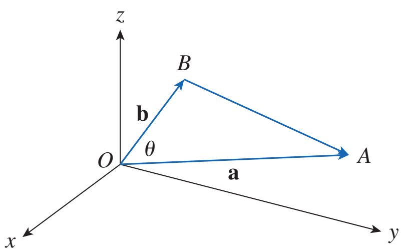


Figure 9.45
The angle θ between **a** and **b**.

The Dot Product in Component Form

Suppose we are given two vectors in component form:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \qquad \mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

We would like to find a convenient expression for $\mathbf{a} \cdot \mathbf{b}$ in terms of these components. If we apply the Law of Cosines to the triangle in Figure 9.45, we get

$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,\mathbf{a} \cdot \mathbf{b} \end{aligned}$$

Solve for the dot product in this expression:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \frac{1}{2}(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2) \\ &= \frac{1}{2}[a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - (a_1 - b_1)^2 - (a_2 - b_2)^2 - (a_3 - b_3)^2] \\ &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned}$$

The dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Therefore, to find the dot product of **a** and **b**, we multiply corresponding components and add. The dot product of two-dimensional vectors is found in a similar manner:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1b_1 + a_2b_2$$

Example 3 Compute Dot Products from Components

Here are some examples of dot products using the component definition.

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = (2)(3) + (4)(-1) = 2$$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\tfrac{1}{2} \rangle = (-1)(6) + (7)(2) + (4)(-\tfrac{1}{2}) = 6$$

$$(\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}) \cdot (2\,\mathbf{j} - \mathbf{k}) = (1)(0) + (2)(2) + (-3)(-1) = 7$$

Example 4 Test for Orthogonality

Show that $2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$ is perpendicular to $5\,\mathbf{i} - 4\,\mathbf{j} + 2\,\mathbf{k}$.

Solution

Find the dot product.

$$(2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}) \cdot (5\,\mathbf{i} - 4\,\mathbf{j} + 2\,\mathbf{k}) = (2)(5) + (2)(-4) + (-1)(2) = 0$$

Therefore, these vectors are perpendicular by Equation 2.

Example 5 Angle Between Two Vectors

Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

Solution

Find the magnitude of each vector.

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3 \quad \text{and} \quad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and the dot product

$$\mathbf{a} \cdot \mathbf{b} = (2)(5) + (2)(-3) + (-1)(2) = 2$$

Use the definition of the dot product.

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

Therefore, the angle between **a** and **b** is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.462\ (= 83.79°).$$

Example 6 Compute the Work Done

A force is given by a vector $\mathbf{F} = 3\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

Solution

The displacement vector is

$$\mathbf{D} = \overrightarrow{PQ} = \langle 4 - 2, 6 - 1, 2 - 0 \rangle = \langle 2, 5, 2 \rangle.$$

Therefore, the work done is

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{D} = \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle \\ &= 6 + 20 + 10 = 36. \end{aligned}$$

If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 J.

Many basic real number properties involving products also hold for the dot product. Here is a list of dot product properties.

Properties of the Dot Product

If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

Properties 1, 2, and 5 are immediate consequences of the definition of a dot product. Here is a proof of Property 3 using components.

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle && \text{Algebraic addition of } \mathbf{b} \text{ and } \mathbf{c}. \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) && \text{Dot product definition.} \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 && \text{Multiply through.} \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) && \text{Group terms carefully.} \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} && \text{Dot product definition.} \end{aligned}$$

The proof of Property 4 is left as an exercise.

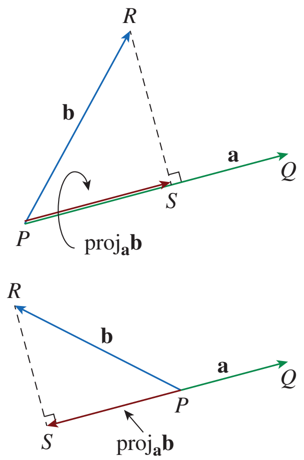


Figure 9.46
Vector projections.

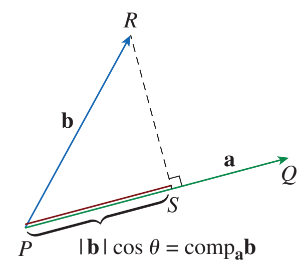


Figure 9.47
Scalar projection.

Projections

Figure 9.46 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector representation $\overrightarrow{PS}$ is called the **vector projection** of $\mathbf{b}$ onto $\mathbf{a}$ and is denoted by $\text{proj}_{\mathbf{a}}\,\mathbf{b}$. You can visualize this as a shadow of $\mathbf{b}$ on the line containing $\overrightarrow{PQ}$ made by a light over $\mathbf{b}$.

The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. See Figure 9.47. This value is denoted by $\text{comp}_{\mathbf{a}}\mathbf{b}$. Note that $\text{comp}_{\mathbf{a}}\mathbf{b}$ is negative if $\frac{\pi}{2} < \theta \le \pi$. (We used the component of the force $\mathbf{F}$ along the displacement $\mathbf{D}$, $\text{comp}_{\mathbf{D}}\,\mathbf{F}$, at the beginning of this section.)

The equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = |\mathbf{a}|(|\mathbf{b}|\cos\theta)$$

shows that the dot product of $\mathbf{a}$ and $\mathbf{b}$ can be interpreted as the length of $\mathbf{a}$ times the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$. Since

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}$$

the component of $\mathbf{b}$ along $\mathbf{a}$ can be computed by taking the dot product of $\mathbf{b}$ with the unit vector in the direction of $\mathbf{a}$. Here is a summary of these ideas.

Scalar and Vector Projections

Scalar projection of **b** onto **a**: $\quad \text{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\quad \text{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

Example 7 Find the Projections

Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.

Solution

Find the magnitude of **a**.

$$|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$$

The scalar projection of **b** onto **a** is

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + (3)(1) + (1)(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}.$$

The vector projection is this scalar projection times the unit vector in the direction of **a**.

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{3}{\sqrt{14}}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle$$

■

At the beginning of this section, we considered one use of projections in physics; we used a scalar projection of a force vector in defining work. Other uses of projections occur in three-dimensional geometry. In Exercises 57, you are asked to use a projection to find the distance from a point to a line, and in Section 9.5, we use a projection to find the distance from a point to a plane.

9.3 Exercises

1. Which of the following expressions are meaningful? Which are meaningless? Explain your reasoning.
(a) $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ (b) $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$
(c) $|\mathbf{a}|(\mathbf{b}\cdot\mathbf{c})$ (d) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$
(e) $\mathbf{a}\cdot\mathbf{b}+\mathbf{c}$ (f) $|\mathbf{a}|\cdot(\mathbf{b}+\mathbf{c})$

Find $\mathbf{a}\cdot\mathbf{b}$.

2. $|\mathbf{a}| = 7$, $|\mathbf{b}| = 4$, the angle between $\mathbf{a}$ and $\mathbf{b}$ is 30°

3. $|\mathbf{a}| = 80$, $|\mathbf{b}| = 50$, the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{3\pi}{4}$

4. $\mathbf{a} = \langle -2, 3\rangle$, $\mathbf{b} = \langle 0.7, 1.2\rangle$

5. $\mathbf{a} = \langle -1, -4\rangle$, $\mathbf{b} = \langle 4, 5\rangle$

6. $\mathbf{a} = \langle -2, \frac{1}{4}\rangle$, $\mathbf{b} = \langle -5, 12\rangle$

7. $\mathbf{a} = \langle 3, 8\rangle$, $\mathbf{b} = \langle 3, 9\rangle$

8. $\mathbf{a} = \langle 6, -2, 3\rangle$, $\mathbf{b} = \langle 2, 5, -1\rangle$

9. $\mathbf{a} = \langle -1, -2, -5\rangle$, $\mathbf{b} = \langle 2, 5, 7\rangle$

10. $\mathbf{a} = \langle 8, 1, \frac{1}{3}\rangle$, $\mathbf{b} = \langle -2, 3, -9\rangle$

11. $\mathbf{a} = \langle p, -p, 2p\rangle$, $\mathbf{b} = \langle 2q, q, -q\rangle$

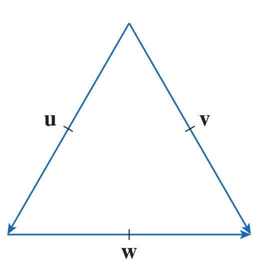
12. $\mathbf{a} = \langle 3, 7, -11\rangle$, $\mathbf{b} = \langle 4, 5, -4\rangle$

13. $\mathbf{a} = 2\,\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

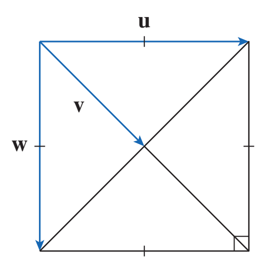
14. $\mathbf{a} = 3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 4\,\mathbf{i} + 5\,\mathbf{k}$

If $\mathbf{u}$ is a unit vector, find $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\cdot\mathbf{w}$.

15.



16.



17. (a) Show that $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$.
(b) Show that $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$.

18. A street vendor sells a hamburgers, b hot dogs, and c soft drinks on a given day. They charge \$3 for a hamburger, \$2.50 for a hot dog, and \$2 for a soft drink. If $\mathbf{A} = \langle a, b, c\rangle$ and $\mathbf{P} = \langle 3, 2.5, 2\rangle$, explain the meaning of the dot product $\mathbf{A}\cdot\mathbf{P}$ in the context of this problem.

19. The number of 2-axle, 3-axle, and 4-axle low-profile vehicles that pass through the Mainline Toll Plaza on Route 476 during a certain day are n_1, n_2, and n_3. The toll for each is \$2.20, \$2.90, and \$3.50. Let $\mathbf{N} = \langle n_1, n_2, n_3\rangle$ and $\mathbf{T} = \langle 2.2, 2.9, 3.5\rangle$. Explain the meaning of the dot product $\mathbf{N}\cdot\mathbf{T}$ in the context of this problem.

Find the angle between the vectors in both radians and degrees.

20. $\mathbf{a} = \langle -8, 6\rangle$, $\mathbf{b} = \langle \sqrt{7}, 3\rangle$

21. $\mathbf{a} = \langle \sqrt{3}, 1\rangle$, $\mathbf{b} = \langle 0, 5\rangle$

22. $\mathbf{a} = \langle 0.1, 1\rangle$, $\mathbf{b} = \langle 1, 0.1\rangle$

23. $\mathbf{a} = \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$

24. $\mathbf{a} = \langle 1, -4, 1\rangle$, $\mathbf{b} = \langle 0, 2, -2\rangle$

25. $\mathbf{a} = \langle -1, 3, 4\rangle$, $\mathbf{b} = \langle 5, 2, 1\rangle$

26. $\mathbf{a} = \mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}$, $\mathbf{b} = 4\,\mathbf{i} - 3\,\mathbf{k}$

27. $\mathbf{a} = \mathbf{i} - 4\,\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -3\,\mathbf{i} + \mathbf{j} + 5\,\mathbf{k}$

28. $\mathbf{a} = 8\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$, $\mathbf{b} = 4\,\mathbf{j} + 2\,\mathbf{k}$

Find, to the nearest degree, the three angles of the triangle with the given vertices.

29. $A(1, 0)$, $B(3, 6)$, $C(-1, 4)$

30. $A(4, 0)$, $B(0, 3)$, $C(2, 5)$

31. $P(0, 1, 1)$, $Q(-2, 4, 3)$, $R(1, 2, -1)$

32. $P(1, 0, 0)$, $Q(0, 3, 0)$, $R(2, 2, 7)$

Determine whether the given vectors are orthogonal, parallel, or neither.

33. (a) $\mathbf{a} = \langle 4, 6\rangle$, $\mathbf{b} = \langle -3, 2\rangle$
(b) $\mathbf{a} = \langle 9, 3\rangle$, $\mathbf{b} = \langle -2, 6\rangle$
(c) $\mathbf{a} = \langle -5, 3, 7\rangle$, $\mathbf{b} = \langle 6, -8, 2\rangle$
(d) $\mathbf{a} = \langle 4, 5, -2\rangle$, $\mathbf{b} = \langle 3, -1, 5\rangle$

34. (a) $\mathbf{a} = -\mathbf{i} + 2\,\mathbf{j} + 5\,\mathbf{k}$, $\mathbf{b} = 3\,\mathbf{i} + 4\,\mathbf{j} - \mathbf{k}$
(b) $\mathbf{a} = 2\,\mathbf{i} + 6\,\mathbf{j} - 4\,\mathbf{k}$, $\mathbf{b} = -3\,\mathbf{i} - 9\,\mathbf{j} + 6\,\mathbf{k}$
(c) $\mathbf{a} = -8\,\mathbf{i} + 12\,\mathbf{j} + 4\,\mathbf{k}$, $\mathbf{b} = 6\,\mathbf{i} - 9\,\mathbf{j} - 3\,\mathbf{k}$
(d) $\mathbf{a} = 3\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$, $\mathbf{b} = 5\,\mathbf{i} + 9\,\mathbf{j} - 2\,\mathbf{k}$

35. (a) $\mathbf{u} = \langle a, b, c\rangle$, $\mathbf{v} = \langle -b, a, 0\rangle$
(b) $\mathbf{u} = \langle c, c, c\rangle$, $\mathbf{v} = \langle c, 0, -c\rangle$
(c) $\mathbf{u} = \langle a, 0, 0\rangle$, $\mathbf{v} = \langle a, a, 0\rangle$, $a > 0$

36. Use vectors to determine whether the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, and $R(6, -2, -5)$ is a right triangle.

37. Find the values of b such that the vectors $\langle -6, b, 2\rangle$ and $\langle b, b^2, b\rangle$ are orthogonal.

38. Find the values of x such that the angle between the vectors $\langle 2, 1, -1\rangle$ and $\langle 1, x, 0\rangle$ is $\dfrac{\pi}{4}$.

39. Find a unit vector that is orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$.

40. Find two unit vectors that make an angle of $\dfrac{\pi}{3}$ with $\mathbf{v} = \langle 3, 4\rangle$.

Find the acute angle between the lines.

41. $2x - y = 3$, $3x + y = 7$

42. $x + 2y = 7$, $5x - y = 2$

Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$.

43. $\mathbf{a} = \langle 3, -4\rangle$, $\mathbf{b} = \langle 5, 0\rangle$

44. $\mathbf{a} = \langle 1, 2\rangle$, $\mathbf{b} = \langle -4, 1\rangle$

45. $\mathbf{a} = \langle -5, 12\rangle$, $\mathbf{b} = \langle 4, 6\rangle$

46. $\mathbf{a} = 2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$, $\mathbf{b} = \mathbf{j} + \frac{1}{2}\,\mathbf{k}$

47. $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

48. $\mathbf{a} = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$, $\mathbf{b} = 5\,\mathbf{i} - \mathbf{k}$

49. The **orthogonal projection** of $\mathbf{b}$ onto $\mathbf{a}$ is denoted $\text{orth}_{\mathbf{a}}\mathbf{b}$. Show that $\text{orth}_{\mathbf{a}}\mathbf{b} = \mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b}$ is orthogonal to $\mathbf{a}$.

50. Let $\mathbf{a} = \langle 1, 2\rangle$ and $\mathbf{b} = \langle -4, 1\rangle$. Find $\text{orth}_{\mathbf{a}}\mathbf{b}$ and illustrate this result by drawing the vectors $\mathbf{a}$, $\mathbf{b}$, $\text{proj}_{\mathbf{a}}\mathbf{b}$, and $\text{orth}_{\mathbf{a}}\mathbf{b}$.

51. If $\mathbf{a} = \langle 3, 0, -1\rangle$, find a vector $\mathbf{b}$ such that $\text{comp}_{\mathbf{a}}\mathbf{b} = 2$.

52. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors.
(a) Under what conditions is $\text{comp}_{\mathbf{a}}\mathbf{b} = \text{comp}_{\mathbf{b}}\mathbf{a}$?
(b) Under what conditions is $\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{a}$?

53. Find the work done by a force $\mathbf{F} = 8\,\mathbf{i} - 6\,\mathbf{j} + 9\,\mathbf{k}$ that moves an object from the point $(0, 10, 8)$ to the point $(6, 12, 20)$ along a straight line. The distance is measured in meters and the force in newtons.

54. A tow truck drags a stalled car along a road. The chain makes an angle of 30° with the road and the tension in the chain is 1500 N. How much work is done by the truck in pulling the car 1 km?

55. A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of 40° above the horizontal moves the sled 80 ft. Find the work done by the force.

56. A boat sails south with the help of a wind blowing in the direction S36°E with magnitude 400 lb. Find the work done by the wind as the boat moves 120 ft.

57. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

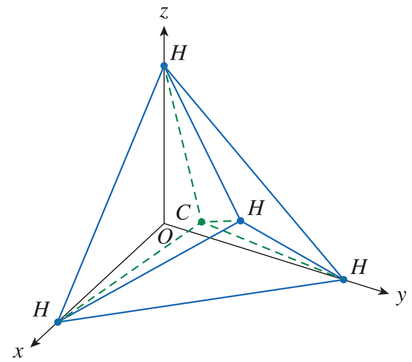
58. If $\mathbf{r} = \langle x, y, z\rangle$, $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a sphere, and find its center and radius.

59. Find the angle between a diagonal of a cube and one of its edges.

60. Find the angle between a diagonal of a cube and a diagonal of one of its faces.

61. A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The *bond angle* is the angle formed by the H-C-H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is approximately 109.5°.

Hint: Take the vertices of the tetrahedron to be the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$ as shown in the figure. The centroid is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.



62. If $\mathbf{c} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are all nonzero vectors, show that $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

63. Prove Property 4 of the dot product. Use either the definition of a dot product (considering the case $c > 0$, $c = 0$, and $c < 0$ separately) or the component form.

64. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

65. Prove the Cauchy–Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

66. The Triangle Inequality for vectors is

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

(a) Give a geometric interpretation of the Triangle Inequality.
(b) Use the Cauchy–Schwarz Inequality from Exercise 65 to prove the Triangle Inequality.

Hint: Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and use Property 3 of the dot product.

67. The Parallelogram Law states that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

(a) Give a geometric interpretation of the Parallelogram Law.
(b) Prove the Parallelogram Law. Consider the hint in Exercise 66.

68. Show that if $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal, then the vectors $\mathbf{u}$ and $\mathbf{v}$ must have the same length.

69. If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, show that

$$\text{proj}_{\mathbf{a}}\mathbf{b} \cdot \text{proj}_{\mathbf{b}}\mathbf{a} = (\mathbf{a} \cdot \mathbf{b}) \cos^2 \theta$$

9.4 The Cross Product

The **cross product** of two vectors is another form of vector multiplication. However, unlike the dot product, the result is a vector. Therefore, this is also called the **vector product**. The cross product is useful in geometry because it produces a vector perpendicular to both **a** and **b**. Let's consider this product by looking at a situation where it arises in physics and engineering.

Torque and the Cross Product

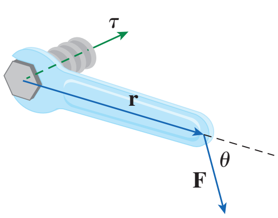


Figure 9.48
An illustration of the definition of torque.

Suppose we consider a force $\mathbf{F}$ acting on a rigid body at a point given by a position vector $\mathbf{r}$. For example, if we tighten a bolt by applying a force to a wrench as in Figure 9.48, we produce a turning effect called a **torque** $\boldsymbol{\tau}$. The magnitude of the torque depends on two things:

(1) The distance from the axis of the bolt to the point where the force is applied. This value is $|\mathbf{r}|$, the length of the position vector $\mathbf{r}$.
(2) The scalar component of the force $\mathbf{F}$ in the direction perpendicular to $\mathbf{r}$ (and in the plane perpendicular to the bolt). This is the only component that can cause a rotation and, from Figure 9.49, we see that it is

$$|\mathbf{F}| \sin \theta$$

where θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{F}$.

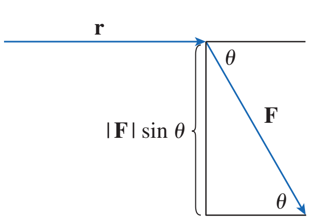


Figure 9.49
An illustration of the scalar component of the force **F**.

We define the magnitude of the torque vector to be the product of these two factors:

$$|\boldsymbol{\tau}| = |\mathbf{r}||\mathbf{F}| \sin \theta$$

The direction is along the axis of rotation. If $\mathbf{n}$ is a unit vector that points in the direction in which a right-threaded bolt moves (see Figure 9.48), we define the **torque** to be the vector

$$\boldsymbol{\tau} = (|\mathbf{r}||\mathbf{F}| \sin \theta)\, \mathbf{n} \tag{1}$$

We denote this torque vector by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ and we call it the *cross product* or *vector product* of $\mathbf{r}$ and $\mathbf{F}$.

The type of expression in Equation 1 occurs frequently in the study of fluid flow, planetary motion, and other areas of physics and engineering. Therefore, we use the following definition for the cross product of *any* pair of three-dimensional vectors **a** and **b**.

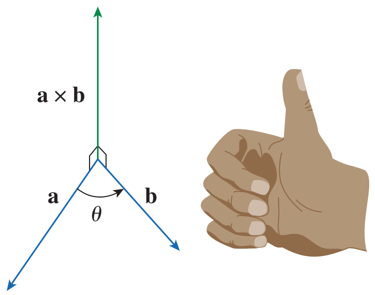


Figure 9.50
The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

Definition • Cross Product

If **a** and **b** are nonzero three-dimensional vectors, the **cross product** of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}|\sin\theta)\,\mathbf{n}$$

where θ is the angle between **a** and **b**, $0 \le \theta \le \pi$, and **n** is a unit vector perpendicular to both **a** and **b** and whose direction is given by the **right-hand rule**: if a person curls the fingers of their right hand through the angle θ from **a** to **b**, then their thumb points in the direction of **n**. See Figure 9.50.

If either **a** or **b** is **0**, then we define $\mathbf{a} \times \mathbf{b}$ to be **0**.

Because $\mathbf{a} \times \mathbf{b}$ is a scalar multiple of **n**, it has the same direction as **n** and so

$$\mathbf{a} \times \mathbf{b} \text{ is orthogonal to both } \mathbf{a} \text{ and } \mathbf{b} \tag{2}$$

Notice that two nonzero vectors **a** and **b** are parallel if and only if the angle between them is 0 or π. In either case, $\sin\theta = 0$ and so $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

$$\text{Two nonzero vectors } \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel if and only if } \mathbf{a} \times \mathbf{b} = \mathbf{0}. \tag{3}$$

This makes sense in the torque interpretation: if we pull or push the wrench in the direction of its handle (so **F** is parallel to **r**), we produce no torque.

Example 1 Find the Magnitude of the Torque

A bolt is tightened by applying a 40-N force to a 0.25-m wrench, as shown in Figure 9.51. Find the magnitude of the torque about the center of the bolt.

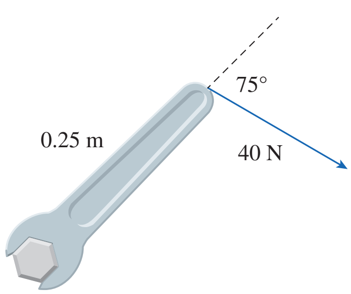


Figure 9.51
The geometry associated with the bolt and wrench.

Solution

The magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}|$$
$$= |\mathbf{r}||\mathbf{F}|\sin 75° = (0.25)(40)\sin 75° \qquad \text{Formula for the magnitude of torque.}$$
$$= 10\sin 75° \approx 9.659 \text{ N} \cdot \text{m}. \qquad \text{Simplify.}$$

If the bolt is right-threaded, then the torque vector is

$$\boldsymbol{\tau} = |\boldsymbol{\tau}|\mathbf{n} \approx \mathbf{9.659\ N \cdot m}.$$

where **n** is a unit vector directed down into the page (by the right-hand rule). ■

Example 2 Cross Product of Standard Basis Vectors

Find $\mathbf{i} \times \mathbf{j}$ and $\mathbf{j} \times \mathbf{i}$.

Solution

The standard basis vectors **i** and **j** both have length 1 and the angle between them is $\pi/2$.

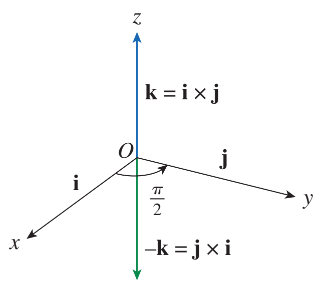


Figure 9.52
The unit vector perpendicular to **i** and **j** is **k**.

By the right-hand rule, the unit vector perpendicular to **i** and **j** is $\mathbf{n} = \mathbf{k}$ (see Figure 9.52). Therefore,

$$\mathbf{i} \times \mathbf{j} = \left(|\mathbf{i}|\,|\mathbf{j}|\sin\left(\frac{\pi}{2}\right)\right)\mathbf{k} = \mathbf{k}.$$

If we apply the right-hand rule to the vectors **j** and **i** (in that order), then the vector **n** points downward, so $\mathbf{n} = -\mathbf{k}$.

Therefore, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$. ■

From Example 2, we see that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

so the cross product is not commutative. Similar reasoning as in Example 2 shows that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

In general, the right-hand rule shows that

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

Another algebraic law that fails for the cross product is the associative law for multiplication: that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

For example, if $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{i}$, and $\mathbf{c} = \mathbf{j}$, then

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

whereas

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{j} = -\mathbf{j}$$

So, the cross product is not associative. However, there are some usual laws of algebra that are true for cross products. Here is a summary of these properties.

Properties of the Cross Product

If **a**, **b**, and **c** are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

Property 2 is proved by applying the definition of a cross product to each of the three expressions. Properties 3 and 4 (the Vector Distributive Laws) are more difficult to establish (see Exercise 4).

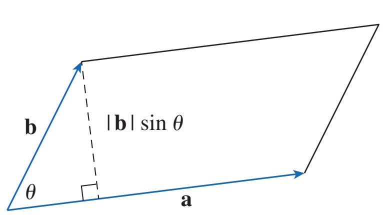


Figure 9.53
The parallelogram determined by **a** and **b**.

A geometric interpretation of the length of the cross product is illustrated in Figure 9.53. If **a** and **b** are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}|\sin\theta$, and area

$$A = |\mathbf{a}|(|\mathbf{b}|\sin\theta) = |\mathbf{a} \times \mathbf{b}|$$

This expression is interpreted as the length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by **a** and **b**.

The Cross Product in Component Form

Suppose **a** and **b** are given in component form:

$$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k} \qquad \mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$$

We can express $\mathbf{a} \times \mathbf{b}$ in component form by using the Vector Distributive Laws together with the results from Example 2:

Note that
$\mathbf{i} \times \mathbf{i} = \mathbf{0},\ \mathbf{j} \times \mathbf{j} = \mathbf{0},\ \mathbf{k} \times \mathbf{k} = \mathbf{0}.$

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}) \times (b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}) \\
&= a_1b_1\,\mathbf{i} \times \mathbf{i} + a_1b_2\,\mathbf{i} \times \mathbf{j} + a_1b_3\,\mathbf{i} \times \mathbf{k} \\
&\quad + a_2b_1\,\mathbf{j} \times \mathbf{i} + a_2b_2\,\mathbf{j} \times \mathbf{j} + a_2b_3\,\mathbf{j} \times \mathbf{k} \\
&\quad + a_3b_1\,\mathbf{k} \times \mathbf{i} + a_3b_2\,\mathbf{k} \times \mathbf{j} + a_3b_3\,\mathbf{k} \times \mathbf{k} \\
&= a_1b_2\,\mathbf{k} + a_1b_3(-\mathbf{j}) + a_2b_1(-\mathbf{k}) + a_2b_3\,\mathbf{i} + a_3b_1\,\mathbf{j} + a_3b_2(-\mathbf{i}) \\
&= (a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}
\end{aligned}$$

Therefore, if $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle. \tag{4}$$

A determinant is a scalar quantity computed using the elements of a square matrix.

In order to make it easier to remember and compute the cross product, we often use the notation associated with determinants. A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

In words, we multiply across the diagonals and subtract. For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = (2)(4) - (1)(-6) = 14$$

A **determinant of order 3** can be defined in terms of second-order determinants:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \tag{5}$$

Observe that each term of the right side of Equation 5 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears. Notice also the minus sign in the second term. For example,

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} &= 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix} \\
&= 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0) = -38
\end{aligned}$$

We can now rewrite the expression for $\mathbf{a} \times \mathbf{b}$ using second-order determinants and the standard basis vectors **i**, **j**, and **k**. The cross product of the vectors $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$ is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} \tag{6}$$

Given the similarity between Equations 5 and 6, we often write

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{7}$$

Although the first row of the symbolic determinant in Equation 7 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 5, we obtain Equation 6. The symbolic formula in Equation 7 is probably the easiest way to remember and compute cross products.

Example 3 Cross Product of Vectors in Component Form

If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k} \\ &= (-15 - 28)\,\mathbf{i} - (-5 - 8)\,\mathbf{j} + (7 - 6)\,\mathbf{k} = -43\,\mathbf{i} + 13\,\mathbf{j} + \mathbf{k}. \end{aligned}$$

■

Example 4 Vector Perpendicular to a Plane

Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Solution

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore perpendicular to the plane through P, Q, and R.

Use Equation 1 from Section 9.2 to write

$$\overrightarrow{PQ} = (-2 - 1)\,\mathbf{i} + (5 - 4)\,\mathbf{j} + (-1 - 6)\,\mathbf{k} = -3\,\mathbf{i} + \mathbf{j} - 7\,\mathbf{k} \quad \text{and}$$

$$\overrightarrow{PR} = (1 - 1)\,\mathbf{i} + (-1 - 4)\,\mathbf{j} + (1 - 6)\,\mathbf{k} = -5\,\mathbf{j} - 5\,\mathbf{k}.$$

Compute the cross product of these two vectors.

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} \\ &= (-5 - 35)\,\mathbf{i} - (15 - 0)\,\mathbf{j} + (15 - 0)\,\mathbf{k} = -40\,\mathbf{i} - 15\,\mathbf{j} + 15\,\mathbf{k} \end{aligned}$$

Therefore, the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane. Any nonzero scalar multiple of this vector, for example $\langle -8, -3, 3 \rangle$, is also perpendicular to the plane. ■

Example 5 Area of a Triangle

Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Solution

In Example 4, we found $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$.

The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product.

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

The area of the triangle PQR is half the area of this parallelogram, that is $\frac{5}{2}\sqrt{82}$. ■

Triple Products

The expression $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the **scalar triple product** of the vectors **a**, **b**, and **c**. The geometric interpretation of the scalar triple product can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c**. See Figure 9.54. The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between **a** and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. Note that we need to use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$. Therefore, the volume of the parallelepiped is

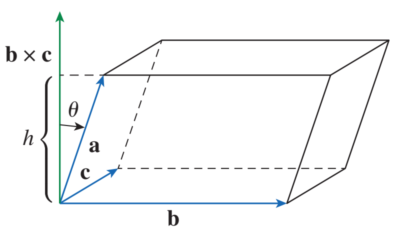


Figure 9.54
The scalar triple product is the volume of the parallelepiped.

$$\begin{aligned} V = Ah &= |\mathbf{b} \times \mathbf{c}|(|\mathbf{a}||\cos\theta|) && h \text{ is the length of a vector projection.} \\ &= |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|. && \text{Definition of dot product.} \end{aligned}$$

This proves the following formula.

Volume of a Parallelepiped

The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Instead of thinking of the parallelepiped as having its base parallelogram determined by **b** and **c**, we can think of it with base parallelogram determined by **a** and **b**. In this way, we see that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

Because the dot product is commutative, we can write

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \tag{8}$$

Suppose that **a**, **b**, and **c** are given in component form:

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \qquad \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

Then

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a \cdot \left[\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k} \right] \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \end{aligned}$$

This shows that we can write the scalar triple product of **a**, **b**, and **c** as the determinant whose rows are the components of these vectors:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{9}$$

Example 6 Coplanar Vectors

Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7\rangle$, $\mathbf{b} = \langle 2, -1, 4\rangle$, and $\mathbf{c} = \langle 0, -9, 18\rangle$ are coplanar, that is, they lie in the same plane.

Solution

Use Equation 9 to compute their scalar triple product.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

Equation 9 for scalar triple product.

$$= 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

Determinant of order 3, Equation 5.

$$= 1(18) - 4(36) - 7(-18) = 0$$

Determinants of order 2; simplify.

Therefore, the volume of the parallelepiped determined by **a**, **b**, and **c** is 0. This means that **a**, **b**, and **c** are coplanar. ■

The product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is called the **vector triple product** of **a**, **b**, and **c**. The proof of the following formula for the vector triple product is left as Exercise 53.

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} \tag{10}$$

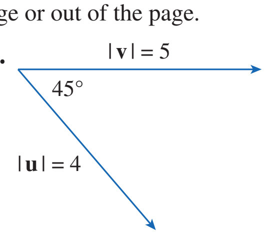
Equation 10 will be used to derive Kepler's First Law of planetary motion in Chapter 10.

9.4 Exercises

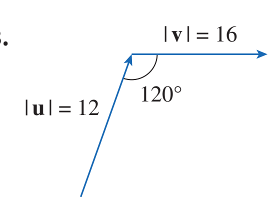
1. State whether each expression is meaningful. If not, explain why. If so, state whether the result is a vector or a scalar.
(a) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ (b) $\mathbf{a}\times(\mathbf{b}\cdot\mathbf{c})$
(c) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ (d) $(\mathbf{a}\cdot\mathbf{b})\times\mathbf{c}$
(e) $(\mathbf{a}\cdot\mathbf{b})\times(\mathbf{c}\cdot\mathbf{d})$ (f) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$

Find $|\mathbf{u}\times\mathbf{v}|$ and determine whether $\mathbf{u}\times\mathbf{v}$ is directed into the page or out of the page.

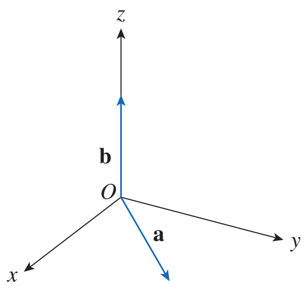
2.



3.

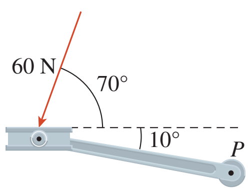


4. The figure shows a vector **a** in the *xy*-plane and a vector **b** in the direction of **k**. Their lengths are $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 2$.

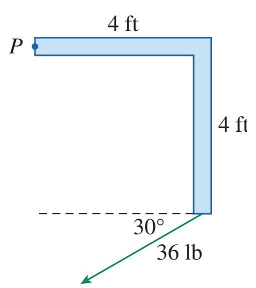


(a) Find $|\mathbf{a} \times \mathbf{b}|$.
(b) Use the right-hand rule to decide whether the components of $\mathbf{a} \times \mathbf{b}$ are positive, negative, or 0.

5. A bicycle pedal is pushed by a foot with a 60-N force as shown in the figure. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about *P*.



6. Find the magnitude of the torque about *P* if a 36-lb force is applied as shown in the figure.



Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both **a** and **b**.

7. $\mathbf{a} = \langle 2, 3, 0 \rangle, \ \mathbf{b} = \langle 1, 0, 5 \rangle$

8. $\mathbf{a} = \langle 4, 3, -2 \rangle, \ \mathbf{b} = \langle 2, -1, 1 \rangle$

9. $\mathbf{a} = \langle 1, 0, 0 \rangle, \ \mathbf{b} = \langle 1, -2, -3 \rangle$

10. $\mathbf{a} = 2\,\mathbf{j} - 4\,\mathbf{k}, \ \mathbf{b} = -\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$

11. $\mathbf{a} = 3\,\mathbf{i} + 3\,\mathbf{j} - 3\,\mathbf{k}, \ \mathbf{b} = 3\,\mathbf{i} - 3\,\mathbf{j} + 3\,\mathbf{k}$

12. $\mathbf{a} = \frac{1}{2}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} + \frac{1}{4}\,\mathbf{k}, \ \mathbf{b} = \mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$

13. $\mathbf{a} = t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}, \quad \mathbf{b} = \mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$

14. $\mathbf{a} = \mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}, \ \mathbf{b} = 2\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}$

15. $\mathbf{a} = \langle t, 1, 1/t \rangle, \ \mathbf{b} = \langle t^2, t^2, 1 \rangle$

16. $\mathbf{a} = \langle t, t^2, t^3 \rangle, \ \mathbf{b} = \langle 1, 2t, 3t^2 \rangle$

17. If $\mathbf{a} = \mathbf{i} - 2\,\mathbf{k}$, and $\mathbf{b} = \mathbf{j} + \mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$. Sketch **a**, **b**, and $\mathbf{a} \times \mathbf{b}$ as vectors with initial points at the origin.

Use properties of cross products to find the resulting vector.

18. $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$

19. $\mathbf{k} \times (\mathbf{i} - 2\,\mathbf{j})$

20. $(\mathbf{j} - \mathbf{k}) \times (\mathbf{k} - \mathbf{i})$

21. $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$

22. If $\mathbf{a} = \langle 2, -1, 3 \rangle$ and $\mathbf{b} = \langle 4, 2, 1 \rangle$, find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$.

23. If $\mathbf{a} = \langle 1, 0, 1 \rangle$, $\mathbf{b} = \langle 2, 1, -1 \rangle$, and $\mathbf{c} = \langle 0, 1, 3 \rangle$, show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

24. Find two unit vectors orthogonal to both $\langle 3, 2, 1 \rangle$ and $\langle -1, 1, 0 \rangle$.

25. Find two unit vectors orthogonal to both $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\,\mathbf{i} + \mathbf{k}$.

Find the area of the parallelogram with given vertices.

26. $A(-3, 0)$, $B(-1, 3)$, $C(5, 2)$, $D(3, -1)$

27. $A(-2, 1)$, $B(0, 4)$, $C(4, 2)$, $D(2, -1)$

28. $P(1, 0, 2)$, $Q(3, 3, 3)$, $R(7, 5, 8)$, $S(5, 2, 7)$

29. $P(1, 2, 3)$, $Q(1, 3, 6)$, $R(3, 8, 6)$, $S(3, 7, 3)$

(a) Find a nonzero vector orthogonal to the plane through the points *P*, *Q*, and *R*.
(b) Find the area of triangle *PQR*.

30. $P(0, -2, 0)$, $Q(4, 1, -2)$, $R(5, 3, 1)$

31. $P(-1, 3, 1)$, $Q(0, 5, 2)$, $R(4, 3, -1)$

32. $P(0, 0, -3)$, $Q(4, 2, 0)$, $R(3, 3, 1)$

33. $P(2, -3, 4)$, $Q(-1, -2, 2)$, $R(3, 1, -3)$

Find the volume of the parallelepiped determined by the vectors **a**, **b**, and **c**.

34. $\mathbf{a} = \langle 6, 3, -1 \rangle, \ \mathbf{b} = \langle 0, 1, 2 \rangle, \ \mathbf{c} = \langle 4, -2, 5 \rangle$

35. $\mathbf{a} = \langle 1, 2, 3 \rangle, \ \mathbf{b} = \langle -1, 1, 2 \rangle, \ \mathbf{c} = \langle 2, 1, 4 \rangle$

36. $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}, \ \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}, \ \mathbf{c} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$

37. $\mathbf{a} = \mathbf{i} + \mathbf{j}, \ \mathbf{b} = \mathbf{j} + \mathbf{k}, \ \mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Find the volume of the parallelepiped with adjacent edges *PQ*, *PR*, and *PS*.

38. $P(2, 0, -1)$, $Q(4, 1, 0)$, $R(3, -1, 1)$, $S(2, -2, 2)$

39. $P(3, 0, 1)$, $Q(-1, 2, 5)$, $R(5, 1, -1)$, $S(0, 4, 2)$

40. $P(-2, 1, 0)$, $Q(2, 3, 2)$, $R(1, 4, -1)$, $S(3, 6, 1)$

41. Use the scalar triple product to verify that the vectors $\mathbf{u} = \mathbf{i} + 5\,\mathbf{j} - 2\,\mathbf{k}$, $\mathbf{v} = 3\,\mathbf{i} - \mathbf{j}$, and $\mathbf{w} = 5\,\mathbf{i} + 9\,\mathbf{j} - 4\,\mathbf{k}$ are coplanar.

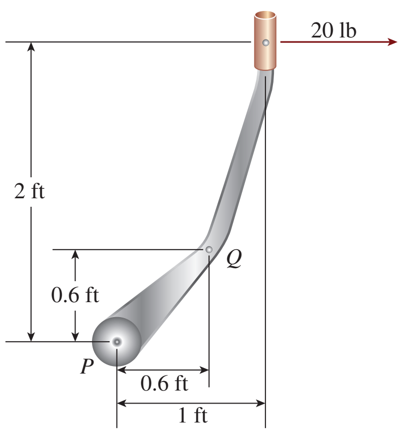
42. Use the scalar triple product to determine whether the points $A(1, 3, 2)$, $B(3, -1, 6)$, $C(5, 2, 0)$, and $D(3, 6, -4)$ lie in the same plane.

43. A wrench 30 cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction $\langle 0, 3, -4 \rangle$ at the end of the wrench. Find the magnitude of the force needed to supply a $100 \text{ N} \cdot \text{m}$ of torque to the bolt.

44. Let $\mathbf{v} = 5\,\mathbf{j}$ and let $\mathbf{u}$ be a vector with length 3 that starts at the origin and rotates in the xy-plane. Find the maximum and minimum values of the length of the vector $\mathbf{u} \times \mathbf{v}$. In what direction does $\mathbf{u} \times \mathbf{v}$ point?

45. (a) A horizontal force of 20 lb is applied to the handle of a gearshift lever as shown in the figure. Find the magnitude of the torque about the pivot point P.

(b) Find the magnitude of the torque about P if the same force is applied at the elbow Q of the lever.



46. If $\mathbf{a} \cdot \mathbf{b} = \sqrt{3}$ and $\mathbf{a} \times \mathbf{b} = \langle 1, 2, 2 \rangle$, find the angle between $\mathbf{a}$ and $\mathbf{b}$.

47. (a) Find all vectors $\mathbf{v}$ such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, -5 \rangle$$

(b) Explain why there is no vector $\mathbf{v}$ such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, 5 \rangle$$

48. Let P be a point not on the line L that passes through the points Q and R.

(a) Show that the distance d from the point P to the line L is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

where $\mathbf{a} = \overrightarrow{QR}$ and $\mathbf{b} = \overrightarrow{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(1, 1, 1)$ to the line through $Q(0, 6, 8)$ and $R(-1, 4, 7)$.

49. Let P be a point on the plane that passes through the noncollinear points Q, R, and S.

(a) Show that the distance d from P to the plane is

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$$

where $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QS}$, and $\mathbf{c} = \overrightarrow{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(2, 1, 4)$ to the plane through the points $Q(1, 0, 0)$, $R(0, 2, 0)$ and $S(0, 0, 3)$.

---

50. Show that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.

51. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, show that

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

52. Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

53. Prove the following formula (10) for the vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

54. Use Exercise 53 to show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

55. Prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

56. Suppose that $\mathbf{a} \neq \mathbf{0}$.

(a) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

(b) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

(c) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

57. (a) If $\mathbf{u}$ is a unit vector and $\mathbf{a}$ is orthogonal to $\mathbf{u}$, show that

$$\mathbf{u} \times (\mathbf{u} \times \mathbf{a}) = -\mathbf{a}$$

(b) If $\mathbf{u}$ is a unit vector and $\mathbf{v}$ is any vector in V_3, show that

$$\mathbf{u} \times (\mathbf{u} \times (\mathbf{u} \times (\mathbf{u} \times \mathbf{v}))) = -\mathbf{u} \times (\mathbf{u} \times \mathbf{v})$$

58. (a) If $\mathbf{u} \cdot \mathbf{r} = \mathbf{v} \cdot \mathbf{r}$ for every vector $\mathbf{r}$ in V_3, show that $\mathbf{u} = \mathbf{v}$.

(b) Prove Property 3 of the cross product

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

by showing that

$$[\mathbf{a} \times (\mathbf{b} + \mathbf{c})] \cdot \mathbf{r} = [\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}] \cdot \mathbf{r}$$

for every vector $\mathbf{r}$ in V_3.

59. If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are noncoplanar vectors, let

$$\mathbf{k}_1 = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \qquad \mathbf{k}_2 = \frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

$$\mathbf{k}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

These vectors occur in the study of crystallography. Vectors of the form $n_1\mathbf{v}_1 + n_2\mathbf{v}_2 + n_3\mathbf{v}_3$, where each n_i is an integer, form a *lattice* for a crystal. Vectors written similarly in terms of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ form a *reciprocal lattice*.

(a) Show that $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_j$ if $i \neq j$.

(b) Show that $\mathbf{k}_i \cdot \mathbf{v}_j = 1$ for $i = 1, 2, 3$.

(c) Show that $\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \dfrac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$.

Discovery Project | The Geometry of a Tetrahedron

A tetrahedron is a solid with four vertices, P, Q, R, and S, and four triangular faces, as shown in Figure 9.55.

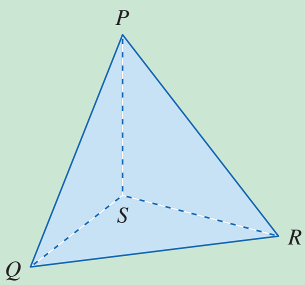


Figure 9.55
The tetrahedron $PQRS$.

1. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be vectors with lengths equal to the areas of the faces opposite the vertices P, Q, R, and S, respectively, and directions perpendicular to the respective faces and pointing outward. Show that
$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$$
2. The volume V of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face.
   (a) Find a formula for the volume of a tetrahedron in terms of the coordinates of its vertices P, Q, R, and S.
   (b) Find the volume of the tetrahedron whose vertices are $P(1, 1, 1)$, $Q(1, 2, 3)$, $R(1, 1, 2)$, and $S(3, -1, 2)$.
3. Suppose the tetrahedron in the figure has a trirectangular vertex S. (This means that the three angles at S are all right angles.) Let A, B, and C be the areas of the three faces that meet at S, and let D be the area of the opposite face PQR. Use the result of Problem 1 or any other method to show that
$$D^2 = A^2 + B^2 + C^2$$
This is a three-dimensional version of the Pythagorean Theorem.

9.5 Equations of Lines and Planes

Lines

A line in the xy-plane is completely determined by a point on the line and the direction of the line, described by its slope or angle of inclination. The equation of the line can then be written using point-slope form.

Similarly, a line L in three-dimensional space is determined by a point $P_0(x_0, y_0, z_0)$ on L and a direction for L, which is described by a vector $\mathbf{v}$ parallel to the line. Let $P(x, y, z)$ be an arbitrary point on L and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P (that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$). If $\mathbf{a}$ is the vector with representation $\overrightarrow{P_0P}$, as in Figure 9.56, then the Triangle Law for vector addition gives $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$.

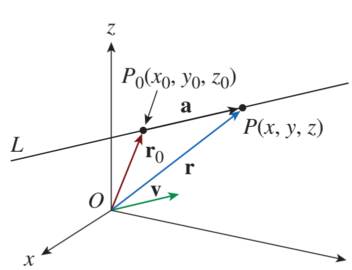


Figure 9.56
A line L in three-dimensional space in the direction of $\mathbf{v}$.

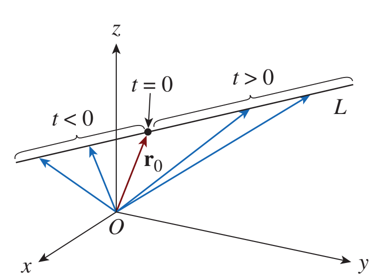


Figure 9.57
The vector equation $\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v}$ describes the line L.

Since **a** and **v** are parallel vectors, there is a scalar t such that $\mathbf{a} = t\,\mathbf{v}$. Therefore,

$$\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v} \tag{1}$$

which is a **vector equation** of L. Each value of the **parameter** t gives the position vector **r** of a point on L. This means, as t varies, the line is traced out by the tip of the vector **r**. As Figure 9.57 indicates, positive values of t correspond to points on L that lie on one side of P_0, whereas negative values of t correspond to points that lie on the other side of P_0.

If the vector **v** that gives the direction of the line L is written in component form as $\mathbf{v} = \langle a, b, c\rangle$, then $t\,\mathbf{v} = \langle ta, tb, tc\rangle$. We can also write $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$. So, the vector equation (1) becomes

$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$

Two vectors are equal if and only if their corresponding components are equal. Therefore, this vector equation yields the three scalar equations:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct \tag{2}$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$. Each value of the parameter t gives a point (x, y, z) on L.

Example 1 Equation of a Line with Given Direction

(a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\mathbf{i} + 4\,\mathbf{j} - 2\,\mathbf{k}$.
(b) Find two other points on the line.

Solution

(a) The position vector of P_0 is $\mathbf{r}_0 = \langle 5, 1, 3\rangle = 5\,\mathbf{i} + \mathbf{j} + 3\,\mathbf{k}$.

The line is in the direction of $\mathbf{v} = \mathbf{i} + 4\,\mathbf{j} - 2\,\mathbf{k}$.

The vector equation for the line is

$$\begin{aligned}\mathbf{r} &= (5\,\mathbf{i} + \mathbf{j} + 3\,\mathbf{k}) + t(\mathbf{i} + 4\,\mathbf{j} - 2\,\mathbf{k})\\ &= (5 + t)\,\mathbf{i} + (1 + 4t)\,\mathbf{j} + (3 - 2t)\,\mathbf{k}.\end{aligned}$$

The parametric equations are

$x = 5 + t,\ \ y = 1 + 4t,\ \ z = 3 - 2t.$

(b) Pick any two values for t.

Let $t = 1$: $x = 6$, $y = 5$, $z = 1$. So, (6, 5, 1) is a point on the line.

Let $t = -1$: $x = 4$, $y = -3$, $z = 5$. The point $(4, -3, 5)$ is on the line.

Figure 9.58 shows a graph of the line. ■

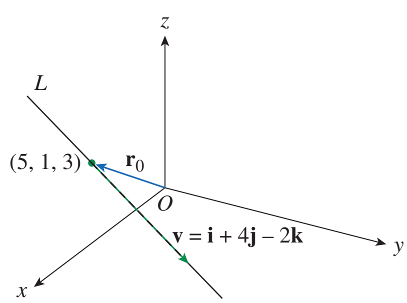


Figure 9.58
The line L determined by the point (5, 1, 3) in the direction of the vector **v**.

The vector equation and the parametric equations of a line are not unique. If we change the point, or the parameter, or choose a different parallel vector, then the equations are different. For example, if, instead of (5, 1, 3), we choose the point (6, 5, 1) in Example 1, then the parametric equations of the line become

$$x = 6 + t \qquad y = 5 + 4t \qquad z = 1 - 2t$$

Or, if we use the original point (5, 1, 3) but choose the parallel vector $2\,\mathbf{i} + 8\,\mathbf{j} - 4\,\mathbf{k}$, the parametric equations are

$$x = 5 + 2t \qquad y = 1 + 8t \qquad z = 3 - 4t$$

In general, if a vector $\mathbf{v} = \langle a, b, c\rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L. Since any vector parallel to $\mathbf{v}$ could be used to describe L, any three numbers proportional to a, b, and c could also be used as a set of direction numbers for L.

Another way to describe a line L is to eliminate the parameter t from the equations in (2). If none of a, b, or c is 0, then we can solve each of these equations for t:

$$t = \frac{x - x_0}{a} \qquad t = \frac{y - y_0}{b} \qquad t = \frac{z - z_0}{c}$$

Equate the results to obtain

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{3}$$

These equations are called the **symmetric equations** of L. Notice that the numbers a, b, and c that appear in the denominators of Equation 3 are direction numbers of L, that is, components of a vector parallel to L. If one of a, b, or c is 0, we can still eliminate t. For example, if $a = 0$, we could write the equations of L as

$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This means that L lies in the vertical plane $x = x_0$.

Example 2 Equations of a Line Through Two Points

(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.

(b) At what point does this line intersect the xy-plane?

Solution

(a) We are not explicitly given a vector parallel to the line.

However, the vector $\mathbf{v}$ with representation $\overrightarrow{AB}$ is parallel to the line.

$\mathbf{v} = \langle 3 - 2, -1 - 4, 1 - (-3)\rangle = \langle 1, -5, 4\rangle$

Therefore, direction numbers are $a = 1$, $b = -5$, and $c = 4$. Let P_0 be the point $(2, 4, -3)$. The parametric equations are

$x = 2 + t, \quad y = 4 - 5t, \quad z = -3 + 4t.$

The symmetric equations are $\dfrac{x - 2}{1} = \dfrac{y - 4}{-5} = \dfrac{z + 3}{4}$.

(b) The line intersects the xy-plane when $z = 0$.
Use the parametric equations, set $z = 0$, and solve for t.

$$z = -3 + 4t = 0 \qquad \Rightarrow \qquad t = \frac{3}{4}$$

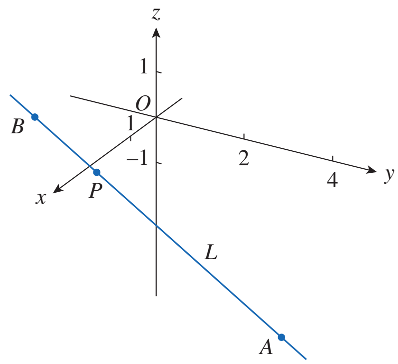


Figure 9.59
Graph of the line L and the point P where it intersects the xy-plane.

Use this value of t to find the remaining coordinates of the point.

$$x = 2 + \frac{3}{4} = \frac{11}{4}, \qquad y = 4 - 5\left(\frac{3}{4}\right) = \frac{1}{4}$$

Therefore, the line intersects the xy-plane at the point $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$.

Alternatively, we can use $z = 0$ in the symmetric equations to obtain

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{3}{4}.$$

This also yields $x = \frac{11}{4}$ and $y = \frac{1}{4}$.

Figure 9.59 shows a graph of the line and the point P where it intersects the xy-plane. ■

In general, the procedure in Example 2 shows that direction numbers of the line L through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are $x_1 - x_0$, $y_1 - y_0$, and $z_1 - z_0$. Therefore, symmetric equations of L are

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Often we need a description of just a line segment, not of an entire line. For example, suppose we want to describe the line segment AB in Example 2. Notice that if we let $t = 0$ in the parametric equations in Example 2(a), we get the point $(2, 4, -3)$, and if we let $t = 1$ we get $(3, -1, 1)$. Therefore, the line segment AB is described by the parametric equations

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t \qquad 0 \le t \le 1$$

or by the corresponding vector equation

$$\mathbf{r}(t) = \langle 2 + t, 4 - 5t, -3 + 4t \rangle \quad 0 \le t \le 1$$

We know from Equation 1 that the vector equation of a line through the (tip of the) vector $\mathbf{r}_0$ in the direction of a vector $\mathbf{v}$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. If the line also passes through the tip of $\mathbf{r}_1$, then we can take $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$ and so its vector equation is

$$\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$$

The line segment from the tip of $\mathbf{r}_0$ to the tip of $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1 \tag{4}$$

Example 3 Skew Lines

Show that the lines L_1 and L_2 with parametric equations

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are **skew lines**; that is, they do not intersect and are not parallel, and therefore do not lie in the same plane.

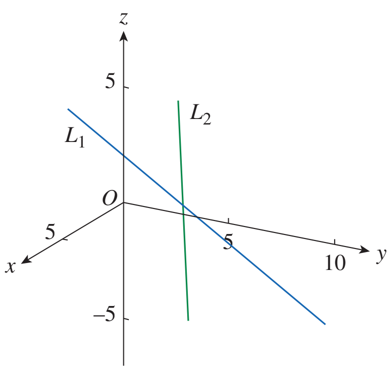


Figure 9.60
The lines L_1 and L_2 are skew lines.

Solution

The direction vectors $\langle 1, 3, -1\rangle$ and $\langle 2, 1, 4\rangle$ are not parallel; their components are not proportional. Therefore, the lines are not parallel.

If L_1 and L_2 intersect, there must be values of t and s such that

$$\begin{aligned} 1 + t &= 2s, \\ -2 + 3t &= 3 + s, \\ 4 - t &= -3 + 4s. \end{aligned}$$

If we solve the first two equations (simultaneously), we get $t = \dfrac{11}{5}$ and $s = \dfrac{8}{5}$.

These values do not satisfy the third equation.

So, there are no values of t and s that satisfy the three equations, so L_1 and L_2 do not intersect.

Therefore, L_1 and L_2 are skew lines. Figure 9.60 shows a graph of L_1 and L_2. ■

Planes

Although a line in space is completely determined by a point and a direction, a plane in space is just a little more complicated to describe. A single vector parallel to a plane is not enough to convey the *direction* of the plane, but a vector perpendicular to the plane is sufficient. Therefore, a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. The orthogonal vector $\mathbf{n}$ is called a **normal vector**.

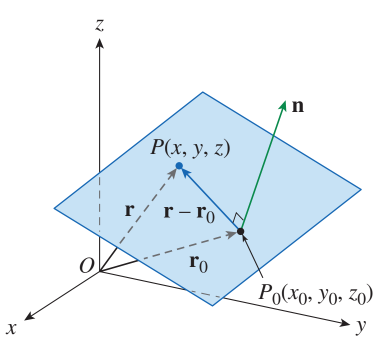


Figure 9.61
The geometric definition of a plane in terms of $\mathbf{r}$, $\mathbf{r}_0$, and $\mathbf{n}$.

Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P. Then the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$. See Figure 9.61. The normal vector $\mathbf{n}$ is orthogonal to every vector in the given plane. In particular, $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r}_0$; the dot product is 0.

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \tag{5}$$

Using a property of the dot product, this equation can be written as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0 \tag{6}$$

Either Equation 5 or Equation 6 is called a **vector equation of the plane**.

To obtain a scalar equation for a plane, write $\mathbf{n} = \langle a, b, c\rangle$, $\mathbf{r} = \langle x, y, z\rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$. Then the vector equation (5) becomes

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

If we expand the left side of this equation using the definition of a dot product, we obtain the following.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{7}$$

Equation 7 is the **scalar equation of the plane through** $P_0(x_0, y_0, z_0)$ **with normal vector** $\mathbf{n} = \langle a, b, c\rangle$.

Example 4 Equation of a Plane

Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4\rangle$. Find the intercepts and sketch the plane.

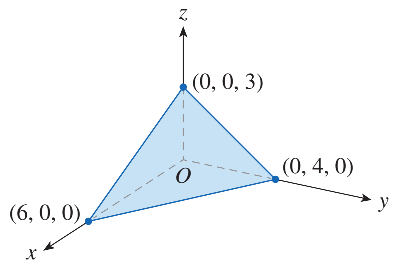


Figure 9.62
Graph of the plane described by $2x + 3y + 4z = 12$.

Solution

Use $\mathbf{n} = \langle 2, 3, 4 \rangle \quad \Rightarrow \quad a = 2, b = 3, c = 4$ and
$P_0(2, 4, -1) \quad \Rightarrow \quad x_0 = 2, y_0 = 4, z_0 = -1$ in Equation 7.

The equation of the plane is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0 \qquad \text{Equation 7.}$$

or

$$2x + 3y + 4z = 12 \qquad \text{Simplify.}$$

A graph of the plane is shown in Figure 9.62.

To find the x-intercept, we set $y = z = 0$ in this equation to obtain $x = 6$.

Similarly, the y-intercept is 4 and the z-intercept is 3. ■

If we multiply through and collect terms in Equation 7, as we did in Example 4, we can rewrite the equation of a plane as

$$ax + by + cz + d = 0 \tag{8}$$

where $d = -(ax_0 + by_0 + cz_0)$. Equation 8 is called a **linear equation** in x, y, and z. Conversely, it can be shown that if a, b, and c are not all 0, then the linear equation 8 represents a plane with a normal vector $\langle a, b, c \rangle$. (See Exercise 81.)

Example 5 The Plane Through Three Points

Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

Solution

The vectors **a** and **b** corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\mathbf{a} = \langle 3 - 1, -1 - 3, 6 - 2 \rangle = \langle 2, -4, 4 \rangle, \qquad \mathbf{b} = \langle 5 - 1, 2 - 3, 0 - 2 \rangle = \langle 4, -1, -2 \rangle.$$

Since both **a** and **b** lie in the plane, their cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane. Therefore, we can use the cross product as the normal vector.

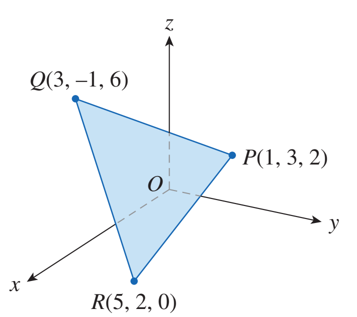


Figure 9.63
Graph of the plane determined by P, Q, and R.

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\,\mathbf{i} + 20\,\mathbf{j} + 14\,\mathbf{k}$$

Use the point $P(1, 3, 2)$ and the normal vector **n** to write an equation of the plane.

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$

Figure 9.63 shows a graph of the plane and the points P, Q, and R. ■

Example 6 Find a Point of Intersection

Find the point at which the line with parametric equations
$x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

Solution

Substitute the expressions for x, y, and z from the parametric equations into the equation of the plane.

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

$$-10t = 20 \quad \Rightarrow \quad t = -2 \qquad \text{Solve for } t.$$

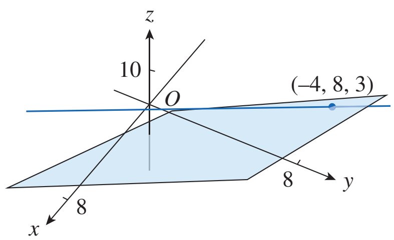


Figure 9.64
Graph of the plane, the line, and the point of intersection.

The point of intersection occurs when the parameter value is $t = -2$.

$x = 2 + 3(-2) = -4, \quad y = -4(-2) = 8, \quad z = 5 - 2 = 3$

Figure 9.64 shows the plane, the line, and the point of intersection $(-4, 8, 3)$. ■

Two planes are **parallel** if their normal vectors are parallel. For example, the planes $x + 2y - 3z = 4$ and $2x + 4y - 6z = 3$ are parallel because their normal vectors are $\mathbf{n}_1 = \langle 1, 2, -3\rangle$ and $\mathbf{n}_2 = \langle 2, 4, -6\rangle$ and $\mathbf{n}_2 = 2\mathbf{n}_1$. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors. The angle θ is illustrated in Figure 9.65.

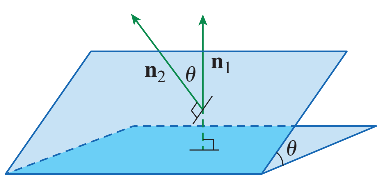


Figure 9.65
The angle θ is determined by the two normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$.

Example 7 Angle Between Planes; Line of Intersection of Planes

(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

(b) Find symmetric equations for the line of intersection L of these two planes.

Solution

(a) The normal vectors of these planes are $\mathbf{n}_1 = \langle 1, 1, 1\rangle$ and $\mathbf{n}_2 = \langle 1, -2, 3\rangle$.

If θ is the angle between the planes, then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

Section 11.3, Corollary 1.

$$= \frac{(1)(1) + (1)(-2) + (1)(3)}{\sqrt{1+1+1}\sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$

Use normal vectors; dot product; vector magnitudes.

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 1.257\ (= 72.025°).$$

Use $\cos^{-1}$ to find θ.

Another way to find the line of intersection is to solve the equations of the planes for two of the variables in terms of the third, which can be taken as the parameter.

(b) First, find a point on the line L.

One way to do this is to find the point where the line intersects the xy-plane.
Let $z = 0$ in the equations of both planes.

$$x + y = 1,\ x - 2y = 1 \qquad \Rightarrow \qquad x = 1,\ y = 0$$

Therefore, the point $(1, 0, 0)$ lies on L.

Next, since L lies in both planes, it is perpendicular to both of the normal vectors.

Therefore, a vector $\mathbf{v}$ parallel to L is given by the cross product

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\,\mathbf{i} - 2\,\mathbf{j} - 3\,\mathbf{k}.$$

The symmetric equations of L can be written as

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}.$$

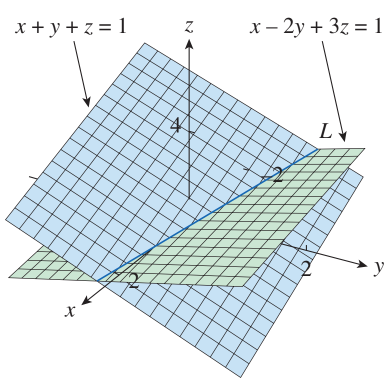


Figure 9.66
The planes and their line of intersection.

Figure 9.66 shows the two planes and the line of intersection. ■

Note: Since a linear equation in x, y, and z represents a plane and two nonparallel planes intersect in a line, it follows that two linear equations can represent a line. The points (x, y, z) that satisfy both $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ lie in

both planes. Therefore, the pair of linear equations represents the line of intersection of the planes (if they are not parallel).

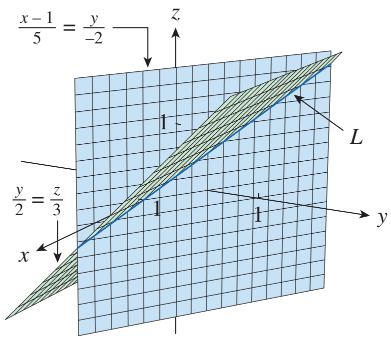


Figure 9.67
The planes and their line of intersection.

Here's an example. The line L in Example 7 is the intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. The symmetric equations found for L could be written as

$$\frac{x-1}{5} = \frac{y}{-2} \quad \text{and} \quad \frac{y}{-2} = \frac{z}{-3}$$

which is a pair of linear equations. They describe L as the line of intersection of the planes

$$\frac{x-1}{5} = \frac{y}{-2} \quad \text{and} \quad \frac{y}{-2} = \frac{z}{-3}.$$

See Figure 9.67.

In general, when we write the equations of a line in the symmetric form

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

we can think of this line as the line of intersection of the two planes described by

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} \quad \text{and} \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

Distances

We can use the concept of vector projections to find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Example 8 Distance from a Point to a Plane

Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

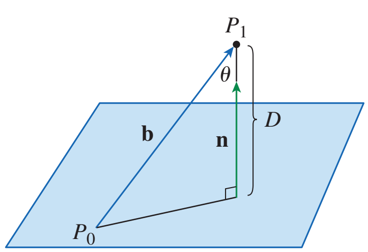


Figure 9.68
Illustration for finding the distance from a point to a plane.

Solution

Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and **b** be the vector corresponding to $\overrightarrow{P_0P_1}$.

Then $\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$.

Using Figure 9.68, the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of **b** onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$ (Section 9.3). Therefore,

$$\begin{aligned} D = |\text{comp}_{\mathbf{n}}\mathbf{b}| &= \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

Since P_0 lies in the plane, its coordinates satisfy the equation of the plane. Therefore, $ax_0 + by_0 + cz_0 + d = 0$. This results in the following formula.

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{9}$$

Example 9 Distance Between Planes

Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Solution

A quick check confirms that the planes are indeed parallel. Their normal vectors are $\langle 10, 2, -2\rangle$ and $\langle 5, 1, -1\rangle$ and $\langle 10, 2, -2\rangle = 2\langle 5, 1, -1\rangle$.

Choose any point on one plane and find its distance to the other plane.

Let $y = z = 0$ in the equation of the first plane.

$10x = 5 \qquad \Rightarrow \qquad \left(\frac{1}{2}, 0, 0\right)$ is a point in this plane.

Use the formula in 9 to find the distance between $\left(\frac{1}{2}, 0, 0\right)$ and the plane $5x + y - z - 1 = 0$.

$$D = \frac{\left|5\left(\frac{1}{2}\right) + 1(0) - 1(0) - 1\right|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

The distance between the planes is $\dfrac{\sqrt{3}}{6}$. ■

Example 10 Distance Between Lines

In Example 3, we showed that the lines

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$

$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are skew. Find the distance between them.

Solution

Since the two lines L_1 and L_2 are skew, we can think of these as lying on two parallel planes P_1 and P_2. The distance between L_1 and L_2 is the same as the distance between P_1 and P_2.

The common normal vector to both planes must be orthogonal to both $\mathbf{v}_1 = \langle 1, 3, -1\rangle$ (the direction of L_1) and $\mathbf{v}_2 = \langle 2, 1, 4\rangle$ (the direction of L_2).

Find a normal vector using the cross product.

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\,\mathbf{i} - 6\,\mathbf{j} - 5\,\mathbf{k}$$

Let $s = 0$ in the equations of L_2: the point $(0, 3, -3)$ is on L_2. Therefore, an equation for P_2 is

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0 \quad \text{or} \quad 13x - 6y - 5z + 3 = 0.$$

Let $t = 0$ in the equations for L_1: the point $(1, -2, 4)$ is on L_1 and in P_1. The distance between L_1 and L_2 is the same as the distance from $(1, -2, 4)$ to $13x - 6y - 5z + 3 = 0$.

Use the distance formula in 9.

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0.528$$

■

9.5 Exercises

1. Determine whether each statement is True or False in $\mathbb{R}^3$.
(a) Two lines parallel to a third line are parallel.
(b) Two lines perpendicular to a third line are parallel.
(c) Two planes parallel to a third plane are parallel.
(d) Two planes perpendicular to a third plane are parallel.
(e) Two lines parallel to a plane are parallel.
(f) Two lines perpendicular to a plane are parallel.
(g) Two planes parallel to a line are parallel.
(h) Two planes perpendicular to a line are parallel.
(i) Two planes either intersect or are parallel.
(j) Two lines either intersect or are parallel.
(k) A plane and a line either intersect or are parallel.

Find a vector equation and parametric equations for the line.

2. The line through the point $(6, -5, 2)$ and parallel to the vector $\langle 1, 3, -\frac{2}{3}\rangle$

3. The line through the point $(2, 2.4, 3.5)$ and parallel to the vector $3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$

4. The line through the point $(0, 14, -10)$ and parallel to the line $x = -1 + 2t$, $y = 6 - 3t$, $z = 3 + 9t$

5. The line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$

Find parametric equations and symmetric equations for the line.

6. The line through the points $(6, 1, -3)$ and $(2, 4, 5)$

7. The line through the points $(0, 1/2, 2)$ and $(2, 1, -3)$

8. The line through the points $(1, 2.4, 4.6)$ and $(2.6, 1.2, 0.3)$

9. The line through the points $(-8, 1, 4)$ and $(3, -2, 4)$

10. The line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$

11. The line through $(1, -1, 1)$ and parallel to the line
$$x + 2 = \frac{1}{2}y = z - 3$$

12. The line through $(-6, 2, 3)$ and parallel to the line
$$\frac{1}{2}x = \frac{1}{3}y = z + 1$$

13. The line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$

14. Is the line through $(-4, -6, 1)$ and $(-2, 0, -3)$ parallel to the line through $(10, 18, 4)$ and $(5, 3, 14)$?

15. Is the line through $(4, 1, -1)$ and $(2, 5, 3)$ perpendicular to the line through $(-3, 2, 0)$ and $(5, 1, 4)$?

16. Is the line through $(-2, 4, 0)$ and $(1, 1, 1)$ perpendicular to the line through $(2, 3, 4)$ and $(3, -1, -8)$?

17. (a) Find symmetric equations for the line that passes through the point $(1, -5, 6)$ and is parallel to the vector $\langle -1, 2, -3\rangle$.
(b) Find the points in which the required line in part (a) intersects the coordinate planes.

18. (a) Find parametric equations for the line through $(2, 4, 6)$ that is perpendicular to the plane $x - y + 3z = 7$.
(b) Find the points at which this line intersects the coordinate planes.

19. Find a vector equation for the line segment from $(2, -1, 4)$ to $(4, 6, 1)$.

20. Find a vector equation for the line segment from $(-3, -2, -5)$ to $(1, -4, 3)$.

21. Find parametric equations for the line segment from $(10, 3, 1)$ to $(5, 6, -3)$.

22. Find parametric equations for the line segment from $(-2, 11, -11)$ to $(3, 4, 8)$.

Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

23. L_1: $x = -6t, \quad y = 1 + 9t, \quad z = -3t$
L_2: $x = 1 + 2s, \quad y = 4 - 3s, \quad z = s$

24. L_1: $x = 1 + 2t, \quad y = 3t, \quad z = 2 - t$
L_2: $x = -1 + s, \quad y = 4 + s, \quad z = 1 + 3s$

25. L_1: $\dfrac{x}{1} = \dfrac{y - 1}{2} = \dfrac{z - 2}{3}$
L_2: $\dfrac{x - 3}{4} = \dfrac{y - 2}{-3} = \dfrac{z - 1}{2}$

26. L_1: $\dfrac{x - 1}{2} = \dfrac{y - 3}{2} = \dfrac{z - 2}{-1}$
L_2: $\dfrac{x - 2}{1} = \dfrac{y - 6}{-1} = \dfrac{z + 2}{3}$

Find an equation of the plane.

27. The plane through the point $(6, 3, 2)$ and perpendicular to the vector $\langle -2, 1, 5\rangle$

28. The plane through the origin and perpendicular to the vector $\langle 1, -2, 5\rangle$

29. The plane through the point $(4, 0, -3)$ and with normal vector $\mathbf{j} + 2\,\mathbf{k}$

30. The plane through the point $\left(-1, \frac{1}{2}, 3\right)$ and with normal vector $\mathbf{i} + 4\,\mathbf{j} + \mathbf{k}$

31. The plane through the point $(4, -2, 3)$ and parallel to the plane $3x - 7z = 12$

32. The plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$

33. The plane through the point $\left(1, \frac{1}{2}, \frac{1}{3}\right)$ and parallel to the plane $x + y + z = 0$

34. The plane that contains the line $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$

35. The plane through the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$

36. The plane through the points $(3, 0, -1)$, $(-2, -2, 3)$, and $(7, 1, -4)$

37. The plane that passes through the origin and the points $(3, -2, 1)$ and $(1, -1, 1)$

38. The plane that passes through the point $(6, 0, -2)$ and contains the line $x = 4 - 2t$, $y = 3 + 5t$, $z = 7 + 4t$

39. The plane that passes through the point $(1, -1, 1)$ and contains the line with symmetric equations $x = 2y = 3z$

40. The plane that passes through the point $(-1, 2, 1)$ and contains the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$

41. The plane that passes through the points $(0, -2, 5)$ and $(-1, 3, 1)$ and is perpendicular to the plane $2z = 5x + 4y$

42. The plane that passes through the point $(1, 5, 1)$ and is perpendicular to the planes $2x + y - 2z = 2$ and $x + 3z = 4$

43. The plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$

Find the intercepts and use these points to help sketch the plane.

44. $2x + 5y + z = 10$

45. $3x + y + 2z = 6$

46. $6x - 3y + 4z = 6$

47. $6x + 5y - 3z = 15$

48. Find the point at which the line $x = 3 - t$, $y = 2 + t$, $z = 5t$ intersects the plane $x - y + 2z = 9$

49. Where does the line through $(1, 0, 1)$ and $(4, -2, 2)$ intersect the plane $x + y + z = 6$?

50. Where does the line through $(-3, 1, 0)$ and $(-1, 5, 6)$ intersect the plane $2x + y - z = -2$?

Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.

51. $x + 4y - 3z = 1$, $-3x + 6y + 7z = 0$

52. $x + 2y + 2z = 1$, $2x - y + 2z = 1$

53. $x + y + z = 1$, $x - y + z = 1$

54. $2z = 4y - x$, $3x - 12 + 6z = 1$

55. $2x - 3y = z$, $4x = 3 + 6y + 2z$

56. $5x + 2y + 3z = 2$, $y = 4x - 6z$

(a) Find parametric equations for the line of intersection of the planes.
(b) Find the angle, in radians and degrees, between the planes.

57. $x + y + z = 1$, $x + 2y + 2z = 1$

58. $3x - 2y + z = 1$, $2x + y - 3z = 3$

Find symmetric equations for the line of intersection of the planes.

59. $5x - 2y - 2z = 1$, $4x + y + z = 6$

60. $z = 2x - y - 5$, $z = 4x + 3y - 5$

61. Find an equation for the plane consisting of all points that are equidistant from the points $(2, 5, 5)$ and $(-6, 3, 1)$.

62. Find an equation for the plane consisting of all points that are equidistant from the points $(1, 0, -2)$ and $(3, 4, 0)$.

63. Find an equation for the plane with x-intercept a, y-intercept b, and z-intercept c.

64. (a) Find the point at which the given lines intersect:

$$\mathbf{r} = \langle 1, 1, 0 \rangle + t\langle 1, -1, 2 \rangle$$

$$\mathbf{r} = \langle 2, 0, 2 \rangle + s\langle -1, 1, 0 \rangle$$

(b) Find an equation of the plane that contains these lines.

65. Find parametric equations for the line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$.

66. Find parametric equations for the line through the point $(0, 1, 2)$ that is perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$ and intersects this line.

67. Which of the following four planes are parallel? Are any of them identical?

P_1: $3x + 6y - 3z = 6$ $\quad$ P_2: $4x - 12y + 8z = 5$

P_3: $9y = 1 + 3x + 6z$ $\quad$ P_4: $z = x + 2y - 2$

68. Which of the following four lines are parallel? Are any of them identical?

L_1: $x = 1 + 6t, \quad y = 1 - 3t, \quad z = 12t + 5$

L_2: $x = 1 + 2t, \quad y = t, \quad z = 1 + 4t$

L_3: $2x - 2 = 4 - 4y = z + 1$

L_4: $\mathbf{r} = \langle 3, 1, 5 \rangle + t\langle 4, 2, 8 \rangle$

Use the formula Exercise 48 in Section 9.4 to find the distance from the point to the given line.

69. $(4, 1, -2)$; $\quad x = 1 + t, \quad y = 3 - 2t, \quad z = 4 - 3t$

70. $(0, 1, 3)$; $\quad z = 2t, \quad y = 6 - 2t, \quad z = 3 + t$

Find the distance from the point to the given plane.

71. $(1, -2, 4)$, $\quad 3x + 2y + 6z = 5$

72. $(-6, 3, 5)$, $\quad x - 2y - 4z = 8$

Find the distance between the given parallel planes.

73. $2x - 3y + z = 4$, $\quad 4x - 6y + 2z = 3$

74. $6z = 4y - 2x$, $\quad 9z = 1 - 3x + 6y$

75. Show that the distance between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

76. Find equations of the planes that are parallel to the plane $x + 2y - 2z = 1$ and two units away from it.

77. Show that the lines with symmetric equations $x = y = z$ and $x + 1 = \dfrac{y}{2} = \dfrac{z}{3}$ are skew, and find the distance between these lines.

78. Find the distance between the skew lines with parametric equations $x = 1 + t$, $y = 1 + 6t$, $z = 2t$, and $x = 1 + 2s$, $y = 5 + 15s$, $z = -2 + 6s$.

79. Let L_1 be the line through the origin and the point $(2, 0, -1)$. Let L_2 be the line through the points $(1, -1, 1)$ and $(4, 1, 3)$. Find the distance between L_1 and L_2.

80. Let L_1 be the line through the points $(1, 2, 6)$ and $(2, 4, 8)$. Let L_2 be the line of intersection of the planes P_1 and P_2, where P_1 is the plane $x - y + 2z + 1 = 0$ and P_2 is the plane through the points $(3, 2, -1)$, $(0, 0, 1)$ and $(1, 2, 1)$. Find the distance between L_1 and L_2.

81. If a, b, and c are not all 0, show that the equation $ax + by + cz + d = 0$ represents a plane and $\langle a, b, c\rangle$ is a normal vector to the plane.

Hint: Suppose $a \neq 0$ and rewrite the equation in the form

$$a\left(x + \frac{d}{a}\right) + b(y - 0) + c(z - 0) = 0$$

82. Give a geometric description of each family of planes.

(a) $x + y + z = c$

(b) $x + y + cz = 1$

(c) $y\cos\theta + z\sin\theta = 1$

Laboratory Project | Putting 3D in Perspective

Computer graphics programmers face the same challenge as the great painters of the past: how to represent a three-dimensional scene as a flat image on a two-dimensional plane (a screen or a canvas). To create the illusion of perspective, in which closer objects appear larger than those farther away, three-dimensional objects in the computer's memory are projected onto a rectangular screen window from a viewpoint where the eye, or camera, is located. The viewing volume, the portion of space that will be visible, is the region contained by the four planes that pass through the viewpoint and an edge of the screen window. If objects in the scene extend beyond these four planes, they must be truncated before pixel data are sent to the screen. These planes are therefore called *clipping planes*.

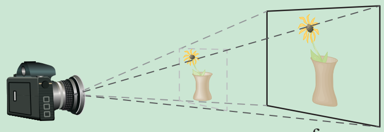


Figure 9.69
Clipping planes.

1. Suppose the screen is represented by a rectangle in the yz-plane with vertices $(0, \pm400, 0)$ and $(0, \pm400, 600)$, and the camera is placed at $(1000, 0, 0)$. A line L in the scene passes through the points $(230, -285, 102)$ and $(860, 105, 264)$. At what points should L be clipped by the clipping planes?
2. If the clipped line segment is projected onto the screen window, identify the resulting line segment.
3. Use parametric equations to plot the edges of the screen window, the clipped line segment, and its projection onto the screen window. Then add sight lines connecting the viewpoint to each end of the clipped segments to verify that the projection is correct.
4. A rectangle with vertices $(621, -147, 206)$, $(563, 31, 242)$, $(657, -111, 86)$, and $(599, 67, 122)$ is added to the scene. The line L intersects this rectangle. To make the rectangle appear opaque, a programmer can use *hidden line rendering*, which removes portions of objects that are behind other objects. Identify the portion of L that should be removed.

9.6 Functions and Surfaces

In this section, we will begin to consider functions of two variables and their graphs, which are surfaces in three-dimensional space. A more complete presentation of these types of functions is given in Chapter 11.

Functions of Two Variables

The temperature T at a point on the surface of Earth at any given time depends on the longitude x and latitude y of the point. We can think of T as being a function of the two variables x and y, or as a function of the pair (x, y). This idea of a function is written as $T = f(x, y)$.

The volume V of a circular cylinder depends on its radius r and its height h. In fact, we know that $V = \pi r^2 h$. In this case, V is a function of r and h, and we write $V(r, h) = \pi r^2 h$.

Definition • Function of Two Variables

A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

This is just a consistent extension of the notation used to represent functions of a single variable: $y = f(x)$.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general ordered pair (x, y). The variables x and y are **independent variables** and z is the **dependent variable**.

The domain is a subset of $\mathbb{R}^2$, the xy-plane. As usual we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs. If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x, y) for which the given expression is a well-defined real number.

Example 1 Domain and Range

If $f(x, y) = 4x^2 + y^2$, then $f(x, y)$ is defined for all possible ordered pairs of real numbers (x, y).

Therefore, the domain is $\mathbb{R}^2$, the entire xy-plane.

Since $x^2 \geq 0$ and $y^2 \geq 0$, then $f(x, y) \geq 0$ for all x and y.

Therefore, the range of f is the set $[0, \infty)$; all nonnegative real numbers. ■

Example 2 Sketching Domains

For each of the following functions, evaluate $f(3, 2)$ and find and sketch the domain.

(a) $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$ (b) $f(x, y) = x \ln(y^2 - x)$

Solution

(a) $f(3, 2) = \dfrac{\sqrt{3 + 2 + 1}}{3 - 1} = \dfrac{\sqrt{6}}{2}$

We can evaluate the expression for f if the denominator is not 0 and the quantity under the square root sign is nonnegative.

Therefore, the domain of f is $D = \{(x, y) \mid x + y + 1 \geq 0, x \neq 1\}$.

The inequality $x + y + 1 \geq 0$, or $y \geq -x - 1$, describes the points that lie on or above the line $y = -x - 1$. But $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain.

Figure 9.70 shows the domain of f.

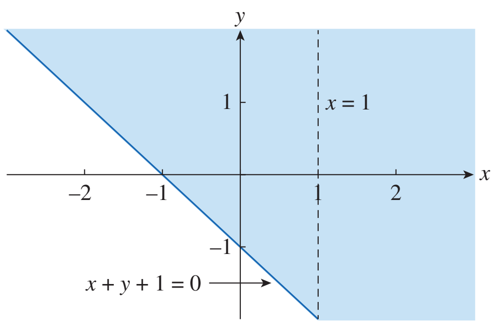


Figure 9.70

Domain of $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$.

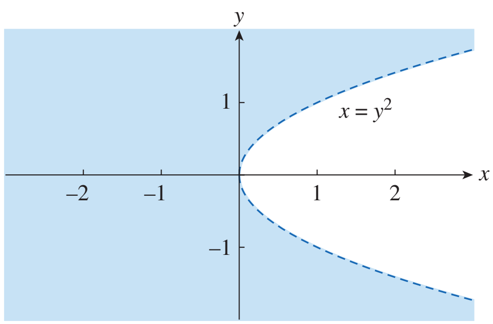


Figure 9.71

Domain of $f(x, y) = x \ln(y^2 - x)$.

(b) $f(3, 2) = 3 \ln(2^2 - 3) = 3 \ln 1 = 0$

The expression $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$.

Therefore, the domain of f is $D = \{(x, y) \mid x < y^2\}$.

This is the set of points to the left of the parabola $x = y^2$.

Figure 9.71 shows the domain of f. ■

Recall that a function can be represented by a formula, a table, a graph, or even verbally. The function in the next example is described verbally and by numerical estimates of its values.

Example 3 Wave Height as a Function of Wind Speed and Time

The wave heights h (in feet) in the open sea depend mainly on the speed v of the wind (in knots) and the length of time t (in hours) that the wind has been blowing at that speed. So h is a function of v and t and we can write $h = f(v, t)$. Observations and measurements have been made by meteorologists and oceanographers and are recorded in Table 9.1.

Duration (hours)

Wind speed (knots)

v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

Table 9.1

Wave heights (in feet) produced by different wind speeds for various lengths of time.

For instance, the table indicates that if the wind has been blowing at 50 knots for 30 hours, then the wave heights are estimated to be 45 ft, that is, $f(50, 30) = 45$.

The domain of this function h is given by $v \geq 0$ and $t \geq 0$.

Although there is no exact formula for h in terms of v and t, we will see that the operations of calculus can still be carried out for such an experimentally defined function. ■

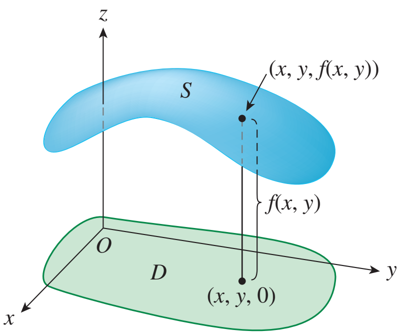


Figure 9.72
A visualization of the graph S of the function f.

Graphs

One way to visualize the behavior of a function of two variables is to consider its graph.

> **Definition • Graph of a Function of Two Variables**
> If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Recall that the graph of a function f of one variable is a curve C with equation $y = f(x)$. Analogously, the graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane. See Figure 9.72.

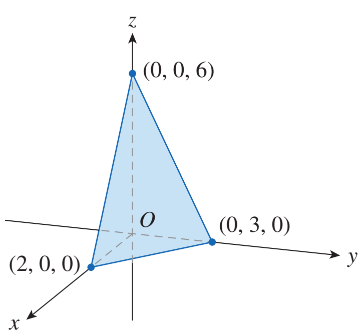


Figure 9.73
Graph of the surface S described by $f(x, y) = 6 - 3x - 2y$.

Example 4 Graph of a Linear Function

Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

Solution

The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$.

This equation represents a plane.

To sketch the graph of the plane, start by finding the intercepts.

Let $y = z = 0$ in this equation, we get $x = 2$.

Similarly, the y-intercept is 3, and the x-intercept is 6.

Figure 9.73 shows a sketch of the plane through the three intercepts. ■

The function in Example 4 is a special case of the function

$$f(x, y) = ax + by + c$$

which is called a **linear function**. The graph of a linear function has the equation

$$z = ax + by + c \quad \text{or} \quad ax + by - z + c = 0$$

and is therefore a plane. Just as linear functions of one variable are important in single-variable calculus, we will see that linear functions of two variables play a central role in multivariable calculus.

Example 5 Parabolic Cylinder

Sketch the graph of the function $f(x, y) = x^2$.

Solution

The function f depends only on x; no matter what the value of y, the value of f is always x^2.

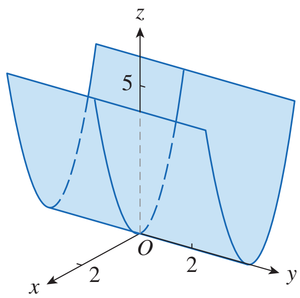


Figure 9.74
The graph of $f(x) = x^2$ is a parabolic cylinder.

Since the equation $z = x^2$ does not involve y, any vertical plane with equation $y = k$ (parallel to the xz-plane) intersects the graph in a curve with equation $z = x^2$, which is a parabola.

Figure 9.74 shows how the graph is formed by taking the parabola $z = x^2$ in the xz-plane and moving it in the direction of the y-axis.

The graph of this surface is called a **parabolic cylinder**, made up of infinitely many shifted copies of the same parabola. ■

In order to sketch the graph of a function of two variables, it is often useful to start by determining the shapes of cross-sections (slices) of the graph. For example, if x is held fixed, $x = k$ (a constant), and y varies, the result is a function of one variable $z = f(k, y)$. The graph of this function is the intersection of the surface $z = f(x, y)$ and the vertical plane $x = k$. Similarly, we can slice the surface with the vertical plane $y = k$ and consider the curves $z = f(x, k)$. We can, of course, also slice the surface with horizontal planes $z = k$. All three types of curves like this are called **traces** (or cross-sections) of the surface $z = f(x, y)$.

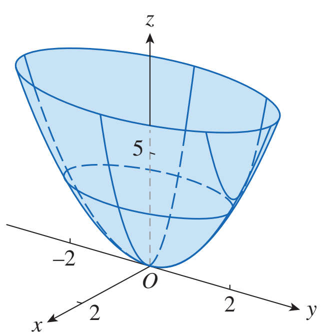


Figure 9.75
The graph of $f(x, y) = 4x^2 + y^2$ is the elliptic paraboloid $z = 4x^2 + y^2$. Horizontal traces are ellipses; vertical traces are parabolas.

Example 6 Use Traces to Sketch an Elliptic Paraboloid

Use traces to sketch the graph of the function $f(x, y) = 4x^2 + y^2$.

Solution

The equation of the graph is $z = 4x^2 + y^2$.

Let $x = 0$, then $z = y^2$; the yz-plane intersects the surface in a parabola.

Let $x = k$, then $z = y^2 + 4k^2$; the slice of the graph with any plane parallel to the yz-plane is a parabola that opens upward.

Let $y = k$, then the trace is $z = 4x^2 + k^2$; this is also a parabola that opens upward.

Let $z = k$, then the horizontal traces are $4x^2 + y^2 = k$; this is an ellipse.

Using the shapes of the traces, a sketch of the graph of f is shown in Figure 9.75.

Because the traces involve ellipses and parabolas, the surface $z = 4x^2 + y^2$ is called an **elliptic paraboloid**. ■

Example 7 Hyperbolic Paraboloid

Sketch the graph of $f(x, y) = y^2 - x^2$.

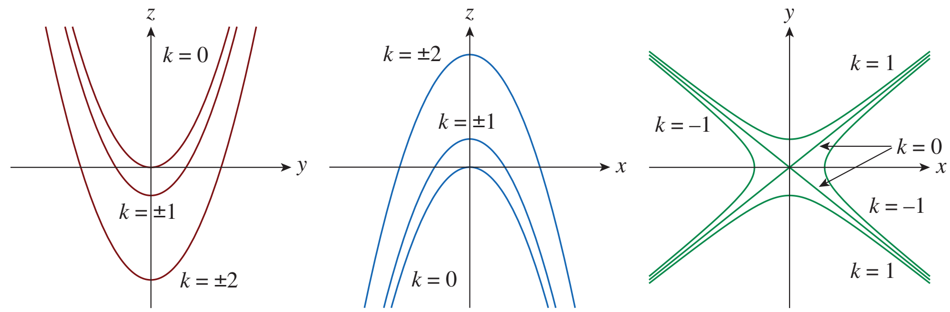
Solution

Let $x = k$, then $z = y^2 - k^2$; the traces in the vertical planes $x = k$ are parabolas that open upward.

Let $y = k$, then $z = k^2 - x^2$; the traces in the planes $y = k$ are parabolas that open downward.

Let $z = k$, then $k = y^2 - x^2$; the horizontal traces are hyperbolas.

Figure 9.76 shows graphs of these traces, and Figure 9.77 show how these traces appear in their correct planes.

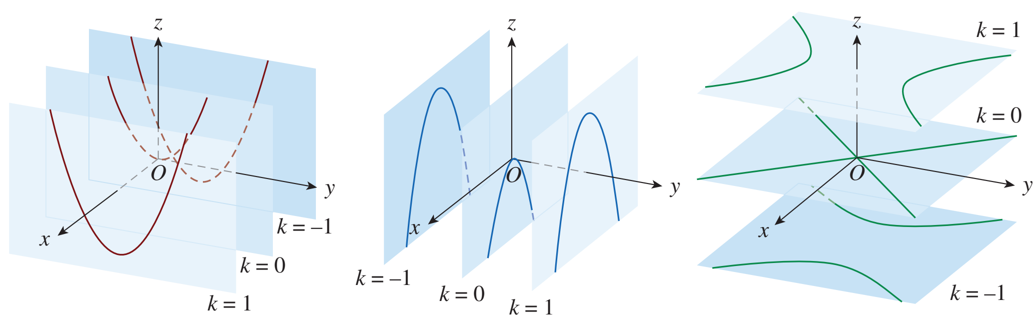


Traces in $x = k$ are $z = y^2 - k^2$.

Traces in $y = k$ are $z = k^2 - x^2$.

Traces in $z = k$ are $k = y^2 - x^2$.

Figure 9.76
Vertical traces are parabolas; horizontal traces are hyperbolas.



Traces in $x = k$

Traces in $y = k$

Traces in $z = k$

Figure 9.77
Graphs of traces in their correct planes.

In Figure 9.78, we put all of this information together to form the surface $z = y^2 - x^2$, called a **hyperbolic paraboloid**.

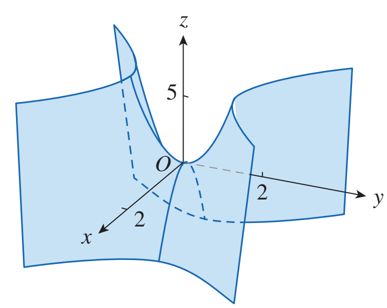
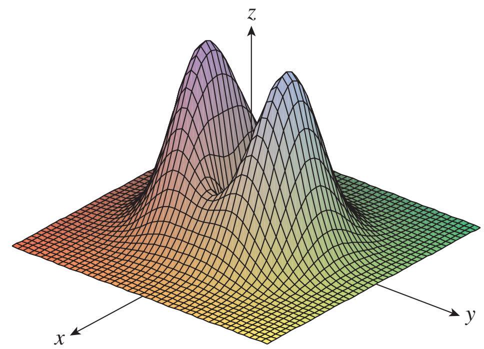


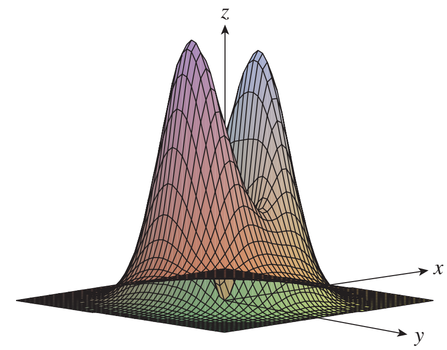
Figure 9.78
The graph of $f(x, y) = y^2 - x^2$ is the hyperbolic paraboloid $z = y^2 - x^2$.

Notice that the shape of the surface near the origin looks like a saddle. We will study this surface more in Section 11.7, when we consider saddle points. ■

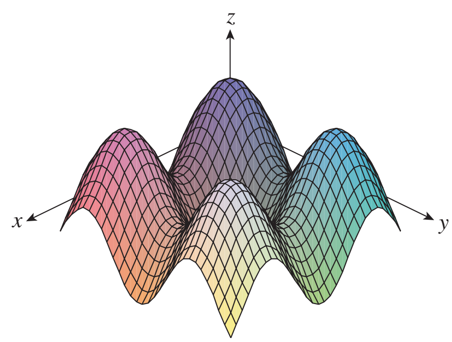
Most three-dimensional graphing software uses the idea of traces to sketch a surface. In most cases, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k and parts of the graph are eliminated using hidden line removal. Figure 9.79 shows computer-generated graphs of several functions. Using different viewpoints allows us to see various characteristics of a surface. In Figure 9.79(a) and (b), the graph of f is very flat and close to the xy-plane except near the origin; this is because $e^{-x^2-y^2}$ is very small when x or y is large.



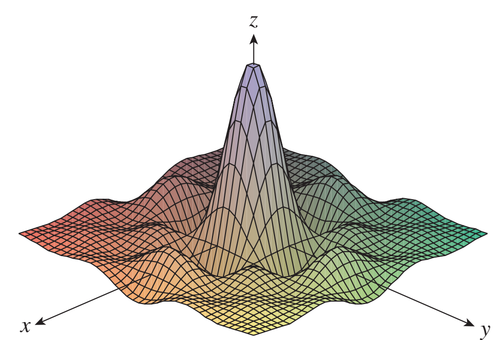
(a) Graph of $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



(b) Graph of $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$ from another viewpoint



(c) Graph of $f(x, y) = \sin x + \sin y$



(d) Graph of $f(x, y) = \dfrac{\sin x \sin y}{xy}$

Figure 9.79
Computer-generated graphs of various surfaces.

Quadric Surfaces

The graph of a second-degree equation in three variables x, y, and z is called a **quadric surface**. In Examples 6 and 7, we sketched the quadric surfaces $z = 4x^2 + y^2$ (an elliptic paraboloid) and $z = y^2 - x^2$ (a hyperbolic paraboloid). The next example involves a quadric surface called an *ellipsoid.*

Example 8 Ellipsoid

Sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

Solution

Let $z = 0$; the trace in the xy-plane is $x^2 + \dfrac{y^2}{9} = 1$.

This is the equation of an ellipse.

In general, the horizontal trace in the plane $z = k$ is $x^2 + \dfrac{y^2}{9} = 1 - \dfrac{k^2}{4}$.

This is the equation of an ellipse, provided that $k^2 < 4$, that is, $-2 < k < 2$.

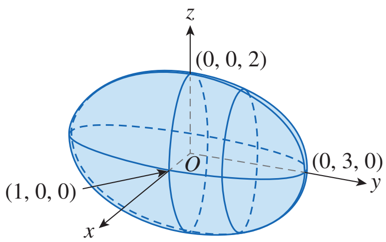


Figure 9.80
Graph of the ellipsoid
$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1.$

Similarly, the vertical traces are also ellipses.

Let $x = k$, then $\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2$, if $-1 < k < 1$.

Let $y = k$, then $x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9}$, if $-3 < k < 3$.

Figure 9.80 shows how drawing some traces help to indicate the shape of the surface. This figure is called an **ellipsoid** because all of its traces are ellipses.

Notice that this surface is symmetric with respect to each coordinate plane; this is a reflection of the fact that its equation involves only even powers of x, y, and z. ■

The ellipsoid in Example 8 is *not* the graph of a function because there are two values of z that correspond to at least one pair (x, y) where $(-1 < x < 1,\ -3 < y < 3)$. Geometrically, some vertical lines, for example, the z-axis, intersect the graph more than once. However, the top and bottom halves are graphs of functions. In fact, if we solve the equation of the ellipsoid for z, we get

$$z^2 = 4\left(1 - x^2 - \frac{2y^2}{9}\right) \quad \Rightarrow \quad z = \pm 2\sqrt{1 - x^2 - \frac{y^2}{9}}$$

So the graphs of the functions

$$f(x, y) = 2\sqrt{1 - x^2 - \frac{y^2}{9}} \quad \text{and} \quad g(x, y) = -2\sqrt{1 - x^2 - \frac{y^2}{9}}$$

are the top and bottom halves of the ellipsoid. See Figure 9.81.

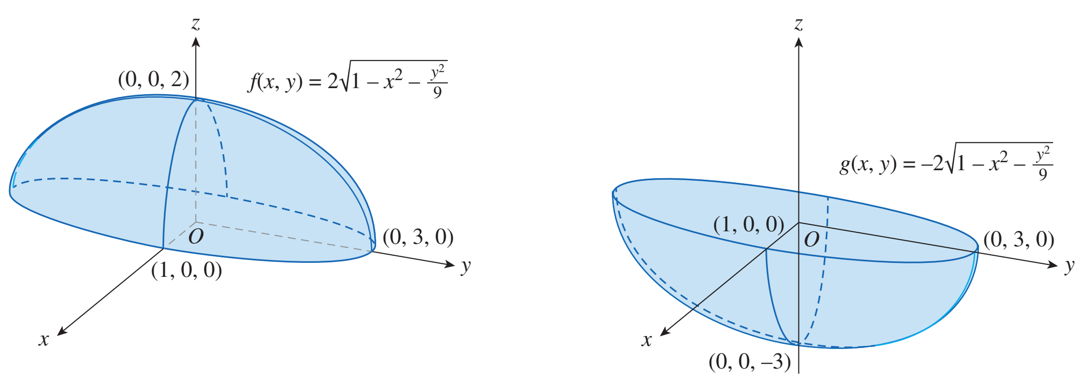


Figure 9.81
The top and bottom halves of the ellipsoid are graphs of functions.

The domain of both f and g is the set of all points (x, y) such that

$$1 - x^2 - \frac{y^2}{9} \geq 0 \quad \Leftrightarrow \quad x^2 + \frac{y^2}{9} \leq 1$$

Therefore, the domain is the set of all points that lie on or inside the ellipse $x^2 + \frac{y^2}{9} = 1$.

Table 9.2 shows computer-drawn graphs of the six basic types of quadric surfaces in standard form. All of these surfaces are symmetric with respect to the z-axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.

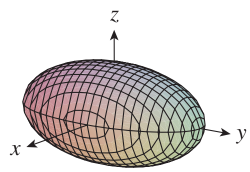
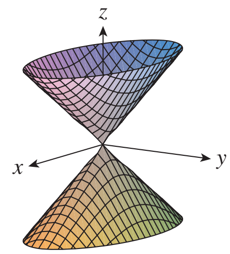
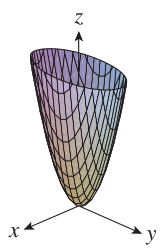
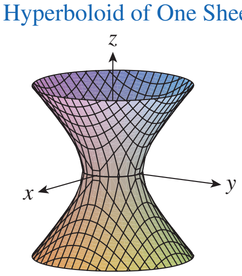
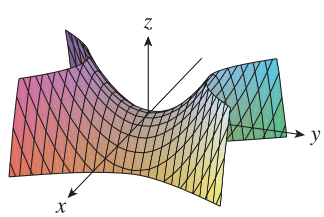
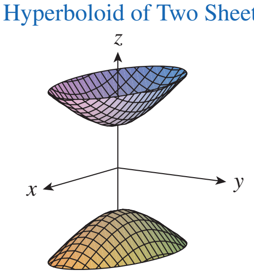
Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case $c<0$ is illustrated here.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Table 9.2
Graphs of quadric surfaces.

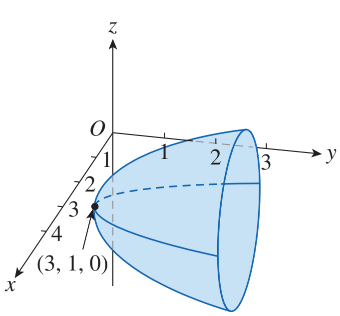


Figure 9.82
Graph of the elliptic paraboloid described by $x^2+2z^2-6x-y+10=0$.

Example 9 Identify and Sketch a Quadric Surface

Classify the quadric surface $x^2+2z^2-6x-y+10=0$.

Solution

Complete the square to rewrite the equation in a recognizable form.

$$(x^2-6x+9)+2z^2=y-10+9$$
$$(x-3)^2+2z^2=y-1$$

Compare this equation with those in Table 9.2.
The equation represents an elliptic paraboloid.

In this case, the axis of the paraboloid is parallel to the y-axis, and the surface is shifted so that its vertex is the point $(3, 1, 0)$.

The traces in the plane $y=k\,(k>1)$ are the ellipses $(x-3)^2+2z^2=k-1$.

The trace in the xy-plane is the parabola with equation $y=1+(x-3)^2$, $z=0$.

The paraboloid is shown in Figure 9.82. ■

9.6 Exercises

1. In Example 3, we considered the function $h = f(v, t)$, where h is the height of waves produced by wind at speed v for a time t. Use Table 9.1 to answer the following.

(a) Find the value of $f(40, 15)$ and explain the meaning of this value in the context of this example.

(b) Explain the meaning of the function $h = f(30, t)$ in the context of this example. Describe the behavior of this function.

(c) Explain the meaning of the function $h = f(v, 30)$ in the context of this example. Describe the behavior of this function.

2. Let $f(x, y) = y^4 e^{x/y}$.

(a) Evaluate $f(0, 2)$ and $f(-1, -1)$.

(b) Find the domain of f.

(c) Find the range of f.

3. Let $g(x, y) = \cos(x + 2y)$.

(a) Evaluate $g(2, -1)$ and $g(0, \pi)$.

(b) Find the domain of g.

(c) Find the range of g.

4. Let $h(x, y) = x \ln y$.

(a) Evaluate $h(1, 1)$ and $h(2, 2)$.

(b) Find and sketch the domain of h.

(c) Find the range of h.

5. Let $F(x, y) = 1 + \sqrt{4 - y^2}$.

(a) Evaluate $F(3, 1)$ and $F(-2, 2)$.

(b) Find and sketch the domain of F.

(c) Find the range of F.

Find and sketch the domain of the function.

6. $f(x, y) = \dfrac{\sqrt{y - x^2}}{1 - x^2}$ **7.** $f(x, y) = \sqrt{xy}$

8. $f(x, y) = \sqrt{1 - x^2} - \sqrt{1 - y^2}$

9. $f(x, y) = \ln(x^2 + y^2 - 2)$

Sketch the graph of the function.

10. $f(x, y) = 3$ **11.** $f(x, y) = y$

12. $f(x, y) = 6 - 3x - 2y$ **13.** $f(x, y) = \cos x$

14. $f(x, y) = y^2 + 1$ **15.** $f(x, y) = e^{-y^2}$

16. (a) Find traces of the function $f(x, y) = x^2 + y^2$ in the planes $x = k$, $y = k$, and $z = k$.Use these traces to sketch the graph.

(b) Sketch the graph of $g(x, y) = -x^2 - y^2$. How is it related to the graph of f?

(c) Sketch the graph of $h(x, y) = 3 - x^2 - y^2$. How is it related to the graph of g?

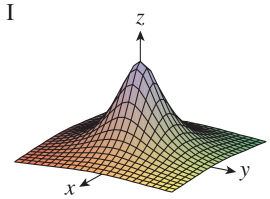
17. Match the function with its graph (labeled I–VI). Give reasons for your choices.

(a) $f(x, y) = |x| + |y|$ (b) $f(x, y) = |xy|$

(c) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$ (d) $f(x, y) = (x^2 - y^2)^2$

(e) $f(x, y) = (x - y)^2$ (f) $f(x, y) = \sin(|x| + |y|)$

I



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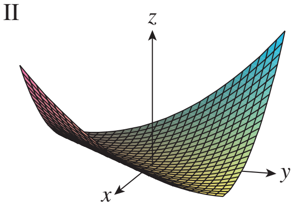

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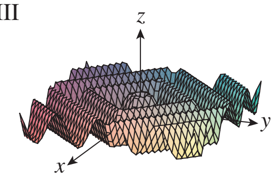
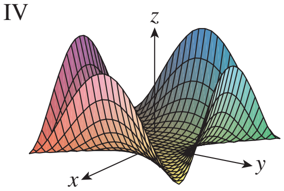
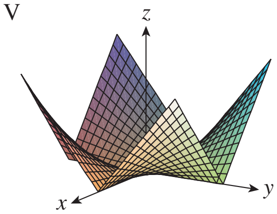


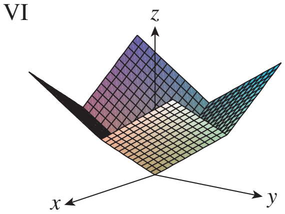
IV



V



VI



Use traces to sketch the graph of the function.

18. $f(x, y) = \sqrt{16 - x^2 - 16y^2}$

19. $f(x, y) = \sqrt{4x^2 + y^2}$

20. $f(x, y) = x^2 - y^2$

Use traces to sketch and identify the surface.

21. $y = z^2 - x^2$

22. $x = y^2 + 4z^2$

23. $x^2 = 4y^2 + z^2$

24. $3x^2 - y^2 + 3z^2 = 0$

Classify the surface by comparing with one of the standard forms in Table 2. Then sketch its graph.

25. $4x^2 + y^2 + 4z^2 - 4y - 24z + 36 = 0$

26. $4y^2 + z^2 - x - 16y - 4z + 20 = 0$

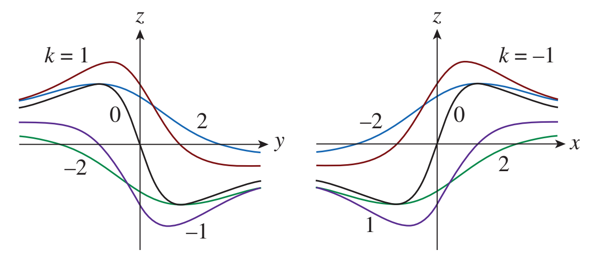
27. (a) Describe the curve represented by $x^2 + y^2 = 1$ in $\mathbb{R}^2$.
(b) Describe the surface represented by $x^2 + y^2 = 1$ in $\mathbb{R}^3$.
(c) Describe the surface represented by $x^2 + z^2 = 1$ in $\mathbb{R}^3$.

28. (a) Describe the traces of the surface represented by $z^2 = x^2 + y^2$.
(b) Sketch the surface.
(c) Sketch the graphs of the functions $f(x, y) = \sqrt{x^2 + y^2}$ and $g(x, y) = -\sqrt{x^2 + y^2}$.

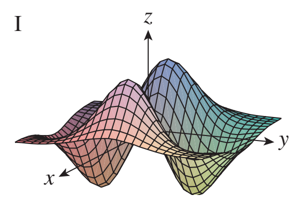
29. (a) Find and identify the traces of the quadric surface $x^2 + y^2 - z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of one sheet in Table 9.2.
(b) If the equation in part (a) is changed to $x^2 - y^2 + z^2 = 1$, explain how the graph changes.
(c) Suppose the equation in part (a) is changed to $x^2 + y^2 + 2y - z^2 = 0$. Explain how the graph changes.

30. (a) Find and identify the traces of the quadric surface $-x^2 - y^2 + z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of two sheets in Table 9.2.
(b) If the equation in part (a) is changed to $x^2 - y^2 - z^2 = 1$, explain how the graph changes, and sketch the new graph.

31. The figure shows vertical traces for a function $z = f(x, y)$. Which one of the graphs I–IV has these traces? Explain your reasoning.



I



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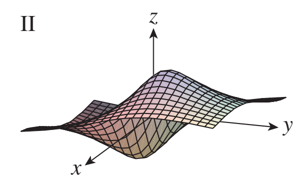
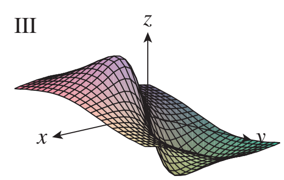
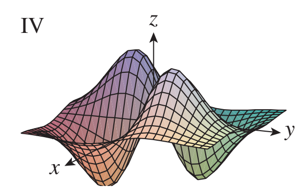


III



IV



Use technology to graph the function using various domains and viewpoints. Estimate the maximum and minimum values of the function geometrically where the highest peaks or lowest valley occurs in the surface. Try to identify any *local maximum values* or *local minimum values*.

32. $f(x, y) = x^2y^2e^{x - 4x^2 - 4y^2}$

33. $f(x, y) = xye^{x + 2y - 9x^2 - 9y^2}$

Use technology to graph the function using various domains and viewpoints. Describe the limiting behavior of the function. What happens as both x and y increase without bound? What happens as (x, y) approaches the origin?

34. $f(x, y) = \dfrac{x + y}{x^2 + y^2}$

35. $f(x, y) = \dfrac{xy}{x^2 + y^2}$

36. Use technology to sketch the surfaces $z = x^2 + y^2$ and $z = 1 - y^2$ in the same coordinate system using the domain $|x| \le 1.2$ and $|y| \le 1.2$. Notice that the surfaces intersect in a curve. Show that the projection of this curve onto the xy-plane is an ellipse.

37. Show that the intersection of the surfaces $x^2 + 2y^2 - z^2 + 3x = 1$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$ is a curve that lies in a plane.

38. Show that if the point (a, b, c) lies on the hyperbolic paraboloid $z = y^2 - x^2$, then the lines with parametric equations $x = a + t,\ y = b + t,\ z = c + 2(b - a)t$ and $x = a + t,\ y = b - t,\ z = c - 2(b + a)t$ both lie entirely on this paraboloid.

This shows that the hyperbolic paraboloid is a **ruled surface**, that is, it can be generated by the motion of a straight line. In fact, this exercise shows that through each point on the hyperbolic paraboloid, there are two generating lines. The only other quadric surfaces that are ruled surfaces are cylinders, cones, and hyperboloids of one sheet.

39. Find an equation for the surface consisting of all points P for which the distance from P to the x-axis is twice the distance from P to the yz-plane. Identify the surface.

9.7 Cylindrical and Spherical Coordinates

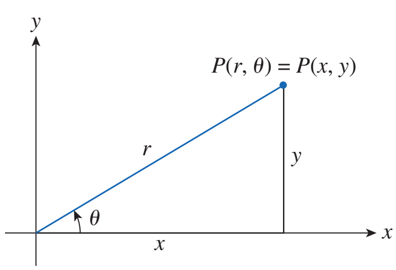


Figure 9.83
Relationship between Cartesian and polar coordinates.

In plane geometry, we often use the polar coordinate system to provide a convenient description of certain curves and regions. Recall the connection between polar and Cartesian coordinates, illustrated in Figure 9.83. If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, using the figure,

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

In three dimensions there are two coordinate systems that are similar to polar coordinates and both provide (often less complicated) descriptions of some commonly occurring surfaces and solids. These coordinate systems are also useful when we compute certain volumes and triple integrals.

Cylindrical Coordinates

In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P. See Figure 9.84.

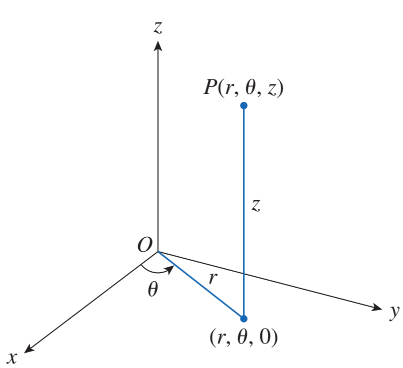


Figure 9.84
The cylindrical coordinates of a point.

To convert from cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \tag{1}$$

and to convert from rectangular to cylindrical coordinates, we use these equations

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z \tag{2}$$

Example 1 Convert Between Cylindrical and Rectangular Coordinates

(a) Plot the point with cylindrical coordinates $\left(2, \frac{2\pi}{3}, 1\right)$ and find its rectangular coordinates.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Solution

(a) The point with cylindrical coordinates $\left(2, \frac{2\pi}{3}, 1\right)$ is plotted in Figure 9.85.

Use Equation 1 to find its rectangular coordinates.

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

Therefore, the point is represented by $(-1, \sqrt{3}, 1)$ in rectangular coordinates.

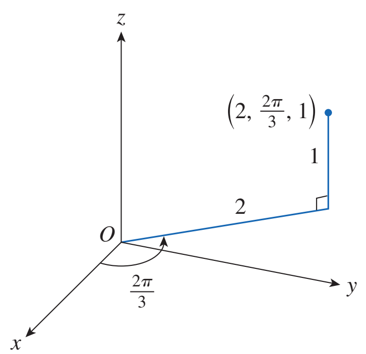


Figure 9.85
The point with cylindrical coordinates $\left(2, \frac{2\pi}{3}, 1\right)$.

(b) Use Equation 2 to find the cylindrical coordinates.

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \quad \Rightarrow \quad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore, one set of cylindrical coordinates is $\left(3\sqrt{2}, \frac{7\pi}{4}, -7\right)$.

Another is $\left(3\sqrt{2}, -\frac{\pi}{4}, -7\right)$.

As with polar coordinates, there are infinitely many ways to represent this point in cylindrical coordinates. ■

Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry. For example, the axis of the circular cylinder with Cartesian equation $x^2 + y^2 = c^2$ is the z-axis. In cylindrical coordinates, this cylinder has a very simple equation; $r = c$. The graph is shown in Figure 9.86. This is the reason for the name *cylindrical* coordinates.

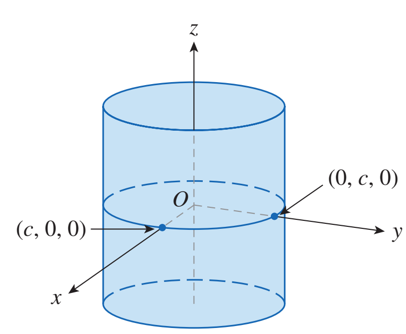


Figure 9.86
Graph of a cylinder described by $r = c$ in cylindrical coordinates.

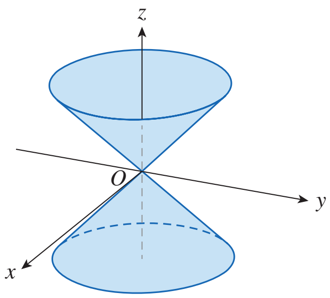


Figure 9.87
Graph of the circular cone described by $z = r$ in cylindrical coordinates.

Example 2 Cylindrical Equation for a Cone

Describe the surface whose equation in cylindrical coordinates is $z = r$.

Solution

In words, the equation says that the z-value, or height, of each point on the surface is the same as r, the distance from the point to the z-axis.

Because θ does not appear in the equation, it can vary, or be any value.

Therefore, any horizontal trace in the plane $z = k$ $(k > 0)$ is a circle of radius k. These traces suggest that the surface is a cone.

Confirm this conclusion by converting the equation into rectangular coordinates.

Use Equation 2:

$$z^2 = r^2 = x^2 + y^2$$

Using Table 2 in Section 9.6, the equation $z^2 = x^2 + y^2$ represents a circular cone whose axis is the z axis. Figure 9.87 shows a graph of this surface. ■

Example 3 Cylindrical Equation for an Ellipsoid

Find an equation in cylindrical coordinates for the ellipsoid $4x^2 + 4y^2 + z^2 = 1$.

Solution

Use the equations in (2) that relate rectangular and cylindrical coordinates.

$$z^2 = 1 - 4(x^2 + y^2) = 1 - 4r^2 \qquad \text{Equation in (2); } r^2 = x^2 + y^2.$$

Therefore, an equation of the ellipsoid in cylindrical coordinates is $z^2 = 1 - 4r^2$. ■

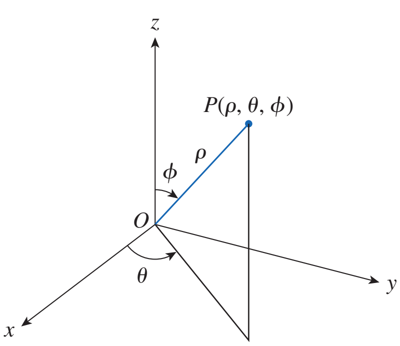


Figure 9.88
The spherical coordinates of a point.

Spherical Coordinates

In the **spherical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (ρ, θ, ϕ), where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. See Figure 9.88. Note that

$$\rho \geq 0 \qquad 0 \leq \phi \leq \pi$$

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point. For example, the sphere with center at the origin and radius c has the simple equation (in spherical coordinates) $\rho = c$. See Figure 9.89. This is the reason for the name *spherical* coordinates. The graph of the equation $\theta = c$ is a vertical half-pane (Figure 9.90), and the equation $\phi = c$ represents a half-cone with the z-axis as its axis (Figure 9.91).

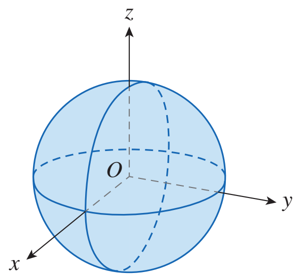


Figure 9.89
Graph of $\rho = c$, a sphere.

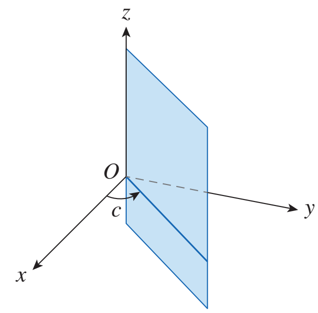


Figure 9.90
Graph of $\theta = c$, a half-plane.

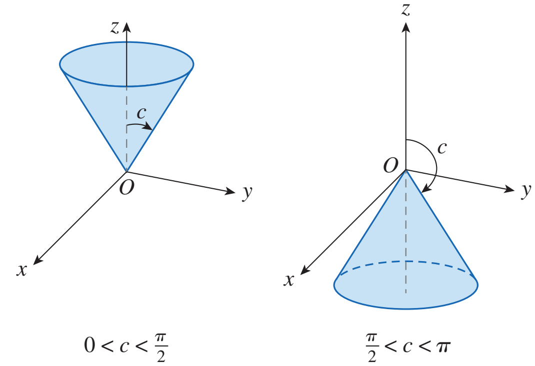


Figure 9.91
Graph of $\phi = c$, a half-cone.

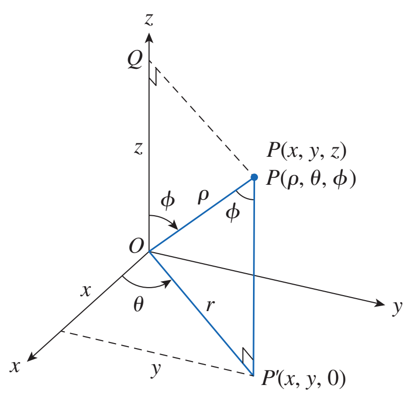


Figure 9.92
Geometric illustration of the relationship between rectangular and spherical coordinates.

The relationship between rectangular and spherical coordinates can be derived from Figure 9.92. Using triangles OPQ and OPP', we have

$$z = \rho\cos\phi \qquad r = \rho\sin\phi$$

Use the equation for r in the formulas $x = r\cos\theta$ and $y = r\sin\theta$. This gives us the equations to convert from spherical to rectangular coordinates.

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi \tag{3}$$

The distance formula shows that

$$\rho^2 = x^2 + y^2 + z^2 \tag{4}$$

We can use this equation to convert from rectangular to spherical coordinates.

Example 4 Convert from Spherical to Rectangular Coordinates

The point $\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

Solution

The point is shown in Figure 9.93.

Use the equations in (3) to find the rectangular coordinates.

$$x = \rho\,\sin\,\phi\,\cos\,\theta = 2\sin\frac{\pi}{3}\,\cos\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho\,\sin\,\phi\,\sin\,\theta = 2\sin\frac{\pi}{3}\,\sin\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho\,\cos\,\phi = 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

Therefore, the point $\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$ is represented by $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)$ in rectangular coordinates. ■

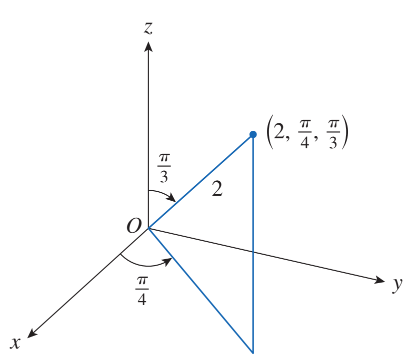


Figure 9.93
The point $\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$ in spherical coordinates.

Example 5 Convert from Rectangular to Spherical Coordinates

The point $(0,\ 2\sqrt{3},\ -2)$ is given in rectangular coordinates. Find the spherical coordinates for this point.

Solution

Use Equation 4 to find the value of ρ.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = \sqrt{16} = 4$$

Use the Equations in (3).

$$\cos\ \phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \qquad \Rightarrow \qquad \phi = \frac{2\pi}{3}$$

$$\cos\ \theta = \frac{x}{\rho\ \sin\ \phi} = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$$

Note that $\theta \neq \dfrac{3\pi}{2}$ because $y = 2\sqrt{3} > 0$.

Therefore, the spherical coordinates of the given point are $\left(4,\ \dfrac{\pi}{2},\ \dfrac{2\pi}{3}\right)$. ■

Example 6 A Spherical Equation for a Hyperboloid

Find an equation in spherical coordinates for the hyperboloid of two sheets with equation $x^2 - y^2 - z^2 = 1$.

Solution

Substitute the expressions in Equation 3 into the given equation.

$$\rho^2\sin^2\phi\ \cos^2\theta - \rho^2\sin^2\phi\ \sin^2\theta - \rho^2\cos^2\phi = 1 \qquad \text{Use Equation 3.}$$

$$\rho^2[\sin^2\phi(\cos^2\theta - \sin^2\theta) - \cos^2\phi] = 1 \qquad \text{Factor.}$$

$$\rho^2(\sin^2\phi\cos 2\theta - \cos^2\phi) = 1 \qquad \text{Trigonometric identity}$$

■

Example 7 Convert a Spherical Equation to Rectangular

Find a rectangular equation for the surface whose spherical equation is $\rho = \sin\theta\ \sin\phi$.

Solution

Use Equations 4 and 3.

$$x^2 + y^2 + z^2 = \rho^2 = \rho \cdot \rho = \rho(\sin\theta\sin\phi) = y \qquad \text{Equations 4, 3, and given equation.}$$

$$x^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = \frac{1}{4} \qquad \text{Complete the square.}$$

This is the equation of a sphere with center $\left(0,\ \dfrac{1}{2},\ 0\right)$ and radius $\dfrac{1}{2}$. ■

Example 8 Technology Sketch

Use technology to draw a picture of the solid that remains when a hole of radius 3 is drilled through the center of a sphere of radius 4.

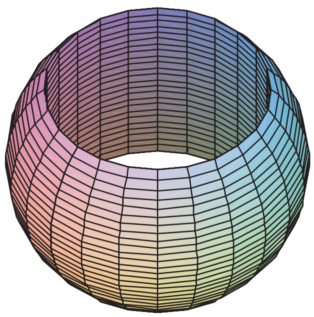


Figure 9.94
The solid left when a hole of radius 3 is drilled through the center of a sphere of radius 4.

Solution

Choose the coordinate system so that the center of the sphere is at the origin and the axis of the cylinder that forms the hole is the z-axis.

We could use either cylindrical or spherical coordinates to describe the solid, but it's probably a little easier to use cylindrical coordinates.

The equation of the cylinder is $r = 3$ and the equation of the sphere is $x^2 + y^2 + z^2 = 16$, or $r^2 + z^2 = 16$.

The points in the solid lie outside the cylinder and inside the sphere. Therefore, $3 \le r \le \sqrt{16 - z^2}$.

To make sure we graph only the appropriate parts of these surfaces, we need to find where they intersect.

Solve the equations $r = 3$ and $r = \sqrt{16 - z^2}$ simultaneously.

$$\sqrt{16 - z^2} = 3 \quad \Rightarrow \quad 16 - z^2 = 9 \quad \Rightarrow \quad z^2 = 7 \quad \Rightarrow \quad z = \pm\sqrt{7}$$

The solid lies between $z = -\sqrt{7}$ and $z = \sqrt{7}$.

We can use the following equations and domains to sketch the solid.

$$r = 3 \qquad 0 \le \theta \le 2\pi \qquad -\sqrt{7} \le z \le \sqrt{7}$$

$$r = \sqrt{16 - z^2} \qquad 0 \le \theta \le 2\pi \qquad -\sqrt{7} \le z \le \sqrt{7}$$

The resulting solid is shown in Figure 9.94. ■

9.7 Exercises

1. Describe the types of surfaces that are conveniently parameterized using cylindrical coordinates.

2. Describe the types of surfaces that are conveniently parameterized using spherical coordinates.

Plot the point whose cylindrical coordinates are given. Then find rectangular coordinates of the point.

3. (a) $\left(2, \frac{\pi}{4}, 1\right)$ (b) $\left(4, -\frac{\pi}{3}, 5\right)$

4. (a) $\left(3, -\frac{2\pi}{3}, 2\right)$ (b) $\left(1, \frac{\pi}{6}, \sqrt{3}\right)$

5. (a) $(1, \pi, e)$ (b) $\left(1, \frac{3\pi}{2}, 2\right)$

Convert each point from rectangular to cylindrical coordinates.

6. (a) $(1, -1, 4)$ (b) $\left(-1, -\sqrt{3}, 2\right)$

7. (a) $\left(2\sqrt{3}, 2, -1\right)$ (b) $(4, -3, 2)$

8. (a) $\left(1, 1, \sqrt{2}\right)$ (b) $(-2, 3, -4)$

Plot the point whose spherical coordinates are given. Then find the rectangular coordinates of the point.

9. (a) $(1, 0, 0)$ (b) $\left(2, \frac{\pi}{3}, \frac{\pi}{4}\right)$

10. (a) $\left(3, -\frac{\pi}{3}, -\frac{\pi}{6}\right)$ (b) $\left(1, -\frac{\pi}{4}, \frac{\pi}{4}\right)$

11. (a) $\left(5, \pi, \frac{\pi}{2}\right)$ (b) $\left(4, \frac{3\pi}{4}, \frac{\pi}{3}\right)$

Convert each point from rectangular to spherical coordinates.

12. (a) $\left(1, \sqrt{3}, 2\sqrt{3}\right)$ (b) $(0, -1, -1)$

13. (a) $\left(0, \sqrt{3}, 1\right)$ (b) $\left(-1, 1, \sqrt{6}\right)$

14. (a) $\left(-2, -2, \sqrt{24}\right)$ (b) $\left(1, 1, \sqrt{2}\right)$

Identify the surface represented by the given equation in cylindrical coordinates.

15. $\theta = \frac{\pi}{4}$

16. $r = 5$

17. $z = 4 - r^2$

18. $r = 2\cos\theta$

19. $2r^2 + z^2 = 1$

Identify the surface represented by the given equation in spherical coordinates.

20. $\phi = \dfrac{\pi}{3}$

21. $\rho = 3$

22. $\rho \sin\phi = 2$

23. $\rho = \sin\theta \sin\phi$

24. $\rho^2(\sin^2\phi\ \sin^2\theta + \cos^2\phi) = 9$

Write the equation (a) in cylindrical coordinates and (b) in spherical coordinates.

25. $x^2 + y^2 = 2y$

26. $x^2 + y^2 + z^2 = 2$

27. $3x + 2y + z = 6$

28. $x^2 - 2x + y^2 + z^2 = 0$

Sketch the solid described by the given inequalities.

29. $0 \le r \le 2, \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}, \quad 0 \le z \le 1$

30. $0 \le \theta \le \dfrac{\pi}{2}, \quad r \le z \le 2$

31. $\rho \le 2, \quad 0 \le \phi \le \dfrac{\pi}{2}, \quad 0 \le \theta \le \dfrac{\pi}{2}$

32. $2 \le \rho \le 3, \quad \dfrac{\pi}{2} \le \phi \le \pi$

33. $\rho \le 1, \quad \dfrac{3\pi}{4} \le \phi \le \pi$

34. $\rho \le 2, \quad \rho \le \csc\,\phi$

35. A cylindrical shell is 20 cm long, with inner radius 6 cm and outer radius 7 cm. Write inequalities that describe the shell in an appropriate coordinate system. Explain how you have positioned the coordinate system with respect to the shell.

36. (a) Find inequalities that describe a hollow ball with diameter 30 cm and thickness 0.5 cm. Explain how you have positioned the coordinate system that you have chosen.

(b) Suppose the ball is cut in half. Write inequalities that describe one of the halves.

37. A solid lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. Write a description of the solid in terms of inequalities involving spherical coordinates.

38. Use technology to sketch the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 5 - x^2 - y^2$.

39. Use technology to sketch a silo consisting of a cylinder with radius 3 and height 10 surmounted by a hemisphere.

40. Use technology to sketch and describe the graph of $\rho = 1 + 2\cos 2\theta$.

41. The latitude and longitude of a point P in the Northern Hemisphere are related to spherical coordinates ρ, θ, ϕ as follows. We take the origin to be the center of Earth and the positive z-axis to pass through the North Pole. The positive x-axis passes through the point where the prime meridian (the meridian through Greenwich, England) intersects the equator. Then the latitude of P is $\alpha = 90° - \phi°$ and the longitude is $\beta = 360° - \theta°$.

Find the great-circle distance from Los Angeles (lat. 34.06°N, long. 118.25°W) to Montréal (lat. 45.50°N, long. 73.60°W). Take the radius of Earth to be 3960 mi. (A *great circle* is the circle of intersection of a sphere and a plane through the center of the sphere.)

Laboratory Project | Families of Surfaces

The purpose of this project is to discover some interesting shapes associated with families of surfaces. You will learn how the shape of a surface evolves as constants describing the family change.

1. Use technology to investigate the family of functions
$$f(x,\ y) = (ax^2 + by^2)\, e^{-x^2-y^2}$$
Describe how the shape of the graph varies as the numbers a and b change.

2. Use technology to investigate the family of surfaces $z = x^2 + y^2 + cxy$. In particular, you should determine the transitional values of c for which the surface changes from one type of quadric surface to another.

3. Members of the family of surfaces given in spherical coordinates by the equation
$$\rho = 1 + 0.2\sin m\theta\ \sin n\phi$$
have been suggested as models for tumors and have been called *bumpy spheres* and *wrinkled spheres.* Use technology to investigate this family of surfaces, assuming that m and n are positive integers. What roles do the values of m and n play in the shape of the surface?

9 Review

Concepts and Vocabulary

1. Explain the difference between a vector and a scalar.
2. Explain how to add two vectors geometrically. Explain how to add two vectors algebraically.
3. If **a** is a vector and c is a scalar, explain how $c\mathbf{a}$ is related to **a** geometrically. How do you find $c\mathbf{a}$ algebraically?
4. Explain how to find the vector from one point to another.
5. Explain how to find the dot product $\mathbf{a} \cdot \mathbf{b}$ of two vectors if their lengths and the angle between them are known. What if you know their components?
6. Describe some practical uses for the dot product.
7. Write expressions for the scalar and vector projections of **b** onto **a**. Illustrate these expressions with diagrams.
8. Explain how to find the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors if their lengths and the angle between them are known. What if you know their components?
9. Describe some practical uses for the cross product.
10. (a) Explain how to find the area of the parallelogram determined by **a** and **b**.
    (b) How do you find the volume of the parallelepiped determined by **a**, **b**, and **c**?
11. How do you find a vector perpendicular to a plane?
12. How do you find the angle between two intersecting planes?
13. Write a vector equation, parametric equations, and symmetric equations for a line.
14. Write a vector equation and a scalar equation for a plane.
15. (a) How do you tell if two vectors are parallel?
    (b) How do you tell if two vectors are perpendicular?
    (c) How do you tell if two planes are parallel?
16. (a) Describe a method for determining whether three points P, Q, and R lie on the same line.
    (b) Describe a method for determining whether four points P, Q, R, and S lie in the same plane.
17. (a) How do you find the distance from a point to a line?
    (b) How do you find the distance from a point to a plane?
    (c) How do you find the distance between two lines?
18. Explain how you would sketch the graph of a function of two variables.
19. What are the traces of a surface? How do you find them?
20. Write equations in standard form of the six types of quadric surfaces.
21. (a) Write the equations for converting from cylindrical to rectangular coordinates. In what situation would you use cylindrical coordinates?
    (b) Write the equations for converting from spherical to rectangular coordinates. In what situation would you use spherical coordinates?

True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, then $\mathbf{u} \cdot \mathbf{v} = \langle u_1v_1, u_2v_2 \rangle$.
2. For any vectors **u** and **v** in V_3, $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$.
3. For any vectors **u** and **v** in V_3, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
4. For any vectors **u** and **v** in V_3, $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|$
5. For any vectors **u** and **v** in V_3, $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.
6. For any vectors **u** and **v** in V_3, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$
7. For any vectors **u** and **v** in V_3, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|$
8. For any vectors **u** and **v** in V_3 and any scalar k, $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$.
9. For any vectors **u** and **v** in V_3 and any scalar k, $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v}$.
10. For any vectors **u**, **v**, and **w** in V_3, $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$.
11. For any vectors **u**, **v**, and **w** in V_3, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.
12. For any vectors **u**, **v**, and **w** in V_3, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.
13. For any vectors **u** and **v** in V_3, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$.
14. For any vectors **u** and **v** in V_3, $(\mathbf{u} + \mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v}$.
15. The cross product of two unit vectors is a unit vector.
16. A linear equation $Ax + By + Cz + D = 0$ represents a line in space.
17. The vector $\langle -2, 1, 3 \rangle$ is parallel to the plane $-6x + 3y + 9z = 5$.
18. The set of points $\{(x, y, z) \mid x^2 + y^2 = 1\}$ is a circle.
19. In $\mathbb{R}^3$, the graph of $y = x^2$ is a paraboloid.
20. If $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
21. If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
22. If $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
23. If **u** and **v** are in V_3, then $|\mathbf{u} \cdot \mathbf{v}| \le |\mathbf{u}|\,|\mathbf{v}|$.

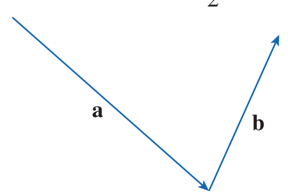
Exercises

1. (a) Find an equation of the sphere that passes through the point $(6, -2, 3)$ and has center $(-1, 2, 1)$.
(b) Find the curve in which this sphere intersects the yz-plane.
(c) Find the center and radius of the sphere

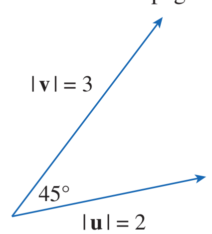
$$x^2 + y^2 + z^2 - 8x + 2y + 6z + 1 = 0$$

2. Copy the vectors in the figure and use them to draw each of the following vectors.

(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$ (c) $-\frac{1}{2}\mathbf{a}$ (d) $2\mathbf{a} + \mathbf{b}$



3. If $\mathbf{u}$ and $\mathbf{v}$ are the vectors shown in the figure, find $\mathbf{u} \cdot \mathbf{v}$ and $|\mathbf{u} \times \mathbf{v}|$. Is $\mathbf{u} \times \mathbf{v}$ directed into the page or out of it?



4. Suppose the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in V_3 are defined as
$\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$, $\mathbf{b} = 3\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{j} - 5\,\mathbf{k}$.
Evaluate each expression.

(a) $2\mathbf{a} + 3\mathbf{b}$ (b) $|\mathbf{b}|$
(c) $\mathbf{a} \cdot \mathbf{b}$ (d) $\mathbf{a} \times \mathbf{b}$
(e) $|\mathbf{b} \times \mathbf{c}|$ (f) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
(g) $\mathbf{c} \times \mathbf{c}$ (h) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
(i) $\text{comp}_{\mathbf{a}}\mathbf{b}$ (j) $\text{proj}_{\mathbf{a}}\mathbf{b}$
(k) The angle between $\mathbf{a}$ and $\mathbf{b}$ (correct to the nearest degree)

5. Find the values of x such that the vectors $\langle 3, 2, x \rangle$ and $\langle 2x, 4, x \rangle$ are orthogonal.

6. Find two unit vectors that are orthogonal to both $\mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}$.

7. Suppose that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 2$. Evaluate each expression.

(a) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ (b) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$
(c) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$ (d) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$

8. Show that if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are in V_3, then

$$(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2$$

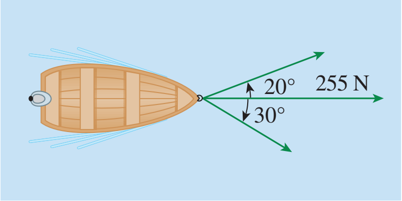
9. Find the acute angle between two diagonals of a cube.

10. Given the points $A(1, 0, 1)$, $B(2, 3, 0)$, $C(-1, 1, 4)$, and $D(0, 3, 2)$, find the volume of the parallelepiped with adjacent edges AB, AC, and AD.

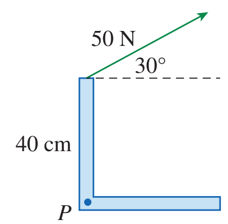
11. (a) Find a vector perpendicular to the plane through the points $A(1, 0, 0)$, $B(2, 0, -1)$, and $C(1, 4, 3)$.
(b) Find the area of triangle ABC.

12. A constant force $\mathbf{F} = 3\,\mathbf{i} + 5\,\mathbf{j} + 10\,\mathbf{k}$ moves an object along the line segment from $(1, 0, 2)$ to $(5, 3, 8)$. Find the work done if the distance is measured in meters and the force in newtons.

13. A boat is pulled onto shore using two ropes, as shown in the diagram. If a force of 255 N is needed, find the magnitude of the force in each rope.



14. Find the magnitude of the torque about P if a 50-N force is applied as shown in the figure.



Find parametric equations for the line.

15. The line through $(4, -1, 2)$ and $(1, 1, 5)$

16. The line through $(1, 0, -1)$ and parallel to the line $\frac{1}{3}(x - 4) = \frac{1}{2}y = z + 2$

17. The line through $(-2, 2, 4)$ and perpendicular to the plane $2x - y + 5z = 12$

Find an equation of the plane.

18. The plane through $(2, 1, 0)$ and parallel to $x + 4y - 3z = 1$

19. The plane through $(3, -1, 1)$, $(4, 0, 2)$, and $(6, 3, 1)$

20. The plane through $(1, 2, -2)$ that contains the line $x = 2t$, $y = 3 - t$, $z = 1 + 3t$

21. The plane through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and perpendicular to the plane $x + y - 2z = 1$

22. Find the point in which the line with parametric equations $x = 2 - t$, $y = 1 + 3t$, $z = 4t$ intersects the plane $2x - y + z = 2$.

23. Determine whether the lines given by the symmetric equations

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

and
$$\frac{x+1}{6} = \frac{y-3}{-1} = \frac{z+5}{2}$$

are parallel, skew, or intersecting.

24. (a) Show that the planes $x + y - z = 1$ and $2x - 3y + 4z = 5$ are neither parallel nor perpendicular.
(b) Find the angle between these planes.

25. (a) Find the distance between the planes $3x + y - 4z = 2$ and $3x + y - 4z = 24$.
(b) Find the distance from the origin to the line $x = 1 + t$, $y = 2 - t$, $z = -1 + 2t$.

26. (a) Find an equation of the plane that passes through the points $A(2, 1, 1)$, $B(-1, -1, 10)$, and $C(1, 3, -4)$.
(b) Find symmetric equations for the line through B that is perpendicular to the plane in part (a).
(c) A second plane passes through $(2, 0, 4)$ and has normal vector $\langle 2, -4, -3 \rangle$. Show that the acute angle between the planes is approximately 43°.
(d) Find parametric equations for the line of intersection of the two planes.

Find and sketch the domain of the function.

27. $f(x, y) = x \ln(x - y^2)$

28. $f(x, y) = \sqrt{\sin \pi(x^2 + y^2)}$

Sketch the graph of the function.

29. $f(x, y) = 6 - 2x - 3y$

30. $f(x, y) = \cos y$

31. $f(x, y) = 4 - x^2 - 4y^2$

32. $f(x, y) = \sqrt{4 - x^2 - 4y^2}$

Identify and sketch the graph of the surface.

33. $x = 3$

34. $x = z$

35. $y = z^2$

36. $x^2 = y^2 + 4z^2$

37. $y^2 + z^2 = 1 - 4x^2$

38. $y^2 + z^2 = x$

39. $y^2 + z^2 = 1$

40. $y^2 + z^2 = 1 + x^2$

41. The cylindrical coordinates of a point are $\left(2\sqrt{3}, \frac{\pi}{3}, 2\right)$. Find the rectangular and spherical coordinates of the point.

42. The rectangular coordinates of a point are $(2, 2, -1)$. Find the cylindrical and spherical coordinates of the point.

43. The spherical coordinates of a point are $\left(8, \frac{\pi}{4}, \frac{\pi}{6}\right)$. Find the rectangular and cylindrical coordinates of the point.

44. Identify the surfaces whose equations are given.
(a) $\theta = \frac{\pi}{4}$ (b) $\phi = \frac{\pi}{4}$

Write the equation in cylindrical coordinates and in spherical coordinates.

45. $x^2 + y^2 + z^2 = 4$

46. $x^2 + y^2 = 4$

47. The parabola $z = 4y^2$, $x = 0$, is rotated about the z-axis. Write an equation of the resulting surface in cylindrical coordinates.

48. Sketch the solid consisting of all points with spherical coordinates (ρ, θ, ϕ) such that $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{6}$, and $0 \le \rho \le 2 \cos \phi$.

Focus on Problem Solving

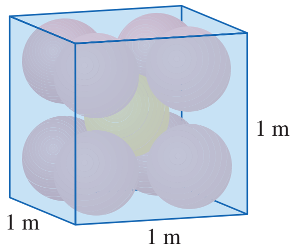


Figure 9.95
Nine tightly packed balls, each with radius *r*.

1. Each edge of a cubical box has length 1 m. The box contains nine spherical balls with the same radius r. The center of one ball is at the center of the cube and it touches the other eight balls. Each of the other eight balls touches three sides of the box. Therefore, the balls are tightly packed in the box, as illustrated in Figure 9.95. Find the value of r.

2. Let B be a solid box with length L, width W, and height H. Let S be the set of all points that are a distance at most 1 from some point of B. Find an expression for the volume of S in terms of L, W, and H.

3. Let L be the line of intersection of the planes $cx + y + z = c$ and $x - cy + cz = -1$, where c is a real number.
(a) Find symmetric equations for L.
(b) As the number c varies, the line L sweeps out a surface S. Find an equation for the curve of intersection of S with the horizontal plane $z = t$ (the trace of S in the plane $z = t$).
(c) Find the volume of the solid bounded by S and the planes $z = 0$ and $z = 1$.

4. A plane is capable of flying at a speed of 180 km/h in still air. The pilot takes off from an airfield and heads due north according to the plane's compass. After 30 minutes of flight time, the pilot notices that, due to the wind, the plane has actually traveled 80 km at an angle 5° east of north.
(a) What is the wind velocity?
(b) In what direction should the pilot have headed to reach the intended destination?

5. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors with $|\mathbf{v}_1| = 2$, $|\mathbf{v}_2| = 3$, and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 5$. Let $\mathbf{v}_3 = \text{proj}_{\mathbf{v}_1} \mathbf{v}_2$, $\mathbf{v}_4 = \text{proj}_{\mathbf{v}_2} \mathbf{v}_3$, $\mathbf{v}_5 = \text{proj}_{\mathbf{v}_3} \mathbf{v}_4$, and so on. Compute $\sum_{n=1}^{\infty} |\mathbf{v}_n|$.

6. Find an equation of the largest sphere that passes through the point $(-1, 1, 4)$ such that each of the points (x, y, z) inside the sphere satisfies the condition

$$x^2 + y^2 + z^2 < 136 + 2(x + 2y + 3z)$$

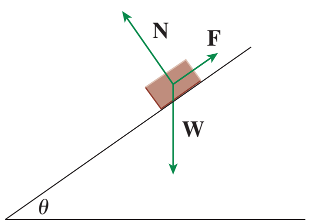


Figure 9.96
A block of mass *m* on an inclined plane.

7. Suppose a block of mass m is placed on an inclined plane, as shown in Figure 9.96. The block's descent down the plane is slowed by friction; if θ is not too large, friction will prevent the block from moving at all. The forces acting on the block are the weight $\mathbf{W}$, where $|\mathbf{W}| = mg$ (g is the acceleration due to gravity); the normal force $\mathbf{N}$ (the normal component of the reactionary force of the plane on the block), where $|\mathbf{N}| = n$; and the force $\mathbf{F}$ due to friction, which acts parallel to the inclined plane, opposing the direction of motion.

If the block is at rest and θ is increased, $|\mathbf{F}|$ must also increase until ultimately $|\mathbf{F}|$ reaches its maximum, beyond which the block begins to slide. At this angle θ_s, it has been observed that $|\mathbf{F}|$ is proportional to n. Therefore, when $|\mathbf{F}|$ is maximal, we can say that $|\mathbf{F}| = \mu_s n$, where μ_s is called the *coefficient of static friction* and depends on the materials that are in contact.
(a) Observe that $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$ and deduce that $\mu_s = \tan(\theta_s)$.
(b) Suppose that, for $\theta > \theta_s$, an additional outside force $\mathbf{H}$ is applied to the block, horizontally from the left, and let $|\mathbf{H}| = h$. If h is small, the block may still slide down the plane; if h is large enough, the block will move up the plane. Let $h_{\min}$ be the smallest value of h that allows the block to remain motionless (so that $|\mathbf{F}|$ is maximal).

By choosing coordinate axes so that $\mathbf{F}$ lies along the x-axis, resolve each force into components parallel and perpendicular to the inclined plane and show that

$$h_{\min} \sin\theta + mg\cos\theta = n \quad \text{and} \quad h_{\min}\cos\theta + \mu_s n = mg\sin\theta$$

(c) Show that

$$h_{\min} = mg\tan(\theta - \theta_s)$$

Does this equation seem reasonable? Does it make sense for $\theta = \theta_s$? Does it make sense as $\theta \to 90°$?

(d) Let $h_{\max}$ be the largest value of h that allows the block to remain motionless. (In which direction is **F** heading?) Show that

$$h_{\max} = mg\tan(\theta + \theta_s)$$

Does this equation seem reasonable? Explain.

8. A solid has the following properties. When illuminated by rays parallel to the z-axis, its shadow is a circular disk. If the rays are parallel to the y-axis, its shadow is a square. If the rays are parallel to the x-axis, its shadow is an isosceles triangle. Assume that the projection on the xz-plane is a square whose sides have length 1.

(a) What is the volume of the largest such solid?

(b) Is there a smallest volume?

Trong Nguyen/Shutterstock.com

An Euler Spiral, or Cornu Spiral, has curvature that changes linearly with its curve length. These curves are used in the construction of railroad tracks, applications of optics, and typography.

Many freeway interchanges are also designed in the shape of a Cornu Spiral. This 3D transition curve between two straight roads provides a smooth ride for passengers in vehicles, especially if they are traveling at high rates of speed. These curves are designed so that there is little sideways acceleration.

10 Vector Functions

Contents

The functions that we have studied so far have been real-valued functions. In this chapter, we will consider functions whose values are vectors. These kinds of functions are used to describe curves and surfaces in space. We will also use vector-valued functions to describe motion of objects through space; we will use them to derive Kepler's laws of planetary motion.

10.1 Vector Functions and Space Curves

Vector Functions

In general, a function is a rule that assigns to each element in the domain an element in the range. A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors. We are most interested in vector functions $\mathbf{r}$ whose values are three-dimensional vectors. This means that for every number t in the domain of $\mathbf{r}$, there is a unique vector denoted by $\mathbf{r}(t)$. If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f, g, and h are real-value functions called the **component functions** of $\mathbf{r}$, and we can write

$$\mathbf{r}(t) = \langle f(t),\ g(t),\ h(t)\rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

The letter t is often used to denote the independent variable and represents time in many applications of vector functions.

Example 1 Domain of a Vector Function

Suppose $\mathbf{r}(t) = \langle t^3,\ \ln(3-t),\ \sqrt{t}\rangle$ is a vector function.

Then the component functions are

$f(t) = t^3, \quad g(t) = \ln(3-t), \quad h(t) = \sqrt{t}.$

Using the usual convention, the domain of $\mathbf{r}$ consists of all values of t for which the expression for $\mathbf{r}(t)$ is defined.

The component function t^3 is defined for all real numbers; the component $\ln(3-t)$ is defined when $3 - t > 0 \quad \Rightarrow \quad t < 3$; and the component $\sqrt{t}$ is defined when $t \geq 0$.

Therefore, the domain of $\mathbf{r}$ is the interval $[0, 3)$. ■

Limits and Continuity

The **limit** of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions. Here's a formal definition.

If $\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$, this definition is equivalent to saying that the length and direction of the vector $\mathbf{r}(t)$ approach the length and direction of the vector $\mathbf{L}$.

Limit of a Vector Function

If $\mathbf{r}(t) = \langle f(t),\ g(t),\ h(t)\rangle$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) \right\rangle \tag{1}$$

provided the limits of the component functions exist.

Limits of vector functions follow the same rules as limits of real-valued functions.

Example 2 Find the Limit of a Vector Function

Suppose $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \dfrac{\sin t}{t}\,\mathbf{k}$. Find $\lim_{t\to 0} \mathbf{r}(t)$.

Solution

The limit of **r** is the vector whose components are the limits of the component functions of **r**.

$$\begin{aligned}\lim_{t\to 0}\mathbf{r}(t) &= \left[\lim_{t\to 0}(1+t^3)\right]\mathbf{i} + \left[\lim_{t\to 0} te^{-t}\right]\mathbf{j} + \left[\lim_{t\to 0}\frac{\sin t}{t}\right]\mathbf{k} && \text{Equation 1.}\\ &= 1\mathbf{i} + 0\mathbf{j} + 1\mathbf{k} && \text{Direct substitution; L'Hospital's Rule; known limit.}\\ &= \mathbf{i} + \mathbf{k} && \text{Simplify.}\end{aligned}$$

A vector function **r** is **continuous at *a*** if

$$\lim_{t\to a}\mathbf{r}(t) = \mathbf{r}(a)$$

Given the definition for finding the limit of a vector function, this means that **r** is continuous at a if and only if its component functions f, g, and h are continuous at a.

Space Curves

There is a close connection between continuous vector functions and space curves. Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

$$x = f(t) \qquad y = g(t) \qquad z = h(t) \tag{2}$$

and t varies throughout the interval I, is called a **space curve**. The equations in (2) are called **parametric equations of *C*** and t is called a **parameter**. We can think of C as being traced out by a moving particle whose position at time t is $(f(t), g(t), h(t))$. If we now consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C. Therefore, any continuous vector function **r** defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$, as shown in Figure 10.1.

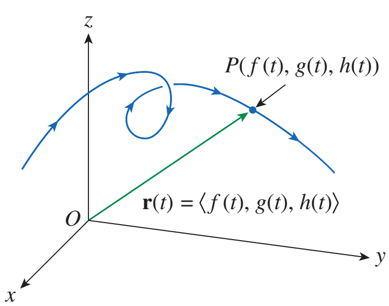


Figure 10.1
The space curve C is traced out by the tip of a moving position vector $\mathbf{r}(t)$.

Example 3 Space Curve Description

Describe the curve defined by the vector function

$$\mathbf{r}(t) = \langle 1+t,\ 2+5t,\ -1+6t\rangle$$

Solution

The corresponding parametric equations are

$x = 1 + t, \quad y = 2 + 5t, \quad z = -1 + 6t.$

These are the parametric equations of a line passing through the point $(1, 2, -1)$ and parallel to the vector $\mathbf{v} = \langle 1, 5, 6\rangle$.

Note that we can also write this equation as

$\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v} = \langle 1, 2, -1\rangle + t\langle 1, 5, 6\rangle$; this is the vector equation of a line.

Plane curves can also be represented in vector notation. For example, the curve given by the parametric equations $x = t^2 - 2t$ and $y = t + 1$ could also be described by the vector equation

$$\mathbf{r}(t) = \langle t^2 - 2t,\ t+1\rangle = (t^2 - 2t)\,\mathbf{i} + (t+1)\,\mathbf{j}$$

where $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$.

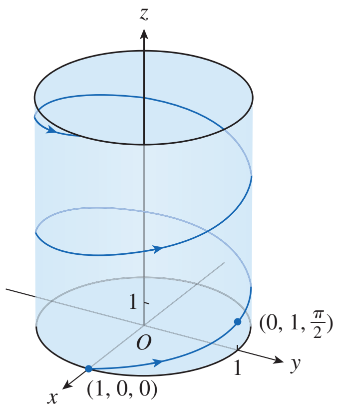


Figure 10.2
The curve spirals upward around the cylinder.

Figure 10.3
A double helix.

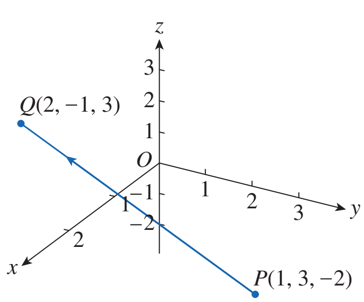


Figure 10.4
Graph of the line segment from P to Q.

Example 4 Sketch a Helix

Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

Solution

The parametric equations for this curve are

$x = \cos\ t, \quad y = \sin\ t, \quad z = t.$

Since $x^2 + y^2 = \cos^2\ t + \sin^2\ t = 1$, the curve must lie on the circular cylinder $x^2 + y^2 = 1$.

The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$ in the xy-plane.

Alternatively, the projection of the curve onto the xy-plane has vector equation $\mathbf{r}(t) = \langle \cos\ t,\ \sin\ t, 0\rangle$.

Since $z = t$, the curve spirals upward around the cylinder as t increases.

The curve is shown in Figure 10.2 and is called a **helix**. ■

The corkscrew shape of the helix in Example 4 is probably familiar. It looks like a coiled spring. It also occurs in the model of DNA (deoxyribonucleic acid, the genetic material of living cells). In 1953, James Watson and Francis Crick showed that the structure of the DNA molecule is that of two linked, parallel helixes that are intertwined as in Figure 10.3.

In Examples 3 and 4, we were given vector equations of curves and asked for a geometric description or sketch. In the next two examples, we are given a geometric description of a curve and are asked to find parametric equations for the curve.

Example 5 Find an Equation

Find a vector equation and parametric equations for the line segment that joins the point $P(1,\ 3,\ -2)$ to the point $Q(2,\ -1,\ 3)$.

Solution

A vector equation for the line segment that joins the tip of the vector $\mathbf{r}_0$ to the tip of vector $\mathbf{r}_1$ is

$\mathbf{r}(t) = (1 - t)\,\mathbf{r}_0 + t\,\mathbf{r}_1, \quad 0 \le t \le 1.$

Let $\mathbf{r}_0 = \langle 1,\ 3,\ -2\rangle$ and $\mathbf{r}_1 = \langle 2,\ -1,\ 3\rangle$.

The vector equation of the line segment from P to Q is:

$$\mathbf{r}(t) = (1 - t)\langle 1,\ 3,\ -2\rangle + t\langle 2,\ -1,\ 3\rangle \qquad 0 \le t \le 1$$

or $$\mathbf{r}(t) = \langle 1 + t,\ 3 - 4t,\ -2 + 5t\rangle \qquad 0 \le t \le 1$$

Figure 10.4 shows a graph of this line segment.

The corresponding parametric equations are

$x = 1 + t, \quad y = 3 - 4t, \quad z = -2 + 5t, \quad 0 \le t \le 1.$ ■

Example 6 The Intersection of Two Surfaces Is a Space Curve

Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Solution

Figure 10.5 shows how the plane and the cylinder intersect, and Figure 10.6 shows the curve of intersection C, which is an ellipse.

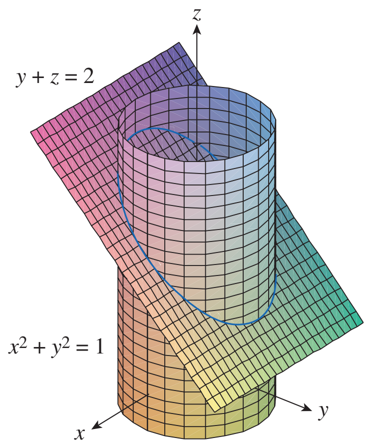


Figure 10.5
A graph of the plane and the cylinder.

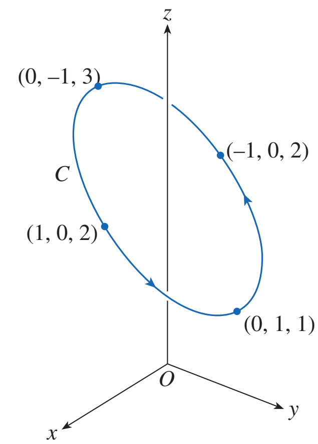


Figure 10.6
The curve of intersection C is an ellipse.

The projection of C onto the xy-plane is the circle $x^2 + y^2 = 1,\ z = 0$.

The equation of the circle can be written as:
$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi.$

Use this expression for y in the equation of the plane:
$z = 2 - y = 2 - \sin t$

Therefore, the parametric equations for C are:
$x = \cos t, \quad y = \sin t, \quad z = 2 - \sin t, \quad 0 \le t \le 2\pi.$

The corresponding vector equation is
$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + (2 - \sin t)\,\mathbf{k}, \quad 0 \le t \le 2\pi.$

This equation is called a *parametrization* of the curve C.
The arrows in Figure 10.6 indicate the direction in which C is traced as the parameter t increases.

Using Technology to Draw Space Curves

Space curves are inherently more difficult to draw by hand than plane curves; for an accurate representation, we need to use technology. For instance, Figure 10.7 shows a computer-generated graph of the curve with parametric equations

$$x = (4 + \sin 20t)\cos t \qquad y = (4 + \sin 20t)\sin t \qquad z = \cos 20t$$

This is called a **toroidal spiral** because the curve lies on a torus. Another interesting curve, the **trefoil knot**, with equations

$$x = (2 + \cos 1.5t)\cos t \qquad y = (2 + \cos 1.5t)\sin t \qquad z = \sin 1.5t$$

is shown in Figure 10.8. It's not easy to plot either of these curves by hand.

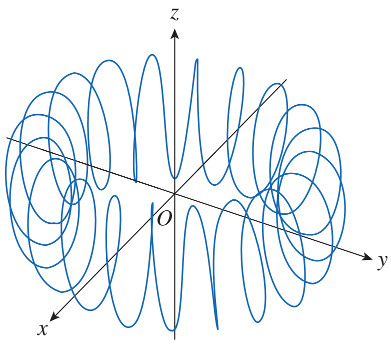


Figure 10.7
A toroidal spiral.

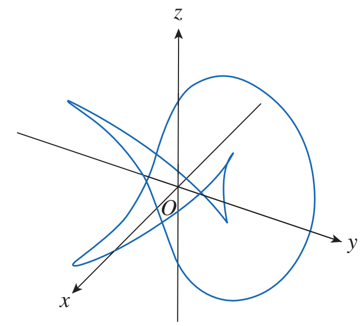


Figure 10.8
A trefoil knot.

Even when we use technology to draw a space curve, optical illusions and different viewpoints often make it difficult to really understand what the curve looks like. The next example shows a method for dealing with these issues.

Example 7 Twisted Cubic

Use technology to sketch the curve with vector equation $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. This curve is called a **twisted cubic**.

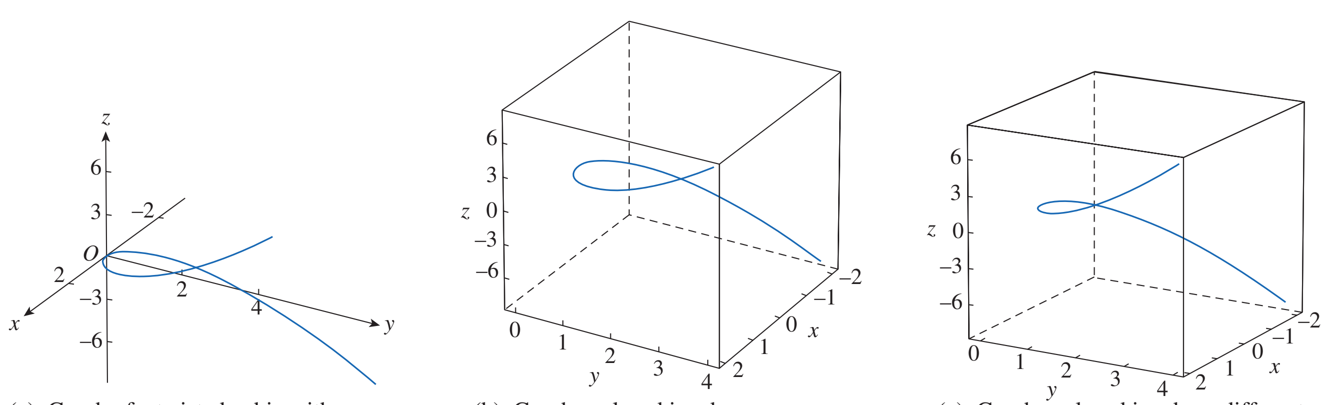
Solution

Start by plotting the curve with parametric equations

$$x = t, \quad y = t^2, \quad z = t^3, \quad -2 \le t \le 2.$$

The result is shown in Figure 10.9(a), but it's hard to see the true characteristics of this curve from this graph alone.

Most three-dimensional graphing programs allow the user to enclose a curve or surface in a box instead of, or in addition to, displaying the coordinate axes. The same curve in a box in Figure 10.9(b) presents a clearer picture of the curve. We can see that it climbs from a lower corner of the box to the upper corner nearest us, and it twists as it climbs. We get an even better idea of the curve when we view it from different viewpoints. Figure 10.9(c) shows the curve enclosed in a box but from another viewpoint.



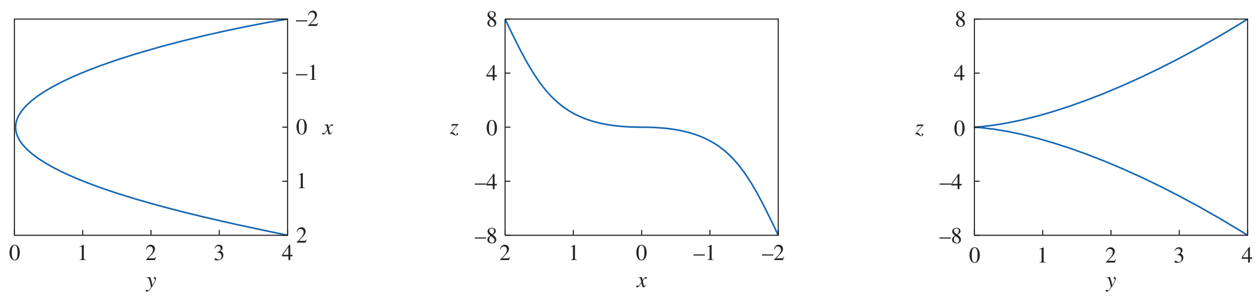
(a) Graph of a twisted cubic with coordinate axes.

(b) Graph enclosed in a box.

(c) Graph enclosed in a box; different viewpoint.

Figure 10.9
Computer-generated graphs of a twisted cubic; different viewpoints.

Figures 10.10(a), (b), and (c) show the curve when we look directly at a face of the box. In particular, Figure 10.10(a) shows the view from directly above the box. It is the projection of the curve on the xy-plane, namely, the parabola $y = x^2$. Figure 10.10(b) shows the projection on the xz-plane, the cubic curve $z = x^3$.



(a) Projection of the twisted cubic on the xy-plane.

(b) Projection of the twisted cubic on the xz-plane.

(c) Projection of the twisted cubic on the yz-plane.

Figure 10.10
The views of the twisted cubic looking at a face of the box.

All of these viewpoints suggest the name for this curve; a twisted cubic. ■

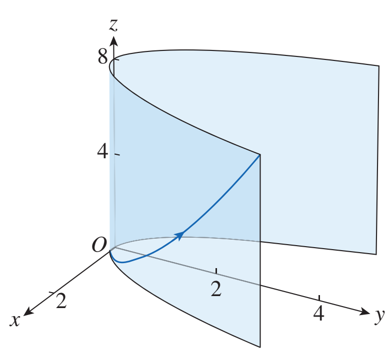


Figure 10.11
The twisted cubic on the parabolic cylinder.

Another method of visualizing a space curve is to draw it on a surface. For example, the twisted cubic in Example 7 lies on the parabolic cylinder $y = x^2$. To verify this, eliminate the parameter from the first two parametric equations, $x = t$ and $y = t^2$.

Figure 10.11 shows both the cylinder and the twisted cubic; the curve moves upward from the origin along the surface of the cylinder. We used this method in Example 4 to visualize the helix lying on the circular cylinder (see Figure 10.2).

A third method for visualizing the twisted cubic is to recognize that it also lies on the cylinder $z = x^3$. Therefore, it can be interpreted and visualized as the curve of intersection of the cylinders $y = x^2$ and $z = x^3$. See Figure 10.12.

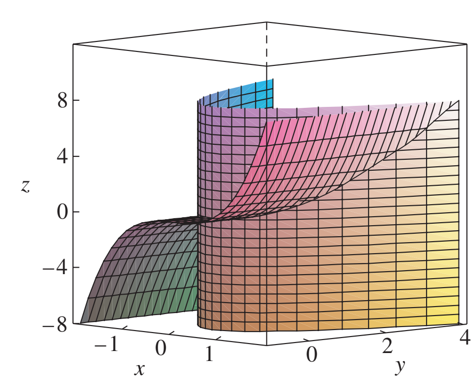


Figure 10.12
The twisted cubic visualized as the intersection of surfaces: the cylinders $y = x^2$ and $z = x^3$.

We have already seen an interesting space curve, the helix, which is used as a model of DNA. Another neat example of a space curve in science is the trajectory of a positively charged particle in orthogonally oriented electric and magnetic fields **E** and **B**.

Depending on the initial velocity given the particle at the origin, the path of the particle is either a space curve whose projection on the horizontal plane is the cycloid (Figure 10.13) or a curve whose projection is the trochoid (Figure 10.14).

Often we can visualize a space curve better by enclosing it in a tube. This type of plot enables us to see whether one part of a curve passes in front of or behind another part of the curve. For example, Figure 10.15 shows the curve from Figure 10.14 rendered as a tube.

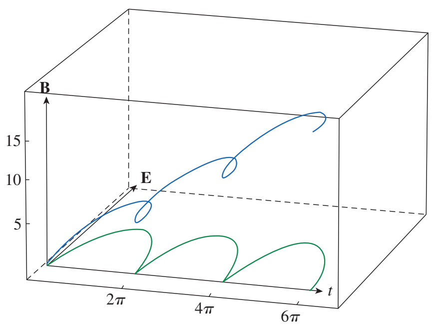


Figure 10.13
Plot of the space curve defined by the vector function $\mathbf{r}(t) = \langle t - \sin t,\ 1 - \cos t,\ t\rangle$. The projection on the horizontal plane is a cycloid.

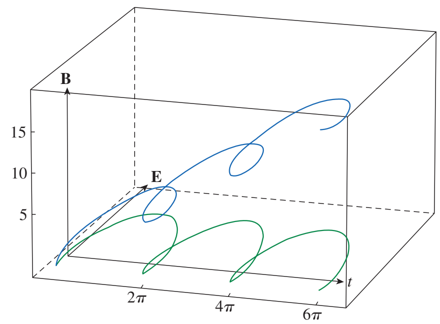


Figure 10.14
Plot of the space curve defined by the vector function $\mathbf{r}(t) = \left\langle t - \frac{3}{2}\sin t,\ 1 - \frac{3}{2}\cos t,\ t\right\rangle$. The projection on the horizontal plane is a trochoid.

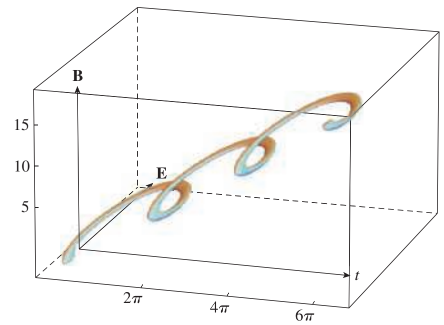


Figure 10.15
A tube plot of the space curve defined in Figure 10.14.

10.1 Exercises

Find the domain of the vector function.

1. $\mathbf{r}(t) = \langle \sqrt{4 - t^2},\ e^{-3t},\ \ln(t+1)\rangle$

2. $\mathbf{r}(t) = \left\langle \ln(t+1),\ \dfrac{t}{\sqrt{9 - t^2}},\ 2^t \right\rangle$

3. $\mathbf{r}(t) = \dfrac{t-2}{t+2}\,\mathbf{i} + \sin t\,\mathbf{j} + \ln(9 - t^2)\,\mathbf{k}$

4. $\mathbf{r}(t) = \tan^{-1}t\,\mathbf{i} + |t - 1|\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$

5. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \ln t\,\mathbf{j} + \dfrac{1}{t-2}\,\mathbf{k}$

Find the limit.

6. $\lim\limits_{t\to\infty} \left\langle \dfrac{1+t^2}{1-t^2},\ \tan^{-1}t,\ \dfrac{1 - e^{-2t}}{t} \right\rangle$

7. $\lim\limits_{t\to 0} \left\langle \dfrac{e^t - 1}{t},\ \dfrac{\sqrt{1+t} - 1}{t},\ \dfrac{3}{1+t} \right\rangle$

8. $\lim\limits_{t\to 0} \left(e^{-3t}\,\mathbf{i} + \dfrac{t^2}{\sin^2 t}\,\mathbf{j} + \cos 2t\,\mathbf{k} \right)$

9. $\lim\limits_{t\to 1} \left(\dfrac{t^2 - t}{t - 1}\,\mathbf{i} + \sqrt{t + 8}\,\mathbf{j} + \dfrac{\sin \pi t}{\ln t}\,\mathbf{k} \right)$

Sketch the curve with the given vector equation. Indicate with an arrow the direction in which t increases.

10. $\mathbf{r}(t) = \langle \sin t,\ t\rangle$

11. $\mathbf{r}(t) = \langle t^2 - 1,\ t\rangle$

12. $\mathbf{r}(t) = \langle t^3,\ t^2\rangle$

13. $\mathbf{r}(t) = \langle t,\ 2 - t,\ 2t\rangle$

14. $\mathbf{r}(t) = \langle \sin \pi t,\ t,\ \cos \pi t\rangle$

15. $\mathbf{r}(t) = \langle 3,\ t,\ 2 - t^2\rangle$

16. $\mathbf{r}(t) = \langle 1,\ \cos t,\ 2\sin t\rangle$

17. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$

18. $\mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{j} + 2\,\mathbf{k}$

19. $\mathbf{r}(t) = t^2\,\mathbf{i} + t^4\,\mathbf{j} + t^6\,\mathbf{k}$

20. $\mathbf{r}(t) = \cos t\,\mathbf{i} - \cos t\,\mathbf{j} + \sin t\,\mathbf{k}$

Draw the projections of the curve onto the three coordinate planes. Use these projections to help sketch the curve.

21. $\mathbf{r}(t) = \langle t, \sin t, 2\cos t \rangle$ **22.** $\mathbf{r}(t) = \langle t, t, t^2 \rangle$

Find a vector equation and parametric equations for the line segment that joins P to Q.

23. $P(0, 0, 0)$, $Q(1, 2, 3)$

24. $P(1, 0, 1)$, $Q(2, 3, 1)$

25. $P(1, -1, 2)$, $Q(4, 1, 7)$

26. $P(-2, 4, 0)$, $Q(6, -1, 2)$

27. $P(-1, 2, -2)$, $Q(-3, 5, 1)$

28. $P(a, b, c)$, $Q(u, v, w)$

Match the parametric equations with the graphs labeled I–VI. Give reasons for your choices.

29. $x = t\cos t, \quad y = t, \quad z = t\sin t, \quad t \geq 0$

30. $x = \cos t, \quad y = \sin t, \quad z = \dfrac{1}{1 + t^2}$

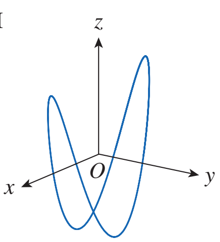
31. $x = t, \quad y = \dfrac{1}{1 + t^2}, \quad z = t^2$

32. $x = \cos t, \quad y = \sin t, \quad z = \cos 2t$

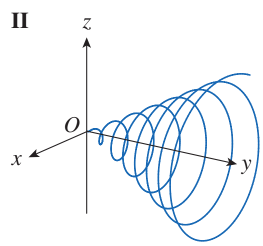
33. $x = \cos 8t, \quad y = \sin 8t, \quad z = e^{0.8t}, \quad t \geq 0$

34. $x = \cos^2 t, \quad y = \sin^2 t, \quad z = t$

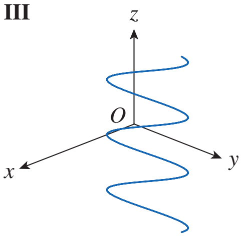
I



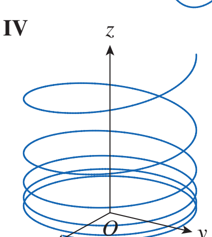
II



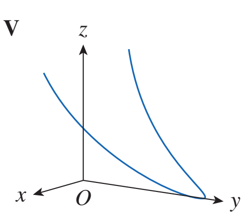
III



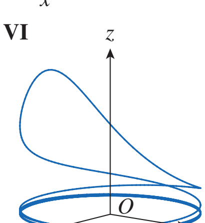
IV



V



VI



35. Show that the curve with parametric equations $x = t\cos t$, $y = t\sin t$, $z = t$ lies on the cone $z^2 = x^2 + y^2$, and use this fact to help sketch the curve.

36. Show that the curve with parametric equations $x = \sin t$, $y = \cos t$, $z = \sin^2 t$ is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$. Use this fact to help sketch the curve.

37. Find three different surfaces that contain the curve

$$\mathbf{r}(t) = 2t\,\mathbf{i} + e^t\,\mathbf{j} + e^{2t}\,\mathbf{k}$$

38. Find three different surfaces that contain the curve

$$\mathbf{r}(t) = t^2\,\mathbf{i} + \ln t\,\mathbf{j} + \frac{1}{t}\,\mathbf{k}$$

39. At what point does the curve $\mathbf{r}(t) = t\,\mathbf{i} + (2t - t^2)\,\mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?

40. At what point does the helix $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

Use technology to graph the curve with the given vector equation. Choose a parameter domain and viewpoints that reveal the important characteristics of the curve.

41. $\mathbf{r}(t) = \langle \cos t \sin 2t, \sin t \sin 2t, \cos 2t \rangle$

42. $\mathbf{r}(t) = \langle t, e^t, \cos t \rangle$

43. $\mathbf{r}(t) = \left\langle \sin 3t \cos t, \frac{1}{4}t, \sin 3t \sin t \right\rangle$

44. $\mathbf{r}(t) = \langle \cos(8\cos t)\sin t, \sin(8\cos t)\sin t, \cos t \rangle$

45. $\mathbf{r}(t) = \langle \cos 2t, \cos 3t, \cos 4t \rangle$

---

46. Graph the curve with parametric equations

$$x = \sin t \quad y = \sin 2t \quad z = \cos 4t$$

Explain its shape by graphing projections onto the three coordinate planes.

47. Graph the curve with parametric equations

$$x = (1 + \cos 16t)\cos t$$
$$y = (1 + \cos 16t)\sin t$$
$$z = 1 + \cos 16t$$

Explain the appearance of the graph by showing that it lies on a cone.

48. Graph the curve with parametric equations

$$x = \sqrt{1 - 0.25\cos^2 10t}\,\cos t$$
$$y = \sqrt{1 - 0.25\cos^2 10t}\,\sin t$$
$$z = 0.5\cos 10t$$

Explain the appearance of the graph by showing that it lies on a sphere.

49. Show that the curve with parametric equations $x = t^2$, $y = 1 - 3t$, $z = 1 + t^3$ passes through the points $(1, 4, 0)$ and $(9, -8, 28)$ but not through the point $(4, 7, -6)$.

Find a vector function that represents the curve of intersection of the two surfaces.

50. The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

51. The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$.

52. The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$.

53. The hyperboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$.

54. The semiellipsoid $x^2 + y^2 + 4z^2 = 4$, $y \geq 0$, and the cylinder $x^2 + z^2 = 1$.

---

55. Try to sketch the curve of intersection of the circular cylinder $x^2 + y^2 = 4$ and the parabolic cylinder $z = x^2$. Then find parametric equations for this curve and use these equations and technology to graph the curve.

56. Try to sketch the curve of intersection of the parabolic cylinder $y = x^2$ and the top half of the ellipsoid $x^2 + 4y^2 + 4z^2 = 16$. Then find parametric equations for this curve and use technology to graph the curve.

57. If two objects travel through space along two different curves, it is often important to know whether they will collide. For example, will two aircrafts collide? Or will two satellites in orbit hit each other? The curves (of travel) might intersect, but we need to know whether the objects are in the same position *at the same time.*

Suppose the trajectories of two particles are given by the vector functions

$$\mathbf{r}_1(t) = \langle t^2,\ 7t - 12,\ t^2 \rangle \qquad \mathbf{r}_2(t) = \langle 4t - 3,\ t^2,\ 5t - 6 \rangle$$

for $t \geq 0$. Do these particles collide?

58. Two particles travel along the space curves

$$\mathbf{r}_1(t) = \langle t,\ t^2,\ t^3 \rangle \qquad \mathbf{r}_2(t) = \langle 1 + 2t,\ 1 + 6t,\ 1 + 14t \rangle$$

Do the particles collide? Do their paths intersect?

59. (a) Graph the curve with parametric equations

$$x = \frac{27}{26}\sin 8t - \frac{8}{39}\sin 18t$$
$$y = -\frac{27}{26}\cos 8t + \frac{8}{39}\cos 18t$$
$$z = \frac{144}{65}\sin 5t$$

(b) Show that the curve lies on the hyperboloid of one sheet $144x^2 + 144y^2 - 25z^2 = 100$.

60. The view of the trefoil knot shown in Figure 10.8 is accurate, but it does not reveal all of the characteristics of this curve. Use the parametric equations

$$x = (2 + \cos 1.5t)\ \cos t$$
$$y = (2 + \cos 1.5t)\ \sin t$$
$$z = \sin 1.5t$$

to sketch the curve as viewed from above, with gaps indicating where the curve passes over itself. Start by showing that the projection of the curve onto the xy-plane has polar coordinates $r = 2 + \cos\ 1.5t$ and $\theta = t$, so r varies between 1 and 3. Then show that z has maximum and minimum values when the projection is halfway between $r = 1$ and $r = 3$.
Use technology to draw the curve with viewpoint directly above the curve and compare with your sketch. Use technology to draw the curve from several other viewpoints. Try to sketch the curve with a tube in order to obtain a better illustration of this curve.

61. Suppose $\mathbf{u}$ and $\mathbf{v}$ are vector functions whose limits exist as $t \to a$, and let c be a constant. Prove the following properties of limits.

(a) $\lim_{t\to a} [\mathbf{u}(t) + \mathbf{v}(t)] = \lim_{t\to a} \mathbf{u}(t) + \lim_{t\to a} \mathbf{v}(t)$

(b) $\lim_{t\to a} c\mathbf{u}(t) = c \lim_{t\to a} \mathbf{u}(t)$

(c) $\lim_{t\to a} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \lim_{t\to a} \mathbf{u}(t) \cdot \lim_{t\to a} \mathbf{v}(t)$

(d) $\lim_{t\to a} [\mathbf{u}(t) \times \mathbf{v}(t)] = \lim_{t\to a} \mathbf{u}(t) \times \lim_{t\to a} \mathbf{v}(t)$

10.2 Derivatives and Integrals of Vector Functions

Later in this chapter we are going to use vector functions to describe the motion of planets and other objects through space. The purpose of this section is to develop the necessary calculus of vector functions.

Derivatives

The **derivative** $\mathbf{r}'$ of a vector $\mathbf{r}$ is defined in much the same way as for real-valued functions.

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h} \tag{1}$$

if this limit exists. The geometric interpretation of this definition is shown in Figures 10.16 and 10.17. If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$, then $\overrightarrow{PQ}$ represents the vector $\mathbf{r}(t+h) - \mathbf{r}(t)$, which can be thought of as a secant vector (Figure 10.16).

If $h > 0$, the scalar multiple $\left(\dfrac{1}{h}\right)(\mathbf{r}(t+h) - \mathbf{r}(t))$ has the same direction as $\mathbf{r}(t+h) - \mathbf{r}(t)$. As $h \to 0$, it appears that this vector approaches a vector that lies on the tangent line. For this reason, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$. See Figure 10.17.

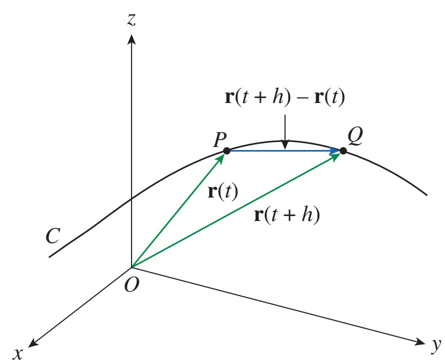


Figure 10.16
The secant vector: $\mathbf{r}(t+h) - \mathbf{r}$.

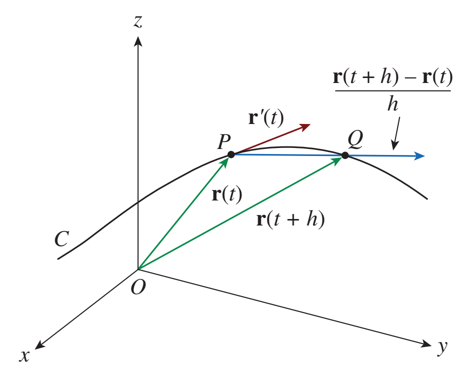


Figure 10.17
The tangent vector: $\mathbf{r}'(t)$.

The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. Often we need to consider the **unit tangent vector** which is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The next theorem provides a method for computing the derivative of a vector function $\mathbf{r}$: just differentiate each component of $\mathbf{r}$.

Theorem • Derivative of a Vector Function

If $\mathbf{r}(t) = \langle f(t),\ g(t),\ h(t)\rangle = f(t)\ \mathbf{i} + g(t)\ \mathbf{j} + h(t)\ \mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t),\ g'(t),\ h'(t)\rangle = f'(t)\ \mathbf{i} + g'(t)\ \mathbf{j} + h'(t)\ \mathbf{k}$$

Proof

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)]$$ Definition of $\mathbf{r}'(t)$.

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\langle f(t+\Delta t),\ g(t+\Delta t),\ h(t+\Delta t)\rangle - \langle f(t),\ g(t),\ h(t)\rangle]$$ Use component functions.

$$= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$ Component operations.

$$= \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \langle f'(t), g'(t), h'(t) \rangle$$ Definition of derivative.

■

Example 1 Find a Unit Tangent Vector

(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\mathbf{j} + \sin 2t\,\mathbf{k}$.
(b) Find the unit tangent vector at the point where $t = 0$.

Solution

(a) Differentiate each component of $\mathbf{r}$.

$$\mathbf{r}'(t) = (3t^2)\,\mathbf{i} + (t \cdot e^{-t} \cdot (-1) + 1 \cdot e^{-t})\,\mathbf{j} + (\cos 2t \cdot 2)\mathbf{k}$$ Power Rule; Product Rule; Chain Rule.

$$= 3t^2\,\mathbf{i} + (1 - t)e^{-t}\mathbf{j} + 2\cos 2t\,\mathbf{k}$$ Simplify.

(b) $\mathbf{r}(0) = (1 + 0^3)\mathbf{i} + 0 \cdot e^{-0}\mathbf{j} + \sin 0\,\mathbf{k} = \mathbf{i} = \langle 1, 0, 0 \rangle$

$\mathbf{r}'(0) = 3 \cdot 0^2\mathbf{i} + (1 - 0)e^{-0}\,\mathbf{j} + 2\cos 0\,\mathbf{k} = \mathbf{j} + 2\mathbf{k}$

The unit tangent vector at the point (1, 0, 0) is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}.$$ ■

Example 2 Find the Derivative of a Vector Function

For the curve $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (2 - t)\,\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

Solution

$$\mathbf{r}(1) = \sqrt{1}\,\mathbf{i} + (2 - 1)\,\mathbf{j} = \mathbf{i} + \mathbf{j}$$

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} - \mathbf{j} \quad \Rightarrow \quad \mathbf{r}'(1) = \frac{1}{2}\mathbf{i} - \mathbf{j}$$

The curve described by $\mathbf{r}$ is a plane curve.

Use the equations $x = \sqrt{t}$ and $y = 2 - t$.

Eliminate the parameter t to obtain $y = 2 - x^2$, $x \geq 0$.

Figure 10.18 shows the position vector $r(1)$ starting at the origin and the tangent vector $\mathbf{r}'(1)$ starting at the corresponding point (1, 1). ■

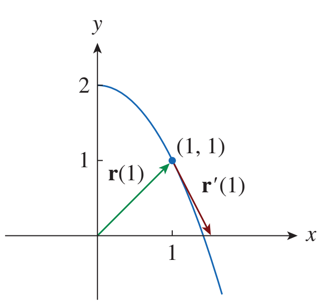


Figure 10.18
Graph of the curve, the position vector $\mathbf{r}(1)$, and the tangent vector $\mathbf{r}'(1)$.

Example 3 Find a Tangent Line to a Space Curve

Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $\left(0, 1, \frac{\pi}{2}\right)$.

Solution

The vector equation of the helix is $\mathbf{r}(t) = \langle 2\cos t, \sin t, t \rangle$.

Therefore, $\mathbf{r}'(t) = \langle -2\sin t, \cos t, 1\rangle$.

The value $t = \dfrac{\pi}{2}$ corresponds to the point $\left(0, 1, \dfrac{\pi}{2}\right)$.

The tangent vector is $\mathbf{r}'\left(\dfrac{\pi}{2}\right) = \left\langle -2\sin\dfrac{\pi}{2}, \cos\dfrac{\pi}{2}, 1\right\rangle = \langle -2, 0, 1\rangle$.

The tangent line is the line through $\left(0, 1, \dfrac{\pi}{2}\right)$ parallel to the vector $\langle -2, 0, 1\rangle$.

The parametric equations are

$$x = -2t, \quad y = 1, \quad z = \frac{\pi}{2} + t.$$

Figure 10.19 shows the helix and the tangent line.

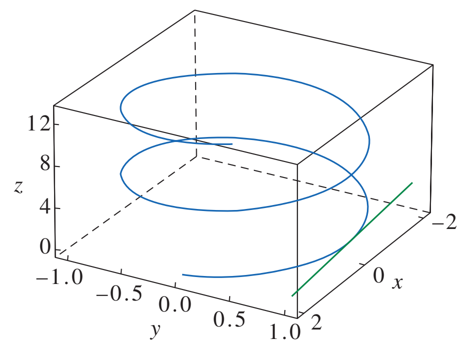


Figure 10.19
Graph of the helix and the tangent line.

Just as for real-valued functions, the **second derivative** of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$. For instance, the second derivative of the function in Example 3 is

$$\mathbf{r}''(t) = \langle -2\cos t, -\sin t, 0\rangle$$

Differentiation Rules

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-value functions.

Theorem • Differentiation Formulas

Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

This theorem can be proved either by using the definition of the derivative of a vector-valued function or by using the Derivative of a Vector Function Theorem and the corresponding differentiation formulas for real-valued functions. The proof of Formula 4 follows; the remaining proofs are left as exercises.

Proof of Formula 4

Let $\mathbf{u}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle$ and $\mathbf{v}(t) = \langle g_1(t), g_2(t), g_3(t)\rangle$.

Then $\mathbf{u}(t) \cdot \mathbf{v}(t) = f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t) = \sum_{i=1}^{3} f_i(t)g_i(t)$.

Use the Product Rule for real-valued functions.

$$\begin{aligned}
\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] &= \frac{d}{dt}\sum_{i=1}^{3} f_i(t)g_i(t) = \sum_{i=1}^{3} \frac{d}{dt}\Big[f_i(t)g_i(t)\Big] && \text{Derivative of a sum is the sum of the derivatives.}\\
&= \sum_{i=1}^{3}[f'_i(t)g_i(t) + f_i(t)g'_i(t)] && \text{Product Rule.}\\
&= \sum_{i=1}^{3} f'_i(t)g_i(t) + \sum_{i=1}^{3} f_i(t)g'_i(t) && \text{Summation property.}\\
&= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) && \text{Definition of dot product and derivative.}
\end{aligned}$$

■

Example 4 Position Vector, Tangent Vector, and Orthogonality

Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Solution

Consider the dot product of $\mathbf{r}$ with itself.

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

Use Differentiation Formula 4, and the fact that c^2 is a constant.

$$\frac{d}{dt}c^2 = 0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t)$$

Therefore, $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, which means that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Geometrically, this result says that if a curve lies on a sphere with center at the origin, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector $\mathbf{r}(t)$. ■

Integrals

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be defined in much the same way as for real-valued functions except that the resulting integral is a vector. We can express the integral of $\mathbf{r}$ in terms of the integrals of its component functions f, g, and h as follows.

$$\int_a^b \mathbf{r}(t)\ dt = \lim_{n\to\infty} \sum_{i=1}^n \mathbf{r}(t_i^*)\ \Delta t$$

$$= \lim_{n\to\infty}\left[\left(\sum_{i=1}^n f(t_i^*)\ \Delta t\right)\mathbf{i} + \left(\sum_{i=1}^n g(t_i^*)\ \Delta t\right)\mathbf{j} + \left(\sum_{i=1}^n h(t_i^*)\ \Delta t\right)\mathbf{k}\right]$$

Therefore,

$$\int_a^b \mathbf{r}(t)\ dt = \left(\int_a^b f(t)\ dt\right)\mathbf{i} + \left(\int_a^b g(t)\ dt\right)\mathbf{j} + \left(\int_a^b h(t)\ dt\right)\mathbf{k} \tag{2}$$

This means that we can evaluate an integral of a vector function by integrating each component function.

We can even extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\ dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a) \tag{3}$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$, that is $\mathbf{R}'(t) = \mathbf{r}(t)$. And as you might expect, we use the notation $\int \mathbf{r}(t)\ dt$ for indefinite integrals (antiderivatives).

Example 5 Integral of a Vector Function

Let $\mathbf{r}(t) = 2\ \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + 2t\ \mathbf{k}$. Then

$$\int \mathbf{r}(t)\ dt = \left(\int 2\cos t\ dt\right)\mathbf{i} + \left(\int \sin t\ dt\right)\mathbf{j} + \left(\int 2t\ dt\right)\mathbf{k}$$ Integrate component functions.

$$= 2\sin t\ \mathbf{i} - \cos t\ \mathbf{j} + t^2\ \mathbf{k} + \mathbf{C}$$ Antiderivatives.

where $\mathbf{C}$ is a vector constant of integration.

Here is an example of a definite integral.

$$\int_0^{\pi/2} \mathbf{r}(t)\ dt = \left[2\ \sin t\ \mathbf{i} - \cos t\ \mathbf{j} + t^2\ \mathbf{k}\right]_0^{\pi/2}$$ Apply FTC.

$$= \left[2\ \sin\frac{\pi}{2}\ \mathbf{i} - \cos\frac{\pi}{2}\ \mathbf{j} + \left(\frac{\pi}{2}\right)^2\mathbf{k}\right] - [2\sin 0\ \mathbf{i} - \cos 0\ \mathbf{j} + 0^2\ \mathbf{k}]$$

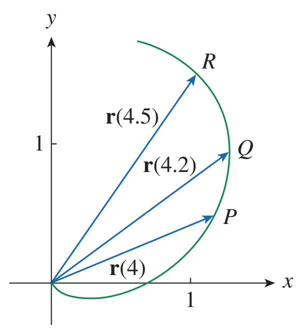
$$= \left[(2)(1)\ \mathbf{i} - 0\ \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}\right] - [(2)(0)\ \mathbf{i} - 1\ \mathbf{j} + 0\ \mathbf{k}]$$

$$= 2\ \mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\ \mathbf{k}$$ Simplify.

■

10.2 Exercises

1. The figure shows a curve C given by a vector function $\mathbf{r}(t)$.



(a) Draw the vectors $\mathbf{r}(4.5) - \mathbf{r}(4)$ and $\mathbf{r}(4.2) - \mathbf{r}(4)$.

(b) Draw the vectors

$$\frac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} \quad \text{and} \quad \frac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2}$$

(c) Write expressions for $\mathbf{r}'(4)$ and the unit tangent vector $\mathbf{T}(4)$.

(d) Draw the vector $\mathbf{T}(4)$.

2. (a) Sketch the curve described by the vector function $\mathbf{r}(t) = \langle t^2, t\rangle$, $0 \le t \le 2$, and draw the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.

(b) Draw the vector $\mathbf{r}'(1)$ starting at (1, 1), and compare it with the vector

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$$

Explain why these vectors are so close to each other in length and direction.

3–9

(a) Sketch the plane curve with the given vector equation.

(b) Find $\mathbf{r}'(t)$.

(c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the given value of t.

3. $\mathbf{r}(t) = \langle t - 2, t^2 + 1\rangle, \quad t = -1$

4. $\mathbf{r}(t) = \langle 1 + t, \sqrt{t}\rangle, \quad t = 1$

5. $\mathbf{r}(t) = \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}, \quad t = \dfrac{\pi}{4}$

6. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}, \quad t = 0$

7. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{3t}\,\mathbf{j}, \quad t = 0$

8. $\mathbf{r}(t) = t\,\mathbf{i} + t\sin t\,\mathbf{j}, \quad t = \dfrac{3\pi}{4}$

9. $\mathbf{r}(t) = (1 + \cos t)\,\mathbf{i} + (2 + \sin t)\,\mathbf{j}, \quad t = \dfrac{\pi}{6}$

Find the derivative of the vector function.

10. $\mathbf{r}(t) = \left\langle \sqrt{t-2}, 4, \dfrac{1}{t^2}\right\rangle$

11. $\mathbf{r}(t) = \langle e^{-t}, t - t^3, \ln t\rangle$

12. $\mathbf{r}(t) = \langle \tan t, \sec t, \tan t \sec t\rangle$

13. $\mathbf{r}(t) = \langle t \sin t, t^2 \cos t, t^3\rangle$

14. $\mathbf{r}(t) = \left\langle t^2, \dfrac{1}{1+t^2}, \arctan t\right\rangle$

15. $\mathbf{r}(t) = t^2\,\mathbf{i} + \cos(t^2)\,\mathbf{j} + \sin^2 t\,\mathbf{k}$

16. $\mathbf{r}(t) = \dfrac{1}{1+t}\,\mathbf{i} + \dfrac{t}{1+t}\,\mathbf{j} + \dfrac{t^2}{1+t}\,\mathbf{k}$

17. $\mathbf{r}(t) = t\sin t\,\mathbf{i} + e^t\cos t\,\mathbf{j} + \sin t\cos t\,\mathbf{k}$

18. $\mathbf{r}(t) = \sin^2 at\,\mathbf{i} + te^{bt}\,\mathbf{j} + \cos^2 ct\,\mathbf{k}$

19. $\mathbf{r}(t) = \mathbf{a} + t\,\mathbf{b} + t^2\,\mathbf{c}$

20. $\mathbf{r}(t) = t\,\mathbf{a} \times (\mathbf{b} + t\,\mathbf{c})$

Find the unit tangent vector $\mathbf{T}(t)$ at the point with the given value of the parameter t.

21. $\mathbf{r}(t) = \langle t\,e^{-t}, 2\arctan t, 2e^t\rangle, \quad t = 0$

22. $\mathbf{r}(t) = 4\sqrt{t}\,\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k}, \quad t = 1$

23. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 3t\,\mathbf{j} + 2\sin 2t\,\mathbf{k}, \quad t = 0$

24. $\mathbf{r}(t) = 2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \tan t\,\mathbf{k}, \quad t = \dfrac{\pi}{4}$

25. $\mathbf{r}(t) = \dfrac{e^t}{t}\,\mathbf{i} + \ln t\,\mathbf{j} + \dfrac{1}{1+t^2}\,\mathbf{k}, \quad t = 1$

26. If $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$, find $\mathbf{r}'(t)$, $\mathbf{T}(1)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

27. If $\mathbf{r}(t) = \langle t, e^t, t\,e^t\rangle$, find $\mathbf{r}'(t)$, $\mathbf{T}(0)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

28. If $\mathbf{r}(t) = \langle \cos t, 3\sin t, 4t\rangle$, find $\mathbf{T}(0)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

29. $x = 1 + 2\sqrt{t}, \quad y = t^3 - t, \quad z = t^3 + t; \quad (3, 0, 2)$

30. $x = e^t, \quad y = te^t, \quad z = te^{t^2}; \quad (1, 0, 0)$

31. $x = e^{-t}\cos t, \quad y = e^{-t}\sin t, \quad z = e^{-t}; \quad (1, 0, 1)$

32. $x = \ln t, \quad y = 2\sqrt{t}, \quad z = t^2; \quad (0, 2, 1)$

33. $x = t\cos t, \quad y = t, \quad z = t\sin t; \quad (-\pi, \pi, 0)$

34. Find a vector equation for the tangent line to the curve of intersection of the cylinders $x^2 + y^2 = 25$ and $y^2 + z^2 = 20$ at the point (3, 4, 2).

35. Find the point on the curve $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, e^t\rangle$, $0 \le t \le \pi$, where the tangent line is parallel to the plane $\sqrt{3}x + y = 1$.

Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point. Illustrate by graphing both the curve and the tangent line in the same viewing rectangle.

36. $x = t, \quad y = e^{-t}, \quad z = 2t - t^2; \quad (0, 1, 0)$

37. $x = 2 \cos t, \quad y = 2 \sin t, \quad z = 4 \cos 2t; \quad (\sqrt{3}, 1, 2)$

38. $x = t \cos t, \quad y = t, \quad z = t \sin t; \quad (-\pi, \pi, 0)$

---

39. (a) Find the point of intersection of the tangent lines to the curve $\mathbf{r}(t) = \langle \sin \pi t, 2 \sin \pi t, \cos \pi t \rangle$ at the points where $t = 0$ and $t = 0.5$.
(b) Illustrate by graphing the curve and both tangent lines.

40. The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection.

41. Find the point of intersection of the curves $\mathbf{r}_1(t) = \langle t, 1 - t, 3 + t^2 \rangle$ and $\mathbf{r}_2(t) = \langle 3 - s, s - 2, s^2 \rangle$ and their angle of intersection.

Evaluate the integral.

42. $\displaystyle\int_0^1 (16t^3\, \mathbf{i} - 9t^2\, \mathbf{j} + 25t^4\, \mathbf{k})\, dt$

43. $\displaystyle\int_1^4 (2t^{3/2}\, \mathbf{i} + (t + 1)\sqrt{t}\, \mathbf{k})\, dt$

44. $\displaystyle\int_0^1 \left(\frac{1}{1+t}\mathbf{i} + \frac{1}{1+t^2}\mathbf{j} + \frac{t}{1+t^2}\mathbf{k} \right) dt$

45. $\displaystyle\int_0^{\pi/2} (3 \sin^2 t \cos t\, \mathbf{i} + 3 \sin t \cos^2 t\, \mathbf{j} + 2 \sin t \cos t\, \mathbf{k})\, dt$

46. $\displaystyle\int_1^2 (t^2\, \mathbf{i} + t\sqrt{t-1}\, \mathbf{j} + t \sin \pi t\, \mathbf{k})\, dt$

47. $\displaystyle\int (\sec^2 t\, \mathbf{i} + t(t^2 + 1)^3\, \mathbf{j} + t^2 \ln t\, \mathbf{k})\, dt$

48. $\displaystyle\int \left(te^{2t}\, \mathbf{i} + \frac{t}{1-t}\mathbf{j} + \frac{1}{\sqrt{1-t^2}}\mathbf{k} \right) dt$

---

49. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 2t\, \mathbf{i} + 3t^2\, \mathbf{j} + \sqrt{t}\, \mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$.

50. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = t\, \mathbf{i} + e^t\, \mathbf{j} + te^t\, \mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

51. Prove Differentiation Formula 1.

52. Prove Differentiation Formula 3.

53. Prove Differentiation Formula 5.

54. Prove Differentiation Formula 6.

55. If $\mathbf{u}(t) = \langle \sin t, \cos t, t \rangle$ and $\mathbf{v} = \langle t, \cos t, \sin t \rangle$, use Formula 4 of Theorem 3 to find

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$$

56. If $\mathbf{u}(t) = \langle \sin t, \cos t, t \rangle$ and $\mathbf{v} = \langle t, \cos t, \sin t \rangle$, use Formula 5 of Theorem 3 to find

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)]$$

---

57. Find $f'(2)$, where $f(t) = \mathbf{u}(t) \cdot \mathbf{v}(t)$, $\mathbf{u}(2) = \langle 1, 2, -1 \rangle$, $\mathbf{u}'(2) = \langle 3, 0, 4 \rangle$, and $\mathbf{v}(t) = \langle t, t^2, t^3 \rangle$.

58. If $\mathbf{r}(t) = \mathbf{u}(t) \times \mathbf{v}(t)$, where $\mathbf{u}$ and $\mathbf{v}$ are the vector functions in Exercise 57, find $\mathbf{r}'(2)$.

59. If $\mathbf{r}(t) = \mathbf{a} \cos \omega t + \mathbf{b} \sin \omega t$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, show that $\mathbf{r}(t) \times \mathbf{r}'(t) = \omega\, \mathbf{a} \times \mathbf{b}$.

60. If $\mathbf{r}$ is the vector function defined in Exercise 59, show that $\mathbf{r}''(t) + \omega^2 \mathbf{r}(t) = \mathbf{0}$.

61. Show that if $\mathbf{r}$ is a vector function such that $\mathbf{r}''$ exists, then

$$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$$

62. Find an expression for $\dfrac{d}{dt}[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t))]$.

63. If $\mathbf{r}(t) \neq \mathbf{0}$, show that $\dfrac{d}{dt}|\mathbf{r}(t)| = \dfrac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t)$

Hint: $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$

64. If a curve has the property that the position vector $\mathbf{r}(t)$ is always perpendicular to the tangent vector $\mathbf{r}'(t)$, show that the curve lies on a sphere with center at the origin.

65. If $\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]$, show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)]$$

66. Show that the tangent vector to a curve defined by a vector function $\mathbf{r}(t)$ points in the direction of increasing t.
Hint: Refer to Figures 10.16 and 10.17 and consider the cases $h > 0$ and $h < 0$ separately.

10.3 Arc Length and Curvature

Arc Length

In Section 6.4, we defined the length of a plane curve with parametric equations $x = f(t)$, $y = g(t)$, $a \le t \le b$, as the limit of lengths of polygonal paths and, for the case where f' and g' are continuous, we derived the formula

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{1}$$

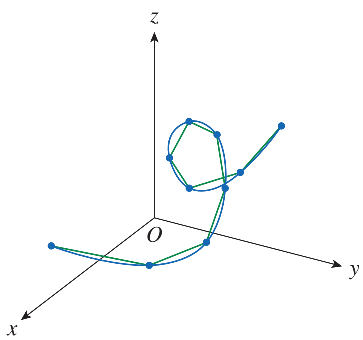


Figure 10.20
The length of the space curve is the limit of the lengths of polygonal paths.

The length of a space curve is defined in exactly the same way (see Figure 10.20).

Suppose that the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \le t \le b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f', g', and h' are continuous. If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \end{aligned} \tag{2}$$

Notice that both of the arc length formulas (1) and (2) can be written more compactly as

$$L = \int_a^b |\mathbf{r}'(t)|\, dt \tag{3}$$

because, for the plane curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$,

$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves defined by $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Example 1 Helix Length

Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

Solution

Find $\mathbf{r}'(t)$ and use this in Equation 3.

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$$

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}$$

The arc from $(1, 0, 0)$ to $(1, 0, 2\pi)$ is described by the parameter interval $0 \le t \le 2\pi$.

Therefore, the length of the arc is

$$L = \int_0^{2\pi} |\mathbf{r}'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$$

Figure 10.21 shows the arc of the helix from $t = 0$ to $t = 2\pi$. ■

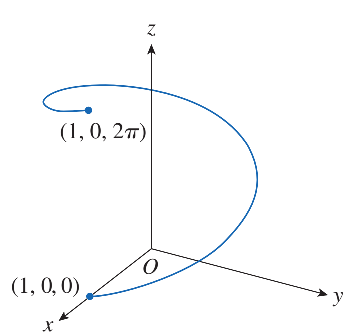


Figure 10.21
The arc of the helix.

A single curve C can be represented by more than one vector equation. For example, the twisted cubic

$$\mathbf{r}_1(t) = \langle t, t^2, t^3\rangle \quad 1 \le t \le 2 \tag{4}$$

could also be represented by the vector function

$$\mathbf{r}_2(t) = \langle e^u, e^{2u}, e^{3u} \rangle \quad 0 \le u \le \ln 2 \tag{5}$$

where the connection between the parameters t and u is given by $t = e^u$. We say that Equations 4 and 5 are **parametrizations** of the curve C. If we were to use Equation 3 to compute the length of C using Equations 4 and 5, we would get the same answer. In general, it can be shown that when Equation 3 is used to compute arc length, the answer is independent of the parametrization that is used.

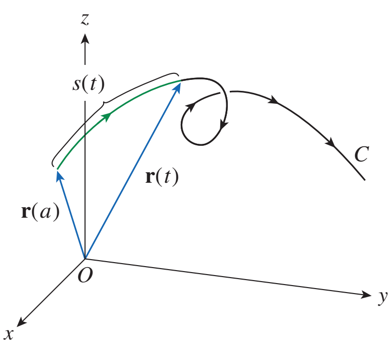


Figure 10.22
An illustration of $s(t)$; the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$.

Suppose that C is a curve given by the vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad a \le t \le b$$

where $\mathbf{r}'$ is continuous and C is traversed exactly once as t increases from a to b. We define its **arc length function** s by

$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,du \tag{6}$$

Therefore, $s(t)$ is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, as illustrated in Figure 10.22. If we differentiate both sides of Equation 6 using Part 1 of the Fundamental Theorem of Calculus, we obtain

$$\frac{ds}{dt} = |\mathbf{r}'(t)| \tag{7}$$

It is often useful to **parametrize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system. If a curve $\mathbf{r}(t)$ is given in terms of a parameter t, and $s(t)$ is the arc length function defined in Equation 6, then it might be possible to solve for t as a function of s: $t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for t: $\mathbf{r} = \mathbf{r}(t(s))$. Therefore, if $s = 3$ for example, $\mathbf{r}(t(3))$ is the position vector of the point 3 units of length along the curve from its starting point.

Example 2 Find an Arc Length Parametrization

Reparametrize the helix with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

Solution

The initial point $(1, 0, 0)$ corresponds to the parameter value $t = 0$.

From Example 1: $\dfrac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{2}$.

Integrate to find an expression for s in terms of t.

$$s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}\,t$$

Solve for t in terms of s: $t = \dfrac{s}{\sqrt{2}}$.

The reparametrization is obtained by substituting for t:

$$\mathbf{r}(t(s)) = \cos\left(\frac{s}{\sqrt{2}}\right)\mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\mathbf{j} + \left(\frac{s}{\sqrt{2}}\right)\mathbf{k}$$ ■

Curvature

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \ne \mathbf{0}$ on I. A curve is called **smooth** if it has a smooth parametrization. A smooth curve has

no sharp edges or corners; when the tangent vector turns, it does so in a continuous (smooth) manner.

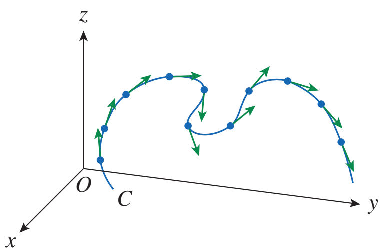


Figure 10.23
The unit tangent vectors change more quickly when C turns more sharply.

If C is a smooth curve defined by the vector function $\mathbf{r}$, recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and indicates the direction of the curve. Figure 10.23 suggests, and it seems reasonable, that $\mathbf{T}(t)$ changes direction very slowly when C is fairly straight, but changes direction more quickly when C bends or twists sharply.

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Analytically it is defined to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. Arc length is used so that the curvature will be independent of the parametrization.

Definition • Curvature

The **curvature** of a curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

where $\mathbf{T}$ is the unit tangent vector.

The curvature is easier to compute if it is expressed in terms of the parameter t instead of s. We can use the Chain Rule (Differentiation Formula 6, Section 10.2) to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \quad \text{and} \quad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right|$$

Using Equation 7, $\dfrac{ds}{dt} = |\mathbf{r}'(t)|$, the curvature can be written as

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \tag{8}$$

Example 3 A Circle Has Constant Curvature

Show that the curvature of a circle of radius a is $\dfrac{1}{a}$.

Solution

Let the circle have center at the origin. Then a parametrization is $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$.

Find the derivative $\mathbf{r}'(t)$ and its magnitude.

$\mathbf{r}'(t) = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}$ Differentiate each component of $\mathbf{r}$.

$|\mathbf{r}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$ Magnitude of $\mathbf{r}'(t)$.

Write an expression for the unit tangent vector.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}}{a} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Find the derivative $\mathbf{T}'(t)$ (differentiate each component), and its magnitude.

$$\mathbf{T}'(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$$

$$\|\mathbf{T}'(t)\| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1$$

Use Equation 8.

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{a}, \text{ which is a constant}$$

The result in Example 3 shows that small circles have large curvature and large circles have small curvature. In addition, using the definition of curvature, the curvature of a straight line is always 0 because the tangent vector is constant.

Although Equation 8 can be used in all cases to compute the curvature, the formula given in the following theorem is often more convenient to apply.

Theorem • Curvature Formula

The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

Proof

Use the formulas $\mathbf{T} = \dfrac{\mathbf{r}'}{\|\mathbf{r}'\|}$ and $\dfrac{ds}{dt} = \|\mathbf{r}'\|$.

$$\mathbf{r}' = \|\mathbf{r}'\|\mathbf{T} = \frac{ds}{dt}\mathbf{T}$$

Use the Product Rule (Theorem 10.2.3, Formula 3) to find $\mathbf{r}''$.

$$\mathbf{r}'' = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'$$

Consider the cross product $\mathbf{r}' \times \mathbf{r}''$.

$$\begin{aligned}
\mathbf{r}' \times \mathbf{r}'' &= \left(\frac{ds}{dt}\mathbf{T}\right) \times \left(\frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'\right) \\
&= \left(\frac{ds}{dt}\mathbf{T}\right) \times \left(\frac{d^2s}{dt^2}\mathbf{T}\right) + \left(\frac{ds}{dt}\mathbf{T}\right) \times \left(\frac{ds}{dt}\mathbf{T}'\right) && \text{Properties of the Cross Product.} \\
&= \left(\frac{ds}{dt}\right)^2 (\mathbf{T} \times \mathbf{T}') && \mathbf{T} \times \mathbf{T} = 0.
\end{aligned}$$

Since $\|\mathbf{T}(t)\| = 1$ for all t, that is, $\|\mathbf{T}(t)\|$ is constant, then $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal (Section 10.2, Example 4).

Use the definition of the cross product.

$$\|\mathbf{r}' \times \mathbf{r}''\| = \left(\frac{ds}{dt}\right)^2 \|\mathbf{T} \times \mathbf{T}'\| = \left(\frac{ds}{dt}\right)^2 \|\mathbf{T}\|\,\|\mathbf{T}'\| = \left(\frac{ds}{dt}\right)^2 \|\mathbf{T}'\|$$

$$\|\mathbf{T}'\| = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{(ds/dt)^2} = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^2} \qquad \text{Solve for } \|\mathbf{T}'\|\text{; use } \|\mathbf{r}'\| = \frac{ds}{dt}.$$

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{\dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2}}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$

Formula for κ; expression for $|\mathbf{T}'|$; simplify.

Example 4 Twisted Cubic Curvature

Find the curvature of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at a general point and at (0, 0, 0).

Solution

Find all of the parts of Equation 10.

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle \quad \Rightarrow \quad \mathbf{r}''(t) = \langle 0, 2, 6t \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\ \mathbf{i} - 6t\ \mathbf{j} + 2\ \mathbf{k}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$$

Use these results in Theorem 10.

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

At the origin, where $t = 0$, the curvature is

$\kappa(0) = 2$.

Consider the special case of a plane curve with equation $y = f(x)$. Let x be the parameter and write $\mathbf{r}(x) = x\,\mathbf{i} + f(x)\mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\ \mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\ \mathbf{j}$.

Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, then $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\ \mathbf{k}$. We can also find the magnitude $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$. Using the curvature formula:

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} \tag{9}$$

Example 5 Parabola Curvature

Find the curvature of the parabola $y = x^2$ at the points (0, 0), (1, 1), and (2, 4).

Solution

$$y = x^2 \quad \Rightarrow \quad y' = 2x \quad \Rightarrow \quad y'' = 2$$

Use Equation 11: $\kappa(x) = \dfrac{|y''|}{[1 + (y')^2]^{3/2}} = \dfrac{2}{(1 + 4x^2)^{3/2}}$

Curvature at (0, 0): $\kappa(0) = 2$

Curvature at (1, 1): $\kappa(1) \approx 0.179$

Curvature at (2, 4): $\kappa(2) \approx 0.029$

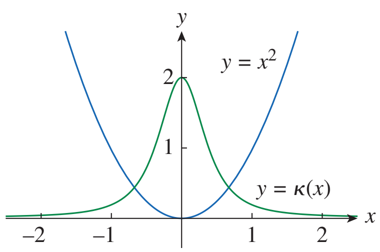


Figure 10.24
Graphs of the parabola $y = x^2$ and its curvature function $y = \kappa(x)$.

Notice from the expression for $\kappa(x)$ that $\kappa(x) \to 0$ as $x \to \pm\infty$. This corresponds to the fact that the parabola appears to become flatter as $x \to \pm\infty$.

Figure 10.24 shows a graph of $y = x^2$ and $y = \kappa(x)$. ■

The Normal and Binormal Vectors

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$. Let's consider a specific one. Since $|\mathbf{T}(t)| = 1$ for all t, then $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$ by Example 4 in Section 10.2; therefore, $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. Note that $\mathbf{T}'(t)$ is not necessarily a unit vector.

At any point where $\kappa \neq 0$, we can define the **principal unit normal vector** $\mathbf{N}(t)$, or simply **unit normal**, as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the **binormal vector**. It is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and is also a unit vector. Figure 10.25 illustrates the geometric relationship among $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$.

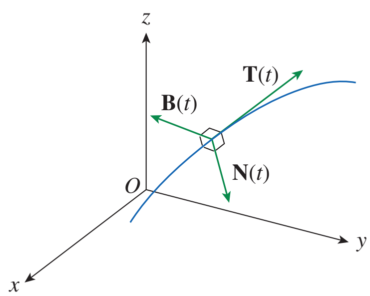


Figure 10.25
The geometric relationship among **T**, **N**, and **B**. The normal vector indicates the direction in which the curve is turning at each point.

Example 6 Normal and Binormal Vectors

Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

Solution

Find all of the parts needed for the unit normal vector.

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}}(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j}) \quad \Rightarrow \quad |\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}$$

Use the definition of the unit normal vector.

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\frac{1}{\sqrt{2}}(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j})}{\frac{1}{\sqrt{2}}} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j} = \langle -\cos t, \ -\sin t, \ 0 \rangle$$

Since the z-component of $\mathbf{N}(t)$ is 0, the normal vector at a point on the helix is horizontal. And since the x- and y-components are opposite of those in $\mathbf{r}(t)$, the normal vector points toward the z-axis.

The binormal vector is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}\langle \sin t, \ -\cos t, \ 1 \rangle.$$

Figure 10.26 shows the vectors **T**, **N**, and **B** at two locations on the helix. ■

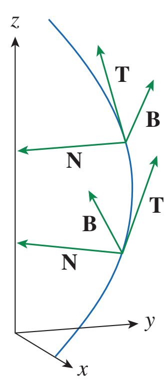


Figure 10.26
In general, the vectors **T**, **N**, and **B**, starting at the various points on a curve, form a set of orthogonal vectors, called the **TNB** frame, that moves along the curve as t varies. This **TNB** frame plays an important role in the branch of mathematics known as differential geometry and in its applications to the motion of spacecraft.

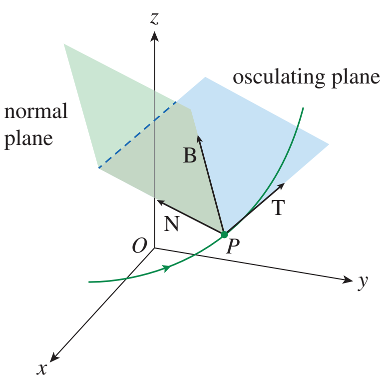


Figure 10.27
The osculating plane.

The plane determined by the normal and binormal vectors **N** and **B** at a point P on a curve C is called the **normal plane** of C at P. It consists of all lines that are orthogonal to the tangent vector **T**. The plane determined by the vectors **T** and **N** is called the **osculating plane** of C at P. See Figure 10.27. The name comes from the Latin *osculum,* meaning "kiss." It is the plane that comes closest to containing the part of the curve near P. For a plane curve, the osculating plane is simply the plane that contains the curve.

The **circle of curvature**, or the **osculating circle**, of C at P is the circle in the osculating plane that passes through P with radius $1/\kappa$ and center at a distance $1/\kappa$ from P along the vector **N**. The center of the circle is called the **center of curvature** of C at P. We can think of the circle of curvature as the circle that best describes how C behaves near P; it shares the same tangent, normal, and curvature at P. Figure 10.28 illustrates two circles of curvature for a plane curve.

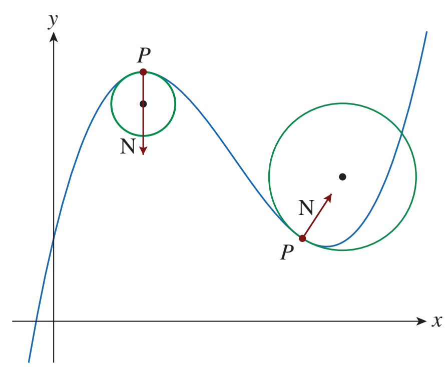


Figure 10.28
An illustration of two circles of curvature for a curve in the plane.

Example 7 Normal and Osculating Planes

Find the equations of the normal plane and osculating plane of the helix described by the vector equation in Example 6 at the point $P\left(0,\ 1,\ \dfrac{\pi}{2}\right)$.

Solution

The normal plane at P has normal vector $\mathbf{r}'\left(\dfrac{\pi}{2}\right) = \langle -1,\ 0,\ 1\rangle$.

An equation of the normal plane is

$$-1(x-0) + 0(y-1) + 1\left(z - \frac{\pi}{2}\right) = 0 \quad \text{or} \quad z = x + \frac{\pi}{2}.$$

The osculating plane at P contains the vectors **T** and **N**; its normal vector is $\mathbf{T} \times \mathbf{N} = \mathbf{B}$.

Use the expression for **B** from Example 6.

$$\mathbf{B}(t) = \frac{1}{\sqrt{2}}\langle \sin t,\ -\cos t,\ 1\rangle \quad \Rightarrow \quad \mathbf{B}\left(\frac{\pi}{2}\right) = \left\langle \frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right\rangle$$

An equivalent and simpler normal vector is $\langle 1,\ 0,\ 1\rangle$.

An equation of the osculating plane is

$$1(x-0) + 0(y-1) + 1\left(z - \frac{\pi}{2}\right) = 0 \quad \text{or} \quad z = -x + \frac{\pi}{2}.$$

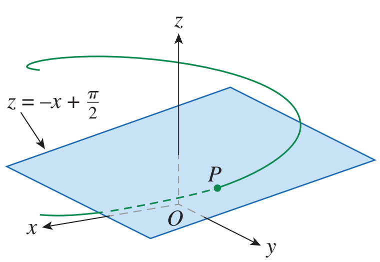


Figure 10.29
The helix and the osculating plane at the point $P(0,\ 1,\ \pi/2)$.

Figure 10.29 shows a graph of the helix, the point P, and the osculating plane at P. ■

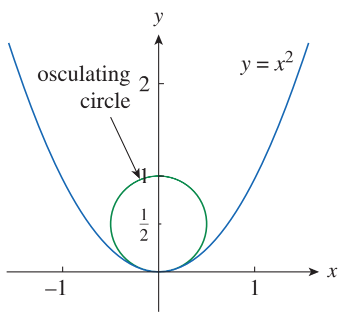


Figure 10.30
Graph of the osculating circle of the parabola $y = x^2$ at the origin.

Example 8 Osculating Circle in the Plane

Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

Solution

From Example 5, the curvature of the parabola at the origin is $\kappa(0) = 2$.

Therefore, the radius of the osculating circle at the origin is $\dfrac{1}{\kappa} = \dfrac{1}{2}$ and its center is $\left(0, \dfrac{1}{2}\right)$.

The equation of the circle is therefore $x^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{1}{4}$.

Figure 10.30 shows a graph of the parabola and the osculating circle at the origin. ■

Here is a summary of the formulas for the unit tangent, unit normal and binomial vectors, and curvature.

Summary Formulas

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

10.3 Exercises

Find the length of the curve.

1. $\mathbf{r}(t) = \langle 2 \sin t, 5t, 2 \cos t \rangle, \quad -10 \le t \le 10$

2. $\mathbf{r}(t) = \left\langle 2t, t^2, \dfrac{1}{3}t^3 \right\rangle, \quad 0 \le t \le 1$

3. $\mathbf{r}(t) = \sqrt{2}t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}, \quad 0 \le t \le 1$

4. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \ln(\cos t)\,\mathbf{k}, \quad 0 \le t \le \dfrac{\pi}{4}$

5. $\mathbf{r}(t) = \mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}, \quad 0 \le t \le 1$

6. $\mathbf{r}(t) = 12t\,\mathbf{i} + 8t^{3/2}\,\mathbf{j} + 3t^2\,\mathbf{k}, \quad 0 \le t \le 1$

7. $\mathbf{r}(t) = t\,\mathbf{i} + 2e^t\,\mathbf{j} + e^{2t}\,\mathbf{k}, \quad 0 \le t \le 1$

Use technology to find the length of the curve.

8. $\mathbf{r}(t) = \langle \sqrt{t}, t, t^2 \rangle, \quad 1 \le t \le 4$

9. $\mathbf{r}(t) = \langle t, \ln t, t \ln t \rangle, \quad 1 \le t \le 2$

10. $\mathbf{r}(t) = \langle \sin t, \cos t, \tan t \rangle, \quad 0 \le t \le \dfrac{\pi}{4}$

11. $\mathbf{r}(t) = \langle t\,e^t, t^2 e^{-t}, \ln t \rangle, \quad 1 \le t \le 2$

12. Use technology to graph the curve with parametric equations $x = \sin t$, $y = \sin 2t$, $z = \sin 3t$. Find the total length of this curve.

13. Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point (6, 18, 36).

14. Use technology to find the length of the curve of intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

(a) Find the arc length function for the curve measured from the point P where $t = 0$ in the direction of increasing t and then reparametrize the curve with respect to arc length starting from P.
(b) Find the point 3 units along the curve (in the direction of increasing t) from P.

15. $\mathbf{r}(t) = 2t\,\mathbf{i} + (1 - 3t)\,\mathbf{j} + (5 + 4t)\,\mathbf{k}$

16. $\mathbf{r}(t) = e^{2t} \cos 2t\,\mathbf{i} + 2\,\mathbf{j} + e^{2t} \sin 2t\,\mathbf{k}$

17. Consider the curve with parametric equations $x = 3 \sin t$, $y = 4t$, $z = 3 \cos t$. Starting at the point (0, 0, 3) and moving in the positive direction, find the point 5 units along the curve.

18. Reparametrize the curve

$$\mathbf{r}(t) = \left(\frac{2}{t^2+1} - 1\right)\mathbf{i} + \frac{2t}{t^2+1}\mathbf{j}$$

with respect to arc length measured from the point (1, 0) in the direction of increasing t. Express the reparametrization in its simplest form. What can you conclude about this curve?

(a) Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

(b) Use Formula 9 to find the curvature.

19. $\mathbf{r}(t) = \langle 2 \sin t,\ 5t,\ 2 \cos t \rangle$

20. $\mathbf{r}(t) = \langle t^2,\ \sin t - t \cos t,\ \cos t + t \sin t \rangle, \quad t > 0.$

21. $\mathbf{r}(t) = \langle \sqrt{2}t,\ e^t,\ e^{-t} \rangle$

22. $\mathbf{r}(t) = \left\langle t,\ \frac{1}{2}t^2,\ t^2 \right\rangle$

Use Theorem 10 to find the curvature.

23. $\mathbf{r}(t) = t^3\,\mathbf{j} + t^2\,\mathbf{k}$

24. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + e^t\,\mathbf{k}$

25. $\mathbf{r}(t) = \sqrt{6}t^2\,\mathbf{i} + 2t\,\mathbf{j} + 2t^3\,\mathbf{k}$

26. $\mathbf{r}(t) = 3t\,\mathbf{i} + 4 \sin t\,\mathbf{j} + 4 \cos t\,\mathbf{k}$

---

27. Find the curvature of $\mathbf{r}(t) = \langle t^2,\ \ln t,\ t \ln t \rangle$ at the point (1, 0, 0).

28. Find the curvature of $\mathbf{r}(t) = \langle t,\ t^2,\ t^3 \rangle$ at the point (1, 1, 1).

29. Use technology to graph the curve with parametric equations $x = \cos t$, $y = \sin t$, $z = \sin 5t$ and find the curvature at the point (1, 0, 0).

Use Equation 11 to find the curvature.

30. $y = x^4$

31. $y = \tan x$

32. $y = xe^x$

33. $y = x^2 \ln x$

At what point does the curve have maximum curvature? Explain what happens to the curvature as $x \to \infty$.

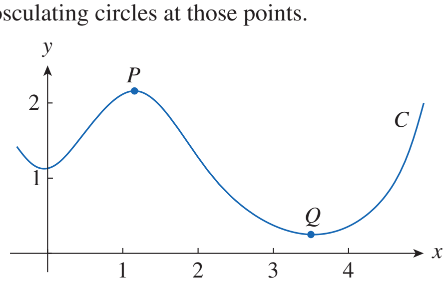
34. $y = \ln x$

35. $y = e^x$

---

36. Find an equation of a parabola that has curvature 4 at the origin.

37. (a) Is the curvature of the curve C shown in the figure greater at P or Q? Explain your reasoning.
(b) Estimate the curvature at P and Q by sketching the osculating circles at those points.



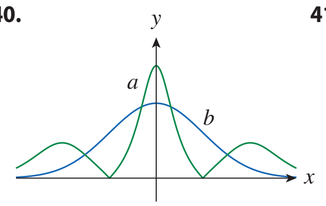
Use technology to graph both the curve and its curvature function $\kappa(x)$ in the same viewing rectangle. Is the graph of κ what you would expect? Explain your reasoning.

38. $y = x^4 - 2x^2$

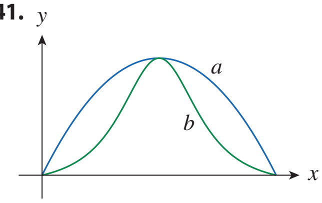
39. $y = x^{-2}$

Two graphs, a and b, are shown. One is the graph of a curve $y = f(x)$ and the other is the graph of its curvature function $y = \kappa(x)$. Identify each curve and explain your choices.

40.



41.



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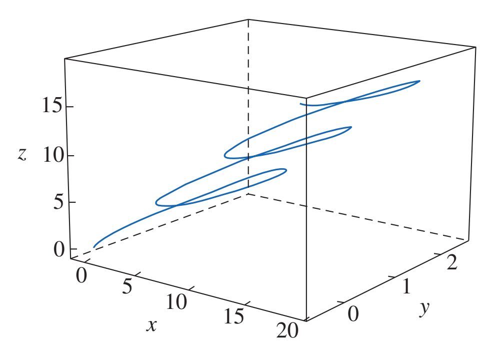
42. (a) Graph the curve $\mathbf{r}(t) = \langle \sin 3t,\ \sin 2t,\ \sin 3t \rangle$. At how many points on the curve does it appear that the curvature has a local or absolute maximum?
(b) Use technology to find and graph the curvature function. Does this graph confirm your conclusion from part (a)? Justify your answer.

43. The graph of $\mathbf{r}(t) = \left\langle t - \frac{3}{2} \sin t,\ 1 - \frac{3}{2} \cos t,\ t \right\rangle$ is shown in the figure. Where do you think the curvature is largest? Use technology to find and graph the curvature function. For which values of t is the curvature the largest?



The curvature of a plane curve described by the parametric equations $x = f(t)$, $y = g(t)$ is given by

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t.

44. Use Theorem 10 to derive this formula for curvature.

45. Find the curvature of the curve $x = t^2$, $y = t^3$.

46. Find the curvature of the curve $x = a \cos \omega t$, $y = b \sin \omega t$.

47. Find the curvature of the curve $x = e^t \cos t$, $y = e^t \sin t$.

---

48. Consider the curvature at $x = 0$ for each member of the family of functions $f(x) = e^{cx}$. For which members is $\kappa(0)$ largest?

Find the vectors **T**, **N**, and **B** at the given point.

49. $\mathbf{r}(t) = \left\langle t^2, \frac{2}{3}t^3, t \right\rangle, \quad \left(1, \frac{2}{3}, 1\right)$

50. $\mathbf{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle, \quad (1, 0, 0)$

Find equations of the normal plane and osculating plane of the curve at the given point.

51. $x = 2 \sin 3t,\ y = t,\ z = 2 \cos 3t; \quad (0, \pi, -2)$

52. $x = t,\ y = t^2,\ z = t^3; \quad (1, 1, 1)$

53. $x = \ln t,\ y = 2t,\ z = t^2; \quad (0, 2, 1)$

54. Find equations of the osculating circles of the ellipse $9x^2 + 4y^2 = 36$ at the points $(2, 0)$ and $(0, 3)$. Use technology to graph the ellipse and both osculating circles in the same viewing rectangle.

55. Find equations of the osculating circles of the parabola $y = \frac{1}{2}x^2$ at the points $(0, 0)$ and $\left(1, \frac{1}{2}\right)$. Graph both osculating circles and the parabola in the same viewing rectangle.

56. At what point on the curve $x = t^3$, $y = 3t$, $z = t^4$ is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

57. Is there a point on the curve $x = t^3$, $y = 3t$, $z = t^4$ where the osculating plane is parallel to the plane $x + y + z = 1$?

Hint: Use technology for differentiating, simplifying, and for computing a cross product.

58. Find equations of the normal and osculating planes of the curve of intersection of the parabolic cylinders $x = y^2$ and $z = x^2$ at the point $(1, 1, 1)$.

59. Show that the osculating plane at every point on the curve $\mathbf{r}(t) = \left\langle t + 2, 1 - t, \frac{1}{2}t^2 \right\rangle$ is the same plane. What can you conclude about this curve?

60. Show that at every point on the curve

$$\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$$

the angle between the unit tangent vector and the z-axis is the same. Then show that the same result holds true for the unit normal and binormal vectors.

61. The *rectifying plane* of a curve at a point is the plane that contains the vectors **T** and **B** at that point. Find the rectifying plane of the curve $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \tan t\,\mathbf{k}$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right)$.

62. Show that the curvature κ is related to the tangent and normal vectors by the equation

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$$

63. Show that the curvature of a plane curve is $\kappa = \left|\frac{d\phi}{ds}\right|$, where ϕ is the angle between **T** and **i**; that is, ϕ is the angle of inclination of the tangent line.

64. (a) Show that $\frac{d\mathbf{B}}{ds}$ is perpendicular to **B**.

(b) Show that $\frac{d\mathbf{B}}{ds}$ is perpendicular to **T**.

(c) Deduce from parts (a) and (b) that $\frac{d\mathbf{B}}{ds} = -\tau(s)\mathbf{N}$ for some number $\tau(s)$ called the **torsion** of the curve. (The torsion measures the degree of twisting of a curve.)

(d) Show that for a plane curve the torsion is $\tau(s) = 0$.

65. The following formulas, called the **Frenet–Serret formulas**, are of fundamental importance in differential geometry:

1. $\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$
2. $\frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B}$
3. $\frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$

Use the fact that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ to deduce Formula 2 from Formulas 1 and 3.

66. Use the Frenet–Serret formulas to prove each of the following. (Primes denote derivatives with respect to t. Start each proof as in Theorem 10.)

(a) $\mathbf{r}'' = s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}$

(b) $\mathbf{r}' \times \mathbf{r}'' = \kappa(s')^3\,\mathbf{B}$

(c) $\mathbf{r}''' = [s''' - \kappa^2(s')^3]\,\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\,\mathbf{N} + \kappa\tau(s')^3\,\mathbf{B}$

(d) $\tau = \dfrac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$

67. Show that the circular helix $\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle$, where a and b are constants, has a constant curvature. [Use the result of Exercise 66(d).]

68. The DNA molecule has the shape of a double helix (see Figure 10.3). The radius of each helix is about 10 angstroms ($1\ \text{Å} = 10^{-8}$ cm). Each helix rises about 34 Å during each complete turn, and there are about 2.9×10^8 complete turns. Estimate the length of each helix.

69. Consider the problem of designing a railroad track to make a smooth transition between sections of straight track. Existing track along the negative x-axis is to be joined smoothly to a track along the line $y = 1$ for $x \geq 1$.

(a) Find a polynomial $P = P(x)$ of degree 5 such that the function F defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

is continuous and has continuous slope and continuous curvature.

(b) Use technology to sketch the graph of F.

10.4 Motion in Space: Velocity and Acceleration

In this section, we will use the ideas of tangent and normal vectors, and curvature, to study the motion of an object, including its velocity and acceleration, along a space curve. In particular, we will follow the footsteps of Newton by using these methods to derive Kepler's First Law of planetary motion.

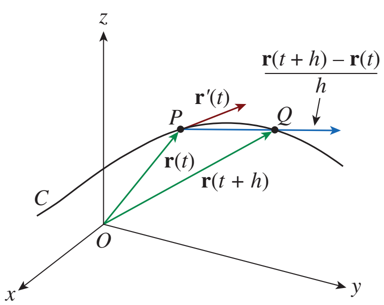


Figure 10.31
The position of the particle at time t is $\mathbf{r}(t)$.

Velocity, Speed, and Acceleration

Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. Figure 10.31 suggests that for small values of h the vector

$$\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} \tag{1}$$

approximates the direction of the particle moving along the curve $\mathbf{r}(t)$. Its magnitude measures the size of the displacement vector per unit time. The vector in Equation 1 represents the average velocity over a time interval of length h and its limit is the **velocity vector** $\mathbf{v}(t)$ at time t:

$$\mathbf{v}(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} = \mathbf{r}'(t) \tag{2}$$

Therefore, the velocity vector is the tangent vector and points in the direction of the tangent line.

The **speed** of the particle at time t is the magnitude of the velocity vector, that is, $|\mathbf{v}(t)|$. This is reasonable because, using Equations 2 and 10.3.7, we have

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt} = \text{rate of change of distance with respect to time}$$

Similar to motion along a line, the **acceleration** of the particle is defined as the derivative of the velocity:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

Example 1 Find Velocity, Speed, and Acceleration

The position vector of an object moving in a plane is given by $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}$. Find its velocity, speed, and acceleration when $t = 1$ and sketch a graph to illustrate the velocity and acceleration vectors.

Solution

Find the velocity and acceleration at time t.

$\mathbf{v}(t) = \mathbf{r}'(t) = 3t^2\,\mathbf{i} + 2t\,\mathbf{j}$

$\mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\mathbf{i} + 2\,\mathbf{j}$

The speed of the object is $|\mathbf{v}(t)| = \sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2}$.

When $t = 1$:

$\mathbf{v}(1) = 3\,\mathbf{i} + 2\,\mathbf{j}$

$\mathbf{a}(1) = 6\,\mathbf{i} + 2\,\mathbf{j}$

$|\mathbf{v}(1)| = \sqrt{13}$

The velocity and acceleration vectors are shown in Figure 10.32. ■

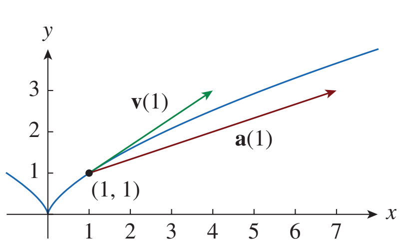


Figure 10.32
The velocity and acceleration vectors when $t = 1$.

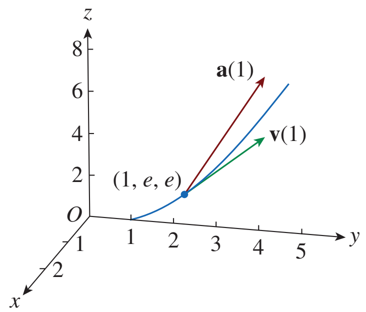


Figure 10.33
The velocity and acceleration vectors when $t = 1$.

Example 2 Find Velocity, Speed, and Acceleration of a Particle in Space

Find the velocity, acceleration, and speed of a particle with position vector $\mathbf{r}(t) = \langle t^2, e^t, t\,e^t \rangle$.

Solution

Find the first and second derivative of $\mathbf{r}$ and the magnitude of $\mathbf{v}$.

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, e^t, (1+t)\,e^t \rangle$ — Product Rule for third component.

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, e^t, (2+t)\,e^t \rangle$ — Product Rule again.

$$|\mathbf{v}(t)| = \sqrt{4t^2 + e^{2t} + (1+t)^2 e^{2t}}$$

Figure 10.33 shows the velocity and acceleration vector when $t = 1$. ■

The vector integrals that were introduced in Section 10.2 can be used to find position vectors when velocity or acceleration vectors are known. This concept is illustrated in the next example.

Example 3 Finding Position, Given Acceleration

A particle moving along a curve starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t\,\mathbf{i} + 6t\,\mathbf{j} + \mathbf{k}$. Find its velocity and position at time t.

Solution

Since $\mathbf{a}(t) = \mathbf{v}'(t)$, integrate to find $\mathbf{v}(t)$.

$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (4t\,\mathbf{i} + 6t\,\mathbf{j} + \mathbf{k})\,dt$ — Use expression for $\mathbf{a}(t)$.

$\quad = 2t^2\,\mathbf{i} + 3t^2\,\mathbf{j} + t\,\mathbf{k} + \mathbf{C}$ — Integrate components.

Use the given value of $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$ to find $\mathbf{C}$.

$$\mathbf{v}(0) = \mathbf{C} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

Use this value of $\mathbf{C}$ in the expression for $\mathbf{v}(t)$.

$$\begin{aligned}\mathbf{v}(t) &= 2t^2\,\mathbf{i} + 3t^2\,\mathbf{j} + t\,\mathbf{k} + \mathbf{i} - \mathbf{j} + \mathbf{k}\\ &= (2t^2+1)\,\mathbf{i} + (3t^2-1)\,\mathbf{j} + (t+1)\,\mathbf{k}\end{aligned}$$

Similarly, since $\mathbf{v}(t) = \mathbf{r}'(t)$:

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt$ — $\mathbf{r}(t)$ is an antiderivative of $\mathbf{v}(t)$.

$\quad = \int [(2t^2+1)\,\mathbf{i} + (3t^2-1)\,\mathbf{j} + (t+1)\,\mathbf{k}]\,dt$ — Use expression for $\mathbf{v}(t)$.

$\quad = \left(\frac{2}{3}t^3 + t\right)\mathbf{i} + (t^3 - t)\,\mathbf{j} + \left(\frac{1}{2}t^2 + t\right)\mathbf{k} + \mathbf{D}$ — Integrate components.

Use the value of $\mathbf{r}(0) = \mathbf{i}$ to find $\mathbf{D}$.

$$\mathbf{r}(0) = \mathbf{D} = \mathbf{i}$$

Therefore,

$$r(t) = \left(\frac{2}{3}t^3 + t + 1\right)\mathbf{i} + (t^3 - t)\,\mathbf{j} + \left(\frac{1}{2}t^2 + t\right)\mathbf{k}.$$

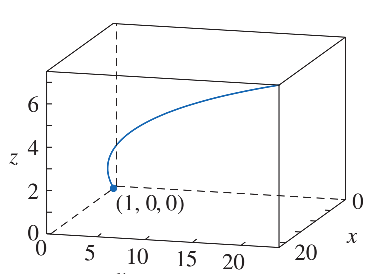


Figure 10.34
The path of the particle for $0 \le t \le 3$.

Figure 10.34 shows the path of the particle for $0 \le t \le 3$. ■

In general, using vector integrals we can find the velocity when acceleration is known for a specific value of t, and position when velocity is known for a specific value of t.

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(u)\,du \qquad \mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(u)\,du$$

The relationship between $\mathbf{r}(t)$, $\mathbf{v}(t)$, and $\mathbf{a}(t)$ can be used to investigate the force acting on an object. If the force that acts on a particle is known, then the acceleration can be found from **Newton's Second Law of Motion**. The vector version of this law states that if, at any time t, a force $\mathbf{F}(t)$ acts on an object of mass m producing an acceleration $\mathbf{a}(t)$, then

$$\mathbf{F}(t) = m\,\mathbf{a}(t)$$

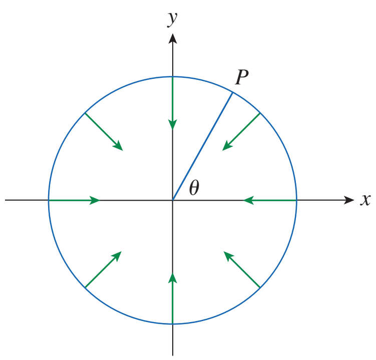


Figure 10.35
The force vector points toward the origin for every value of t. Note that the angular speed of the object at position P is $\omega = \dfrac{d\theta}{dt}$, where θ is the angle shown.

Example 4 Uniform Circular Motion

An object with mass m that moves in a circular path with constant angular speed ω has position vector $\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + a\sin\omega t\,\mathbf{j}$. Find the force acting on the object and show that it is directed toward the origin.

Solution

Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

$\mathbf{v}(t) = \mathbf{r}'(t) = -a\omega\,\sin\omega t\,\mathbf{i} + a\omega\,\cos\omega t\,\mathbf{j}$

$\mathbf{a}(t) = \mathbf{v}'(t) = -a\omega^2\cos\omega t\,\mathbf{i} - a\omega^2\sin\omega t\,\mathbf{j}$

Newton's Second Law gives the force as

$\mathbf{F}(t) = m\,\mathbf{a}(t) = -m\omega^2(a\cos\omega t\,\mathbf{i} + a\sin\omega t\,\mathbf{j}) = -m\omega^2\mathbf{r}(t)$.

Since $\mathbf{F}(t) = -m\omega^2\mathbf{r}(t)$, the force acts in the direction opposite to the radius vector $\mathbf{r}(t)$ and therefore points toward the origin. Several force vectors are illustrated in Figure 10.35.

This type of force is called a *centripetal* (center-seeking) force. ■

Example 5 Projectile Motion

A projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$ as illustrated in Figure 10.36. Assuming that air resistance is negligible and the only external force is due to gravity, find the position function $\mathbf{r}(t)$ of the projectile. What value of α maximizes the range (the horizontal distance traveled)?

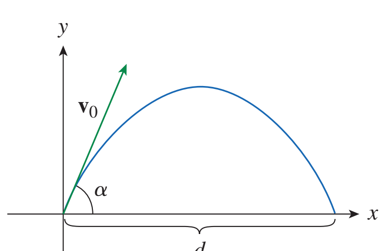


Figure 10.36
Path of the projectile fired with angle of elevation α and initial velocity $\mathbf{v}_0$.

Solution

Suppose the projectile is fired from, or starts at, the origin, as in Figure 10.36.

Since the force due to gravity acts downward:

$\mathbf{F} = m\,\mathbf{a} = -mg\,\mathbf{j}$, where $g = |\mathbf{a}| = 9.8\ \text{m/s}^2 \quad\Rightarrow\quad \mathbf{a} = -g\,\mathbf{j}$.

Since $\mathbf{v}'(t) = \mathbf{a} \quad\Rightarrow\quad \mathbf{v}(t) = \int(-g\,\mathbf{j})\,dt = -gt\,\mathbf{j} + \mathbf{C}$.

$\mathbf{v}(0) = -g\cdot 0\,\mathbf{j} + \mathbf{C} = \mathbf{v}_0 \quad\Rightarrow\quad \mathbf{C} = \mathbf{v}_0$

Therefore, $\mathbf{r}'(t) = \mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{v}_0$.

$\mathbf{r}(t) = \int(-gt\,\mathbf{j} + \mathbf{v}_0)\,dt = -\frac{1}{2}gt^2\,\mathbf{j} + t\mathbf{v}_0 + \mathbf{D}$ Integrate to find $\mathbf{r}(t)$.

The projectile starts at the origin, so $\mathbf{r}(0) = \mathbf{0} = \mathbf{D}$.

Therefore, the position vector of the projectile is given by

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\,\mathbf{j} + t\,\mathbf{v}_0. \tag{3}$$

If we let $|\mathbf{v}_0| = v_0$ (the initial speed of the projectile), then $\mathbf{v}_0 = v_0 \cos\alpha\,\mathbf{i} + v_0 \sin\alpha\,\mathbf{j}$.

Use this expression for $\mathbf{v}_0$ to rewrite the position vector.

$$\mathbf{r}(t) = (v_0 \cos\alpha)\, t\,\mathbf{i} + \left[(v_0 \sin\alpha)\, t - \frac{1}{2}gt^2\right]\mathbf{j}$$

The parametric equations of the trajectory are therefore

If we eliminate the variable t, then y is a quadratic function of x. This means the path of the projectile is part of a parabola.

$$x = (v_0 \cos\alpha)\, t, \quad y = (v_0 \sin\alpha)\, t - \frac{1}{2}gt^2. \qquad (4)$$

The horizontal distance d traveled by the projectile is the value of x when $y = 0$.

$$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 = 0 \quad \Rightarrow \quad t = 0 \quad \text{or} \quad t = \frac{2v_0 \sin\alpha}{g}$$

For $t = \dfrac{2v_0 \sin\alpha}{g}$:

$$d = x = (v_0 \cos\alpha)\frac{2v_0 \sin\alpha}{g} = \frac{v_0^2(2\sin\alpha\cos\alpha)}{g} = \frac{v_0^2 \sin 2\alpha}{g},$$

v_0^2 and g are constant in this expression for d.

Therefore, d is a maximum when $\sin 2\alpha = 1$, that is, when $\alpha = \dfrac{\pi}{4}$. ■

Example 6 Initial Position Above Ground Level

A projectile is fired with muzzle speed 150 m/s and angle of elevation 45° from a position 10 m above ground level. Where does the projectile hit the ground, and with what speed?

Solution

Place the origin at ground level. Then the initial position of the projectile is (0, 10).

Adjust the parametric equations for the trajectory by adding 10 to the expression for y.

Use $v_0 = 150$ m/s, $\alpha = 45°$, and $g = 9.8$ m/s:

$$x = 150 \cos\left(\frac{\pi}{4}\right)t = 75\sqrt{2}\, t$$

$$y = 10 + 150 \sin\left(\frac{\pi}{4}\right)t - \frac{1}{2}(9.8)t^2 = 10 + 75\sqrt{2}\, t - 4.9t^2$$

The projectile hits the ground when $y = 0$, that is, when $4.9t^2 - 75\sqrt{2}\, t - 10 = 0$.

Solve this quadratic equation and use only the positive value of t.

$$t = \frac{75\sqrt{2} + \sqrt{11{,}250 + 196}}{9.8} \approx 21.74$$

Use this value of t to find the horizontal distance traveled by the projectile.

$$x \approx 75\sqrt{2}(21.74) \approx 2305.88$$

Therefore, the projectile hits the ground approximately 2306 m away.

The velocity of the projectile is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 75\sqrt{2}\,\mathbf{i} + (75\sqrt{2} - 9.8t)\,\mathbf{j}.$$

The speed at impact is

$$|\mathbf{v}(21.74)| = \sqrt{(75\sqrt{2})^2 + (75\sqrt{2} - 9.8 \cdot 21.74)^2} \approx 150.652 \text{ m/s}.$$

Tangential and Normal Components of Acceleration

In the study of particle motion, it is often useful to resolve the acceleration into two components, one in the direction of the tangent and the other in the direction of the normal. Let $v = |\mathbf{v}|$ be the speed of the particle, then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v}$$

and therefore $\mathbf{v} = v\ \mathbf{T}$. Differentiate both sides of this equation with respect to t:

$$\mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}' \tag{5}$$

If we use the expression for the curvature given in Equation 10.3.9, then we have

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v} \quad \Rightarrow \quad |\mathbf{T}'| = \kappa v \tag{6}$$

The definition of the unit normal vector was given in Section 10.3 as $\mathbf{N} = \dfrac{\mathbf{T}'}{|\mathbf{T}'|}$. Solve for $\mathbf{T}'$ and use Equation 6.

$$\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa v \mathbf{N}$$

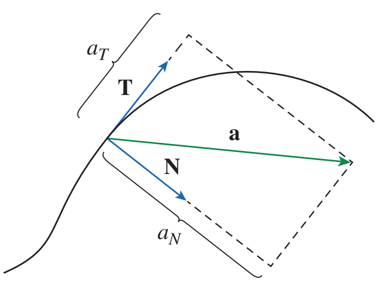


Figure 10.37
The resolution of the acceleration vector into tangent and normal components.

We can now write Equation 5 as

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N} \tag{7}$$

Let a_T and a_N be the tangential and normal components of acceleration, then

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where

$$a_T = v' \quad \text{and} \quad a_N = \kappa v^2 \tag{8}$$

This resolution of the acceleration vector is illustrated in Figure 10.37.

Let's try to interpret Equation 7. The first thing to notice is that the binormal vector **B** is not part of this equation. This means that no matter how an object moves through space, its acceleration always lies in the plane determined by **T** and **N** (the osculating plane). Recall that **T** gives the direction of motion and **N** points in the direction the curve is turning.

Next, notice that the tangential component of acceleration is v', the rate of change of speed, and the normal component of acceleration is κv^2, the curvature times the square of the speed. This seems reasonable if we think about a passenger in a car; a sharp turn in a road means a large value of the curvature κ. Therefore, the component of the acceleration perpendicular to the motion is large and the passenger is tossed against the door. High speed around the turn has the same effect; in fact, if you double the speed, a_N is increased by a factor of 4.

Even though we have expressions for the tangential and normal components of acceleration in Equation 8, it is sometimes desirable to use expressions that depend only on $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}''$. Here's how we can find them. Start by computing the dot product of $\mathbf{v} = v\mathbf{T}$ with $\mathbf{a}$ as given in Equation 7.

$$
\begin{aligned}
\mathbf{v} \cdot \mathbf{a} &= v\mathbf{T} \cdot (v'\mathbf{T} + \kappa v^2 \mathbf{N}) && \text{Use expressions for } \mathbf{v} \text{ and } \mathbf{a}. \\
&= vv'\mathbf{T} \cdot \mathbf{T} + \kappa v^3 \mathbf{T} \cdot \mathbf{N} && \text{Property of dot products.} \\
&= vv' && \mathbf{T} \cdot \mathbf{T} = 1 \text{ and } \mathbf{T} \cdot \mathbf{N} = 0.
\end{aligned}
$$

Solve this expression for v' and rewrite a_T.

$$a_T = v' = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} \tag{9}$$

Using the formula for curvature given in Section 10.3, we can rewrite a_N as

$$a_N = \kappa v^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} |\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} \tag{10}$$

Here is an example in which we use the equations in (9) and (10).

Example 7 Find the Components of Acceleration

A particle moves with position function $\mathbf{r}(t) = \langle t^2, t^2, t^3 \rangle$. Find the tangential and normal components of acceleration.

Solution

Find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$ and $|\mathbf{r}'(t)|$.

$$
\begin{aligned}
&\mathbf{r}(t) = t^2\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k} \\
&\mathbf{r}'(t) = 2t\,\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k} && \text{Derivative of each component.} \\
&\mathbf{r}''(t) = 2\,\mathbf{i} + 2\,\mathbf{j} + 6t\,\mathbf{k} && \text{Derivative of each component, again.} \\
&|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (2t)^2 + (3t^2)^2} = \sqrt{8t^2 + 9t^4} && \text{Vector magnitude.}
\end{aligned}
$$

Use Equation 9 to find the tangential component.

$$
\begin{aligned}
a_T &= \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} && \text{Equation 9.} \\
&= \frac{2t \cdot 2 + 2t \cdot 2 + 3t^2 \cdot 6t}{\sqrt{8t^2 + 9t^4}} && \text{Dot product definition.} \\
&= \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}} && \text{Simplify.}
\end{aligned}
$$

Find the necessary cross product.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = 6t^2\,\mathbf{i} - 6t^2\,\mathbf{j}$$

Use Equation 10 to find the normal component.

$$a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{(6t^2)^2 + (-6t^2)^2}}{\sqrt{8t^2 + 9t^4}} = \frac{6\sqrt{2}t^2}{\sqrt{8t^2 + 9t^4}}$$

■

Kepler's Laws of Planetary Motion

The material in this chapter can be used to prove Kepler's laws of planetary motion. This is truly a great application of calculus and we will prove the first result.

After 20 years of studying the astronomical observations of the Danish astronomer Tycho Brahe, the German mathematician and astronomer Johannes Kepler (1571–1630) formulated the following three laws.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

In his book *Principia Mathematica* of 1687, Sir Isaac Newton was able to show that these three laws are consequences of two of his own laws, the Second Law of Motion and the Law of Universal Gravitation. In what follows, we prove Kepler's First Law. The remaining laws are left as exercises (with hints).

Since the gravitational force of the sun on a planet is so much larger than the forces exerted by other celestial bodies, we can safely ignore all bodies in the universe except the sun and one planet revolving about it. We use a coordinate system with the sun at the origin and we let $\mathbf{r} = \mathbf{r}(t)$ be the position vector of the planet. (Equally well, $\mathbf{r}$ could be the position vector of the moon or a satellite moving around Earth or a comet moving around a star.) The velocity vector is $\mathbf{v} = \mathbf{r}'$ and the acceleration vector is $\mathbf{a} = \mathbf{r}''$. We use the following laws of Newton:

$$\text{Second Law of Motion:} \quad \mathbf{F} = m\mathbf{a}$$

$$\text{Law of Gravitation:} \quad \mathbf{F} = -\frac{GMm}{r^3}\mathbf{r} = -\frac{GMm}{r^2}\mathbf{u}$$

where $\mathbf{F}$ is the gravitational force on the planet, m and M are the masses of the planet and the sun, G is the gravitational constant, $r = |\mathbf{r}|$, and $\mathbf{u} = \dfrac{\mathbf{r}}{r}$ is the unit vector in the direction of $\mathbf{r}$.

We first show that the planet moves in one plane. Equate the expressions for $\mathbf{F}$ in Newton's two laws, and solve for $\mathbf{a}$.

$$m\,\mathbf{a} = -\frac{GMm}{r^3}\mathbf{r} \quad \Rightarrow \quad \mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

This expression shows that $\mathbf{a}$ is parallel to $\mathbf{r}$. Therefore, $\mathbf{r} \times \mathbf{a} = \mathbf{0}$.

Now use this result and Equation 5 in Theorem 10.2.3 to write

$$\begin{aligned} \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) &= \mathbf{r}' \times \mathbf{v} + \mathbf{r} \times \mathbf{v}' \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = \mathbf{0} + \mathbf{0} = \mathbf{0} \end{aligned}$$

Since this derivative is $\mathbf{0}$, then $\mathbf{r} \times \mathbf{v} = \mathbf{h}$, where $\mathbf{h}$ is a constant vector. (We may assume that $\mathbf{h} \neq \mathbf{0}$; that is, $\mathbf{r}$ and $\mathbf{v}$ are not parallel.) This means that the vector $\mathbf{r} = \mathbf{r}(t)$ is perpendicular to $\mathbf{h}$ for all values of t, so the planet always lies in the plane through the origin perpendicular to $\mathbf{h}$. Therefore, the orbit of the planet is a plane curve.

To prove Kepler's First Law, write the vector $\mathbf{h}$ as follows:

$$\begin{aligned} \mathbf{h} &= \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{r}' = r\,\mathbf{u} \times (r\,\mathbf{u})' \\ &= r\,\mathbf{u} \times (r\,\mathbf{u}' + r'\mathbf{u}) = r^2(\mathbf{u} \times \mathbf{u}') + rr'(\mathbf{u} \times \mathbf{u}) \\ &= r^2(\mathbf{u} \times \mathbf{u}') \end{aligned}$$

Then consider the following cross product:

$$\mathbf{a} \times \mathbf{h} = \frac{-GM}{r^2}\mathbf{u} \times (r^2\mathbf{u} \times \mathbf{u}') = -GM\ \mathbf{u} \times (\mathbf{u} \times \mathbf{u}')$$

$$= -GM[(\mathbf{u} \cdot \mathbf{u}')\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u}'] \qquad \text{Equation 9.4.10.}$$

Since $\mathbf{u}$ is a unit vector, $|\mathbf{u}(t)| = 1$ and $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 = 1$. Therefore, from Example 4 in Section 10.2, $\mathbf{u} \cdot \mathbf{u}' = 0$. Use these results to simplify the cross product.

$$\mathbf{a} \times \mathbf{h} = -GM[0\ \mathbf{u} - 1\ \mathbf{u}'] = GM\ \mathbf{u}'$$

Consider the following derivative.

$$(\mathbf{v} \times \mathbf{h})' = \mathbf{v}' \times \mathbf{h} + \mathbf{v} \times \mathbf{h}' = \mathbf{v}' \times \mathbf{h} \qquad \text{Differentiation rule; } \mathbf{h} \text{ constant.}$$

$$= \mathbf{a} \times \mathbf{h} = GM\ \mathbf{u}' \qquad \mathbf{v}' = \mathbf{a}\text{; expression for } \mathbf{a} \times \mathbf{h}.$$

Integrate both sides of this equation:

$$\mathbf{v} \times \mathbf{h} = GM\ \mathbf{u} + \mathbf{c} \tag{11}$$

where $\mathbf{c}$ is a constant vector.

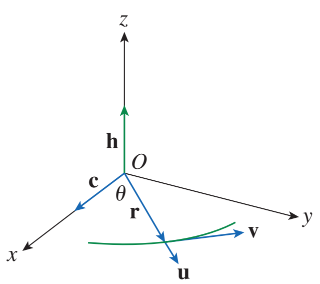


Figure 10.38
Choose coordinate axes so that **k** points in the direction of **h**.

At this point, it is convenient to choose the coordinate axes so that the standard basis vector **k** points in the direction of the vector **h**. Then the planet moves in the xy-plane. Since both $\mathbf{v} \times \mathbf{h}$ and $\mathbf{u}$ are perpendicular to $\mathbf{h}$, Equation 11 shows that $\mathbf{c}$ lies in the xy-plane. This means that we can choose the x- and y-axes so that the vector $\mathbf{i}$ lies in the direction of $\mathbf{c}$, as shown in Figure 10.38.

If θ is the angle between $\mathbf{c}$ and $\mathbf{r}$, then (r, θ) are polar coordinates of the planet. Using Equation 11:

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = \mathbf{r} \cdot (GM\ \mathbf{u} + \mathbf{c}) = GM\ \mathbf{r} \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{c}$$

$$= GM\ r\ \mathbf{u} \cdot \mathbf{u} + |\mathbf{r}||\mathbf{c}|\ \cos\theta = GM\ r + rc\ \cos\theta$$

where $c = |\mathbf{c}|$. Solve this expression for r.

$$r = \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{GM + c\cos\theta} = \frac{1}{GM}\frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{1 + e\cos\theta}$$

where $e = \dfrac{c}{GM}$. Using a property of cross products and $\mathbf{r} \times \mathbf{v} = \mathbf{h}$:

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = |\mathbf{h}|^2 = h^2$$

where $h = |\mathbf{h}|$. Use this result to rewrite the expression for r.

$$r = \frac{h^2/(GM)}{1 + e\cos\theta} = \frac{eh^2/c}{1 + e\cos\theta}$$

If we let $d = h^2/c$, then this equation becomes

$$r = \frac{ed}{1 + e\cos\theta} \tag{12}$$

It can be shown that Equation 12 is the polar equation of a conic section with focus at the origin and eccentricity e. Since the orbit of a planet is a closed curve, then the conic must be an ellipse.

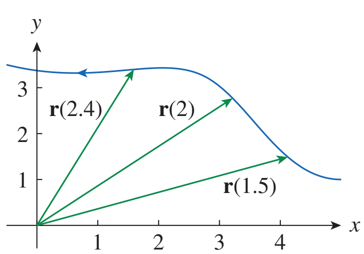
This completes the derivation of Kepler's First Law. The Applied Project at the end of this section will guide you through the derivation of the Second and Third Laws. The proofs of these laws demonstrate that the methods of this chapter provide a powerful tool for describing some of the laws of nature.

10.4 Exercises

1. The table gives coordinates of a particle moving through space along a smooth curve.
(a) Find the average velocities over the time intervals [0, 1], [0.5, 1], [1, 2], and [1, 1.5].
(b) Estimate the velocity and speed of the particle at $t = 1$.

t	x	y	z
0	2.7	9.8	3.7
0.5	3.5	7.2	3.3
1.0	4.5	6.0	3.0
1.5	5.9	6.4	2.8
2.0	7.3	7.8	2.7

2. The figure shows the path of a particle that moves with position vector $\mathbf{r}(t)$ at time t.
(a) Draw a vector that represents the average velocity of the particle over the time interval $2 \le t \le 2.4$.
(b) Draw a vector that represents the average velocity of the time interval $1.5 \le t \le 2$.
(c) Write an expression for the velocity vector $\mathbf{v}(2)$.
(d) Draw an approximation to the vector $\mathbf{v}(2)$ and estimate the speed of the particle at $t = 2$.



Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t.

3. $\mathbf{r}(t) = \left\langle -\frac{1}{2}t^2, t \right\rangle, \quad t = 2$

4. $\mathbf{r}(t) = \left\langle -t^2, \frac{1}{3}t^3 \right\rangle, \quad t = 2$

5. $\mathbf{r}(t) = \langle 2 - t, 4\sqrt{t} \rangle, \quad t = 1$

6. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad t = \frac{\pi}{3}$

7. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{2t}\,\mathbf{j}, \quad t = 0$

8. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle, \quad t = 1$

9. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2\,\mathbf{k}, \quad t = 1$

10. $\mathbf{r}(t) = t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sin t\,\mathbf{k}, \quad t = 0$

Find the velocity, acceleration, and speed of a particle with the given position function.

11. $\mathbf{r}(t) = \langle t^2 + t, t^2 - t, t^3 \rangle$

12. $\mathbf{r}(t) = \langle 2\cos t, 3t, 2\sin t \rangle$

13. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$

14. $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}$

15. $\mathbf{r}(t) = e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k})$

16. $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t - t\sin t \rangle, \quad t \ge 0$

Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

17. $\mathbf{a}(t) = \mathbf{i} + 2\mathbf{j}, \quad \mathbf{v}(0) = \mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i}$

18. $\mathbf{a}(t) = 2\,\mathbf{i} + 2t\,\mathbf{k}, \quad \mathbf{v}(0) = 3\mathbf{i} - \mathbf{j}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

19. $\mathbf{a}(t) = 2\,\mathbf{i} + 6t\,\mathbf{j} + 12t^2\,\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i}, \quad \mathbf{r}(0) = \mathbf{j} - \mathbf{k}$

20. $\mathbf{a}(t) = \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 6t\,\mathbf{k}, \quad \mathbf{v}(0) = -\mathbf{k}, \quad \mathbf{r}(0) = \mathbf{j} - 4\,\mathbf{k}$

(a) Find the position vector of a particle that has the given acceleration and the specified initial velocity and position.
(b) Use technology to graph the path of the particle.

21. $\mathbf{a}(t) = 2t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i}, \quad \mathbf{r}(0) = \mathbf{j}$

22. $\mathbf{a}(t) = t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{k}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

23. The position function of a particle is given by $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. Find the value of t such that the speed of the particle is a minimum.

24. What force is required so that a particle of mass m has the position function $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$?

25. A force with magnitude 20 N acts directly upward from the xy-plane on an object with mass 4 kg. The object starts at the origin with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Find its position function and its speed at time t.

26. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

27. A projectile is fired with an initial speed of 200 m/s and angle of elevation 60°. Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

28. Answer the questions in Exercise 27 if the projectile is fired from a position 100 m above the ground.

29. A ball is thrown at an angle of 45° to the ground. If the ball lands 90 m away, what was the initial speed of the ball?

30. A projectile is fired with angle of elevation 36°. What is the initial speed if the maximum height of the bullet is 1600 ft?

31. Suppose a projectile is fired with muzzle speed of 150 m/s. Find two angles of elevation that can be used to hit a target 800 m away.

32. A batter hits a baseball 3 ft above the ground toward the center field fence, which is 10 ft high and 400 ft from home plate. The ball leaves the bat with speed 115 ft/s at an angle 50° above the horizontal. Is it a home run? (In other words, does the ball clear the fence?)

33. A medieval city has the shape of a square and is protected by walls with length 500 m and height 15 m. You are the commander of an attacking army and the closest you can get to the wall is 100 m. Your plan is to set fire to the city by catapulting heated rocks over the wall (with an initial speed of 80 m/s). At what range of angles should you tell your army to set the catapult? (Assume the path of the rocks is perpendicular to the wall.)

34. Show that a projectile reaches three-quarters of its maximum height in half the time needed to reach its maximum height.

35. A ball is thrown eastward into the air from the origin (in the direction of the positive x-axis). The initial velocity is $50\,\mathbf{i} + 80\,\mathbf{k}$, with speed measured in feet per second. The spin of the ball results in a southward acceleration of 4 ft/s^2, so the acceleration vector is $\mathbf{a} = -4\,\mathbf{j} - 32\,\mathbf{k}$. Where does the ball land and with what speed?

36. A ball with mass 0.8 kg is thrown southward into the air with a speed of 30 m/s at an angle of 30° to the ground. A west wind applies a steady force of 4 N to the ball in an easterly direction. Where does the ball land and with what speed?

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37. Water traveling along a straight portion of a river normally flows fastest in the middle, and the speed slows to almost zero at the banks. Consider a long straight stretch of river flowing north, with parallel banks 40 m apart. If the maximum water speed is 3 m/s, we can use a quadratic function as a basic model for the rate of water flow x units from the west bank:

$$f(x) = \frac{3}{400}x(40 - x).$$

(a) A boat proceeds at a constant speed of 5 m/s from a point on the west bank while maintaining a heading perpendicular to the bank. How far down the river on the opposite bank will the boat touch shore? Graph the path of the boat.

(b) Suppose we would like to pilot the boat to land at point B on the east bank directly opposite A. If we maintain a constant speed of 5 m/s and a constant heading, find the angle at which the boat should head. Then graph the actual path the boat follows. Does the path seem realistic?

38. Another reasonable model for the water speed of the river in Exercise 37 is a sine function:

$$f(x) = 3\sin\left(\frac{\pi x}{40}\right)$$

If a boater would like to cross the river from A to B with constant heading and a constant speed of 5 m/s, determine the angle at which the boat should head.

39. A particle has position function $\mathbf{r}(t)$. If $\mathbf{r}'(t) = \mathbf{c} \times \mathbf{r}(t)$, where $\mathbf{c}$ is a constant vector, describe the path of the particle.

40. (a) If a particle moves along a straight line, what can you say about its acceleration vector?

(b) If a particle moves with constant speed along a curve, what can you say about its acceleration vector?

Find the tangential and normal components of the acceleration vector.

41. $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}$

42. $\mathbf{r}(t) = (1 + t)\,\mathbf{i} + (t^2 - 2t)\,\mathbf{j}$

43. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$

44. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 3t\,\mathbf{k}$

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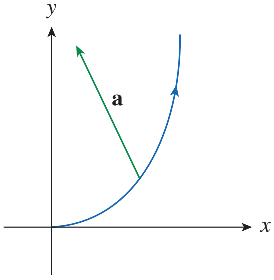
45. The magnitude of the acceleration vector $\mathbf{a}$ is 10 cm/s^2. Use the figure to estimate the tangential and normal components of $\mathbf{a}$.



46. If a particle with mass m moves with position vector $\mathbf{r}(t)$, then its **angular momentum** is defined as $\mathbf{L}(t) = m\,\mathbf{r}(t) \times \mathbf{v}(t)$ and its **torque** as $\boldsymbol{\tau}(t) = m\,\mathbf{r}(t) \times \mathbf{a}(t)$. Show that $\mathbf{L}'(t) = \boldsymbol{\tau}(t)$. Deduce that if $\boldsymbol{\tau}(t) = \mathbf{0}$ for all t, then $\mathbf{L}(t)$ is constant. (This is called the *law of conservation of angular momentum*.)

47. The position function of a spaceship is

$$\mathbf{r}(t) = (3 + t)\,\mathbf{i} + (2 + \ln t)\,\mathbf{j} + \left(7 - \frac{4}{t^2 + 1}\right)\mathbf{k}$$

and the coordinates of a space station are (6, 4, 9). The captain wants the spaceship to coast into the space station. When should the engines be turned off?

48. A rocket burning its onboard fuel while moving through space has velocity $\mathbf{v}(t)$ and mass $m(t)$ at time t. If the exhaust gases escape with velocity $\mathbf{v}_e$ relative to the rocket, it can be deduced from Newton's Second Law of Motion that

$$m\frac{d\mathbf{v}}{dt} = \frac{dm}{dt}\mathbf{v}_e$$

(a) Show that $\mathbf{v}(t) = \mathbf{v}(0) - \ln\dfrac{m(0)}{m(t)}\mathbf{v}_e$.

(b) For the rocket to accelerate in a straight line from rest to twice the speed of its own exhaust gases, what fraction of its initial mass would the rocket have to burn as fuel?

Applied Project | Kepler's Laws

Johannes Kepler stated the following three laws of planetary motion on the basis of masses of data on the positions of the planets at various times.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.

2. The line joining the sun to a planet sweeps out equal areas in equal times.

3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

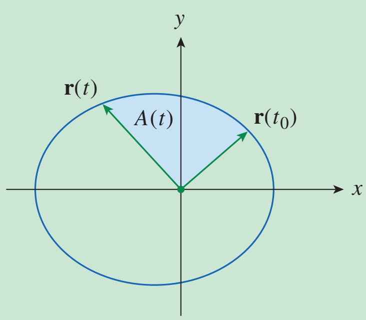


Figure 10.39
$A(t)$ is the area swept out over the time interval $[t_0, t]$.

Kepler formulated these laws because they fit the astronomical data. He wasn't able to see why they were true or how they related to each other. But Sir Isaac Newton, in his *Principia Mathematica* of 1687, showed how to deduce Kepler's three laws from two of Newton's own laws, the Second Law of Motion and the Law of Universal Gravitation. In Section 10.4, we proved Kepler's First Law using the calculus of vector functions. In this project, we'll work through the proofs of Kepler's Second and Third Laws and explore some of their consequences.

1. Use the following steps to prove Kepler's Second Law. The notation is the same as in the proof of the First Law in Section 10.4. In particular, use polar coordinates so that $\mathbf{r} = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}$.

(a) Show that $\mathbf{h} = r^2\dfrac{d\theta}{dt}\mathbf{k}$.

(b) Deduce that $r^2\dfrac{d\theta}{dt} = h$.

(c) If $A = A(t)$ is the area swept out by the radius vector $\mathbf{r} = \mathbf{r}(t)$ in the time interval $[t_0, t]$ as in Figure 10.39, show that

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$$

(d) Deduce that

$$\frac{dA}{dt} = \frac{1}{2}h = \text{constant}$$

This says that the rate at which A is swept out is constant and proves Kepler's Second Law.

2. Let T be the period of a planet about the sun; that is, T is the time required for it to travel once around its elliptical orbit. Suppose that the lengths of the major and minor axes of the ellipse are $2a$ and $2b$.

(a) Use part (d) of Problem 1 to show that $T = 2\pi ab/h$.

(b) Show that $\dfrac{h^2}{GM} = ed = \dfrac{b^2}{a}$.

(c) Use parts (a) and (b) to show that $T^2 = \dfrac{4\pi^2}{GM}a^3$.

(d) This proves Kepler's Third Law. Notice that the proportionality constant $\dfrac{4\pi^2}{GM}$ is independent of the planet.

3. The period of Earth's orbit is approximately 365.25 days. Use this fact and Kepler's Third Law to find the length of the major axis of Earth's orbit. You will need the mass of the sun, $M = 1.99 \times 10^{30}$ kg, and the gravitational constant, $G = 6.67 \times 10^{-11}$ N·m^2/kg^2.

4. It is possible to place a satellite into orbit about Earth so that it remains fixed above a given location on the equator. Compute the altitude that is needed for such a satellite. Earth's mass is 5.98×10^{24} kg; its radius is 6.37×10^6 m. (This orbit is called the Clarke Geosynchronous Orbit after Arthur C. Clark, who first proposed the idea in 1945. The first such satellite, *Syncom II*, was launched in July 1963.)

10.5 Parametric Surfaces

In Section 9.6, we looked at surfaces that are graphs of functions of two variables. In this section, we will use vector functions to discuss more general surfaces, called *parametric surfaces*.

Just as we described a space curve by a vector function $\mathbf{r}(t)$ of a single parameter t, we can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v. We suppose that

$$\mathbf{r}(u, v) = x(u, v)\ \mathbf{i} + y(u, v)\ \mathbf{j} + z(u, v)\ \mathbf{k} \tag{1}$$

is a vector-valued function defined on a region D in the uv-plane. So x, y, and z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D.

The set of all points (x, y, z) in $\mathbb{R}^3$ such that

$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v) \tag{2}$$

and (u, v) varies throughout D, is called a **parametric surface** S and the equations in (2) are called **parametric equations** of S. Each choice of u and v gives a point on S; by using all possible values for u and v we produce the surface S. In other words, the surface S is traced out by the tip of the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region D. Figure 10.40 illustrates this idea.

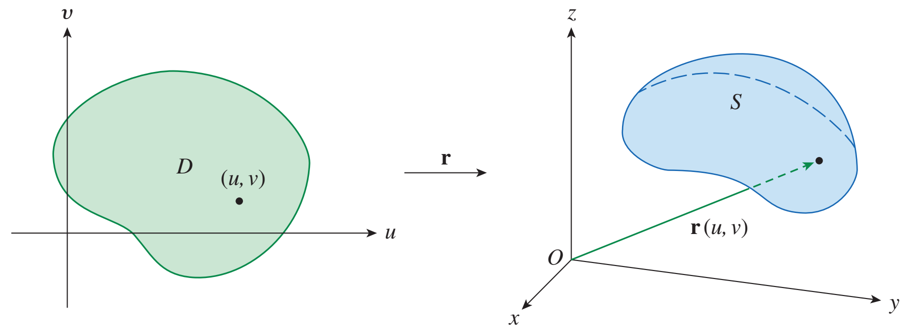


Figure 10.40
The vector-valued function maps points (u, v) in the domain D to points on the parametric surface S.

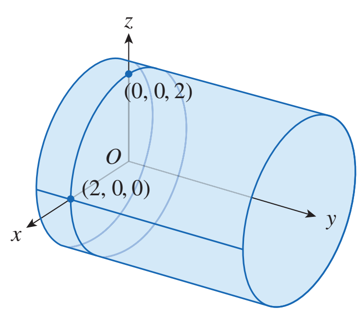


Figure 10.41
Graph of the surface, a circular cylinder with radius 2.

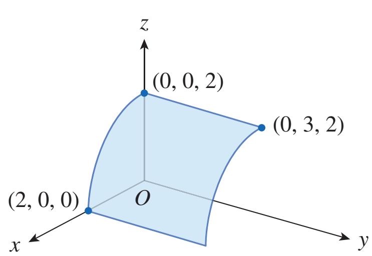


Figure 10.42
Graph of the surface with restrictions on the parameters u and v.

Example 1 Identify and Sketch a Parametric Surface

Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + v\,\mathbf{j} + 2\sin u\,\mathbf{k}$$

Solution

The parametric equations for this surface are

$x = 2\cos u, \quad y = v, \quad z = 2\sin u.$

For any point (x, y, z) on the surface, we have

$x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4.$

Therefore, vertical cross-sections parallel to the xz-plane (that is, with y constant) are circles with radius 2.

Since $y = v$ and no restriction is placed on v, the surface is a circular cylinder with radius 2 whose axis is the y-axis. Figure 10.41 shows a graph of this surface. ■

In Example 1, there were no restrictions on the parameters u and v, so the surface is the entire cylinder. If, for instance, we restrict u nd v by writing the parameter domain as

$$0 \le u \le \frac{\pi}{2} \qquad 0 \le v \le 3$$

then $x \ge 0$, $z \ge 0$, $0 \le y \le 3$, and the resulting surface is a quarter cylinder with length 3 illustrated in Figure 10.42.

Grid Curves

If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$, then there are two useful families of curves that lie on S, one family with u constant and the other with v constant. These families correspond to vertical and horizontal lines in the uv-plane. If we keep u constant by letting $u = u_0$, then $\mathbf{r}(u_0, v)$ is a vector function of the single parameter v and defines a curve C_1 lying on S. Figure 10.43 shows how the line $u = u_0$ is mapped to the curve C_1 on the surface S.

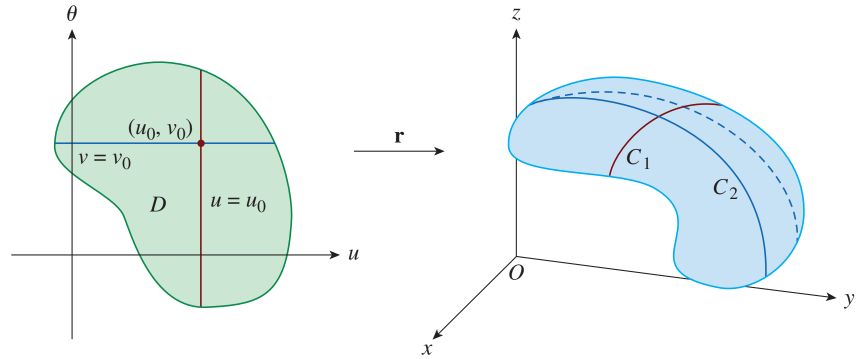


Figure 10.43
The lines $u = u_0$ and $v = v_0$ in the domain D are mapped to the curves C_1 and C_2 on S.

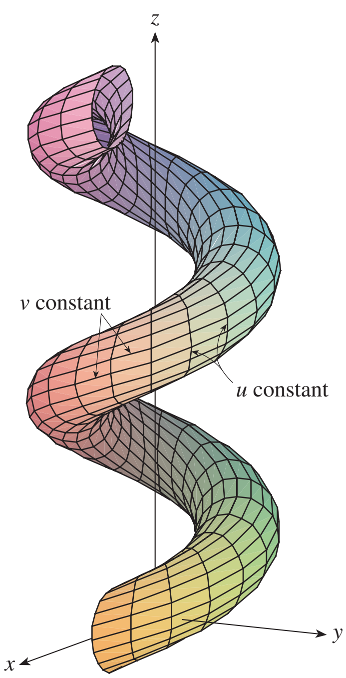


Figure 10.44
If v is held constant, the grid curves are the spiral curves. If u is held constant, the grid curves are the circles.

Similarly, if we keep v constant by letting $v = v_0$, we get the curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S. These curves, C_1 and C_2, are called **grid curves**. In Example 1, the grid curves obtained by letting u be constant are horizontal lines and the grid curves holding v constant are circles. Many graphing calculators and computer software packages display parametric surfaces by plotting grid curves, as illustrated in the next example.

Example 2 Graph a Surface and Identify the Grid Curves

Use technology to graph the surface described by

$$\mathbf{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle$$

Which grid curves have u constant? Which have v constant?

Solution

The graph of a portion of the surface is shown in Figure 10.44, with parameter domain $0 \le u \le 4\pi$, $0 \le v \le 2\pi$. The surface has the appearance of a spiral tube.

To identify the grid curves, consider the corresponding parametric equations.

$$x = (2 + \sin v) \cos u \qquad y = (2 + \sin v) \sin u \qquad z = u + \cos v$$

If v is held constant, then $\sin v$ and $\cos v$ are also constant. Then the parametric equations are similar to those of the helix in Example 4, Section 10.1.

So, the grid curves with v constant are the spiral curves as shown in Figure 10.44.

If u is held constant, then the grid curves must be those that look like circles in the figure. If $u = u_0$ is constant, then the equation $z = u_0 + \cos v$ shows that the z-values vary from $u_0 - 1$ to $u_0 + 1$. ■

Finding Parametric Representations

In Examples 1 and 2, we were given a vector equation and were able to graph the corresponding parametric surface. The following examples illustrate the more challenging problem of finding a vector function to represent a given surface. In later chapters, we will often need to do this.

Example 3 Parametric Equations for a Plane

Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.

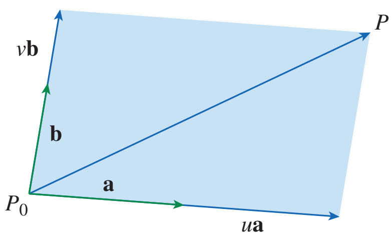


Figure 10.45
There exist scalars u and v such that $\overrightarrow{P_0P} = u\mathbf{a} + v\mathbf{b}$.

Solution

Suppose P is any point in the plane.

We can get from P_0 to P by moving a certain distance in the direction of **a** and another distance in the direction of **b**.

Therefore, there exist scalars u and v such that $\overrightarrow{P_0P} = u\,\mathbf{a} + v\,\mathbf{b}$.

Figure 10.45 shows how this works using the Parallelogram Law, for the case where u and v are positive.

If **r** is the position vector of P, then

$$\mathbf{r} = \overrightarrow{OP_0} + \overrightarrow{P_0P} = \mathbf{r}_0 + u\,\mathbf{a} + v\,\mathbf{b}.$$

Therefore, the vector equation of the plane can be written as

$$\mathbf{r}(u,\ v) = \mathbf{r}_0 + u\,\mathbf{a} + v\,\mathbf{b}, \quad \text{where } u \text{ and } v \text{ are real numbers.}$$

If we write $\mathbf{r} = \langle x,\ y,\ z\rangle$, $\mathbf{r}_0 = \langle x_0,\ y_0,\ z_0\rangle$, $\mathbf{a} = \langle a_1,\ a_2,\ a_3\rangle$, and $\mathbf{b} = \langle b_1,\ b_2,\ b_3\rangle$, then we can write the parametric equations of the plane through the point $(x_0,\ y_0,\ z_0)$ as follows:

$$x = x_0 + ua_1 + vb_1 \qquad y = y_0 + ua_2 + vb_2 \qquad z = z_0 + ua_3 + vb_3$$

Example 4 Parametrizing a Sphere

Find a parametric representation of the sphere described by

$$x^2 + y^2 + z^2 = a^2$$

Solution

In spherical coordinates, the sphere has the representation $\rho = a$.

So, it seems reasonable to choose the angles ϕ and θ in spherical coordinates as the parameters.

Convert the equation $\rho = a$ from spherical to rectangular coordinates (Section 9.7). The parametric equations of the sphere are:

$$x = a\sin\,\phi\,\cos\theta, \quad y = a\sin\,\phi\,\sin\theta, \quad z = a\cos\phi.$$

The corresponding vector equation is

$$\mathbf{r}(\phi,\ \theta) = a\sin\phi\,\cos\theta\ \mathbf{i} + a\sin\phi\,\sin\theta\ \mathbf{j} + a\cos\phi\ \mathbf{k}.$$

The restrictions on the parameters are $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.

Therefore, the parameter domain is the rectangle $D = [0,\ \pi] \times [0,\ 2\pi]$.

The grid curves with ϕ constant are the circles of constant latitude (including the equator).

The grid curves with θ constant are the meridians (semi-circles), which connect the north and south poles. See Figure 10.46.

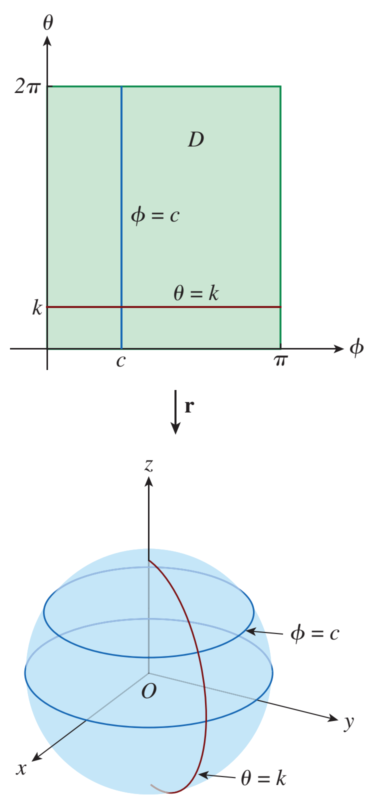


Figure 10.46
The grid curves for ϕ and θ constant.

Example 5 Parametrizing a Cylinder

Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad 0 \le z \le 1$$

Solution

The cylinder is represented by $r = 2$ in cylindrical coordinates.

Therefore, it seems reasonable to choose the parameters θ and z in cylindrical coordinates.

Then the parametric equations of the cylinder are

$x = 2\cos\theta,\quad y = 2\sin\theta,\quad z = z,\quad$ where $0 \le \theta \le 2\pi$ and $0 \le z \le 1$.

Example 6 Vector Function for an Elliptic Paraboloid

Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

Solution

If we use x and y as parameters, then the parametric equations are simply

$$x = x, \qquad y = y, \qquad z = x^2 + 2y^2,$$

and the vector equation is

$\mathbf{r}(x,\ y) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2 + 2y^2)\,\mathbf{k}$.

In general, a surface given as the graph of a function of x and y, that is, with an equation of the form $x = f(x,\ y)$, can always be considered as a parametric surface by using x and y as parameters and writing the parametric equations as

$$x = x \qquad y = y \qquad z = f(x,\ y)$$

Parametric representations (also called parametrizations) of surfaces are not unique. The next example shows two ways to parametrize a cone.

Example 7 Two Ways to Parametrize a Cone

Find a parametric representation for the surface $z = 2\sqrt{x^2 + y^2}$, that is, the top half of the cone described by $z^2 = 4x^2 + 4y^2$.

Solution 1

One straight-forward representation is obtained by choosing x and y as parameters:

$x = x,\quad y = y,\quad z = 2\sqrt{x^2 + y^2}$

Then the vector equation is

$\mathbf{r}(x,\ y) = x\,\mathbf{i} + y\,\mathbf{j} + 2\sqrt{x^2 + y^2}\,\mathbf{k}$.

Solution 2

Another representation can be obtained by using the polar coordinates r and θ as parameters.

A point (x, y, z) on the cone satisfies

$x = r\cos\theta,\ y = r\sin\theta,$ and $z = 2\sqrt{x^2 + y^2} = 2r$.

Therefore, a vector equation for the cone is

$\mathbf{r}(r,\ \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2r\,\mathbf{k} \quad$ where $r \ge 0$ and $0 \le \theta \le 2\pi$.

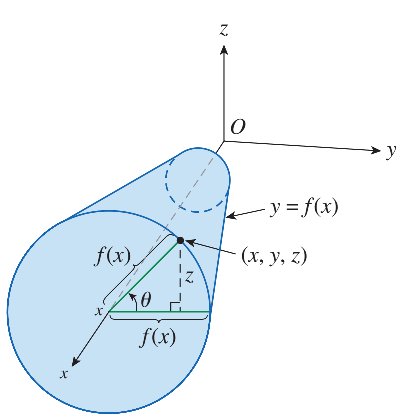


Figure 10.47
Notation to describe a surface of revolution parametrically.

Surfaces of Revolution

Surfaces of revolution can be represented parametrically and therefore we can usually use technology to produce a graph. For example, consider the surface S obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis, where $f(x) \ge 0$. Let θ be the angle of rotation as shown in Figure 10.47. If (x, y, z) is a point on S, then

$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta \tag{3}$$

Therefore, we can choose x and θ as parameters and consider the equations in (3) as parametric equations of S. The parameter domain is given by $a \le x \le b$, $0 \le \theta \le 2\pi$.

Example 8 Graph a Surface of Revolution

Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \le x \le 2\pi$, about the x-axis. Use these equations to graph the surface of revolution.

Solution

Use the equations in (3) to write the parametric equations.

$$x = x, \quad y = \sin x \cos \theta, \quad z = \sin x \sin \theta \quad \text{for } 0 \le x \le 2\pi,\ 0 \le \theta \le 2\pi$$

Figure 10.48 shows a graph of the surface from an advantageous viewpoint. ■

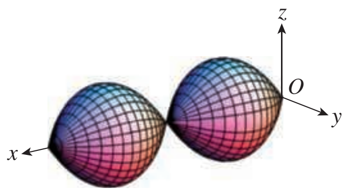


Figure 10.48
Graph of the surface of revolution.

Note that we can adapt the equations in (3) to represent a surface obtained through revolution about the y- or z-axis.

10.5 Exercises

Determine whether the points P and Q lie on the given surface.

1. $\mathbf{r}(u, v) = \langle 2u + 3v,\ 1 + 5u - v,\ 2 + u + v\rangle$
$P(7, 10, 4)$, $Q(5, 22, 5)$

2. $\mathbf{r}(u, v) = \langle u + v,\ u^2 - v,\ u + v^2\rangle$
$P(3, -1, 5)$, $Q(-1, 3, 4)$

3. $\mathbf{r}(u, v) = \langle 1 + u + v,\ u + v^2,\ u^2 - v^2\rangle$
$P(1, 2, 1)$, $Q(2, 3, 3)$

Identify the surface with the given vector equation.

4. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + (3 - v)\,\mathbf{j} + (1 + 4u + 5v)\,\mathbf{k}$

5. $\mathbf{r}(u, v) = 2\sin u\,\mathbf{i} + 3\cos u\,\mathbf{j} + v\,\mathbf{k}, \quad 0 \le v \le 2$

6. $\mathbf{r}(s, t) = \langle s\cos t,\ s\sin t,\ s\rangle$

7. $\mathbf{r}(s, t) = \langle 3\cos t,\ s,\ \sin t\rangle, \quad -1 \le s \le 1$

Use technology to graph the parametric surface. Use a printout of the graph to identify the grid curves that have u constant and those that have v constant.

8. $\mathbf{r}(u, v) = \langle u^2 + 1,\ v^3 + 1,\ u + v\rangle, \quad -1 \le u \le 1,\ -1 \le v \le 1$

9. $\mathbf{r}(u, v) = \langle u + v,\ u^2,\ v^2\rangle, \quad -1 \le u \le 1,\ -1 \le v \le 1$

10. $\mathbf{r}(u, v) = \langle e^u,\ e^{-v},\ \ln(u^2 + v^2)\rangle, \quad -1 \le u \le 1,\ -1 \le v \le 1$

11. $\mathbf{r}(u, v) = \langle u\cos v,\ u\sin v,\ u^5\rangle, \quad -1 \le u \le 1, \quad 0 \le v \le 2\pi$

12. $\mathbf{r}(u, v) = \langle \cos u \sin v,\ \sin u \sin v,\ \cos v + \ln\tan(v/2)\rangle$,
$0 \le u \le 2\pi, \quad 0.1 \le v \le 6.2$

13. $x = \sin v, \quad y = \cos u \sin 4v, \quad z = \sin 2u \sin 4v$,
$0 \le u \le 2\pi, \quad -\dfrac{\pi}{2} \le v \le \dfrac{\pi}{2}$

14. $x = u \sin u \cos v, \quad y = u\cos u \cos v, \quad z = u \sin v$

Match the equations with the graphs labeled I–VI and give reasons for your answers. Determine which families of grid curves have u constant and which have v constant.

15. $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$

16. $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + \sin u\,\mathbf{k}, \quad -\pi \le u \le \pi$

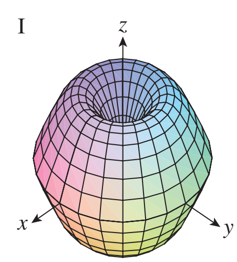
17. $\mathbf{r}(u, v) = \sin v\,\mathbf{i} + \cos u \sin 2v\,\mathbf{j} + \sin u \sin 2v\,\mathbf{k}$

18. $x = (1 - u)(3 + \cos v)\cos 4\pi u$,
$y = (1 - u)(3 + \cos v)\sin 4\pi u$,
$z = 3u + (1 - u)\sin v$

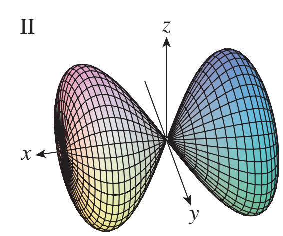
19. $x = \cos^3 u \cos^3 v, \quad y = \sin^3 u \cos^3 v, \quad z = \sin^3 v$

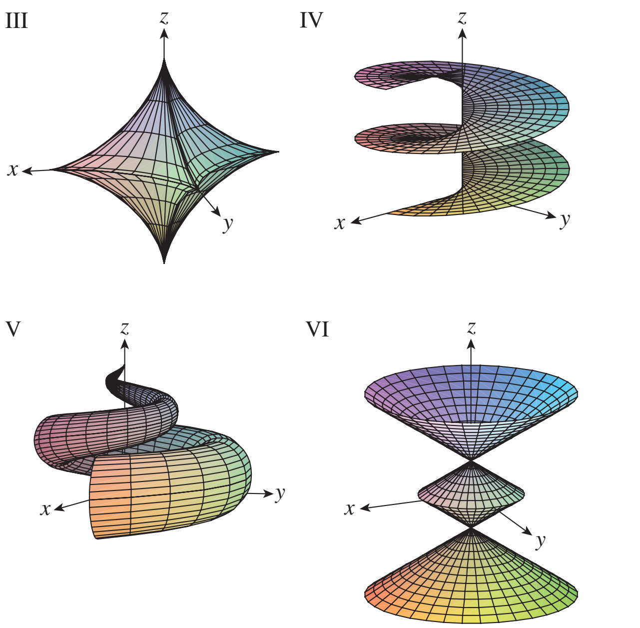
20. $x = (1 - |u|)\cos v, \quad y = (1 - |u|)\sin v, \quad z = u$

I



II





Find a parametric representation for the surface.

21. The plane that passes through the point $(1, 2, -3)$ and contains the vectors $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$

22. The lower half of the ellipsoid $2x^2 + 4y^2 + z^2 = 1$

23. The part of the hyperboloid $x^2 + y^2 - z^2 = 1$ that lies to the right of the xz-plane

24. The part of the elliptic paraboloid $x + y^2 + 2z^2 = 4$ that lies in front of the plane $x = 0$

25. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

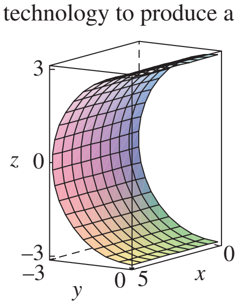
26. The part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = -2$ and $z = 2$

27. The part of the cylinder $y^2 + z^2 = 16$ that lies between the planes $x = 0$ and $x = 5$

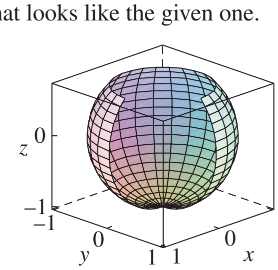
28. The part of the plane $z = x + 3$ that lies inside the cylinder $x^2 + y^2 = 1$

Use technology to produce a graph that looks like the given one.

29.



30.



31. Find parametric equations for the surface obtained by rotating the curve $y = e^{-x}$, $0 \le x \le 3$, about the x-axis and use them to graph the surface.

32. Find parametric equations for the surface obtained by rotating the curve $x = 4y^2 - y^4$, $-2 \le y \le 2$, about the y-axis and use them to graph the surface.

33. (a) Show that the parametric equations $x = a \sin u \cos v$, $y = b \sin u \sin v$, $z = c \cos u$, $0 \le u \le \pi$, $0 \le v \le 2\pi$, represent an ellipsoid.

(b) Use the parametric equations in part (a) to graph the ellipsoid for the case $a = 1$, $b = 2$, $c = 3$.

34. The surface with parametric equations

$$x = 2\cos\theta + r\cos\left(\frac{\theta}{2}\right)$$

$$y = 2\sin\theta + r\cos\left(\frac{\theta}{2}\right)$$

$$z = r\sin\left(\frac{\theta}{2}\right)$$

where $-\frac{1}{2} \le r \le \frac{1}{2}$ and $0 \le \theta \le 2\pi$, is called a **Möbius strip**.

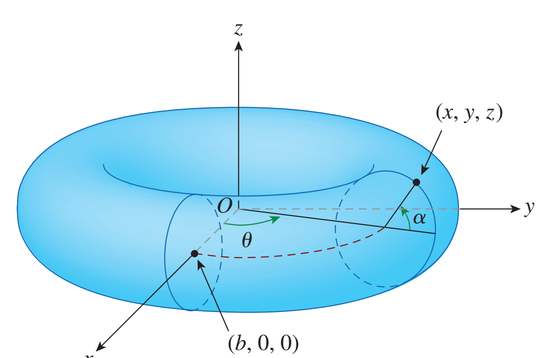
Graph this surface with several viewpoints. Explain any unusual characteristics of this surface.

35. (a) What happens to the spiral tube in Example 2 (see Figure 10.44) if we replace $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$?

(b) What happens if we replace $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$?

36. (a) Find a parametric representation for the torus obtained by rotating about the z-axis the circle in the xz-plane with center $(b, 0, 0)$ and radius $a < b$. [Hint: Take as parameters the angles θ and α as shown in the figure.]

(b) Use the parametric equations found in part (a) to graph the torus for several values of a and b.



10 REVIEW

Concepts and Vocabulary

1. What is a vector function? Explain how to find the derivative and integral of a vector function.
2. Explain the connection between vector functions and space curves.
3. How do you find the tangent vector to a smooth curve at a point? How do you find an equation of the tangent line? Explain how to find the unit tangent vector.
4. If $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function, write the rules for differentiating each vector function.
   (a) $\mathbf{u}(t) + \mathbf{v}(t)$ (b) $c\mathbf{u}(t)$ (c) $f(t)\mathbf{u}(t)$
   (d) $\mathbf{u}(t) \cdot \mathbf{v}(t)$ (e) $\mathbf{u}(t) \times \mathbf{v}(t)$ (f) $\mathbf{u}(f(t))$
5. How do you find the length of a space curve given by a vector function $\mathbf{r}(t)$?
6. (a) What is the definition of curvature?
   (b) Write a formula for curvature in terms of $\mathbf{r}'(t)$ and $\mathbf{T}'(t)$.
   (c) Write a formula for curvature in terms of $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.
   (d) Write a formula for the curvature of a plane curve with equation $y = f(x)$.
7. (a) Write formulas for the unit normal and binormal vectors of a smooth space curve $\mathbf{r}(t)$.
   (b) What is the normal plane of a curve at a point? What is the osculating plane? What is the osculating circle?
8. (a) How do you find the velocity, speed, and acceleration of a particle that moves along a space curve?
   (b) Write the acceleration in terms of its tangential and normal components.
9. State Kepler's Laws.
10. What is a parametric surface? Explain the relationship between its grid curves and the domain.

True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. The curve with vector equation $\mathbf{r}(t) = t^3\,\mathbf{i} + 2t^3\,\mathbf{j} + 3t^3\,\mathbf{k}$ is a line.
2. The curve described by the vector function $\mathbf{r}(t) = \langle 0, t^2, 4t \rangle$ is a parabola.
3. The curve described by the vector function $\mathbf{r}(t) = \langle 2t, 3 - t, 0 \rangle$ is a line that passes through the origin.
4. The derivative of a vector function is obtained by differentiating each component function.
5. If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable vector functions, then
$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}'(t)$$
6. If $\mathbf{r}(t)$ is a differentiable function, then
$$\frac{d}{dt}|\mathbf{r}(t)| = |\mathbf{r}'(t)|$$
7. If $\mathbf{T}(t)$ is the unit tangent vector of a smooth curve, then the curvature is $\kappa = \left|\dfrac{d\mathbf{T}}{dt}\right|$.
8. The binormal vector is $\mathbf{b}(t) = \mathbf{N}(t) \times \mathbf{T}(t)$.
9. Suppose f is twice continuously differentiable. At an inflection point on the graph of $y = f(x)$, the curvature is 0.
10. If $\kappa(t) = 0$ for all t, then the curve is a straight line.
11. If $|\mathbf{r}(t)| = 1$ for all t, then $|\mathbf{r}'(t)|$ is a constant.
12. If $|\mathbf{r}(t)| = 1$ for all t, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.
13. The osculating circle of a curve C at a point has the same tangent vector, normal vector, and curvature as C at that point.
14. Different parametrizations of the same curve result in identical tangent vectors at a given point on the curve.

Exercises

1. (a) Sketch the curve with vector function

$$\mathbf{r}(t) = t\,\mathbf{i} + \cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k} \quad t \geq 0$$

(b) Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

2. Let $\mathbf{r}(t) = \left\langle \sqrt{2-t}, \dfrac{e^t - 1}{t}, \ln(t+1) \right\rangle$.

(a) Find the domain of $\mathbf{r}$.
(b) Find $\lim_{t \to 0} \mathbf{r}(t)$.
(c) Find $\mathbf{r}'(t)$.

3. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 16$ and the plane $x + z = 5$.

4. Find parametric equations for the tangent line to the curve $x = 2 \sin t$, $y = 2 \sin 2t$, $z = 2 \sin 3t$ at the point $(1, \sqrt{3}, 2)$. Graph the curve and the tangent line on the same set of coordinate axes.

5. If $\mathbf{r}(t) = t^2\,\mathbf{i} + t \cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k}$, evaluate $\int_0^1 \mathbf{r}(t)\,dt$.

6. Let C be the curve with equations $x = 2 - t^3$, $y = 2t - 1$, $z = \ln t$. Find (a) the point where C intersects the xz-plane, (b) parametric equations of the tangent line at $(1, 1, 0)$, and (c) an equation of the normal plane to C at $(1, 1, 0)$.

7. Use Simpson's Rule with $n = 6$ to estimate the length of the arc of the curve with equations $x = t^2$, $y = t^3$, $z = t^4$, $0 \leq t \leq 3$.

8. Find the length of the curve $\mathbf{r}(t) = \langle 2t^{3/2}, \cos 2t, \sin 2t \rangle$, $0 \leq t \leq 1$.

9. The helix $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ intersects the curve $\mathbf{r}_2(t) = (1 + t)\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$ at the point $(1, 0, 0)$. Find the angle of intersection of these curves.

10. Reparametrize the curve $\mathbf{r}(t) = e^t\,\mathbf{i} + e^t \sin t\,\mathbf{j} + e^t \cos t\,\mathbf{k}$ with respect to arc length measured from the point $(1, 0, 1)$ in the direction of increasing t.

11. For the curve described by the vector function $\mathbf{r}(t) = \langle \sin^3 t, \cos^3 t, \sin^2 t \rangle$, find

(a) the unit tangent vector,
(b) the unit normal vector,
(c) the unit binormal vector, and
(d) the curvature.

12. Find the curvature of the ellipse $x = 3 \cos t$, $y = 4 \sin t$ at the points $(3, 0)$ and $(0, 4)$.

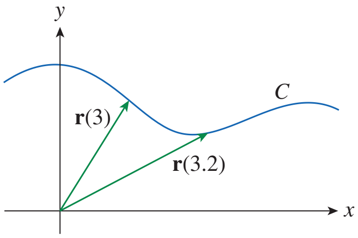
13. Find the curvature of the curve $y = x^4$ at the point $(1, 1)$.

14. Find an equation of the osculating circle of the curve $y = x^4 - x^2$ at the origin. Graph both the curve and its osculating circle.

15. Find an equation of the osculating plane of the curve $x = \sin 2t$, $y = t$, $z = \cos 2t$ at the point $(0, \pi, 1)$.

16. The figure shows the curve C traced by a particle with position vector $\mathbf{r}(t)$ at time t.

(a) Draw a vector that represents the average velocity of the particle over the time interval $3 \leq t \leq 3.2$.
(b) Write an expression for the velocity $\mathbf{v}(3)$.
(c) Write an expression for the unit tangent vector $\mathbf{T}(3)$ and draw it.



17. A particle moves with position function $\mathbf{r}(t) = t \ln t\,\mathbf{i} + t\,\mathbf{j} + e^{-t}\,\mathbf{k}$. Find the velocity, speed, and acceleration of the particle.

18. Find the velocity, speed, and acceleration of a particle moving with position function $\mathbf{r}(t) = (2t^2 - 3)\,\mathbf{i} + 2t\,\mathbf{j}$. Sketch the path of the particle and draw the position, velocity, and acceleration vectors for $t = 1$.

19. A particle starts at the origin with initial velocity $\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 6t\,\mathbf{i} + 12t^2\,\mathbf{j} - 6t\,\mathbf{k}$. Find its position function.

20. An athlete throws a shot at an angle of 45° to the horizontal at an initial speed of 43 ft/s. It leaves their hand 7 ft above the ground.

(a) Where is the shot 2 seconds later?
(b) How high does the shot go?
(c) Where does the shot land?

21. A projectile is launched with an initial speed of 40 m/s from the floor of a tunnel whose height is 30 m. What angle of elevation should be used to achieve the maximum possible horizontal range of the projectile? What is the maximum range?

22. Find the tangential and normal components of the acceleration vector of a particle with position function

$$\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}$$

23. Find a parametric representation for the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies between the planes $z = 1$ and $z = -1$.

24. Use technology to graph the surface with vector equation

$$\mathbf{r}(u, v) = \langle (1 - \cos u) \sin v, u, (u - \sin u) \cos v \rangle$$

Consider several viewpoints and indicate which grid curves have u constant and which have v constant.

25. Find the curvature of the curve with parametric equations

$$x = \int_0^t \sin\left(\frac{1}{2}\pi\theta^2\right) d\theta \quad y = \int_0^t \cos\left(\frac{1}{2}\pi\theta^2\right) d\theta$$

26. A disk of radius 1 is rotating in the counterclockwise direction at a constant angular speed ω. A particle starts at the center of the disk and moves toward the edge along a fixed radius so that its position at time $t \geq 0$ is given by $\mathbf{r}(t) = t\mathbf{R}(t)$, where

$$\mathbf{R}(t) = \cos \omega t \; \mathbf{i} + \sin \omega t \; \mathbf{j}$$

(a) Show that the velocity $\mathbf{v}$ of the particle is

$$\mathbf{v} = \cos \omega t \; \mathbf{i} + \sin \omega t \; \mathbf{j} + t \; \mathbf{v}_d$$

where $\mathbf{v}_d = \mathbf{R}'(t)$ is the velocity of a point on the edge of the disk.

(b) Show that the acceleration $\mathbf{a}$ of the particle is

$$\mathbf{a} = 2\mathbf{v}_d + t \; \mathbf{a}_d$$

where $\mathbf{a}_d = \mathbf{R}''(t)$ is the acceleration of a point on the edge of the disk.

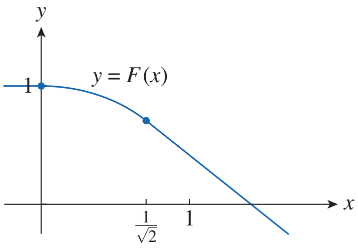
The extra term $2\mathbf{v}_d$ is called the *Coriolis acceleration*; it is the result of the interaction of the rotation of the disk and the motion of the particle. One can obtain a physical demonstration of this acceleration by walking toward the edge of a moving carousel.

(c) Determine the Coriolis acceleration of a particle that moves on a rotating disk according to the equation

$$\mathbf{r}(t) = e^{-t} \cos \omega t \; \mathbf{i} + e^{-t} \sin \omega t \; \mathbf{j}$$

27. In designing *transfer curves* to connect sections of straight railroad tracks, it's important to realize that the acceleration of the train should be continuous so that the reactive force exerted by the train on the track is also continuous. Because of the formulas for the components of acceleration in Section 10.4, this will be the case if the curvature varies continuously.

(a) A logical candidate for a transfer curve to join existing tracks given by $y = 1$ for $x \leq 0$ and $y = \sqrt{2} - x$ for $x \geq \frac{1}{\sqrt{2}}$ might be the function $f(x) = \sqrt{1 - x^2}$, $0 < x < \frac{1}{\sqrt{2}}$, whose graph is the arc of the circle shown in the figure.

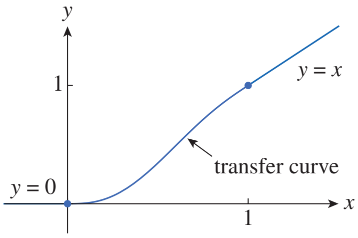


It looks reasonable at first glance. Show that the function

$$F(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ \sqrt{1 - x^2} & \text{if } 0 < x < 1/\sqrt{2} \\ \sqrt{2} - x & \text{if } x \geq 1/\sqrt{2} \end{cases}$$

is continuous and has continuous slope, but does not have continuous curvature. Therefore, f is not an appropriate transfer curve.

(b) Find a fifth-degree polynomial to serve as a transfer curve between the following straight line segments: $y = 0$ for $x \leq 0$ and $y = x$ for $x \geq 1$. Could this be done with a fourth-degree polynomial? Use technology to sketch the graph of the *connected* function and check to see that it looks like the one in the figure.



Focus on Problem Solving

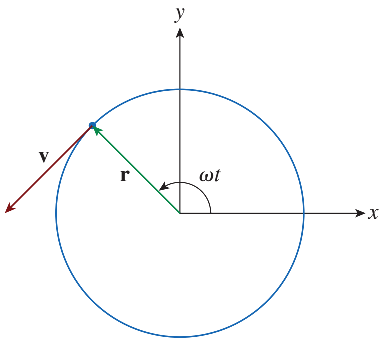


Figure 10.49
The particle moves around a circle centered at the origin with radius R.

1. A particle P moves with constant angular speed ω around a circle whose center is at the origin and whose radius is R. The particle is said to be in *uniform circular motion.* Assume that the motion is counterclockwise and that the particle is at the point $(R, 0)$ when $t = 0$. The position vector at time $t \geq 0$ is $\mathbf{r}(t) = R\cos\omega t\ \mathbf{i} + R\sin\omega t\ \mathbf{j}$. See Figure 10.49.

(a) Find the velocity vector $\mathbf{v}$ and show that $\mathbf{v} \cdot \mathbf{r} = 0$. Conclude that $\mathbf{v}$ is tangent to the circle and points in the direction of the motion.

(b) Show that the speed $|\mathbf{v}|$ of the particle is the constant ωR. The *period* T of the particle is the time required for one complete revolution. Conclude that

$$T = \frac{2\pi R}{|\mathbf{v}|} = \frac{2\pi}{\omega}$$

(c) Find the acceleration vector $\mathbf{a}$. Show that it is proportional to $\mathbf{r}$ and that it points toward the origin. An acceleration with this property is called *centripetal acceleration.* Show that the magnitude of the acceleration vector is $|\mathbf{a}| = R\omega^2$.

(d) Suppose that the particle has mass m. Show that the magnitude of the force $\mathbf{F}$ that is required to produce this motion, called a *centripetal force*, is

$$|\mathbf{F}| = \frac{m|\mathbf{v}|^2}{R}$$

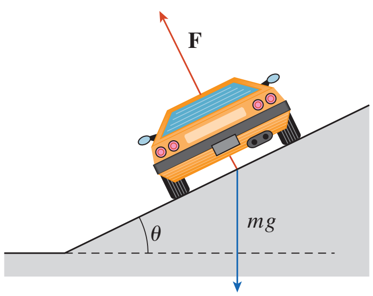


Figure 10.50
A highway banked at an angle θ.

2. A circular curve of radius R on a highway is banked at an angle θ so that a car can safely traverse the curve without skidding when there is no friction between the road and the tires. The loss of friction could occur, for example, if the road is covered with a film of water or ice. The rated speed v_R of the curve is the maximum speed that a car can attain without skidding. Suppose a car of mass m is traversing the curve at the rated speed v_R. Two forces are acting on the car: the vertical force, mg, due to the weight of the car, and a force $\mathbf{F}$ exerted by, and normal to, the road (see Figure 10.50).

The vertical component of $\mathbf{F}$ balances the weight of the car, so that $|\mathbf{F}|\cos\theta = mg$.

The horizontal component of $\mathbf{F}$ produces a centripetal force on the car so that, by Newton's Second Law and part (d) of Problem 1,

$$|\mathbf{F}|\sin\theta = \frac{mv_R^2}{R}$$

(a) Show that $v_R^2 = Rg\tan\theta$.

(b) Find the rated speed of a circular curve with radius 400 ft that is banked at an angle of 12°.

(c) Suppose the design engineers want to keep the banking at 12°, but wish to increase the rated speed by 50%. What should the radius of the curve be?

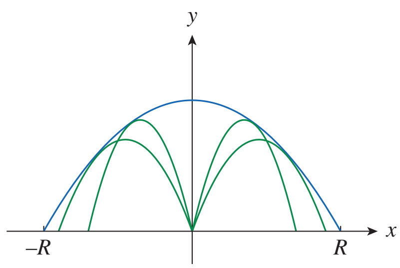


Figure 10.51
The projectile can hit any target inside or on the boundary of the region bounded by the parabola and the x-axis.

3. A projectile is fired from the origin with angle of elevation α and initial speed v_0. Assuming that air resistance is negligible and that the only force acting on the projectile is gravity, g, we showed in Example 10.4.5 that the position vector of the projectile is

$$\mathbf{r}(t) = (v_0\cos\alpha)t\ \mathbf{i} + \left[(v_0\sin\alpha)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

We also showed that the maximum horizontal distance of the projectile is achieved when $\alpha = 45°$ and in this case the range is $R = \dfrac{v_0^2}{g}$.

(a) At what angle should the projectile be fired to achieve maximum height and what is the maximum height?

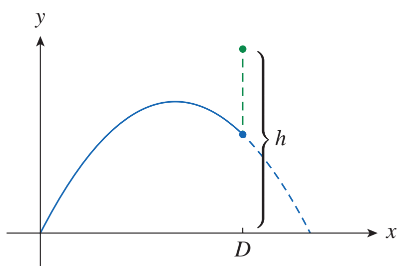


Figure 10.52
The projectile always hits the target.

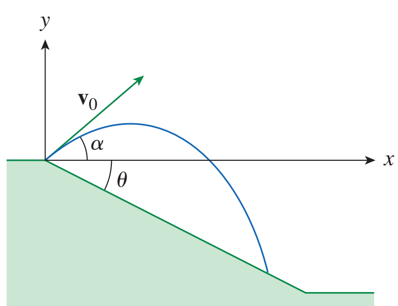


Figure 10.53
A projectile fired down an inclined plane.

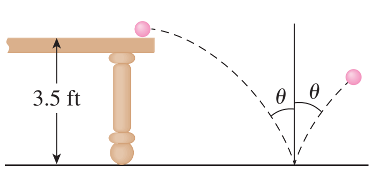


Figure 10.54
A ball rolling off a table.

(b) Fix the initial speed v_0 and consider the parabola $x^2 + 2Ry - R^2$, whose graph is shown in Figure 10.51. Show that the projectile can hit any target inside or on the boundary of the region bounded by the parabola and the x-axis, and it can't hit any target outside this region.

(c) Suppose that the gun is elevated to an angle of inclination α in order to aim at a target that is suspended at a height h directly over a point D units downrange (see Figure 10.52). The target is released at the instant the gun is fired. Show that the projectile always hits the target, regardless of the value v_0, provided the projectile does not hit the ground *before* D.

4. (a) A projectile is fired from the origin down an inclined plane that makes an angle θ with the horizontal. The angle of elevation of the gun and the initial speed of the projectile are α and v_0, respectively. Find the position vector of the projectile and the parametric equations of the path of the projectile as functions of the time t. See Figure 10.53. (Ignore air resistance.)

(b) Show that the angle of elevation α that will maximize the downhill range is the angle halfway between the plane and the vertical.

(c) Suppose the projectile is fired up an inclined plane whose angle of inclination is θ. Show that, in order to maximize the (uphill) range, the projectile should be fired in the direction halfway between the plane and the vertical.

(d) In a paper presented in 1686, Edmond Halley summarized the laws of gravity and projectile motion and applied them to gunnery. One problem he posed involved firing a projectile to hit a target a distance R up an inclined plane. Show that the angle at which the projectile should be fired to hit the target but use the least amount of energy is the same as the angle in part (c). (Use the fact that the energy needed to fire the projectile is proportional to the square of the initial speed, so minimizing the energy is equivalent to minimizing the initial speed.)

5. A ball rolls off a table with a speed of 2 ft/s. The table is 3.5 ft high. See Figure 10.54.

(a) Determine the point at which the ball hits the floor and find its speed at the instant of impact.

(b) Find the angle θ between the path of the ball and the vertical line drawn through the point of impact as shown in Figure 10.54.

(c) Suppose the ball rebounds from the floor at the same angle with which it hits the floor, but loses 20% of its speed due to energy absorbed by the ball on impact. Where does the ball strike the floor on the second bounce?

6. Investigate the shape of the surface with parametric equations

$$x = \sin u \qquad y = \sin v \qquad z = \sin(u + v)$$

Start by graphing the surface from several viewpoints. Explain the appearance of the graphs by determining the traces in the horizontal planes $z = 0$, $z = \pm 1$, and $z = \pm\frac{1}{2}$.

7. If a projectile is fired with angle of elevation α and initial speed v, then parametric equations for its trajectory are

$$x = (v \cos \alpha)t \qquad y = (v \sin \alpha)t - \frac{1}{2}gt^2$$

(See Example 10.4.5.) We know that the range (horizontal distance traveled) is maximized when $\alpha = 45°$. What value of α maximizes the total distance traveled by the projectile?

8. A cable has radius r and length L and is wound around a spool with radius R without overlapping. What is the shortest length along the spool that is covered by the cable?

9. Show that the curve with vector equation

$$\mathbf{r}(t) = \langle a_1t^2 + b_1t + c_1,\ a_2t^2 + b_2t + c_2,\ a_3t^2 + b_3t + c_3 \rangle$$

lies in a plane and find an equation of the plane.

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A function of two variables can be used to describe the shape of a surface like the one formed by this iceberg in Antarctica. We can then use this function to estimate the volume of the iceberg, the maximum height of the iceberg, and even find a point on the iceberg where the slope is the steepest.

11 Partial Derivatives

The majority of functions that we have studied so far have been functions of a single variable. However, in real-world applications, physical quantities often depend upon two or more variables. Therefore, in this chapter, we will study functions of several variables and extend the basic ideas of differential calculus to these functions.

Contents

11.1 Functions of Several Variables

In Section 9.6, we studied functions of two variables and their graphs. In this chapter, we will consider functions of two or more variables from four points of view:

- verbally (by a description in words)
- numerically (by a table of values)
- algebraically (by an explicit formula)
- visually (by a graph or level curves)

Functions of Two Variables

Recall that a function f of two variables is a rule that assigns to each ordered pair (x, y) of real numbers in its domain a unique real number denoted by $f(x, y)$. In Example 9.6.3, we considered wave heights h in the open sea as a function of the wind speed v and the length of time t the wind has been blowing at that speed. We looked at a table of observed wave heights that represent the function $h = f(v, t)$ numerically. The function in the next example is also described verbally and numerically.

Example 1 Wind Chill as a Function of Wind Speed and Temperature

In regions with severe winter weather, the *wind-chill index* is often used to describe the apparent severity of the cold. This index W is a subjective temperature that depends on the wind speed v and the actual temperature T. So, W is a function of v and T, and we can write $W = f(v, T)$. Table 11.1 shows values of W compiled by the US National Weather Service.

Temperature (°F)

Wind speed (mph)

v \ T	5	0	−5	−10	−15	−20	−25	−30	−35	−40
5	−5	−11	−16	−22	−28	−34	−40	−46	−52	−57
10	−10	−16	−22	−28	−35	−41	−47	−53	−59	−66
15	−13	−19	−26	−32	−39	−45	−51	−58	−64	−71
20	−15	−22	−29	−35	−42	−48	−55	−61	−58	−74
25	−17	−24	−31	−37	−44	−51	−58	−64	−71	−78
30	−19	−26	−33	−39	−46	−53	−60	−67	−73	−80
35	−21	−27	−34	−41	−48	−55	−62	−69	−76	−82
40	−22	−29	−36	−43	−50	−57	−64	−71	−78	−84
45	−23	−30	−37	−44	−51	−58	−65	−72	−79	−86
50	−24	−31	−38	−45	−52	−60	−67	−74	−81	−88
55	−25	−32	−39	−46	−54	−61	−68	−75	−82	−89
60	−26	−33	−40	−48	−55	−62	−69	−76	−84	−91

Table 11.1
Wind-chill index as a function of wind speed and air temperature.

For instance, the table shows that if the wind speed is 20 mph and the temperature is -10°F, then subjectively it would feel as cold as a temperature of about -35°F with no wind.

Therefore, we can write $f(20, -10) = -35$. ■

Example 2 Cobb–Douglas Production Function

In 1928, Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899–1922. They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested. While there are many other factors affecting economic performance, their model proved to be remarkably accurate. The function they used to model production was of the form

$$P(L, K) = bL^{\alpha}K^{1-\alpha} \tag{1}$$

where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings). In Section 11.3, we'll see how the form of Equation 1 follows from certain economic assumptions.

Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

Table 11.2
Economic data used by Cobb and Douglas.

Cobb and Douglas used economic data published by the government to obtain Table 11.2. They took the year 1899 as a baseline and P, L, and K for 1899 were each assigned the value 100. The values for other years were expressed as percentages of the 1899 figures.

Cobb and Douglas used the method of least squares to fit the data of Table 11.2 to the function

$$P(L, K) = 1.01L^{0.75}K^{0.25} \tag{2}$$

If we use the model given by the function in Equation 2 to compute the production in the years 1910 and 1920, we get the values

$$P(147, 208) = 1.01(147)^{0.75}(208)^{0.25} = 161.929$$
$$P(194, 407) = 1.01(194)^{0.75}(407)^{0.25} = 235.815$$

which are pretty close to the actual values, 159 and 231.

The production function in Equation 1 has subsequently been used in many settings, including individual firms and global economics. It has become known as the **Cobb–Douglas production function**. ■

The domain of the production function in Example 2 is $\{(L, K) \mid L \geq 0, K \geq 0\}$ because L and K represent labor and capital percentages of 1899 figures and are therefore never negative. For a function of two variables f given by an algebraic formula, if the domain is not explicitly specified, then recall that the domain consists of all pairs (x, y) for which the expression $f(x, y)$ is a well-defined real number.

Example 3 Domain and Range

Find the domain and range of the function g defined by $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution

For $g(x, y)$ to be a well-defined real number, the radicand must be nonnegative.

Therefore, the domain of g is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geq 0\} = \{(x, y) \mid x^2 + y^2 \leq 9\}.$$

This is a disk with center (0, 0) and radius 3. Figure 11.1 illustrates the domain.

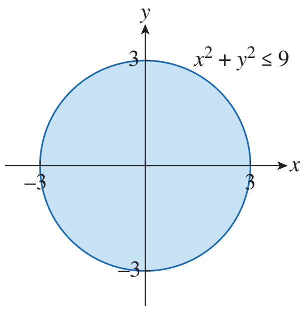


Figure 11.1
The domain of the function g.

The range of g is $\{z \mid z = \sqrt{9 - x^2 - y^2},\ (x, y) \in D\}$.

Since z is a positive square root, $z \geq 0$. Also,

$9 - x^2 - y^2 \le 9 \quad \Rightarrow \quad \sqrt{9 - x^2 - y^2} \le 3.$

So, the range is $\{z \mid 0 \leq z \leq 3\} = [0, 3]$. ■

Visual Representations

One way to visualize a function of two variables is through its graph. Recall from Section 9.6 that the graph of f is the surface with equation $z = f(x, y)$.

Example 4 Graph of a Function

Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution

The graph has the equation $z = \sqrt{9 - x^2 - y^2}$.

Square both sides of this equation: $z^2 = 9 - x^2 - y^2$ or $x^2 + y^2 + z^2 = 9$.

This is the equation of a sphere with center at the origin and radius 3.

But, since $z \geq 0$, the graph of g is the top half of this sphere, as illustrated in Figure 11.2. ■

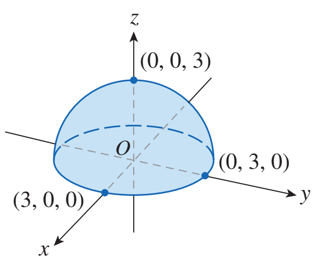


Figure 11.2
Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Note: Since a function returns a unique value for every set of input pairs, an entire sphere cannot be represented by a single function of x and y. However, as in Example 4, the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$ is represented by the function $g(x, y) = \sqrt{9 - x^2 - y^2}$. The lower hemisphere is represented by the function $h(x, y) = -\sqrt{9 - x^2 - y^2}$.

Example 5 Cobb–Douglas Function Graph

Use technology to sketch the graph of the Cobb–Douglas production function $P(L, K) = 1.01L^{0.75}K^{0.25}$.

Solution

Figure 11.3 shows the graph of P for values of the labor L and capital K that lie between 0 and 300.

The grid curves shown are vertical traces on the surface. These curves indicate that the value of the production P increases as either L or K increases, as seems reasonable.

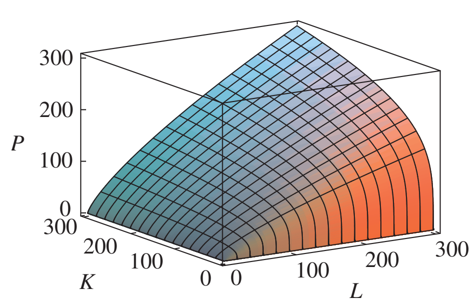


Figure 11.3
The graph of P as a function of labor L and capital K.

■

Another method for visualizing functions, similar to a method used on certain maps, is a contour plot on which points of constant elevation are joined to form *contour lines,* or *level curves.*

Definition • Level Curves

The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A Closer Look

1. A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k. Therefore, a level curve shows where the graph of f has height k.
2. Figure 11.4 illustrates the relationship between level curves and horizontal traces. The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane.

   Therefore, we can piece together a picture of the graph by drawing the level curves of a function and visualize them being lifted up to the surface at the indicated height.

   The surface is steep where the level curves are close together, and relatively flatter where they are farther apart.

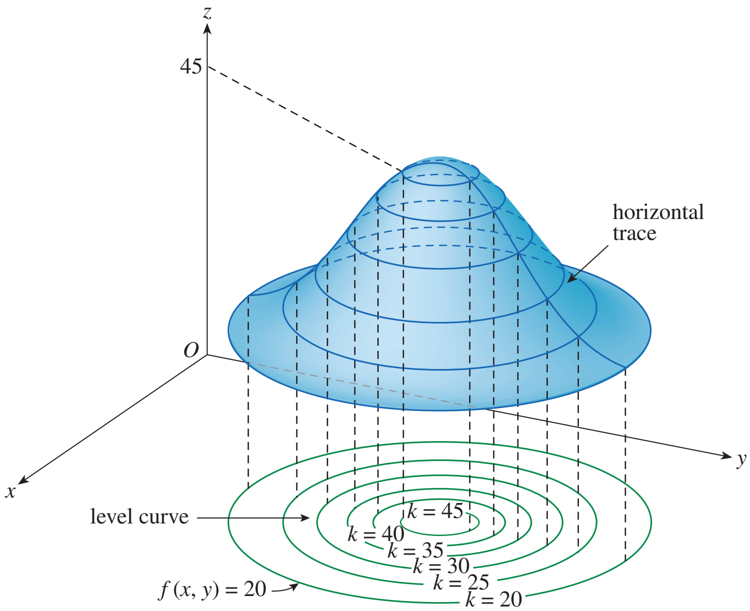


Figure 11.4
The relationship between level curves and horizontal traces.

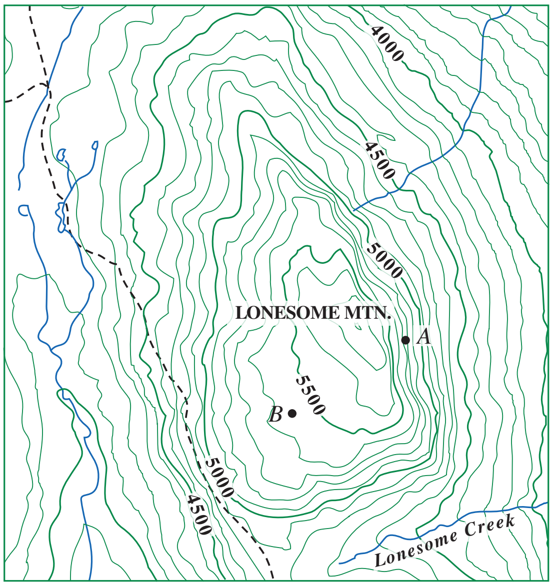


Figure 11.5
A topographic map.

One common example of level curves occurs in topographic maps of mountainous regions, as illustrated in Figure 11.5. The level curves represent points of constant elevation above sea level. If you were to walk along one of these contour lines, you would neither ascend nor descend.

Another common example is the temperature at locations (x, y) with longitude x and latitude y. Here the level curves are called **isothermals** and join locations with the same temperature. Figure 11.6 shows a weather map of the world indicating the average temperature in July. The isothermals are the curves that separate the colored bands.

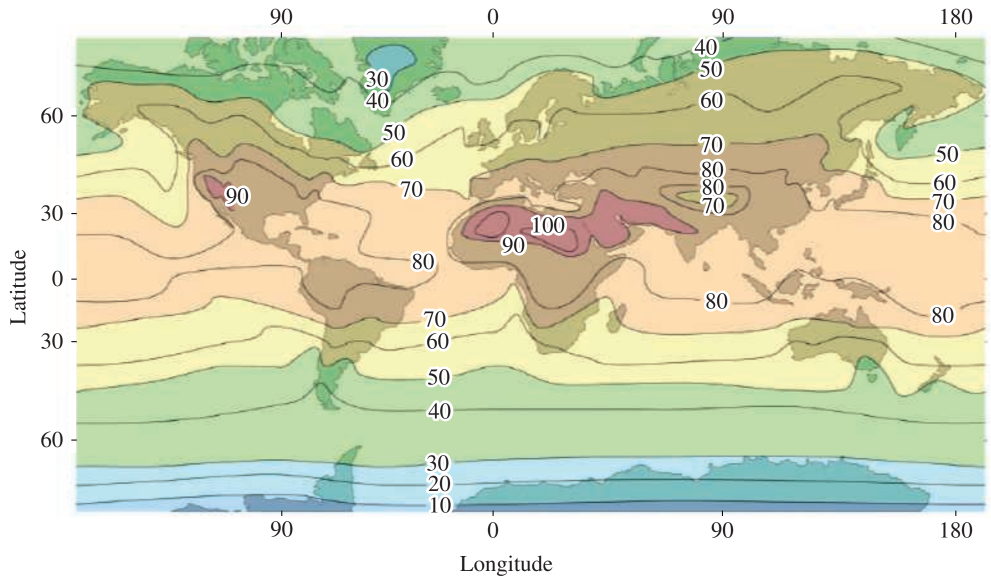


Figure 11.6
The level curves join locations with the same temperature.

In weather maps of atmospheric pressure at a given time as a function of longitude and latitude, the level curves are called **isobars**. They join locations with the same pressure (see Exercise 13). Surface winds tend to flow from areas of high pressure across the isobars toward areas of low pressure, and are strongest where the isobars are tightly packed.

Example 6 Using Level Curves to Estimate Function Values

A contour map for a function f is shown in Figure 11.7. Use it to estimate the values of $f(1, 3)$ and $f(4, 5)$.

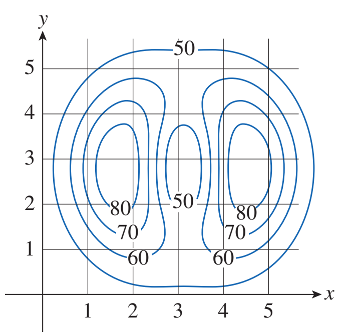


Figure 11.7
Contour map of a function f.

Solution

The point $(1, 3)$ lies partway between the level curves with z-values 70 and 80.

A reasonable estimate is $f(1, 3) \approx 73$.

Similarly, $f(4, 5) \approx 56$. ■

Example 7 Draw a Contour Map

Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

Solution

The level curves are

$$6 - 3x - 2y = k \quad \text{or} \quad 3x + 2y + (k - 6) = 0.$$

This is a family of lines with slope $-\dfrac{3}{2}$.

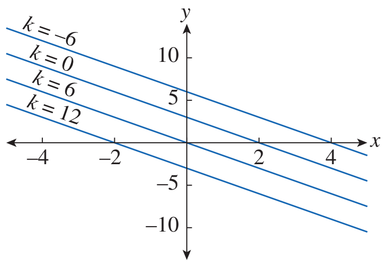


Figure 11.8
A contour map of $f(x, y) = 6 - 3x - 2y$ showing four level curves.

The four particular level curves with $k = -6, 0, 6$, and 12 are

$$3x + 2y - 12 = 0, \quad 3x + 2y - 6 = 0, \quad 3x + 2y = 0, \quad \text{and} \quad 3x + 2y + 6 = 0.$$

These level curves are sketched in Figure 11.8. They are equally spaced parallel lines because the graph of f is a plane.

Example 8 Level Curves for a Hemisphere

Sketch the level curves of the function

$$g(x, y) = \sqrt{9 - x^2 - y^2} \quad \text{for} \quad k = 0, 1, 2, 3$$

Solution

The level curves are

$$\sqrt{9 - x^2 - y^2} = k \quad \text{or} \quad x^2 + y^2 = 9 - k^2.$$

This is a family of concentric circles with center (0, 0) and radius $\sqrt{9 - k^2}$.

The cases $k = 0, 1, 2, 3$ are shown in Figure 11.9.

Try to visualize these level curves lifted up to form a surface and compare with the graph of g (a hemisphere) in Figure 11.2.

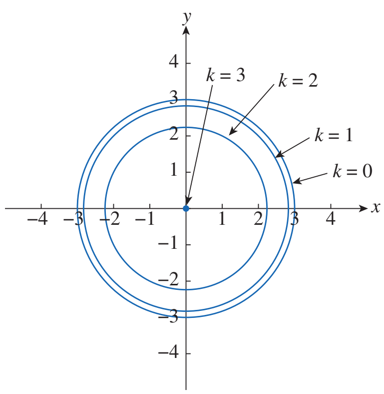


Figure 11.9
A contour map of $g(x, y) = \sqrt{9 - x^2 - y^2}$ showing four level curves.

Example 9 A Family of Ellipses as Level Curves

Sketch several level curves of the function $h(x, y) = 4x^2 + y^2 + 1$.

Solution

The level curves are

$$4x^2 + y^2 + 1 = k \quad \text{or} \quad \frac{x^2}{\frac{1}{4}(k-1)} + \frac{y^2}{k-1} = 1.$$

For $k > 1$, this expression describes a family of ellipses with semiaxes $\frac{1}{2}\sqrt{k-1}$ and $\sqrt{k-1}$.

Figure 11.10(a) shows a contour map of h. Figure 11.10(b) shows these level curves lifted up to the graph of h (an elliptic paraboloid) where they become horizontal traces.

Figure 11.10 suggests how the graph of h is put together from the level curves.

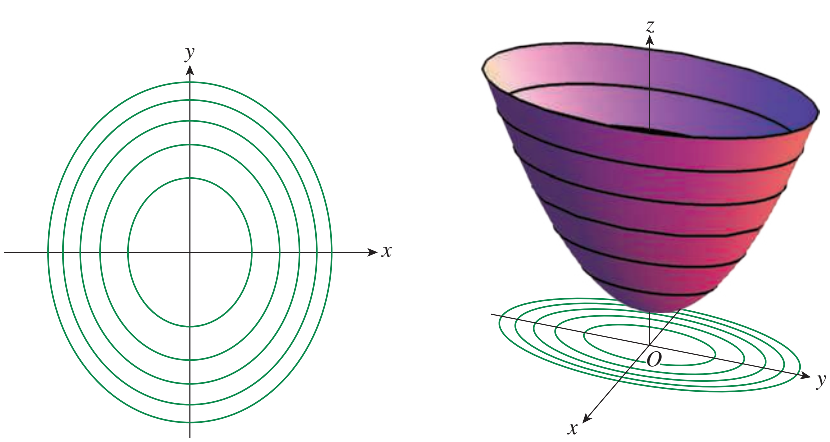


Figure 11.10
The graph of $h(x, y) = 4x^2 + y^2 + 1$ is formed by lifting the level curves.

(a) Contour map

(b) The horizontal traces are raised level curves.

Example 10 Cobb–Douglas Production Function Level Curves

Plot several level curves for the Cobb–Douglas production function given in Example 2.

Solution

We'll use technology to draw a contour plot for the Cobb–Douglas production function $P(L, K) = 1.01L^{0.75}K^{0.25}$.

In Figure 11.11, level curves are labeled with the value of the production P.

For instance, the level curve labeled 140 shows all values of the labor L and capital investment K that result in a production of $P = 140$.

This graph suggests that, for a fixed value of P, as L increases K decreases, and vice versa.

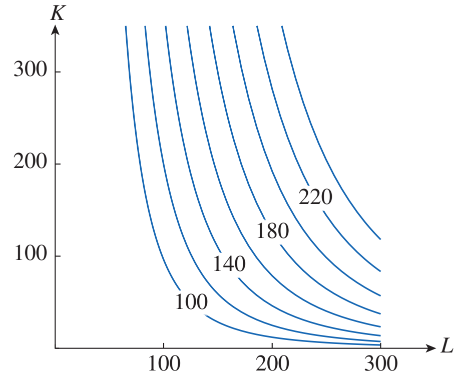
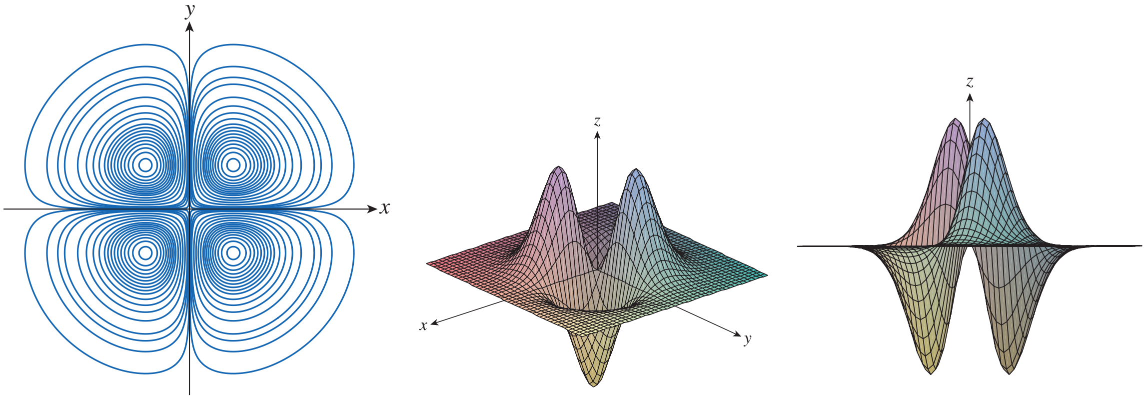


Figure 11.11
A contour plot for the Cobb–Douglas production function.

In some cases, a contour map is actually more useful than a graph. This is certainly true in Example 10; compare Figure 11.11 with Figure 11.3. It is also true in estimating function values, as in Example 6. ■

Figures 11.12 and 11.13 show some level curves together with the corresponding graphs. Notice that the level curves in Figure 11.13(a) crowd together near the origin. This characteristic corresponds to the steepness in the graph in Figure 11.13(b) near the origin.

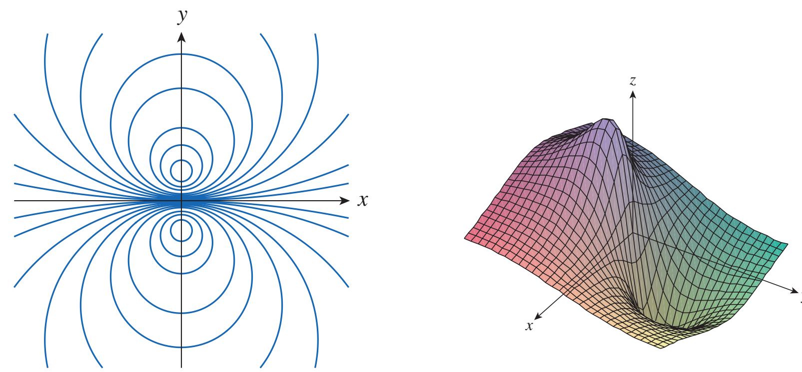


(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$

(b) Graph of $f(x, y) = -xye^{-x^2-y^2}$

(c) Graph of $f(x, y) = -xye^{-x^2-y^2}$; different viewpoint

Figure 11.12
Level curves and graph of $f(x, y) = -xye^{-x^2-y^2}$.



(a) Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$ (b) Graph of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

Figure 11.13
Level curves and graph of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$.

Functions of Three or More Variables

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of Earth depends on the longitude x, latitude y, and elevation z, so we could write $T = f(x, y, z)$.

Example 11 Three-Dimensional Domain

Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

Solution

The expression for $f(x, y, z)$ is defined as long as $z - y > 0$.

Therefore, the domain of f is $D = \{(x, y, z) \in \mathbb{R}^3 \,|\, z > y\}$.

This region is a **half-space** consisting of all points that lie above the plane $z = y$. ■

It's difficult to visualize a function f of three variables by its graph since it would lie in a four-dimensional space. However, we do gain some insight into functions of three variables by examining the **level surfaces**. These are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

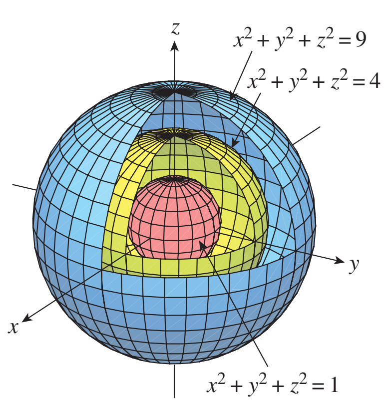


Figure 11.14
Level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$.

Example 12 Level Surfaces

Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

Solution

The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$.

These form a family of concentric spheres with radius $\sqrt{k}$. See Figure 11.14.

Therefore, as (x, y, z) varies over any sphere with center O, the value of $f(x, y, z)$ remains constant. ■

We can extend the concept of a function and consider a function of any number of variables. A **function of n variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote the set of n-tuples by $\mathbb{R}^n$. For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the ith ingredient, and x_i units of the ith ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \ldots, x_n$:

$$C = f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{3}$$

The function f is a real-valued function whose domain is a subset of $\mathbb{R}^n$. Often we can use vector notation in order to denote these functions of n variables more compactly: if $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we write $f(\mathbf{x})$ to denote $f(x_1, x_2, \ldots, x_n)$. Using this notation, we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n.

Since there is a one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and their position vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in V_n, we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, \ldots, x_n$
2. As a function of a single point variable $(x_1, x_2, \ldots, x_n)$
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$

All three characterizations are helpful.

11.1 Exercises

1. In Example 1, we considered the function $W = f(v, T)$, where W is the wind-chill index, v is the wind speed, and T is the actual temperature. A numerical representation of the function f is given in Table 11.1.

(a) Find the value of $f(30, -20)$ and explain the meaning of this value in the context of the problem.

(b) Describe in words the meaning of the question "For what value of v is $f(v, 0) = -31$?" Then answer the question.

(c) Describe in words the meaning of the question "For what value of T is $f(45, T) = -37$?" Then answer the question.

(d) What is the meaning for the function $W = f(10, T)$? Describe the behavior of this function.

(e) What is the meaning of the function $W = f(v, -15)$? Describe the behavior of this function.

2. The *temperature-humidity index* I (or humidex, for short) is the perceived air temperature when the actual temperature is T and the relative humidity is h. Therefore, we can write $I = f(T, h)$. The table of values of I is an excerpt from a table compiled by the National Oceanic & Atmospheric Administration.

Actual Temp (°F) T \ Relative humidity (%) h	20	30	40	50	60	70
80	77	78	79	81	82	83
85	82	84	86	88	90	93
90	87	90	93	96	100	106
95	93	96	101	107	114	124
100	99	104	110	120	132	144

Table 11.3
Apparent temperature as a function of temperature and humidity.

(a) Find the value of $f(95, 70)$ and explain the meaning of this value in the context of the problem.

(b) For what value of h is $f(90, h) = 100$?

(c) For what value of T is $f(T, 50) = 88$?

(d) Explain the meanings of the functions $I = f(80, h)$ and $I = f(100, h)$. Compare the behavior of these two functions of h.

3. (a) For the Cobb–Douglas production function as discussed in Example 2:

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

verify that the production will be doubled if both the amount of labor and the amount of capital are doubled.

(b) Determine whether this is true for the general production function

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

4. A manufacturer has modeled its yearly production function P (the monetary value of its entire production in millions of dollars) as a Cobb–Douglas function

$$P(L, K) = 1.47L^{0.65}\,K^{0.35}$$

where L is the number of labor hours (in thousands) and K is the invested capital (in millions of dollars). Find $P(120, 20)$ and interpret this value.

5. A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.1091w^{0.425}h^{0.725}$$

where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet.

(a) Find $f(160, 70)$ and interpret this value.
(b) Find your own surface area.

6. The wind-chill index W presented in Example 1 has been modeled by the function

$$W(v, T) = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275T\,v^{0.16}$$

Check several values to see how closely this model agrees with the values in Table 11.1.

7. Find and sketch the domain of the function $f(x, y) = \ln(9 - x^2 - 9y^2)$. Find the range of f.

8. Find and sketch the domain of the function $f(x, y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$.

9. Let $f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z} + \ln(4 - x^2 - y^2 - z^2)$.

(a) Evaluate $f(1, 1, 1)$.
(b) Find and describe the domain of f.

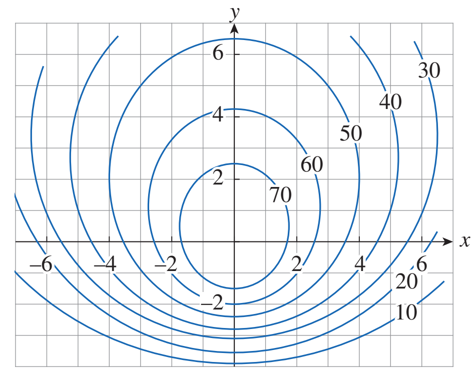
10. Let $g(x, y, z) = x^3y^2z\sqrt{10 - x - y - z}$.

(a) Evaluate $g(1, 2, 3)$.
(b) Find and describe the domain of g.

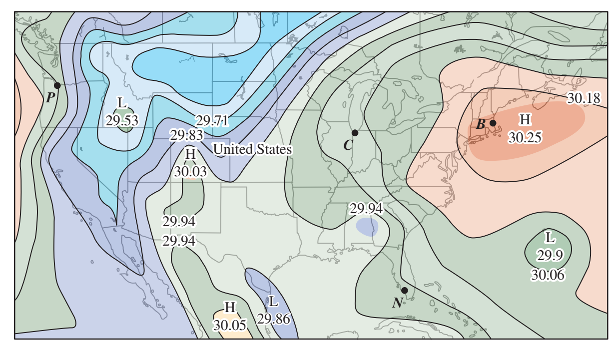
11. A company makes three sizes of cardboard boxes: small, medium, and large. It costs \$2.50 to make a small box, \$4.00 for a medium box, and \$4.50 for a large box. Fixed costs are \$8000.

(a) Express the cost of making x small boxes, y medium boxes, and z large boxes as a function of three variables: $C = f(x, y, z)$.
(b) Find $f(3000, 5000, 4000)$ and interpret this value in the context of the problem.
(c) Find the domain of f.

12. A contour map for a function f is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. Use the contour map to describe the shape of the graph of f.

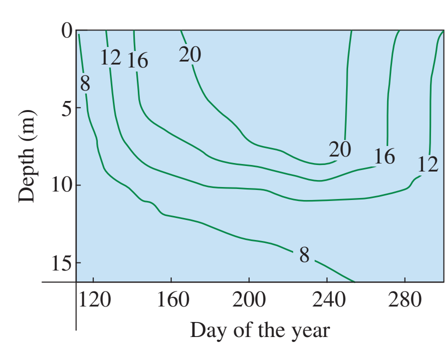


13. A contour map of atmospheric pressure in the United States on August 17, 2021, is shown in the figure (radar-live.com). The pressure is indicated in inches of mercury (in Hg) on the level curves, called isobars.

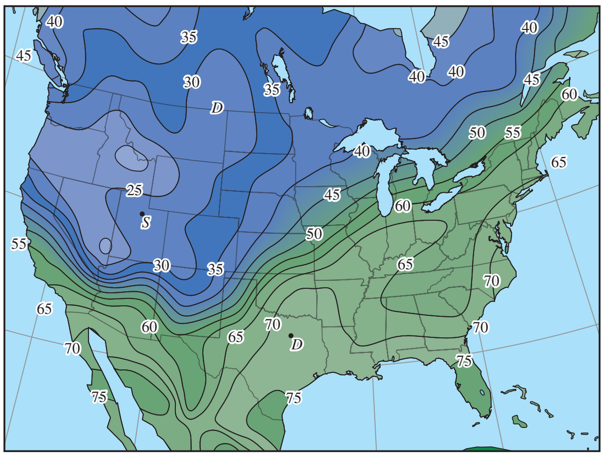


(a) Estimate the pressure at B (Boston), C (Chicago), N (Naples), and P (Portland).
(b) At which of these locations were the winds strongest? Justify your answer.

14. Level curves (isothermals) are shown for the water temperature in Long Lake (Minnesota) in a certain year as a function of depth and time of year. Estimate the temperature in the lake on June 9 (day 160) at a depth of 10 m and on June 29 (day 180) at a depth of 5 m.



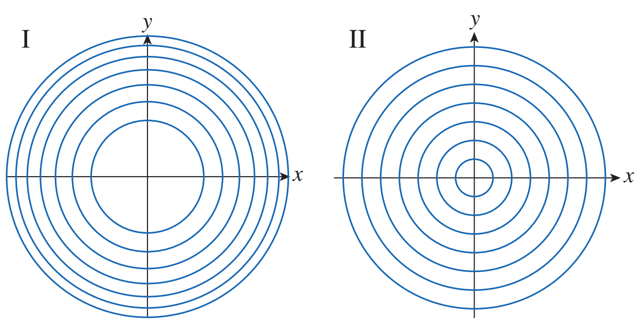
15. The contour map of surface dew point temperature in North America on September 8, 2020, is shown in the figure. The temperature is indicated in °F on the level curves.



Atmospheric Sciences, University of Illinois at Urbana-Champaign

Estimate the dew point temperature at D (Dallas), K (Kennebunkport), and S (Salt Lake City).

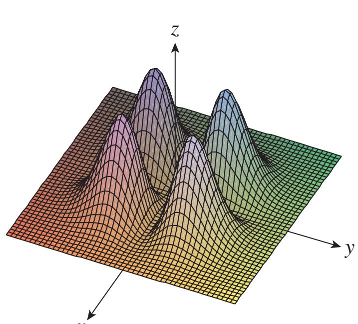
16. Two contour maps are shown. One is for a function f whose graph is a cone. The other is for a function g whose graph is a paraboloid.



Match the contour plot with the function and explain your reasoning.

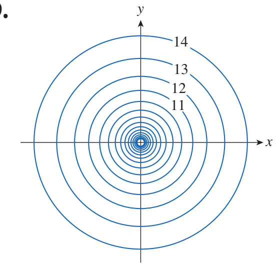
17. Locate the points A and B in the map of Lonesome Mountain (Figure 11.5). Describe the terrain near A and B.

18. Make a rough sketch of a contour map for the function whose graph is shown.

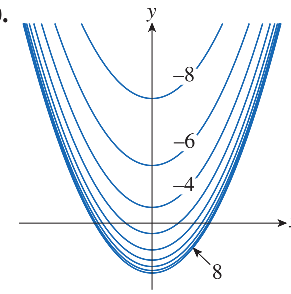


A contour map of a function is shown. Use this figure to make a rough sketch of the graph of f.

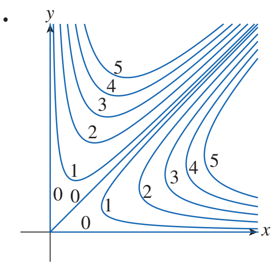
19.



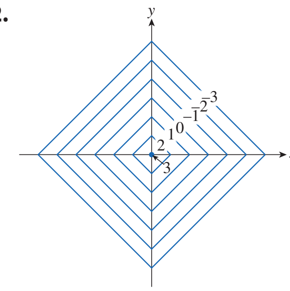
20.



21.



22.



Draw a contour map of the function that includes several level curves.

23. $f(x, y) = (y - 2x)^2$

24. $f(x, y) = x^3 - y$

25. $f(x, y) = x^2 - y^2$

26. $f(x, y) = xy$

27. $f(x, y) = \sqrt{x} + y$

28. $f(x, y) = \ln(x^2 + 4y^2)$

29. $f(x, y) = ye^x$

30. $f(x, y) = y - \arctan x$

31. $f(x, y) = \sqrt{y^2 - x^2}$

32. $f(x, y) = \dfrac{y}{x^2 + y^2}$

Sketch both a contour map and a graph of the function and compare them.

33. $f(x, y) = x^2 + 9y^2$

34. $f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$

35. A thin metal plate, located in the xy-plane, has temperature $T(x, y)$ at the point (x, y). The level curves are called *isothermals* because at all points on an isothermal the temperature is the same. Sketch some isothermals if the temperature function is given by

$$T(x, y) = \frac{100}{1 + x^2 + 2y^2}$$

36. If $V(x, y)$ is the electric potential at a point (x, y) in the xy-plane, then the level curves of V are called *equipotential curves* because at all points on such a curve, the electric potential is the same. Sketch some equipotential curves if

$$V(x, y) = \frac{c}{\sqrt{r^2 - x^2 - y^2}}$$

where c is a positive constant.

Use technology to graph the function using various domains and viewpoints. If possible, include level curves in your graph. Plot several contour lines for the function and explain their relationship to the graph of the function.

37. $f(x, y) = xy^2 - x^3$ (monkey saddle)

38. $f(x, y) = xy^3 - yx^3$ (dog saddle)

39. $f(x, y) = e^{-(x^2+y^2)/3}(\sin(x^2) + \cos(y^2))$

40. $f(x, y) = \cos x \cos y$.

Match the function (a) with its graph (labeled A–F below) and (b) with its contour map (labeled I–VI). Give reasons for your choices.

41. $z = \sin(xy)$

42. $z = e^x \cos y$

43. $z = \sin(x - y)$

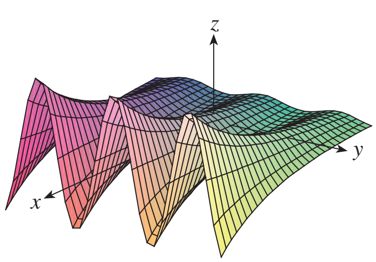
44. $z = \sin x - \sin y$

45. $z = (1 - x^2)(1 - y^2)$

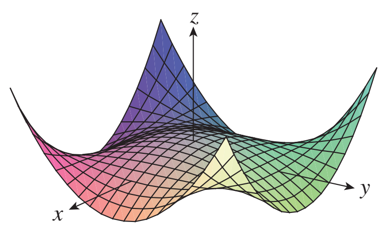
46. $z = \dfrac{x - y}{1 + x^2 + y^2}$

Graphs and Contour Maps for Exercises 41–46.

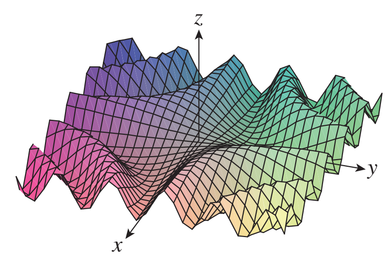
A



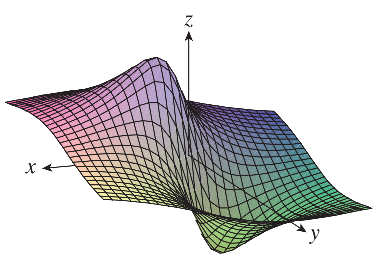
B



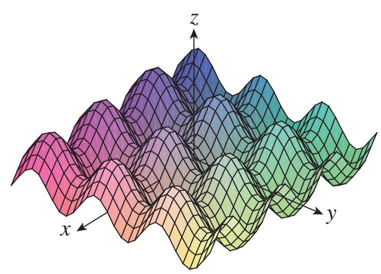
C



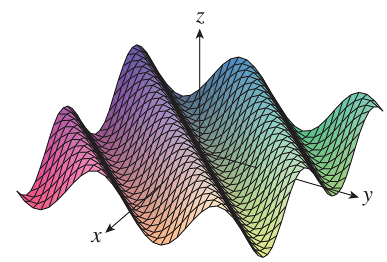
D



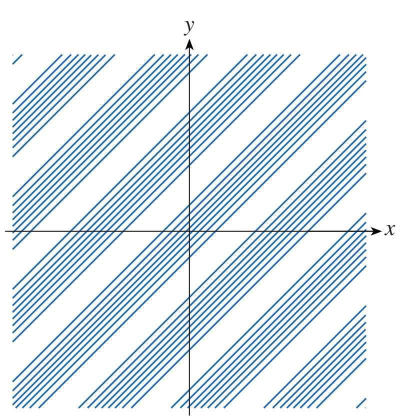
E



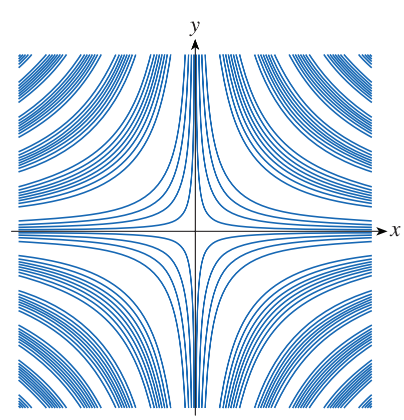
F



I



II



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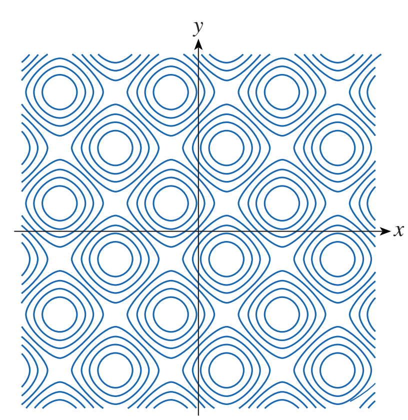

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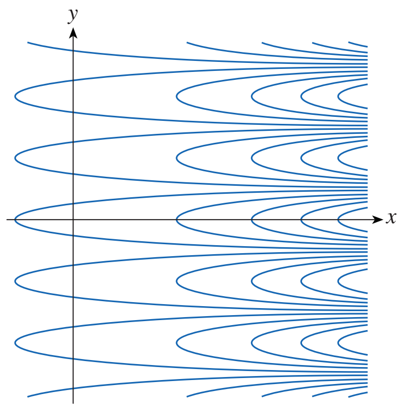

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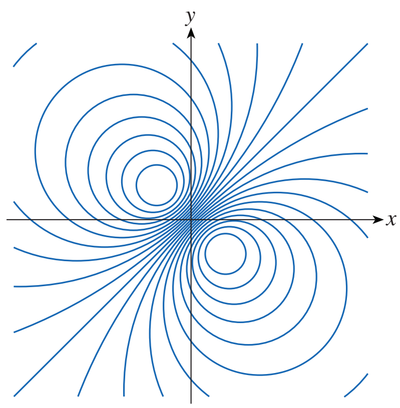

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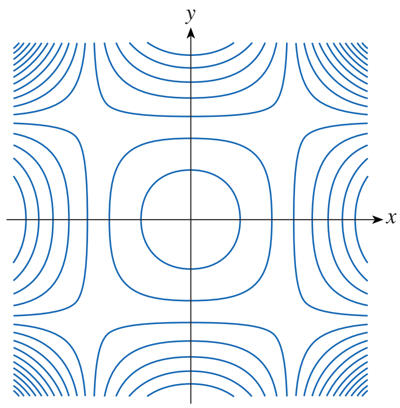


Describe the level surfaces of the function.

47. $f(x, y, z) = x + 3y + 5z$

48. $f(x, y, z) = x^2 + 3y^2 + 5z^2$

49. $f(x, y, z) = y^2 + z^2$

50. $f(x, y, z) = x^2 - y^2 - z^2$

Describe how the graph of g is obtained from the graph of f.

51. (a) $g(x, y) = f(x, y) + 2$ (b) $g(x, y) = 2f(x, y)$
(c) $g(x, y) = -f(x, y)$ (d) $g(x, y) = 2 - f(x, y)$

52. (a) $g(x, y) = f(x - 2, y)$
(b) $g(x, y) = f(x, y + 2)$
(c) $g(x, y) = f(x + 3, y - 4)$

53. Investigate the family of functions $f(x, y) = e^{cx^2 + y^2}$. Explain how the shape of the graph depends on c.

54. Investigate the family of surfaces

$$z = (ax^2 + by^2)e^{-x^2 - y^2}$$

Explain how the shape of the graph depends on the numbers a and b.

55. Investigate the family of surfaces $z = x^2 + y^2 + cxy$. Determine the transitional values of c for which the surface changes from one type of quadric surface to another.

56. Graph the functions

$$f(x, y) = \sqrt{x^2 + y^2} \qquad f(x, y) = e^{\sqrt{x^2 + y^2}}$$
$$f(x, y) = \ln \sqrt{x^2 + y^2} \qquad f(x, y) = \sin(\sqrt{x^2 + y^2})$$
$$\text{and } f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

Suppose g is a function of one variable. Use your results to determine, in general, how the graph of $f(x, y) = g(\sqrt{x^2 + y^2})$ is obtained from the graph of g.

57. (a) Show that, by taking logarithms, the general Cobb–Douglas function $P = bL^{\alpha}K^{1-\alpha}$ can be expressed as

$$\ln \frac{P}{K} = \ln b + \alpha \ln \frac{L}{K}$$

(b) If we let $x = \ln \frac{L}{K}$ and $y = \ln \frac{P}{K}$, then the equation in part (a) can be rewritten as a linear equation $y = \alpha x + \ln b$.

Use Table 11.2 (in Example 2) to construct a table of values of $\ln \frac{L}{K}$ and $\ln \frac{P}{K}$ for the years 1899–1922. Use technology to find the least squares regression line through the points $\left(\ln \frac{L}{K}, \ln \frac{P}{K}\right)$.

(c) Use your results from part (b) to conclude that the Cobb–Douglas production function is $P = 1.01L^{0.75}K^{0.25}$.

11.2 Limits and Continuity

Limits of Functions of Two Variables

Let's begin a discussion of limits by comparing the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \quad \text{and} \quad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 (and therefore the point (x, y) approaches the origin).

Tables 11.4 and 11.5 show values of $f(x, y)$ and $g(x, y)$ for points (x, y) near the origin. Notice that neither function is defined at the origin.

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Table 11.4
Values of $f(x, y)$.

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

Table 11.5
Values of $g(x, y)$.

These tables suggest that as $(x, y) \to (0, 0)$, the values of $f(x, y)$ are approaching 1, whereas the values of $g(x, y)$ do not appear to be approaching any one specific number. It turns out that these guesses based on numerical evidence are correct, and to describe these results, we write

$$\lim_{(x, y)\to(0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \quad \text{and} \quad \lim_{(x, y)\to(0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \text{ does not exist}$$

In general, the notation

$$\lim_{(x, y)\to(a, b)} f(x, y) = L$$

is used to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f. Here is a more formal definition of this concept.

Definition • Limit of a Function of Two Variables

We write

$$\lim_{(x,\, y) \to (a,\, b)} f(x, y) = L$$

and we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L if we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b).

A Closer Look

1. Here is some other notation for the limit in this definition.

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L \quad \text{and} \quad f(x, y) \to L \quad \text{as } (x, y) \to (a, b).$$

2. This definition says that the distance between the numbers $f(x, y)$ and L can be made arbitrarily small by making the distance between the point (x, y) and (a, b) sufficiently small (but not 0). In symbols, we can make the value of $|f(x, y) - L|$ arbitrarily small by making the value of $\sqrt{(x-a)^2 + (y-b)^2}$ sufficiently small.

Figure 11.15 illustrates this interpretation with an arrow diagram. For any $\epsilon > 0$ and interval $(L - \epsilon, L + \epsilon)$ around L, there exists a disk D_δ with center (a, b) and radius $\delta > 0$ such that f maps all points in D_δ [except possibly (a, b)] into the interval $(L - \epsilon, L + \epsilon)$.

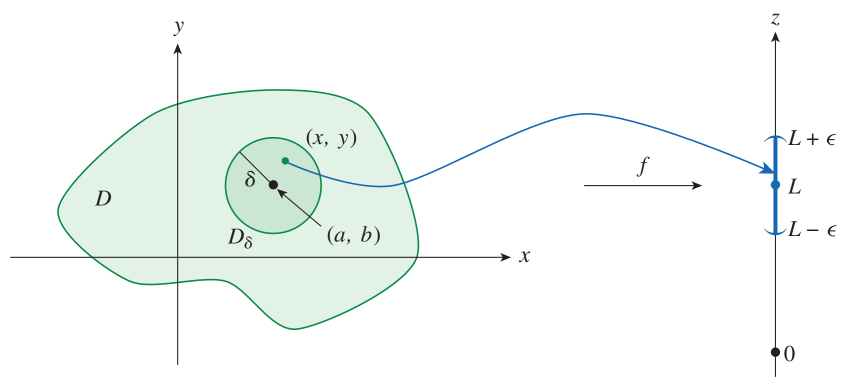


Figure 11.15
The function f maps every (x, y) in the disk D_δ into the interval $(L - \epsilon, L + \epsilon)$.

3. For functions of a single variable, when we let x approach a, there are only two possible directions of approach, from the left or from the right. For functions of two variables, the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever, as long as (x, y) stays within the domain of f. Figure 11.16 shows (x, y) approaching (a, b) along different paths.

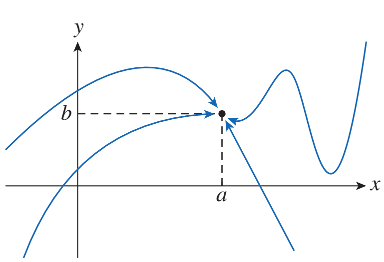


Figure 11.16
For functions of two variables, (x, y) can approach (a, b) along any curve.

4. Recall from Chapter 2, for a function of one variable, if the two one-sided limits are different, then the overall limit does not exist. That is, if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

The limit definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the *direction* of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Therefore,

if we can find two different paths of approach such that the function $f(x, y)$ has different limits, then $\lim_{(x, y)\to(a, b)} f(x, y)$ does not exist.

Here is this conclusion stated more formally.

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2 where $L_1 \neq L_2$, then $\lim_{(x, y)\to(a, b)} f(x, y)$ does not exist. ■

Example 1 Different Paths Lead to Different Limits

Show that $\lim_{(x, y)\to(0, 0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution

Let $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$.

Let's approach (0, 0) along the x-axis. On this path, $y = 0$ for every point (x, y).

So, the function becomes $f(x, 0) = \dfrac{x^2}{x^2} = 1$ for all $x \neq 0$.

Therefore, $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along the x-axis.

If we approach along the y-axis, then $x = 0$.

So, the function is $f(0, y) = \dfrac{-y^2}{y^2} = -1$ for all $y \neq 0$.

Therefore, $f(x, y) \to -1$ as $(x, y) \to (0, 0)$ along the y-axis.

Figure 11.17 illustrates these two paths and limits.

Since f has two different limits as (x, y) approaches (0, 0) along two different lines, the given limit does not exist. This confirms the observation we made using numerical evidence at the beginning of this section. ■

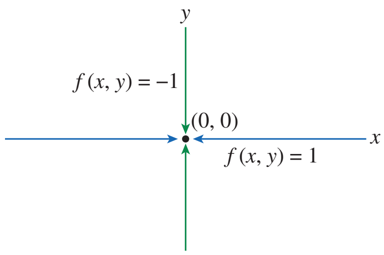


Figure 11.17
Along the x-axis, $f(x, y) = 1$, and along the y-axis, $f(x, y) = -1$.

Example 2 Determine Whether a Limit Exists

If $f(x, y) = \dfrac{xy}{x^2 + y^2}$, does $\lim_{(x, y)\to(0, 0)} f(x, y)$ exist?

Solution

If $y = 0$, then $f(x, 0) = \dfrac{x \cdot 0}{x^2 + 0^2} = \dfrac{0}{x^2} = 0$.

Therefore, $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the x-axis.

If $x = 0$, then $f(0, y) = \dfrac{0 \cdot y}{0^2 + y^2} = \dfrac{0}{y^2} = 0$,

Therefore, $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the y-axis.

Even though we obtained the same limit along the two axes, this does *not* show that the given limit is 0. Remember, we have to obtain the same limit along every possible path to (0, 0).

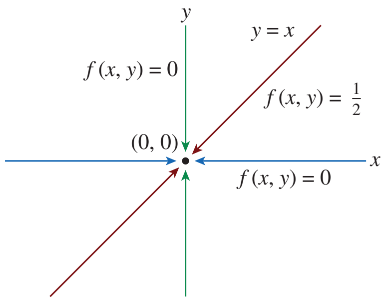


Figure 11.18
Three different paths to (0, 0).

Consider approaching (0, 0) along another line, say $y = x$.

For all $x \neq 0$, $f(x, x) = \dfrac{x \cdot x}{x^2 + x^2} = \dfrac{x^2}{2x^2} = \dfrac{1}{2}$.

Therefore, $f(x, y) \to \dfrac{1}{2}$ as $(x, y) \to (0, 0)$ along the line $y = x$.

Figure 11.18 illustrates these paths and limits.

Since we obtain different limits along different paths, the given limit does not exist. ■

Figure 11.19 helps to understand the result in Example 2. The ridge that occurs above the line $y = x$ corresponds to the fact that $f(x, y) = \frac{1}{2}$ for all points (x, y) on that line except the origin.

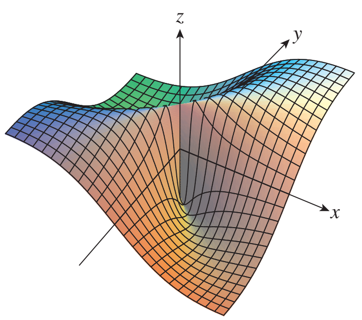


Figure 11.19
Graph of $f(x, y) = \dfrac{xy}{x^2 + y^2}$.

Example 3 Determine Whether a Limit Exists

If $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$, does $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y)$ exist?

Solution

In the previous examples, we found different limits traveling along different paths. So, let's consider $(x, y) \to (0, 0)$ along *any* line through the origin.

If the line is not the y-axis, then $y = mx$, where m is the slope, and

$$f(x, y) = f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}.$$

Therefore, $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along $y = mx$.

We get the same result as $(x, y) \to (0, 0)$ along the line $x = 0$.

Therefore, f has the same limiting value along every line through the origin.
But that doesn't show that the given limit is 0. Remember, we have to consider *all* possible paths to the origin.

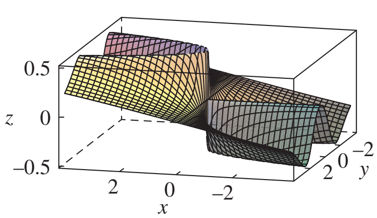


Figure 11.20
Graph of $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$. Notice the ridge above the parabola $x = y^2$.

Consider $(x, y) \to (0, 0)$ along the parabola $x = y^2$. Then

$$f(x, y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}.$$

So, $f(x, y) \to \dfrac{1}{2}$ as $(x, y) \to (0, 0)$ along $x = y^2$.

Since different paths lead to different limiting values, the given limit does not exist. Figure 11.20 helps to justify graphically this result. ■

After these three examples, you're probably thinking that these types of limits never exist; that all we have to do is find the right paths to show a limit does not exist. Let's consider some limits that do exist.

Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits. The Limit Laws listed in Section 2.3 can be extended to functions of two variables: the limit of a sum

is the sum of the limits, the limit of a product is the product of the limits, and so on. In particular, the following equations are true.

$$\lim_{(x,\,y)\to(a,\,b)} x = a \qquad \lim_{(x,\,y)\to(a,\,b)} y = b \qquad \lim_{(x,\,y)\to(a,\,b)} c = c \tag{1}$$

And, the Squeeze Theorem can be extended to functions of two variables.

Example 4 Guess and Prove a Limit

Find $\displaystyle\lim_{(x,\,y)\to(0,\,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Solution

As in Example 3, we could show that the limit along any line through the origin is 0. This doesn't prove that the given limit is 0, but we might consider some other paths.

The limits along the parabolas $y = x^2$ and $x = y^2$ are also 0. So, this suggests that maybe the limit does exist and is equal to 0.

To prove this, consider the distance from $f(x, y)$ to 0.

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| = \left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2}$$

Since $y^2 \geq 0$, then $x^2 \leq x^2 + y^2$ and $\dfrac{x^2}{x^2+y^2} \leq 1$.

Therefore, $0 \leq \dfrac{3x^2|y|}{x^2+y^2} \leq 3|y|$.

Since $\displaystyle\lim_{(x,\,y)\to(0,\,0)} 0 = 0$ and $\displaystyle\lim_{(x,\,y)\to(0,\,0)} 3|y| = 0$, Equation 1.

then by the Squeeze Theorem, $\displaystyle\lim_{(x,\,y)\to(0,\,0)} \frac{3x^2y}{x^2+y^2} = 0$. ■

Continuity

Recall that evaluating the limit of a *continuous* function of a single variable is straightforward. We can use direct substitution because the defining property of a continuous function is $\lim_{x\to a} f(x) = f(a)$. Continuous functions of two variables are defined in a similar manner, using the direct substitution property.

Definition • Continuous Function

A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,\,y)\to(a,\,b)} f(x, y) = f(a, b)$$

We say that f is **continuous on** D if f is continuous at every point (a, b) in D.

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ also changes by a small amount. This means that a surface that is the graph of a continuous function has no hole or break.

Using the properties of limits, we can show that sums, differences, products, and quotients of continuous functions are continuous on their domains. Let's use these results and consider some examples of continuous functions.

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. A **rational function** is a quotient of polynomials. For example,

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, while

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

The limits in Equation 1 show that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ are continuous. Since any polynomial can be constructed out of the simple functions f, g, and h by multiplication and addition, it follows that *all polynomials are continuous on* $\mathbb{R}^2$. Similarly, any rational function is continuous on its domain because it is a quotient of continuous functions. Therefore, using the properties of limits: sums, differences, products, and quotients of continuous functions are continuous on their domains.

Example 5 Use Continuity to Find a Limit

Evaluate $\displaystyle\lim_{(x, y)\to(1, 2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

Solution

$f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial; it is continuous everywhere.

Therefore, we can find the limit by direct substitution.

$$\lim_{(x, y)\to(1, 2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11$$ ■

Example 6 Determine the Region of Continuity

Where is the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

Solution

The function f is discontinuous at (0, 0) because it is not defined there.

Since f is a rational function, it is continuous on its domain.

The domain is $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$. ■

Example 7 A Function That Is Discontinuous at the Origin

Let

$$g(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

The function g is defined at (0, 0) but g is still discontinuous there.

The limit $\displaystyle\lim_{(x, y)\to(0, 0)} g(x, y)$ does not exist, as shown in Example 1. ■

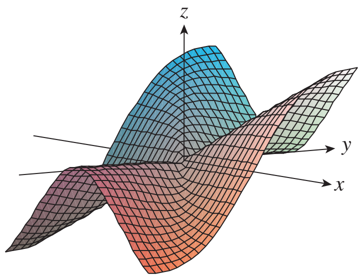


Figure 11.21
Graph of f.

Example 8 A Function That Is Continuous Everywhere

Let

$$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We know f is continuous for $(x, y) \neq (0, 0)$ because it is a rational function for those values.

From Example 4, we have

$$\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{(x, y)\to(0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0).$$

Therefore, f is continuous at $(0, 0)$ and is, therefore, continuous on $\mathbb{R}^2$.

Figure 11.21 shows a graph of the function f. ■

Just as for functions of one variable, composition is another way of combining two continuous functions to construct a new function. In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

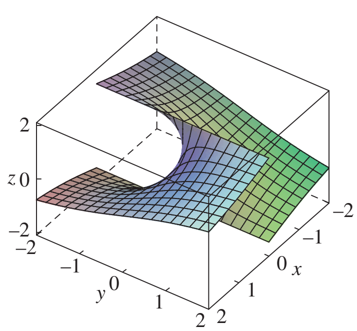


Figure 11.22

The function $h(x, y) = \arctan\left(\dfrac{y}{x}\right)$ is discontinuous where $x = 0$.

Example 9 Composition and Continuity

Where is the function $h(x, y) = \arctan\left(\dfrac{y}{x}\right)$ continuous?

Solution

The function $f(x, y) = \dfrac{y}{x}$ is a rational function.

Therefore, f is continuous except on the line $x = 0$.

The function $g(t) = \arctan t$ is continuous everywhere.

So, the composite function $g(f(x, y)) = \arctan\left(\dfrac{y}{x}\right) = h(x, y)$ is continuous except where $x = 0$.

The graph in Figure 11.22 shows the break in the graph of h above the y-axis. ■

The concepts of limits and continuity presented in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x, y, z)\to(a, b, c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f. This means that we can make the value $f(x, y, z)$ arbitrarily close to L by taking (x, y, z) sufficiently close to (a, b, c).

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z)\to(a, b, c)} f(x, y, z) = f(a, b, c)$$

For example, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^3$ except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center at the origin and radius 1.

11.2 Exercises

1. Suppose that $\lim_{(x, y)\to(3, 1)} f(x, y) = 6$. What can you conclude about the value of $f(3, 1)$? What if f is continuous for all values in its domain?

2. Explain why each function is continuous or discontinuous.

(a) The outdoor temperature as a function of longitude, latitude, and time

(b) Elevation (height above sea level) as a function of longitude, latitude, and time

(c) The cost of a taxi ride as a function of distance traveled and time

(d) The depth of a reservoir as a function of rainfall rate and temperature

(e) The price of a ticket to an NFL game as a function of seating tier and team win-loss record

Use a table of numerical values of $f(x, y)$ for (x, y) near the origin to make a conjecture about the value of the limit of $f(x, y)$ as $(x, y) \to (0, 0)$. Then justify your answer.

3. $f(x, y) = \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$

4. $f(x, y) = \dfrac{2xy}{x^2 + 2y^2}$

Find the limit, if it exists, or show that the limit does not exist.

5. $\lim_{(x, y)\to(2, 3)} (x^2y^3 - 5x^2)$

6. $\lim_{(x, y)\to(-1, 0)} e^{-(x^2 - y^2)} \cos(\pi xy)$

7. $\lim_{(x, y)\to(0, 0)} \dfrac{y^4}{x^4 + 3y^4}$

8. $\lim_{(x, y)\to(2, -1)} \dfrac{x^2y + xy^2}{x^2 - y^2}$

9. $\lim_{(x, y)\to(0, 0)} \dfrac{x^2 + \sin^2 y}{2x^2 + y^2}$

10. $\lim_{(x, y)\to(0, 0)} \dfrac{xy \cos y}{3x^2 + y^2}$

11. $\lim_{(x, y)\to(\pi, \pi/2)} y \sin(x - y)$

12. $\lim_{(x, y)\to(3, 2)} e^{\sqrt{2x - y}}$

13. $\lim_{(x, y)\to(0, 0)} \dfrac{6x^3y}{2x^4 + y^4}$

14. $\lim_{(x, y)\to(0, 0)} \dfrac{x^4 - 4y^2}{x^2 + 2y^2}$

15. $\lim_{(x, y)\to(0, 0)} \dfrac{5y^4 \cos^2 x}{x^4 + y^4}$

16. $\lim_{(x, y)\to(0, 0)} \dfrac{y^2 \sin^2 x}{x^4 + y^4}$

17. $\lim_{(x, y)\to(1, 0)} \dfrac{xy - y}{(x - 1)^2 + y^2}$

18. $\lim_{(x, y)\to(0, 0)} \dfrac{xy}{\sqrt{x^2 + y^2}}$

19. $\lim_{(x, y)\to(0, 0)} \dfrac{x^2ye^y}{x^4 + 4y^2}$

20. $\lim_{(x, y)\to(0, 0)} \dfrac{xy^4}{x^2 + y^8}$

21. $\lim_{(x, y)\to(0, 0)} \dfrac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

22. $\lim_{(x, y)\to(0, 0)} \dfrac{x^4 - y^4}{x^2 + y^2}$

23. $\lim_{(x, y, z)\to(3, 0, 1)} e^{-xy} \sin\left(\dfrac{\pi z}{2}\right)$

24. $\lim_{(x, y, z)\to(\pi, 0, 1/3)} e^{y^2} \tan(xz)$

25. $\lim_{(x, y, z)\to(0, 0, 0)} \dfrac{x^2 + 2y^2 + 3z^2}{x^2 + y^2 + z^2}$

26. $\lim_{(x, y, z)\to(0, 0, 0)} \dfrac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$

27. $\lim_{(x, y, z)\to(0, 0, 0)} \dfrac{yz}{x^2 + 4y^2 + 9z^2}$

Use technology to graph the function and to explain why the limit does not exist.

28. $\lim_{(x, y)\to(0, 0)} \dfrac{2x^2 + 2xy + 4y^2}{3x^2 + 5y^2}$

29. $\lim_{(x, y)\to(0, 0)} \dfrac{xy^3}{x^2 + y^6}$

Find $h(x, y) = g(f(x, y))$ and the set of points at which h is continuous.

30. $g(t) = t^2 + \sqrt{t}, \quad f(x, y) = 2x + 3y - 6$

31. $g(t) = t + \ln t, \quad f(x, y) = \dfrac{1 - xy}{1 + x^2y^2}$

Use the graph of the function to guess where it is discontinuous. Then use the formula to explain and justify your answer.

32. $f(x, y) = e^{1/(x-y)}$

33. $f(x, y) = \dfrac{1}{1 - x^2 - y^2}$

Determine the set of points at which the function is continuous.

34. $F(x, y) = \arctan(x + \sqrt{y})$

35. $F(x, y) = \cos\sqrt{1 + x - y}$

36. $G(x, y) = \ln(x^2 + y^2 - 4)$

37. $G(x, y) = \dfrac{xy}{1 + e^{x-y}}$

38. $H(x, y) = \dfrac{1 + x^2 + y^2}{1 - x^2 - y^2}$

39. $H(x, y) = \dfrac{e^x + e^y}{e^{xy} - 1}$

40. $f(x, y, z) = \dfrac{\sqrt{y}}{x^2 - y^2 + z^2}$

41. $f(x, y, z) = \sqrt{x + y + z}$

42. $f(x, y) = \begin{cases} \dfrac{x^2y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

43. $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Use polar coordinates to find the limit.
[If (r, θ) are polar coordinates of the point (x, y) with $r \geq 0$, note that $r \to 0^+$ as $(x, y) \to (0, 0)$.]

44. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{x^3 + y^3}{x^2 + y^2}$

45. $\displaystyle\lim_{(x, y)\to(0, 0)} (x^2 + y^2)\ln(x^2 + y^2)$

46. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}$

47. Use spherical coordinates to find

$$\lim_{(x, y, z)\to(0, 0, 0)} \frac{xyz}{x^2 + y^2 + z^2}$$

48. At the beginning of this section, we considered the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and guessed that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ on the basis of numerical evidence. Use polar coordinates to confirm the value of this limit. Use technology to graph this function.

49. Graph and discuss the continuity of the function

$$f(x, y) = \begin{cases} \dfrac{\sin xy}{xy} & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

50. Let

$$f(x, y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or } y \geq x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

(a) Show that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along any path through $(0, 0)$ of the form $y = mx^a$ with $a < 4$.
(b) Despite part (a), show that f is discontinuous at $(0, 0)$.
(c) Show that f is discontinuous on two entire curves.

11.3 Partial Derivatives

Partial Derivatives of Functions of Two Variables

On a hot day, extreme humidity makes us think, and feel, that the temperature is higher than it really is, whereas in very dry air, we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex, in some countries) to describe the combined effects of temperature and humidity. The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H. So, I is a function of T and H and we can write $I = f(T, H)$. The following table of values of I is an excerpt from a table compiled by the National Weather Service.

Relative humidity (%)

T (°F) \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	102	106	109	113	117	122
92	99	101	105	108	112	116	121	126	131
94	103	106	110	114	119	124	129	135	141
96	108	112	116	121	126	132	138	145	152
98	113	117	123	128	134	141	148	155	164
100	118	124	129	136	143	150	158	167	176

Actual temperature (°F)

Table 11.6
Heat index I as a function of temperature and humidity.

Let's focus on the highlighted column of the table, which corresponds to a relative humidity of $H = 70\%$; we are considering the heat index as a function of the single variable T for a fixed value of H. If we write $g(T) = f(T, 70)$, then $g(T)$ describes how the heat index I increases as the actual temperature T increases when the relative humidity is 70%. The derivative of g when $T = 96°\text{F}$ is the rate of change of I with respect to T when $T = 96°\text{F}$:

$$g'(96) = \lim_{h\to 0}\frac{g(96+h) - g(96)}{h} = \lim_{h\to 0}\frac{f(96+h, 70) - f(96, 70)}{h}$$

We can approximate $g'(96)$ using the values in Table 11.6 by taking $h = 2$ and -2:

$$g'(96) \approx \frac{g(98) - g(96)}{2} = \frac{f(98, 70) - f(96, 70)}{2} = \frac{134 - 126}{2} = 4$$

$$g'(96) \approx \frac{g(94) - g(96)}{-2} = \frac{f(94, 70) - f(96, 70)}{-2} = \frac{119 - 126}{-2} = 3.5$$

The arithmetic mean of these two values is 3.75, and is another, perhaps better, approximation to the derivative $g'(96)$. This means that, when the actual temperature is 96°F and the relative humidity is 70%, the apparent temperature (heat index) rises by about 3.75°F for every degree that the actual temperature rises.

Now, let's look again at Table 11.6, but this time consider the highlighted row, which corresponds to a fixed temperature of $T = 96°\text{F}$. The numbers in this row are values of the function $G(H) = f(96, H)$, which describes how the heat index increases as the relative humidity H increases when the actual temperature is $T = 96°\text{F}$. The derivative of this function when $H = 70\%$ is the rate of change of I with respect to H when $H = 70\%$:

$$G'(70) = \lim_{h\to 0}\frac{G(70+h) - G(70)}{h} = \lim_{h\to 0}\frac{f(96, 70+h) - f(96, 70)}{h}$$

If we let $h = 5$ and -5, we can approximate $G'(70)$ using values in the table:

$$G'(70) \approx \frac{G(75) - G(70)}{5} = \frac{f(96, 75) - f(96, 70)}{5} = \frac{132 - 126}{5} = 1.2$$

$$G'(70) \approx \frac{G(65) - G(70)}{-5} = \frac{f(96, 65) - f(96, 70)}{-5} = \frac{121 - 126}{-5} = 1$$

If we use the arithmetic mean of these two values, then $G'(70) \approx 1.1$. This says that, when the temperature is 96°F and the relative humidity is 70%, the heat index rises about 1.1°F for every percent that the relative humidity rises.

Let's try to generalize this idea of a derivative down one column or across one row of a table. Suppose f is a function of two variables x and y and we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of f with respect to x at (a, b)** and denote it by $f_x(a, b)$. Therefore,

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b) \tag{1}$$

By the definition of a derivative, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a + h) - g(a)}{h}$$

and so Equation 1 becomes

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h} \tag{2}$$

Similarly, the **partial derivative of f with respect to y at (a, b)**, denoted by $f_y(a, b)$, is obtained by keeping x fixed $(x = a)$ and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h} \tag{3}$$

Using this notation for partial derivatives, we can write the rates of change of the heat index I with respect to the actual temperature T and relative humidity H when $T = 96°\text{F}$ and $H = 70\%$ as follows:

$$f_T(96, 70) \approx 3.75 \qquad f_H(96, 70) \approx 1.1$$

If we let the point (a, b) vary in Equations 2 and 3, then f_x and f_y are functions of two variables.

Definition • Partial Derivatives

If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

There are many alternative notations for partial derivatives. For example, instead of f_x, we often write f_1, or $D_1 f$ to indicate differentiation with respect to the *first* variable.

And although it cannot be interpreted as a ratio of differentials, $\dfrac{\partial f}{\partial x}$ is also used to represent the partial derivative of f with respect to x.

Here is a summary of notation.

Notation for Partial Derivatives

If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

To compute partial derivatives, all we have to do is remember from Equation 1 that the partial derivative with respect to x is the *ordinary* derivative of the function g for a single variable, keeping y fixed. This interpretation leads to the following procedure.

Procedure for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example 1 Evaluating Partial Derivatives

If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Solution

Hold y constant and differentiate with respect to x:

$$f_x(x, y) = 3x^2 + 2xy^3 - 0 = 3x^2 + 2xy^3$$

Evaluate this expression at (2, 1):

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Hold x constant and differentiate with respect to y:

$$f_y(x, y) = 0 + x^2 \cdot 3y^2 - 4y = 3x^2y^2 - 4y$$

Evaluate this expression at (2, 1):

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Interpretations of Partial Derivatives

Let's investigate a geometric interpretation of partial derivatives. Recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S.

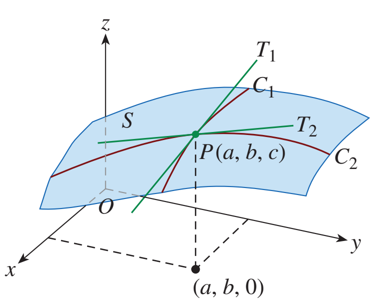


Figure 11.23
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Set $y = b$ and consider the curve C_1 in which the vertical plane $y = b$ intersects S, that is, C_1 is the trace of S in the plane $y = b$. Similarly, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P, and are illustrated in Figure 11.23.

Notice that the curve C_1 is the graph of the function $g(x) = f(x, b)$. Therefore, the slope of the line T_1 tangent to the curve C_1 at P is $g'(a) = f_x(a, b)$. The curve C_2 is the graph of the function $G(y) = f(a, y)$, and the slope of the line T_2 tangent to the curve C_2 at P is $G'(b) = f_y(a, b)$.

Therefore, the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

As we have seen in the example of the heat index function, a partial derivative can also be interpreted as a *rate of change*. If $z = f(x, y)$, then $\dfrac{\partial z}{\partial x}$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\dfrac{\partial z}{\partial y}$ represents the rate of change of z with respect to y when x is fixed.

Example 2 Partial Derivatives Interpreted as Slopes of Tangents

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

Solution

Find each partial derivative and evaluate at the point (1, 1).

$$f_x(x, y) = -2x \quad \Rightarrow \quad f_x(1, 1) = -2$$
$$f_y(x, y) = -4y \quad \Rightarrow \quad f_y(1, 1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$.

The vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. This curve is labeled C_1 in Figure 11.24.

The slope of the tangent line to this parabola at the point (1, 1, 1) is $f_x(1, 1) = -2$.

Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$.

The slope of the tangent line at (1, 1, 1) is $f_y(1, 1) = -4$, illustrated in Figure 11.25.

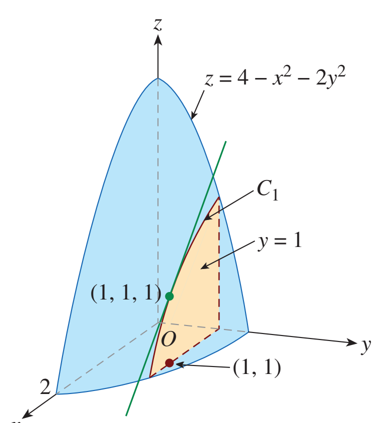


Figure 11.24
The line tangent to the curve C_1 has slope $f_x(1, 1) = -2$.

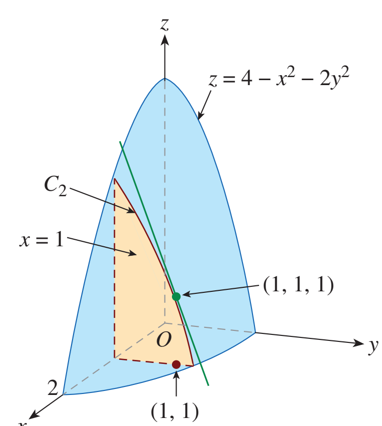


Figure 11.25
The line tangent to the curve C_2 has slope $f_y(1, 1) = -4$.

Figure 11.26 is a computer-generated graph corresponding to Figure 11.24. Part (a) shows the plane $y = 1$ intersecting the surface to form curve C_1 and part (b) shows C_1 and T_1, the line tangent to the curve C_1 at the point (1, 1, 1). Similarly, Figure 11.27 corresponds to Figure 11.25.

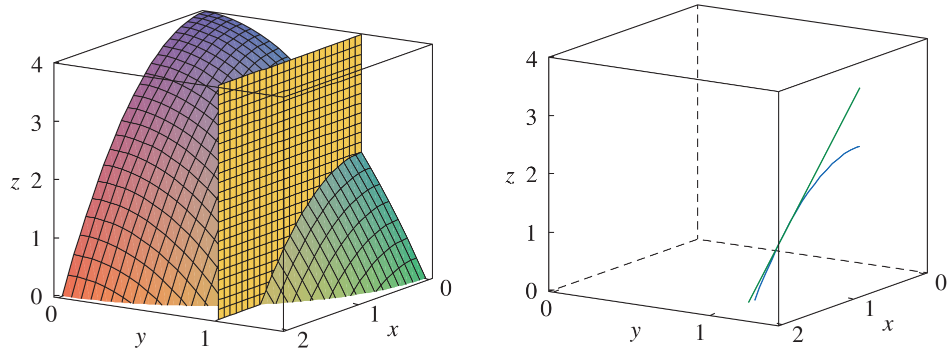


Figure 11.26
Computer-generated graphs corresponding to Figure 11.24.

(a) Graph of the plane $y = 1$ intersecting the surface

(b) Graph of the curve C_1 and the tangent line T_1

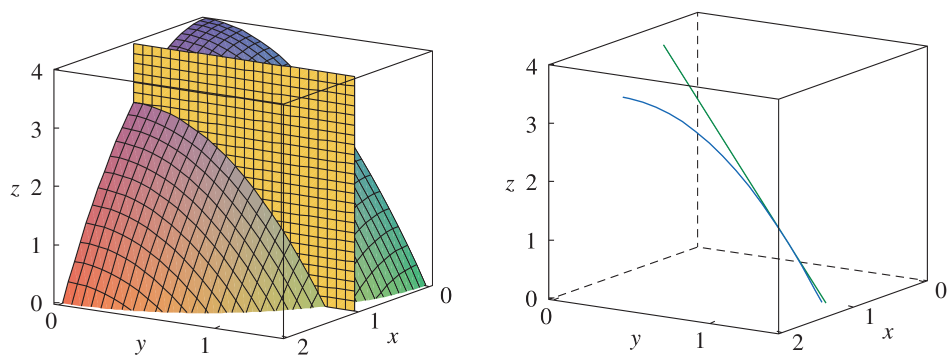


Figure 11.27
Computer-generated graphs corresponding to Figure 11.25.

(a) Graph of the plane $x = 1$ intersecting the surface

(b) Graph of the curve C_2 and the tangent line T_2

Example 3 The Chain Rule and Partial Differentiation

If $f(x, y) = \sin\left(\dfrac{x}{1+y}\right)$, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Solution

Use the Chain Rule for functions of one variable.

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

Example 4 Implicit Partial Differentiation

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

Solution

To find $\frac{\partial z}{\partial x}$, differentiate implicitly with respect to x, and treat y as a constant.

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

Solve for $\frac{\partial z}{\partial x}$ $\quad \Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$

Similarly, differentiate implicitly with respect to y, and solve for $\frac{dz}{dy}$.

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Figure 11.28 shows the surface defined implicitly by this equation. ■

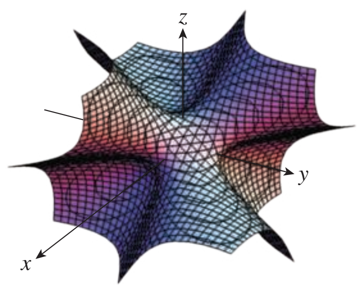


Figure 11.28
Graph of the surface defined implicitly by $x^3 + y^3 + z^3 + 6xyz = 1$.

Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x, y, and z, then its partial derivative with respect to x is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and is found by treating y and z as constants and differentiating $f(x, y, z)$ with respect to x. If $w = f(x, y, z)$, then $f_x = \frac{\partial w}{\partial x}$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed. We can't visualize or interpret this expression geometrically because the graph of f lies in four-dimensional space.

In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

Example 5 Partial Derivatives of a Function of Three Variables

Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

Solution

To find f_x, take the derivative with respect to x, and treat y and z as constants.

$$f_x = ye^{xy} \ln z$$

Similarly,

$$f_y = xe^{xy} \ln z \quad \text{and} \quad f_z = \frac{e^{xy}}{z}.$$ ■

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables. We can now consider the partial derivatives of these functions $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f. If $z = f(x, y)$, here is the standard notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Therefore, the notation f_{xy}, or $\dfrac{\partial^2 f}{\partial y \partial x}$, means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx}, the order is reversed.

Example 6 Second Partial Derivatives

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

Solution

In Example 1, we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

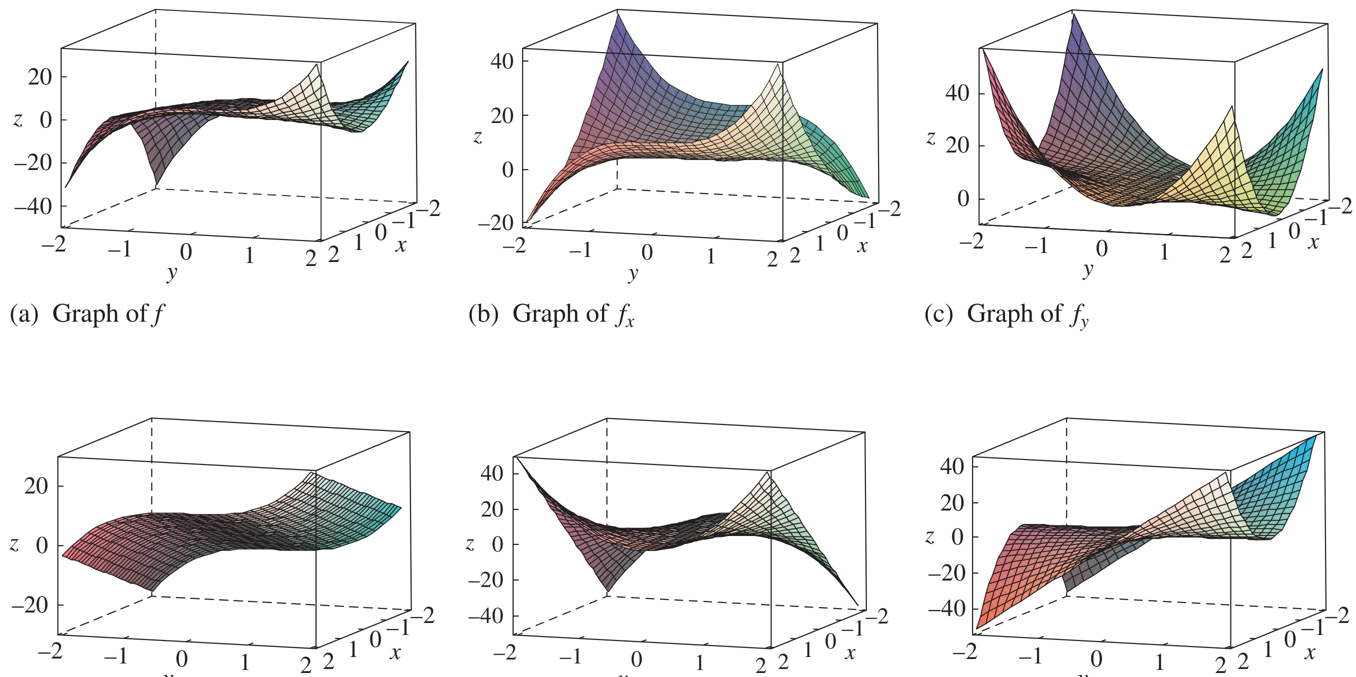
Therefore,

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

Figure 11.29 shows the graph of the function f and the graphs of its first- and second-order partial derivatives for $-2 \le x \le 2$, $-2 \le y \le 2$. These graphs are consistent with our interpretations of f_x and f_y as slopes of tangent lines to traces of the graph of f. For instance, the graph of f decreases if we start at $(0, -2)$ and move in the positive x-direction. This observation is supported by the negative values of f_x.

Consider comparing the graphs of f_{yx} and f_{yy} with the graph of f_y to identify the relationships.



(a) Graph of f (b) Graph of f_x (c) Graph of f_y

(d) Graph of f_{xx} (e) Graph of $f_{xy} = f_{yx}$ (f) Graph of f_{yy}

Figure 11.29
Graph of f, first-, and second-order partial derivatives.

Notice that $f_{xy} = f_{yx}$ in Example 6. This might seem like just a carefully chosen example, but it turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most practical functions. The next theorem, discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can be sure that $f_{xy} = f_{yx}$. The proof is given in Appendix E.

Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

As you might expect, partial derivatives of order 3 or higher can also be defined. For example,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}$$

Using Clairaut's Theorem, it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

Example 7 Higher-Order Derivative

Find f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

Solution

Take the partial derivatives in the correct order. Treat the other variables as constants.

$f_x = 3\cos(3x + yz)$ Derivative of f with respect to x; y and z held constant.

$f_{xx} = -9\sin(3x + yz)$ Derivative of f_x with respect to x; y and z held constant.

$f_{xxy} = -9z\cos(3x + yz)$ Derivative of f_{xx} with respect to y; x and z held constant.

$f_{xxyz} = -9\cos(3x + yz) + 9yz\sin(3x + yz)$ Derivative of f_{xxy} with respect to z; x and y held constant.

Partial Differential Equations

Partial differential equations are often used to model, or express, certain physical laws and, of course, they contain partial derivatives. The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation** after Pierre Laplace (1749–1827). Solutions of this equation are called **harmonic functions**; they play a role in problems of heat conduction, fluid flow, and electric potential.

Example 8 Laplace Solution

Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

Solution

Compute the necessary second-order partial derivatives.

$$u_x = e^x \sin y \quad \Rightarrow \quad u_{xx} = e^x \sin y$$
$$u_y = e^x \cos y \quad \Rightarrow \quad u_{yy} = -e^x \sin y$$

Use the expressions for u_{xx} and u_{yy} in Laplace's equation.

$$u_{xx} + u_{yy} = e^x \sin y + (-e^x \sin y) = 0$$

Therefore, u satisfies Laplace's equation.

The **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or even a wave traveling along a vibrating string. For example, if $u(x, t)$ represents the displacement of a vibrating violin string at time t and at a distance x from one end of the string (as shown in Figure 11.30), then $u(x, t)$ satisfies the wave equation. In this case, the constant a depends on the density of the string and on the tension in the string.

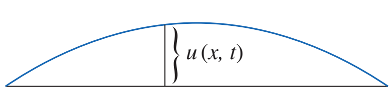


Figure 11.30
An illustration of the value $u(x, t)$.

Example 9 Wave Equation Solution

Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

Solution

Compute the necessary second-order partial derivatives, and simplify.

$$u_x = \cos(x - at) \quad \Rightarrow \quad u_{xx} = -\sin(x - at)$$

$$u_t = -a\cos(x - at) \quad \Rightarrow \quad u_{tt} = -a^2\sin(x - at) = a^2u_{xx}$$

Therefore, u satisfies the wave equation. ■

The Cobb–Douglas Production Function

In Example 2 in Section 11.1, we described the work of Cobb and Douglas in modeling the total production P of an economic system as a function of the amount of labor L and the capital investment K. Now, let's use partial derivatives to show how the particular form of their model follows from certain assumptions they made about the economy.

If the production function is denoted by $P = P(L, K)$, then the partial derivative $\frac{\partial P}{\partial L}$ is interpreted as the rate at which production changes with respect to the amount of labor. Economists refer to this as the marginal production with respect to labor or the *marginal productivity of labor*. Similarly, the partial derivative $\frac{\partial P}{\partial K}$ represents the rate of change of production with respect to capital and is called the *marginal productivity of capital*. In this context, the assumptions made by Cobb and Douglas can be stated as follows.

(i) If either labor or capital vanishes, then so will production.
(ii) The marginal productivity of labor is proportional to the amount of production per unit of labor.
(iii) The marginal productivity of capital is proportional to the amount of production per unit of capital.

The production per unit of labor is $\frac{P}{L}$. Therefore, assumption (ii) can be described by

$$\frac{\partial P}{\partial L} = \alpha\frac{P}{L}$$

for some constant α. If K is held constant ($K = K_0$), then this partial differential equation becomes an ordinary differential equation:

$$\frac{dP}{dL} = \alpha\frac{P}{L} \tag{4}$$

This is a separable differential equation with solution

$$P(L, K_0) = C_1(K_0)L^{\alpha} \tag{5}$$

Notice that the constant C_1 is written as a function of K_0 because it could depend on the value of K_0.

Similarly, assumption (iii) can be described by

$$\frac{\partial P}{\partial L} = \beta\frac{P}{K}$$

This is also a separable differential equation, with solution

$$P(L_0, K) = C_2(L_0)K^{\beta} \tag{6}$$

If we compare Equations 5 and 6, then we can write

$$P(L, K) = bL^{\alpha}K^{\beta} \tag{7}$$

where b is a constant that is independent of both L and K. Assumption (i) shows that $\alpha > 0$ and $\beta > 0$.

Suppose labor and capital both increase by a factor m. Then using Equation 7:

$$P(mL, mK) = b(mL)^{\alpha}(mK)^{\beta} = m^{\alpha+\beta}bL^{\alpha}K^{\beta} = m^{\alpha+\beta}P(L, K)$$

If $\alpha + \beta = 1$, then $P(mL, mK) = mP(L, K)$, which means that production is also increased by a factor of m. This result is why Cobb and Douglas assumed that $\alpha + \beta = 1$ and therefore

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

This is the Cobb–Douglas production function introduced in Section 11.1.

11.3 Exercises

1. The temperature T at a location in the Northern Hemisphere depends on the longitude x, latitude y, and time t, so we can write $T = f(x, y, t)$. Suppose time is measured in hours from the beginning of January.

(a) Describe the meaning of the partial derivatives $\dfrac{\partial T}{\partial x}$, $\dfrac{\partial T}{\partial y}$, and $\dfrac{\partial T}{\partial t}$ in the context of this problem.

(b) Honolulu has longitude 158° W and latitude 21° N. Suppose that at 9:00 AM on January 1 the wind is blowing hot air to the northeast, so the air to the west and south is warm and the air to the north and east is cooler. Would you expect $f_x(158, 21, 9)$, $f_y(158, 21, 9)$, and $f_t(158, 21, 9)$ to be positive or negative? Explain your reasoning.

2. At the beginning of this section, we discussed the function $I = f(T, H)$, where I is the heat index, T is the temperature, and H is the relative humidity. Use Table 11.6 to estimate $f_T(92, 60)$ and $f_H(92, 60)$. Interpret these values in the context of this problem.

3. The wind-chill index W is the perceived temperature that depends on the wind speed v and the actual temperature T, so we can write $W = f(v, T)$. The following table of values is an excerpt from Table 11.1 in Section 11.1.

Temperature (°F)

Wind speed (mph) v \ T	−5	−10	−15	−20	−25
15	−26	−32	−39	−45	−51
20	−29	−35	−42	−48	−55
25	−31	−37	−44	−51	−58
30	−33	−39	−46	−53	−60

(a) Estimate the values of $f_v(20, -15)$ and $f_T(20, -15)$ and interpret these values in the context of the problem.

(b) In general, what can you say about $\dfrac{\partial W}{\partial v}$ and $\dfrac{\partial W}{\partial T}$?

(c) Use the table to make a realistic guess at the following limit:

$$\lim_{v \to \infty} \frac{\partial W}{\partial v}$$

Explain your reasoning.

4. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.

Duration (hours)

Wind speed (knots) v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

(a) Explain the meanings of the partial derivatives $\dfrac{\partial h}{\partial v}$ and $\dfrac{\partial h}{\partial t}$ in the context of the problem.

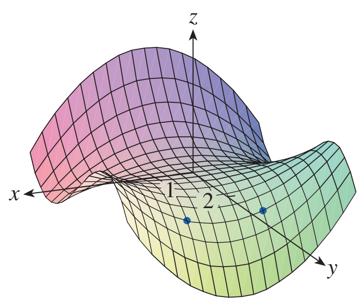
(b) Estimate the values of $f_v(40, 15)$ and $f_t(40, 15)$. Explain the meaning of these values in the context of the problem.

(c) Use the table to make a realistic guess at the following limit:

$$\lim_{t \to \infty} \frac{\partial h}{\partial t}$$

Explain your reasoning.

Determine the sign of the partial derivative for the function f whose graph is shown.



5. (a) $f_x(1, 2)$ (b) $f_y(1, 2)$

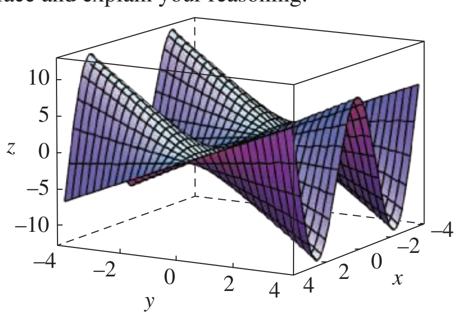
6. (a) $f_x(-1, 2)$ (b) $f_y(-1, 2)$

7. (a) $f_{xx}(-1, 2)$ (b) $f_{yy}(-1, 2)$

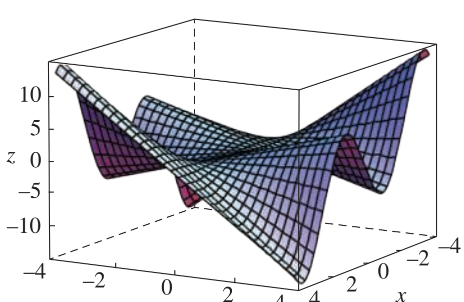
8. (a) $f_{xy}(1, 2)$ (b) $f_{xy}(-1, 2)$

9. The following surfaces, labeled (a), (b), and (c), are graphs of a function f and its partial derivatives f_x and f_y. Identify each surface and explain your reasoning.

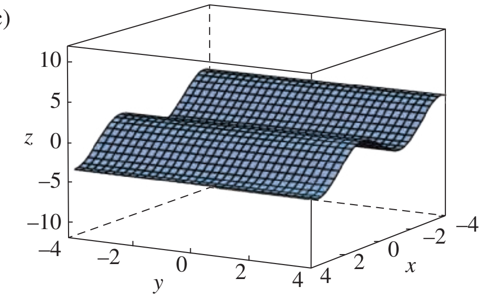
(a)



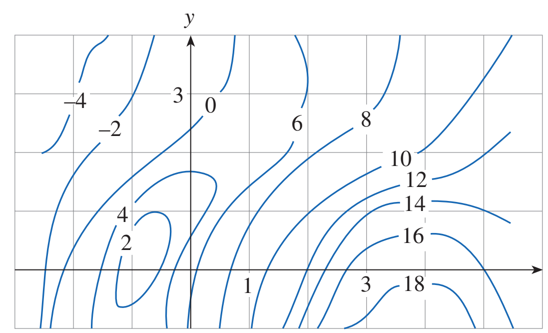
(b)



(c)



10. A contour map is given for a function f. Use this map to estimate $f_x(2, 1)$ and $f_y(2, 1)$.



11. If $f(x, y) = 16 - 4x^2 - y^2$, find $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes. Sketch a graph to illustrate these values.

12. If $f(x, y) = \sqrt{4 - x^2 - 4y^2}$, find $f_x(1, 0)$ and $f_y(1, 0)$. Sketch a graph to illustrate these values.

Find f_x and f_y and use technology to graph f, f_x, and f_y with domains and viewpoints that illustrate the relationships between them.

13. $f(x, y) = x^2 + y^2 + x^2y$ **14.** $f(x, y) = xe^{-(x^2+y^2)}$

Find the first partial derivatives of the function.

15. $f(x, y) = y^5 - 3xy$ **16.** $f(x, y) = x^4y^3 + 8x^2y$

17. $f(x, t) = e^{-t} \cos \pi x$ **18.** $f(x, t) = \sqrt{x} \ln t$

19. $z = (2x + 3y)^{10}$ **20.** $z = \tan xy$

21. $f(x, y) = \dfrac{x - y}{x + y}$ **22.** $f(x, y) = x^y$

23. $w = \sin \alpha \cos \beta$ **24.** $w = \dfrac{e^v}{u + v^2}$

25. $f(x, y) = xye^{xy}$ **26.** $f(x, y) = \sin(xy) + xy$

27. $f(r, s) = r \ln(r^2 + s^2)$ **28.** $f(x, t) = \arctan(x\sqrt{t})$

29. $u = te^{w/t}$ **30.** $f(x, y) = \dfrac{ax + by}{cx + dy}$

31. $f(x, y) = \sin(x \cos y)$ **32.** $f(x, y) = \tan^{-1}(xy^2)$

33. $f(x, y) = \displaystyle\int_y^x \cos(t^2)\, dt$ **34.** $f(x, y) = \displaystyle\int_x^y \sqrt{t^3 + 1}\, dt$

35. $f(x, y, z) = xz - 5x^2y^3z^4$ **36.** $f(x, y, z) = x \sin(y - z)$

37. $w = \ln(x + 2y + 3z)$ **38.** $w = ze^{xyz}$

39. $u = xy \sin^{-1}(yz)$ **40.** $u = x^{y/z}$

41. $f(x, y, z, t) = xyz^2 \tan(yt)$ **42.** $f(x, y, z, t) = \dfrac{xy^2}{t + 2z}$

43. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

44. $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$

Find the indicated partial derivative.

45. $f(x, y) = \ln(x + \sqrt{x^2 + y^2})$; $f_x(3, 4)$

46. $R(s, t) = te^{s/t}$; $R_t(0, 1)$

47. $f(x, y) = \arctan\left(\dfrac{y}{x}\right)$; $f_x(2, 3)$

48. $f(x, y) = y \sin^{-1}(xy)$; $f_y\left(1, \dfrac{1}{2}\right)$

49. $f(x, y, z) = \dfrac{y}{x + y + z}$; $f_y(2, 1, -1)$

50. $f(x, y, z) = \ln\left(\dfrac{1 - \sqrt{x^2 + y^2 + z^2}}{1 + \sqrt{x^2 + y^2 + z^2}}\right)$; $f_y(1, 2, 2)$

51. $f(x, y, z) = x^{yz}$; $f_z(e, 1, 0)$

52. $f(x, y, z) = \sqrt{\sin^2 x + \sin^2 y + \sin^2 z}$; $f_z\left(0, 0, \dfrac{\pi}{4}\right)$

Use the definition of partial derivatives as limits (Definition 4) to find $f_x(x, y)$ and $f_y(x, y)$.

53. $f(x, y) = xy^2 - x^3y$ **54.** $f(x, y) = \dfrac{x}{x + y^2}$

Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

55. $x^2 + y^2 + z^2 = 3xyz$ **56.** $yz = \ln(x + z)$

57. $x - z = \arctan(yz)$ **58.** $\sin(xyz) = x + 2y + 3z$

59. $e^z = xyz$ **60.** $yz + x \ln y = z^2$

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

61. (a) $z = f(x) + g(y)$ (b) $z = f(x + y)$

62. (a) $z = f(x)g(y)$ (b) $z = f(xy)$

(c) $z = f\left(\dfrac{x}{y}\right)$

Find all the second partial derivatives.

63. $f(x, y) = x^3y^5 + 2x^4y$ **64.** $f(x, y) = \sin^2(mx + ny)$

65. $f(x, y) = \ln(ax + by)$ **66.** $f(x, y) = e^{-2x} \cos y$

67. $w = \sqrt{u^2 + v^2}$ **68.** $v = \dfrac{xy}{x - y}$

69. $z = \arctan \dfrac{x + y}{1 - xy}$ **70.** $v = e^{xe^y}$

Verify that the conclusion of Clairaut's Theorem holds, that is, $u_{xy} = u_{yx}$.

71. $u = xe^{xy}$ **72.** $u = \tan(2x + 3y)$

73. $u = \sin(x^2y)$ **74.** $u = \ln(2x + y)$

Find the indicated partial derivative(s).

75. $f(x, y) = 3xy^4 + x^3y^2$; f_{xxy}, f_{yyy}

76. $f(x, y) = \sin(3x - 4y)$; f_{yxy}

77. $f(x, t) = x^2e^{-ct}$; f_{ttt}, f_{txx}

78. $f(x, y, z) = \cos(4x + 3y + 2z)$; f_{xyz}, f_{yzz}

79. $f(x, y, z) = e^x \sin(yz)$; f_{xyz}

80. $f(r, s, t) = r \ln(rs^2t^3)$; f_{rss}, f_{rst}

81. $V = \ln(r + s^2 + t^3)$; $\dfrac{\partial^3 V}{\partial r\, \partial s\, \partial t}$

82. $u = e^{r\theta} \sin \theta$; $\dfrac{\partial^3 u}{\partial r^2 \partial \theta}$

83. $u = x^a y^b z^c$; $\dfrac{\partial^6 u}{\partial x\, \partial y^2\, \partial z^3}$

84. If $f(x, y, z) = xy^2z^3 + \arcsin(x\sqrt{z})$, find f_{xzy}.

Hint: Which order of differentiation is easiest?

85. If $g(x, y, z) = \sqrt{1 + xz} + \sqrt{1 - xy}$, find g_{xyz}

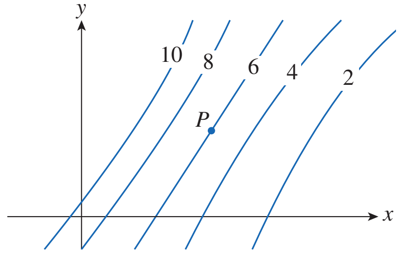
Hint: Use a different order of differentiation for each term.

86. Use the table of values of $f(x, y)$ to estimate the values of $f_x(3, 2)$, $f_x(3, 2.2)$, and $f_{xy}(3, 2)$

x \ y	1.8	2.0	2.2
2.5	12.5	10.2	9.3
3.0	18.1	17.5	15.9
3.5	20.0	22.4	26.1

87. Level curves are shown for a function f. Determine whether the following partial derivatives are positive or negative at the point P.

(a) f_x (b) f_y (c) f_{xx} (d) f_{xy} (e) f_{yy}



88. Verify that the function $u = e^{-\alpha^2 k^2 t} \sin kx$ is a solution of the *heat conduction equation* $u_t = \alpha^2 u_{xx}$.

89. Determine whether each of the following functions is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.

(a) $u = x^2 + y^2$ (b) $u = x^2 - y^2$

(c) $u = x^3 + 3xy^2$ (d) $u = \ln \sqrt{x^2 + y^2}$

(e) $u = e^{-x} \cos y - e^{-y} \cos x$

90. Verify that the function

$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

is a solution of the three-dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.

91. Show that each of the following functions is a solution of the wave equation $u_{tt} = a^2 u_{xx}$.

(a) $u = \sin(kx) \sin(akt)$

(b) $u = \dfrac{t}{a^2 t^2 - x^2}$

(c) $u = (x - at)^6 + (x + at)^6$

(d) $u = \sin(x - at) + \ln(x + at)$

92. If f and g are twice differentiable functions of a single variable, show that the function

$$u(x, t) = f(x + at) + g(x - at)$$

is a solution of the wave equation $u_{tt} = a^2 u_{xx}$.

93. If $u = e^{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n}$, where $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$, show that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = u$$

94. If $u = xe^y + ye^x$, show that

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} = x\frac{\partial^3 u}{\partial x\, \partial y^2} + y\frac{\partial^3 u}{\partial x^2 \partial y}$$

95. Show that the Cobb–Douglas production function $P = bL^\alpha K^\beta$ satisfies the equation

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha + \beta)\, P$$

96. Show that the Cobb–Douglas production function satisfies $P(L, K_0) = C_1(K_0)L^\alpha$ by solving the differential equation

$$\frac{dP}{dL} = \alpha\frac{P}{L}$$

See Equation 4.

97. The temperature at a point (x, y) on a flat metal plate is given by

$$T(x, y) = \frac{60}{1 + x^2 + y^2}$$

where T is measured in °C and x, y in meters. Find the rate of change of temperature with respect to distance at the point $(2, 1)$ in (a) the x-direction and (b) the y-direction.

98. The total resistance R produced by three conductors with resistances R_1, R_2, R_3, connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find $\dfrac{\partial R}{\partial R_1}$.

99. The *van der Waals equation* for n moles of gas is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

where P is pressure, V is volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas. Find $\dfrac{\partial T}{\partial P}$ and $\dfrac{\partial P}{\partial V}$.

100. The gas law for a fixed mass M of an ideal gas at absolute temperature T, pressure P, and volume V is $PV = nRT$, where R is the gas constant.

(a) Show that $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P} = -1$.

(b) Show that $T\dfrac{\partial P}{\partial T}\dfrac{\partial V}{\partial T} = nR$.

101. The wind-chill index is modeled by the function

$$W = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$$

where T is the temperature (°F) and v is the wind speed (mi/h). When $T = -10$°F and $v = 25$ mi/h, by how much would you expect the apparent temperature W to drop if the actual temperature decreases by 1°F? What if the wind speed increases by 1 mi/h?

102. The kinetic energy of a body with mass m and velocity v is $K = \frac{1}{2}mv^2$. Show that

$$\frac{\partial K}{\partial m}\frac{d^2 K}{dv^2} = K$$

---

103. If a, b, c are the sides of a triangle and A, B, C are the opposite angles, find $\dfrac{\partial A}{\partial a}, \dfrac{\partial A}{\partial b}, \dfrac{\partial A}{\partial c}$ by implicit differentiation of the Law of Cosines.

104. Is it possible for a function f to have partial derivatives $f_x(x, y) = x + 4y$ and $f_y(x, y) = 3x - y$? Why or why not?

105. The paraboloid $z = 6 - x - x^2 - 2y^2$ intersects the plane $x = 1$ in a parabola. Find parametric equations for the tangent line to this parabola at the point $(1, 2, -4)$. Use technology to graph the paraboloid, the parabola, and the tangent line on the same set of coordinate axes.

106. The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ intersects the plane $y = 2$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 2)$.

107. A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.1091w^{0.425}h^{0.725}$$

where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet. Calculate and interpret the partial derivatives.

(a) $\dfrac{\partial S}{\partial w}(160, 70)$ (b) $\dfrac{\partial S}{\partial y}(160, 70)$

108. One of Poiseuille's laws states that the resistance of blood flowing through an artery is

$$R = C\frac{L}{r^4}$$

where L and r are the length and radius of the artery and C is a positive constant determined by the viscosity of the blood. Find $\dfrac{\partial R}{\partial L}$ and $\dfrac{\partial R}{\partial r}$ and interpret each expression.

109. The power needed by a bird during its flapping mode can be modeled by

$$P(v, x, m) = Av^3 + \frac{B(mb/x)^2}{v}$$

where A and B are constants specific to a species of bird, v is the velocity of the bird, m is the mass of the bird, and x is the fraction of the flying time spent in flapping mode. Find $\dfrac{\partial P}{\partial v}$, $\dfrac{\partial P}{\partial x}$, and $\dfrac{\partial P}{\partial m}$ and interpret each expression.

110. In a study of frost penetration, it was found that the temperature T at time t (measured in days) at a depth x (measured in feet) can be modeled by the function

$$T(x, t) = T_0 + T_1e^{-\lambda x} \sin(\omega t - \lambda x)$$

where $\omega = \dfrac{2\pi}{365}$ and λ is positive constant.

(a) Find $\dfrac{\partial T}{\partial x}$. What is its physical significance?

(b) Find $\dfrac{\partial T}{\partial t}$. What is its physical significance?

(c) Show that T satisfies the heat equation $T_t = kT_{xx}$ for a certain constant k.

(d) If $\lambda = 0.2$, $T_0 = 0$, and $T_1 = 10$, use technology to graph $T(x, t)$.

(e) What is the physical significance of the term $-\lambda x$ in the expression $\sin(\omega t - \lambda x)$?

111. The average energy E (in kcal) needed for a lizard to walk or run a distance of 1 km has been modeled by the equation

$$E(m, v) = 2.65m^{0.66} + \frac{3.5m^{0.75}}{v}$$

where m is the body mass of the lizard (in grams) and v is its speed (in km/h). Calculate $E_m(400, 8)$ and $E_v(400, 8)$ and explain the meaning of these values in the context of this problem.

Source: C. Robbins, *Wildlife Feeding and Nutrition*, 2d ed. (San Diego: Academic Press, 1993).

112. If

$$f(x, y) = x(x^2 + y^2)^{-3/2}e^{\sin(x^2y)}$$

find $f_x(1, 0)$.

Hint: Instead of finding $f_x(x, y)$ first, it is probably easier to use Equation 1 or Equation 2.

113. If $f(x, y) = \sqrt[3]{x^3 + y^3}$, find $f_x(0, 0)$.

114. Let

$$f(x, y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Use technology to graph f.

(b) Find $f_x(x, y)$ and $f_y(x, y)$ when $(x, y) \neq (0, 0)$.

(c) Find $f_x(0, 0)$ and $f_y(0, 0)$ using Equations 2 and 3.

(d) Show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.

(e) Does the result in part (d) contradict Clairaut's Theorem? Use the graphs of f_{xy} and f_{yx} to illustrate your answer.

11.4 Tangent Planes and Linear Approximations

One of the most important ideas in single-variable calculus is local linearity, that is, if we zoom in toward a point on the graph of a differentiable function, the graph looks like a straight line; it becomes indistinguishable from its tangent line. Therefore, we can approximate the function by a linear function. (See Section 3.9.)

In this section, we will develop a similar idea in three dimensions. As we zoom in toward a point on a surface that is the graph of a differentiable function of two variables, the surface looks more and more like a plane (its tangent plane), and we can approximate the function by a linear function of two variables. We will also extend the idea of a differential to functions of two or more variables.

Tangent Planes

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. As in the preceding section, let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2, as illustrated in Figure 11.31.

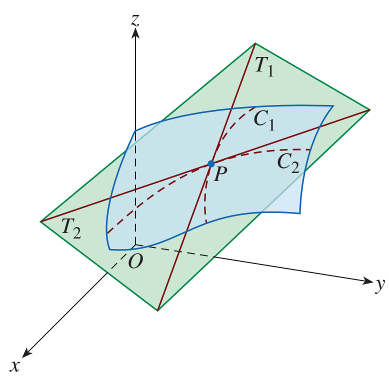


Figure 11.31
The tangent plane contains the tangent lines T_1 and T_2.

We will learn in Section 11.6 that if C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane. Therefore, you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P. The tangent plane at P is the plane that most closely approximates the surface S near the point P.

Recall that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

If we divide this equation by C and let $a = -\dfrac{A}{C}$ and $b = -\dfrac{B}{C}$, then we can write the equation of the plane in the form

$$z - z_0 = a(x - x_0) + b(y - y_0) \tag{1}$$

If Equation 1 represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1. If we set $y = y_0$ in Equation 1, we get

$$z - z_0 = a(x - x_0) \quad \text{where } y = y_0$$

This equation represents a line (in point-slope form) with slope a. In Section 11.3, we learned that the slope of the tangent line T_1 is $f_x(x_0, y_0)$. Therefore, $a = f_x(x_0, y_0)$.

Similarly, if we let $x = x_0$ in Equation 1, we get $z - z_0 = b(y - y_0)$, which must represent the tangent line T_2. Therefore, $b = f_y(x_0, y_0)$. These results lead to a description of the tangent plane in terms of partial derivatives.

Note the similarity between the equation of a tangent plane and the equation of a tangent line:

$y - y_0 = f'(x_0)(x - x_0)$.

Equation of the Tangent Plane

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{2}$$

Example 1 Find a Tangent Plane Equation

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution

Let $f(x, y) = 2x^2 + y^2$.

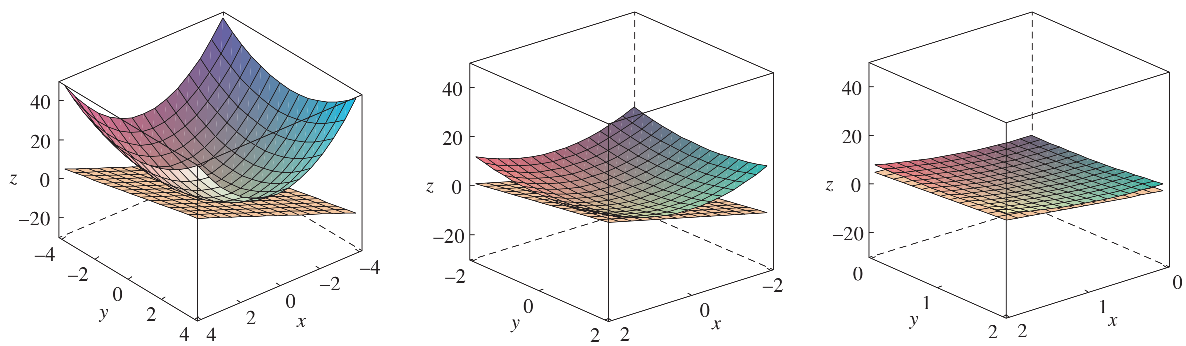
Find the partial derivatives and evaluate each at $(x_0, y_0) = (1, 1)$.

$$f_x(x, y) = 4x \quad \Rightarrow \quad f_x(1, 1) = 4$$
$$f_y(x, y) = 2y \quad \Rightarrow \quad f_y(1, 1) = 2$$

Use Equation 2 to write the equation of the tangent plane at $(1, 1, 3)$.

$$z - 3 = 4(x - 1) + 2(y - 1) \quad \text{or} \quad z = 4x + 2y - 3$$ ■

Figure 11.32(a) shows the elliptic paraboloid and its tangent plane at $(1, 1, 3)$ found in Example 1. In parts (b) and (c), we zoom in toward the point $(1, 1, 3)$ by restricting the domain of the function $f(x, y) = 2x^2 + y^2$. Notice that the more we zoom in, the flatter the graph appears and the more it resembles its tangent plane.



(a) Graph of the elliptic paraboloid and the tangent plane (b) Zoom in near the point $(1, 1, 3)$. (c) Zoom in even more.

Figure 11.32
The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward $(1, 1, 3)$.

In Figure 11.33, we confirm this observation by zooming in toward the point $(1, 1)$ on a contour map of the function $f(x, y) = 2x^2 + y^2$. Notice that the more we zoom in, the more the level curves look like equally spaced parallel lines, which is characteristic of a plane.

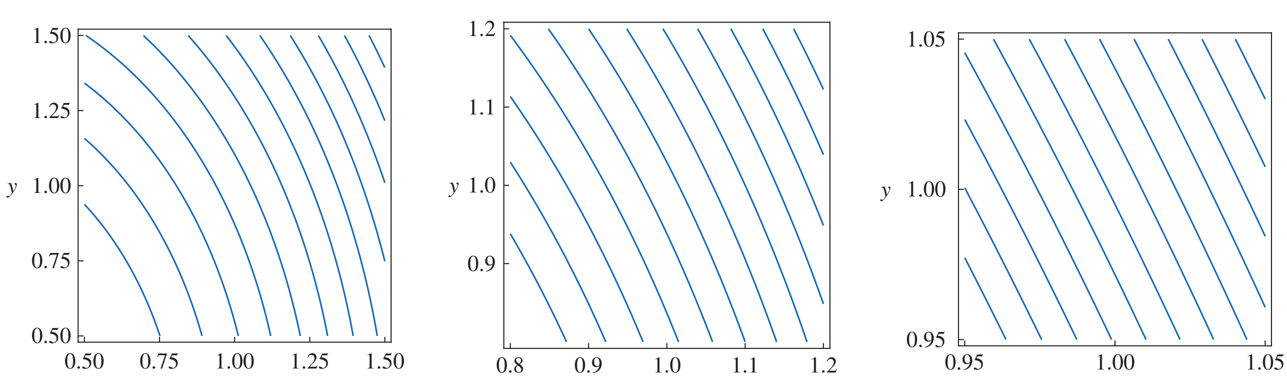


Figure 11.33
Contour maps of $f(x, y) = 2x^2 + y^2$ as we zoom in toward the point $(1, 1)$.

Linear Approximations

In Example 1, we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point (1, 1, 3) is $z = 4x + 2y - 3$. The visual evidence in Figures 11.32 and 11.33 suggests that the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near (1, 1). The function L is called the *linearization* of f at (1, 1) and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the *linear approximation* or *tangent plane approximation* of f at (1, 1).

For example, at the point (1.1, 0.95) the linear approximation is

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$

which is quite close to the true value of $f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225$.

However, as we move farther away from (1, 1), for example, to (2, 3), then the approximation is not as accurate. In fact, in this case $L(2, 3) = 11$, whereas $f(2, 3) = 17$.

In general, using Equation 2, an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{3}$$

is called the **linearization** of f at (a, b) and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{4}$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

We now have a method to find an equation of the tangent plane for surfaces $z = f(x, y)$ where f has continuous first partial derivatives. Let's consider the case where f_x and f_y are not continuous. Figure 11.34 shows the graph of the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

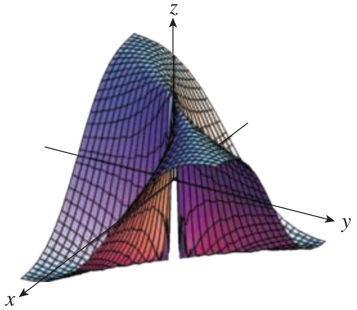


Figure 11.34
Graph of the function f defined by $f(x, y) = \dfrac{xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.

You can verify (Exercise 58) that the partial derivatives exist at the origin and, in fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, but f_x and f_y are not continuous. The linear approximation would be $f(x, y) \approx 0$, but $f(x, y) = \dfrac{1}{2}$ at all points on the line $y = x$.

So, a function of two variables can behave badly even though both of its partial derivatives exist. To recognize this kind of function behavior, we need to consider the idea of a differentiable function of two variables.

Recall that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

In Chapter 3, we showed that if f is differentiable at a, then

$$\Delta y = f'(a)\,\Delta x + \epsilon\,\Delta x \qquad \text{where } \epsilon \to 0 \text{ as } \Delta x \to 0 \tag{5}$$

Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) \tag{6}$$

Therefore the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$. Analogous to Equation 5, we can define the differentiability of a function of two variables as follows.

Definition • Differentiability at a Point

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

This definition means that for a differentiable function, the linear approximation in Equation 4 is pretty accurate when (x, y) is near (a, b). Geometrically, this means that the tangent plane is a good approximation to the graph of f near the point of tangency.

It can be difficult to use this definition explicitly to check the differentiability of a function. However, the next theorem provides a convenient sufficient condition for differentiability.

A proof of this theorem is given in the Appendix.

Theorem • Partial Derivatives and Differentiability at a Point

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example 2 Use a Linearization to Estimate Function Value

Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use this expression to approximate $f(1.1, -0.1)$.

Solution

Find the partial derivatives and evaluate each at $(1, 0)$.

$$f_x(x, y) = e^{xy} + xye^{xy} \quad \Rightarrow \quad f_x(1, 0) = e^{1\cdot 0} + 1 \cdot 0 \cdot e^{1\cdot 0} = 1$$
$$f_y(x, y) = x^2 e^{xy} \quad \Rightarrow \quad f_y(1, 0) = 1^2 \cdot e^{1\cdot 0} = 1$$

Both f_x and f_y are continuous functions, so f is differentiable.

The linearization is

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) \qquad \text{Equation 3.}$$
$$= 1 + 1(x - 1) + 1 \cdot y = x + y \qquad \text{Use values for } f, f_x, f_y\text{; simplify.}$$

The corresponding approximation is $xe^{xy} \approx x + y$.

Therefore, $f(1.1, -0.1) \approx 1.1 - 0.1 = 1$.

Compare this approximation with the actual value:

$$f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$$

Figure 11.35 shows the graphs of the function f and its linearization L. ■

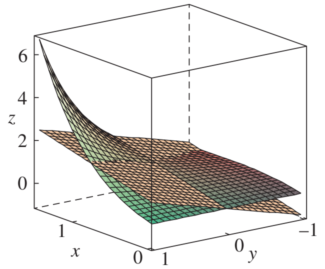


Figure 11.35
The graph of f and its linearization L.

Example 3 Estimate the Heat Index from Tabular Values

At the beginning of Section 11.3, we discussed the heat index (perceived temperature) I as a function of the actual temperature T and the relative humidity H. The following table of values of I was compiled from the National Weather Service.

Relative humidity (%)

Actual temperature (°F)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	102	106	109	113	117	122
92	99	101	105	108	112	116	121	126	131
94	103	106	110	114	119	124	129	135	141
96	108	112	116	121	126	132	138	145	152
98	113	117	123	128	134	141	148	155	164
100	118	124	129	136	143	150	158	167	176

Find a linear approximation for the heat index $I = f(T, H)$, when T is near 96°F and H is near 70%. Use this approximation to estimate the heat index when the temperature is 97°F and the relative humidity is 72%.

Solution

Read from the table: $f(96, 70) = 126$.

In Section 11.3, we used the table to estimate the values: $f_T(96, 70) \approx 3.75$ and $f_H(96, 70) \approx 1.1$.

The linear approximation is:

$$f(T, H) \approx f(96, 70) + f_T(96, 70)(T - 96) + f_H(96, 70)(H - 70) \qquad \text{Equation 3.}$$

$$= 126 + 3.75(T - 96) + 1.1(H - 70) \qquad \text{Use values obtained from the table.}$$

Use this expression to find the desired estimate.

$$f(97, 72) \approx 126 + 3.75(1) + 1.1(2) = 131.95$$

Therefore, when $T = 97$°F and $H = 72$%, the heat index I is approximately 132°F. ■

■ Differentials

The concept of a linear approximation is often developed using the terminology and notation associated with *differentials*. If, $y = f(x)$, where f is a differentiable function, then the differential dx is considered an independent variable; that is, dx can take on the value of any real number. The differential dy is then defined in terms of dx by the equation

$$dy = f'(x)\, dx \tag{7}$$

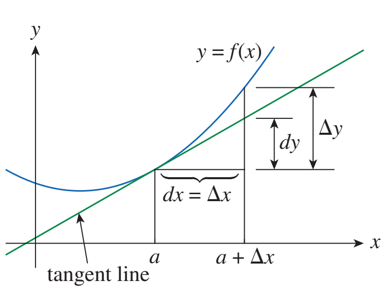


Figure 11.36
A geometric illustration of differentials.

(See Section 3.9.) Figure 11.36 shows the relationship between the increment Δy and the differential dy; Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can take on the value of any real number. The **differential** dz, also called the **total differential**, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \tag{8}$$

Note how this expression is similar to Equation 7, and that the notation df is often used in place of dz.

If we let $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation 8, then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Therefore, using differential notation, the linear approximation in Equation 4 can be written as

$$f(x, y) \approx f(a, b) + dz$$

Figure 11.37 is the three-dimensional extension of Figure 11.36. This figure shows the geometric interpretation of the differential dz and the increment Δz: dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

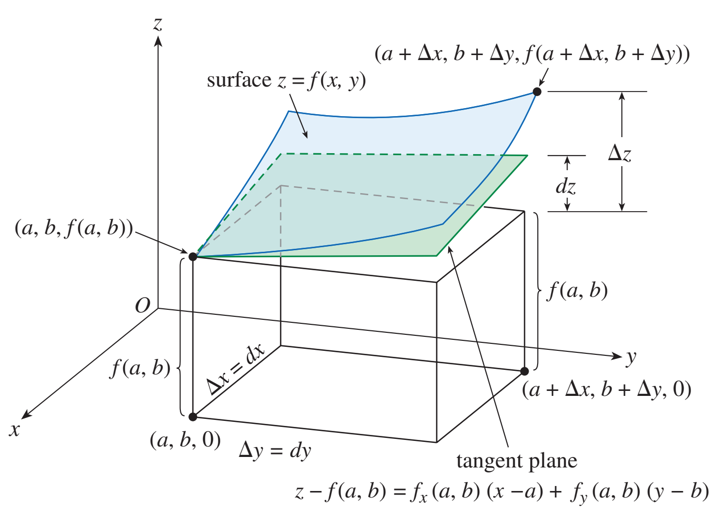


Figure 11.37
A geometric interpretation of differentials in three-dimensions.

Example 4 Differentials Versus Increments

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Solution

(a) Use the definition for dz.

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x + 3y)dx + (3x - 2y)dy$$

(b) Let $x = 2$, $dx = \Delta x = 2.05 - 2 = 0.05$, $y = 3$, and $dy = \Delta y = 2.96 - 3 = -0.04$.

$$dz = [(2(2) + 3(3)](0.05) + [3(2) - 2(3)](-0.04) = 0.65$$

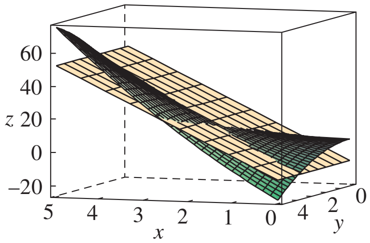


Figure 11.38
The tangent plane is a good approximation to the surface $z = f(x, y)$ near the point (2, 3, 13).

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3) && \text{Definition of } \Delta z.\\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2] && \text{Evaluate } f.\\ &= 0.6449 && \text{Simplify.}\end{aligned}$$

Notice that $\Delta z \approx dz$, but dz is easier to compute.

Figure 11.38 shows the tangent plane to the surface $z = x^2 + 3xy - y^2$ at the point where $(x, y) = (2, 3)$. ■

Example 5 Use Differentials to Estimate an Error

The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Solution

The volume of a cone with base radius r and height h is $V = \frac{1}{3}\pi r^2 h$.

Find the differential of V.

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh$$

Each error is at most 0.1 cm $\Rightarrow$ $|\Delta r| \le 0.1$ and $|\Delta h| \le 0.1$.

To find the largest error in the volume, consider the largest error in the measurement of r and h.

Therefore, let $dr = 0.1$ and $dh = 0.1$ along with $r = 10$ and $h = 25$.

$$\begin{aligned}dV &= \frac{2\pi(10)(25)}{3}(0.1) + \frac{\pi(10)^2}{3}(0.1)\\ &= \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi\end{aligned}$$

The maximum error in the calculated volume is about 20π cm^3 $\approx$ 63 cm^3. ■

Functions of Three or More Variables

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one in the definition of differentiability at a point. For a differentiable function of three variables, the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

If $w = f(x, y, z)$, then the **increment** of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The **differential** dw is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

Example 6 Estimate the Maximum Error

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Solution

If the dimensions of the box are x, y, and z, then the volume is $V = xyz$.

Find the differential dV:

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = yz\,dx + xz\,dy + xy\,dz$$

We are given $|\Delta x| \leq 0.2$, $|\Delta y| \leq 0.2$, and $|\Delta z| \leq 0.2$.

To find the largest error in the volume, use $dx = 0.2$, $dy = 0.2$, and $dz = 0.2$ together with $x = 75$, $y = 60$, and $z = 40$.

$$\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980$$

Thus, an error of only 0.2 cm in measuring each dimension could lead to an error of as much as 1980 cm^3 in the calculated volume! This may seem like a large error, but it's only about 1% of the volume of the box. ■

Tangent Planes to Parametric Surfaces

Parametric surfaces were introduced in Section 10.5. Let's use these concepts to find the tangent plane to a parametric surface S traced out by a vector function

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$. If we keep u constant by letting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1 lying on S. (See Figure 11.39.) The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v:

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\,\mathbf{k}$$

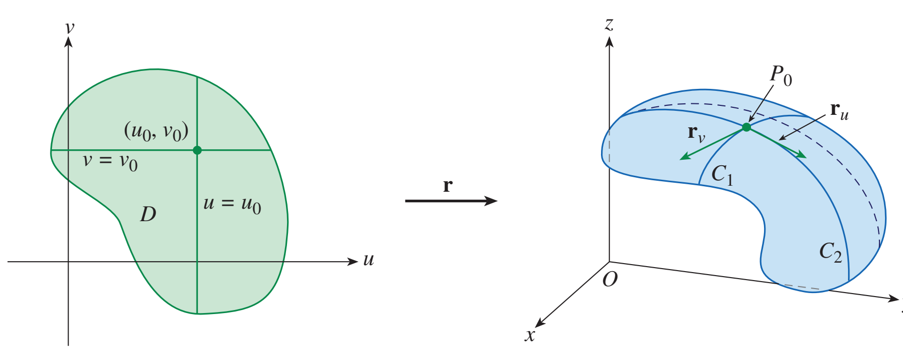


Figure 11.39
The lines $u = u_0$ and $v = v_0$ in the domain D are mapped to the curves C_1 and C_2 on S.

Similarly, if we keep v constant by letting $v = v_0$, we obtain a grid curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S, and its tangent vector at P_0 is

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\mathbf{k}$$

If $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface S is called **smooth**; similar to the graph of a function of a single variable, it has no *corners* or *sharp edges*. For a smooth surface, the **tangent plane** is the plane that contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

Example 7 Find a Tangent Plane

Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1, 1, 3)$.

Solution

Find the tangent vectors.

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k} = 2u\,\mathbf{i} + \mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial y}{\partial v}\,\mathbf{j} + \frac{\partial z}{\partial v}\,\mathbf{k} = 2v\,\mathbf{j} + 2\mathbf{k}$$

Find the normal vector to the tangent plane.

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\,\mathbf{i} - 4u\,\mathbf{j} + 4uv\,\mathbf{k}$$

The point $(1, 1, 3)$ corresponds to the parameter values $u = 1$ and $v = 1$.

Therefore, the normal vector is:

$-2(1)\,\mathbf{i} - 4(1)\,\mathbf{j} + 4(1)(1)\,\mathbf{k} = -2\,\mathbf{i} - 4\,\mathbf{j} + 4\,\mathbf{k}$.

And finally, an equation of the tangent plane at the point $(1, 1, 3)$ is

$-2(x - 1) - 4(y - 1) + 4(z - 3) = 0 \quad \Rightarrow \quad x + 2y - 2z + 3 = 0.$

Figure 11.40 shows a graph of the surface with these parametric equations and the tangent plane. ■

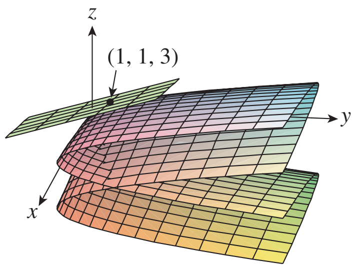


Figure 11.40
Graph of the surface and the tangent plane.

11.4 Exercises

Find an equation of the tangent plane to the given surface at the specified point.

1. $z = 3y^2 - 2x^2 + x$, $\quad(2, -1, -3)$

2. $z = 3(x - 1)^2 + 2(y + 3)^2 + 7$, $\quad(2, -2, 12)$

3. $z = \sqrt{xy}$, $\quad(1, 1, 1)$

4. $z = \dfrac{x}{y^2}$, $\quad(-4, 2, -1)$

5. $z = e^{x-y}$, $\quad(2, 2, 1)$

6. $z = xe^{xy}$, $\quad(2, 0, 2)$

7. $z = y\cos(x - y)$, $\quad(2, 2, 2)$

8. $z = \ln(x - 2y)$, $\quad(3, 1, 0)$

Use technology to graph the surface and the tangent plane at the given point. Choose the domain and viewpoint so that the surface and plane are evident. Zoom in to confirm that the surface and the tangent plane become indistinguishable.

9. $z = x^2 + xy + 3y^2$, $\quad(1, 1, 5)$

10. $z = \sqrt{9 + x^2y^2}$, $\quad(2, 2, 5)$

11. $z = \arctan(xy^2)$, $\quad\left(1, 1, \dfrac{\pi}{4}\right)$

Use technology to compute the partial derivatives and to graph the surface and the tangent plane at the given point. Choose the domain and viewpoint so that the surface and plane are evident. Zoom in to confirm that the surface and the tangent plane become indistinguishable.

12. $f(x, y) = \dfrac{xy\sin(x - y)}{1 + x^2 + y^2}$, $(1, 1, 0)$

13. $f(x, y) = e^{-xy/10}(\sqrt{x} + \sqrt{y} + \sqrt{xy})$, $(1, 1, 3e^{-0.1})$

Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

14. $f(x, y) = x\sqrt{y}$, $(1, 4)$

15. $f(x, y) = x^3y^4$, $(1, 1)$

16. $f(x, y) = \dfrac{x}{x + y}$, $(2, 1)$

17. $f(x, y) = \sqrt{x + e^{4y}}$, $(3, 0)$

18. $f(x, y) = 4\arctan(xy)$, $(1, 1)$

19. $f(x,y) = y + \sin\left(\dfrac{x}{y}\right)$, $(0, 3)$

Verify the linear approximation at $(0, 0)$.

20. $\dfrac{2x + 3}{4y + 1} \approx 3 + 2x - 12y$

21. $\sqrt{y + \cos^2 x} \approx 1 + \frac{1}{2}y$

22. $e^x \ln[(x + 1)(y + 1)] \approx x + y$

23. $\dfrac{e^{x+y}}{\cos(x + y)} \approx 1 + x + y$

24. Given that f is a differentiable function with $f(2, 5) = 6$, $f_x(2, 5) = 1$, and $f_y(2, 5) = -1$, use a linear approximation to estimate $f(2.2, 4.9)$.

25. Find the linear approximation of the function $f(x, y) = \ln(x - 3y)$ at $(7, 2)$ and use it to approximate $f(6.9, 2.06)$. Illustrate this result by graphing f and the tangent plane.

26. Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(3, 2, 6)$ and use it to approximate the value of $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.

27. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table. Use the table to find a linear approximation to the wave height function when v is near 40 knots and t is near 20 hours. Then estimate the wave heights when the wind has been blowing for 24 hours at 43 knots.

Duration (hours)

Wind speed (knots)

v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

28. Use the table in Example 3 to find a linear approximation to the heat index function when the temperature is near 94°F and the relative humidity is near 80%. Then estimate the heat index when the temperature is 95°F and the relative humidity is 78%.

29. The wind-chill index W is the perceived temperature when the wind speed is v and the actual temperature is T, so we can write $W = f(v, T)$. The following table is an excerpt from Table 1, Section 11.1. Use the table to find a linear approximation to the wind-chill index function when v is near 20 mi/h and T is near -10°F. Then estimate the wind-chill index when the wind speed is 22 mph and the temperature is -13°F.

Temperature (°F)

Wind speed (mph)

v \ T	5	0	−5	−10	−15	−20
5	−5	−11	−16	−22	−28	−34
10	−10	−16	−22	−28	−35	−41
15	−13	−19	−26	−32	−39	−45
20	−15	−22	−29	−35	−42	−48
25	−17	−24	−31	−37	−44	−51
30	−19	−26	−33	−39	−46	−53

Find the differential of the function.

30. $z = x^3 \ln(y^2)$

31. $u = e^{-t}\sin(s + 2t)$

32. $m = p^5q^3$

33. $T = \dfrac{v}{1 + uvw}$

34. $R = \alpha\beta^2 \cos\gamma$

35. $w = xye^{xz}$

36. $H = x^4y^2 + y^5z^3$

37. $u = \sqrt{3x^2 + y^4}$

38. If $z = 5x^2 + y^2$ and (x, y) changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of Δz and dz.

39. If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of Δz and dz.

40. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

41. The dimensions of a closed rectangular box are measured as 80 cm, 60 cm, and 50 cm, respectively, with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

42. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

43. The wind-chill index is modeled by the function

$$W = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$$

where T is the temperature (°F) and v is the wind speed (mi/h). Suppose the wind speed is measured as 21 mi/h with a possible error of ± 2 mi/h, and the temperature is measured as -12°F with a possible error of ± 1°F. Use differentials to estimate the maximum error in the calculated value of W due to the measurement errors in v and T.

44. A model for the surface area of a human body is given by $S = 0.1091w^{0.425}h^{0.725}$, where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet. If the errors in measurement of w and h are at most 2%, use differentials to estimate the maximum percentage error in the calculated surface area.

45. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

46. If R is the total resistance of three resistors, connected in parallel, with resistances R_1, R_2, and R_3, then

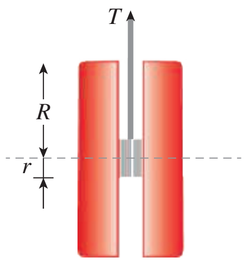
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If the resistances are measured in ohms as $R_1 = 25\ \Omega$, $R_2 = 40\ \Omega$, and $R_3 = 50\ \Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of R.

47. The tension in the string of the yo-yo in the following figure is

$$T = \frac{mgR}{2r^2 + R^2}$$

where m is the mass of the yo-yo and g is the acceleration due to gravity. Use differentials to estimate the change in the tension if R is increased from 3 cm to 3.1 cm and r is increased from 0.7 cm to 0.8 cm. Does the tension increase or decrease?



48. Four positive numbers, each less than 50, are rounded to the first decimal place and then multiplied together. Use differentials to estimate the maximum possible error in the computed product that might result from the rounding.

Find an equation of the tangent plane to the given parametric surface at the specified point. Use technology to sketch the graph of the surface and the tangent plane.

49. $x = u + v, \quad y = 3u^2, \quad z = u - v; \quad (2, 3, 0)$

50. $x = u^2, \quad y = v^2, \quad z = uv; \quad u = 1, \quad v = 1$

51. $\mathbf{r}(u, v) = u^2\,\mathbf{i} + 2u \sin v\,\mathbf{j} + u \cos v\,\mathbf{k}; \quad u = 1, \quad v = 0$

52. $\mathbf{r}(u, v) = uv\,\mathbf{i} + u \sin v\,\mathbf{j} + v \cos u\,\mathbf{k}; \quad u = 0, \quad v = \pi$

53. $\mathbf{r}(u, v) = u\,\mathbf{i} + \ln(uv)\,\mathbf{j} + v\,\mathbf{k}; \quad u = 1, \quad v = 1$

---

54. Suppose you need to know an equation of the tangent plane to a surface S at the point $P(2, 1, 3)$. Although we do not know the equation for S, we do know that the curves

$$\mathbf{r}_1(t) = \langle 2 + 3t, 1 - t^2, 3 - 4t + t^2 \rangle$$
$$\mathbf{r}_2(u) = \langle 1 + u^2, 2u^3 - 1, 2u + 1 \rangle$$

both lie on S. Find an equation of the tangent plane at P.

Show that the function is differentiable by finding values of ϵ_1 and ϵ_2 that satisfy the definition of differentiability of a function.

55. $f(x, y) = x^2 + y^2$

56. $f(x, y) = xy - 5y^2$

---

57. Prove that if f is a function of two variables that is differentiable at (a, b), then f is continuous at (a, b).

Hint: Show that

$$\lim_{(\Delta x, \Delta y) \to (0, 0)} f(a + \Delta x, b + \Delta y) = f(a, b)$$

58. (a) The graph of the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is shown in Figure 11.34. Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but f is not differentiable at $(0, 0)$. [Hint: Use the result of Exercise 57.]

(b) Explain why f_x and f_y are not continuous at $(0, 0)$.

11.5 The Chain Rule

Recall that the Chain Rule for functions of a single variable provides a method for differentiating a composite function: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \tag{1}$$

In this section, we will extend the Chain Rule to functions of more than one variable.

The Chain Rule: Case 1

For functions of more than one variable, the Chain Rule has several versions, and each of them provides a rule for differentiating a composite function. The first version involves the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t. This means that z is indirectly a function of t, $z = f(g(t), h(t))$. The Chain Rule gives us a formula for differentiating z as a function of t. We assume that f is differentiable. Recall that this is the case when f_x and f_y are continuous.

The Chain Rule (Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{2}$$

Proof

A change of Δt in t produces changes of Δx in x and Δy in y. These, in turn, produce a change of Δz in z.

Since f is differentiable, then by definition:

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Note that if the functions ϵ_1 and ϵ_2 are not defined at $(0, 0)$, we can define them to be 0 there.

Divide both sides of this equation by Δt:

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\partial x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}$$

If we now let $\Delta t \to 0$, then $\Delta x = g(t + \Delta t) - g(t) \to 0$ because g is differentiable and, therefore, continuous.

Similarly, let $\Delta y \to 0$. This, in turn, means that $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$.

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \qquad \text{Definition.}$$

$$= \frac{\partial f}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} + \left(\lim_{\Delta t \to 0} \epsilon_1 \right) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \left(\lim_{\Delta t \to 0} \epsilon_2 \right) \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

Limit properties.

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} \qquad \text{Evaluate limits.}$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad \text{Simplify.}$$

Note the similarity to the definition of the differential:

$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

Since we often write $\frac{\partial z}{\partial x}$ in place of $\frac{\partial f}{\partial x}$, we can rewrite the Chain Rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Example 1 Use the Chain Rule

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$.

Solution

Use the Chain Rule.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

We could substitute the expressions for x and y in terms of t. But it's not necessary here. Note that when $t = 0$, $x = \sin 0 = 0$ and $y = \cos 0 = 1$.

Therefore,

$$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 3)(2\cos 0) + (0 + 0)(-\sin 0) = 6.$$

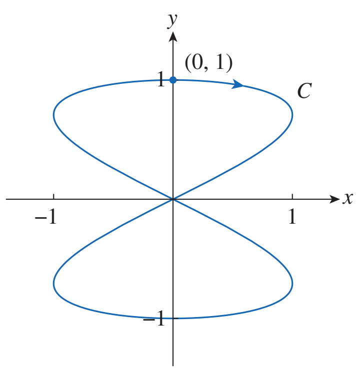


Figure 11.41
Graph of the curve described by the parametric equations $x = \sin 2t$, $y = \cos t$.

The derivative in Example 1 can be interpreted as the rate of change of z with respect to t as the point (x, y) moves along the curve C with parametric equations $x = \sin 2t$, $y = \cos t$. (See Figure 11.41.) In particular, when $t = 0$, the point (x, y) is $(0, 1)$ and $\frac{dz}{dt} = 6$ is the rate of increase as we move along the curve C through $(0, 1)$. If, for instance, $z = T(x, y) = x^2y + 3xy^4$ represents the temperature at the point (x, y), then the composite function $z = T(\sin 2t, \cos t)$ represents the temperature at points on C and the derivative $\frac{dz}{dt}$ represents the rate at which the temperature changes along C.

Example 2 The Chain Rule and Chemistry

The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

Solution

Let t represent the time elapsed, in seconds. At the given instant:

$$T = 300, \quad \frac{dT}{dt} = 0.1, \quad V = 100, \quad \frac{dV}{dt} = 0.2$$

Solve the expression for P: $P = 8.31\dfrac{T}{V}$.

Use the Chain Rule:

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt}$$

$$= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2) = -0.04155$$

The pressure is decreasing at a rate of about 0.042 kPa/s. ■

The Chain Rule: Case 2

Now let's consider the situation in which $z = f(x, y)$ but each of x and y is a function of two variables s and t: $x = g(s, t)$, $y = h(s, t)$. Then z is indirectly a function of s and t and we would like to find expressions for $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Recall that in computing $\dfrac{\partial z}{\partial t}$ we hold s fixed and compute the ordinary derivative of z with respect to t. Therefore, we can use Case 1 of the Chain Rule to obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

A similar result holds for $\dfrac{\partial z}{\partial s}$ and these arguments prove the following version of the Chain Rule.

The Chain Rule (Case 2)

Suppose that $x = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{3}$$

Example 3 The Chain Rule with Two Independent Variables

If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Solution

Apply Case 2 of the Chain Rule.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$ Chain Rule Case 2.

$$= t^2e^{st^2}\sin(s^2t) + 2ste^{st^2}\cos(s^2t)$$ Use expressions for x and y.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$ Chain Rule Case 2.

$$= 2ste^{st^2}\sin(s^2t) + s^2e^{st^2}\cos(s^2t)$$ Use expressions for x and y.

Case 2 of the Chain Rule involves three types of variables: s and t are **independent** variables, x and y are called **intermediate** variables, and z is the **dependent** variable. Notice that the expressions in Equation 3 include one term for each intermediate variable and each of these terms is similar to the one-dimensional Chain Rule in Equation 1.

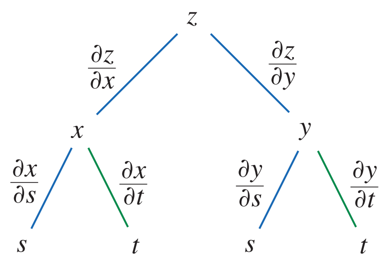


Figure 11.42
A tree diagram to visualize the Chain Rule.

A **tree diagram**, shown in Figure 11.42, is a visual way to remember the Chain Rule. Draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y. Then draw branches from x and y to the independent variables s and t. Write the partial derivative along each branch. To find $\frac{\partial z}{\partial s}$, find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Similarly, we can find $\frac{\partial z}{\partial t}$ by using the paths from z to t.

The Chain Rule: General Version

Now we can consider the general situation in which a dependent variable u is a function of n intermediate variables $x_1, x_2, \ldots, x_n$, each of which is, in turn, a function of m independent variables $t_1, t_2, \ldots, t_m$. Notice that there are n terms, one for each intermediate variable. The proof is similar to that of Case 1.

The Chain Rule (General Version)

Suppose u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i} \tag{4}$$

for each $i = 1, 2, \ldots, m$.

Example 4 The Chain Rule: Two Independent Variables and Four Intermediate Variables

Write out the Chain Rule for the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ and $t = t(u, v)$.

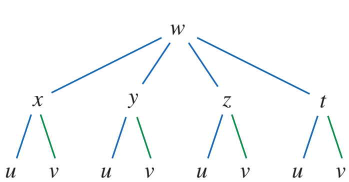


Figure 11.43
A tree diagram to visualize the Chain Rule with $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ and $t = t(u, v)$.

Solution

Use the general version of the Chain Rule with $n = 4$ and $m = 2$.

Figure 11.43 shows the associated tree diagram. Note that the branches are not labeled with the derivatives, but it is understood that if a branch leads from y to u, for example, then the partial derivative for that branch is $\frac{\partial y}{\partial u}$.

With some help from the tree diagram, write the desired expressions.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$ ■

Example 5 The Chain Rule: Three Independent Variables and Three Intermediate Variables

If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s \sin t$, find the value of $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = 0$.

Solution

Figure 11.44 shows the associated tree diagram.

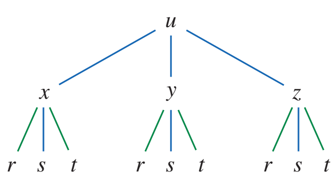


Figure 11.44
A tree diagram to visualize the Chain Rule with $w = f(x, y, z)$ and $x = x(r, s, t)$, $y = y(r, s, t)$, and $z = z(r, s, t)$.

Use the general Chain Rule, find all the partial derivatives, and then evaluate $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = 0$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$ General Chain Rule.

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2 \sin t)$$ Partial derivatives.

When $r = 2$, $s = 1$, and $t = 0$:

$$x = 2 \cdot 1 \cdot e^0 = 2, \quad y = 2 \cdot 1^2 \cdot e^{-0} = 2, \quad z = 2^2 \cdot 1 \cdot \sin 0 = 0$$

$$\frac{\partial u}{\partial s} = (4 \cdot 2^3 \cdot 2)(2 \cdot e^0) + (2^4 + 2 \cdot 2 \cdot 0^3)(2 \cdot 2 \cdot 1 \cdot e^{-0}) + (3 \cdot 2^2 \cdot 0^2)(2^2 \cdot \sin 0)$$

$$= (64)(2) + (16)(4) + (0)(0) = 192$$ ■

Example 6 Partial Differential Equation

If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

Solution

Let $x = s^2 - t^2$ and $y = t^2 - s^2$. Then $g(s, t) = f(x, y)$.

Use the Chain Rule.

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

Use these expressions in the given partial differential equation.

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \left(2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}\right) + \left(-2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}\right) = 0$$ ■

Example 7 Second-Order Partial Derivatives

If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find (a) $\dfrac{\partial z}{\partial r}$ and (b) $\dfrac{\partial^2 z}{\partial r^2}$.

Solution

(a) Use the Chain Rule.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$$

(b) Apply the Product Rule to the expression in part (a).

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)\\ &= 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) \qquad (5)\end{aligned}$$

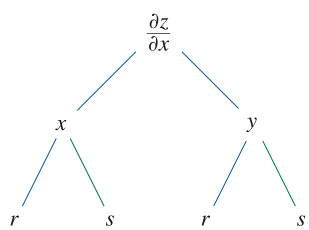


Figure 11.45
A tree diagram to visualize the Chain Rule with dependent variable $\dfrac{\partial z}{\partial x}$.

Figure 11.45 shows the tree diagram associated with the Chain Rule in which the dependent variable is $\dfrac{\partial z}{\partial x}$.

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2}(2r) + \frac{\partial^2 z}{\partial y \partial x}(2s)$$

Similarly, for $\dfrac{\partial z}{\partial y}$.

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x\, \partial y}(2r) + \frac{\partial^2 z}{\partial y^2}(2s)$$

Use these expressions in Equation 5 and the equality of mixed second order derivatives.

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= 2\frac{\partial z}{\partial x} + 2r\left(2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y \partial x}\right) + 2s\left(2r\frac{\partial^2 z}{\partial x \partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right)\\ &= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x \partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Implicit Differentiation

The Chain Rule can be used to provide a more complete description of the process of implicit differentiation that was introduced in Sections 3.5 and 11.3. Suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f. If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x. Since both x and y are functions of x, we obtain,

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

But $\dfrac{dx}{dx} = 1$, and if $\dfrac{\partial F}{\partial y} \neq 0$, we can solve for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \tag{6}$$

To derive this equation we assumed that $F(x, y) = 0$ defines y implicitly as a function of x. The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid: It states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 6.

Example 8 Implicit Differentiation

Find y' if $x^3 + y^3 = 6xy$.

Solution

The given equation can be written as

$F(x, y) = x^3 + y^3 - 6xy = 0$.

Use Equation 6 to find y'.

Compare this solution to the one obtained in Example 2 in Section 3.5.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$ ■

Now suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, f(x, y)) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

But

$$\frac{\partial}{\partial x}(x) = 1 \quad \text{and} \quad \frac{\partial}{\partial x}(y) = 0$$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\dfrac{\partial F}{\partial z} \neq 0$, we can solve for $\dfrac{\partial z}{\partial x}$ to obtain the first formula in Equation 7. The formula for $\dfrac{\partial z}{\partial y}$ is obtained in a similar manner.

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}} \tag{7}$$

Again, a version of the **Implicit Function Theorem** gives conditions under which our assumption is valid: If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by the formulas in Equation 7.

Example 9 Find Partial Derivatives Implicitly

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution

Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Use the formulas in Equation 7.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

11.5 Exercises

Use the Chain Rule to find $\dfrac{dz}{dt}$ or $\dfrac{dw}{dt}$.

1. $z = xy^3 - x^2y, \quad x = t^2 + 1, \quad y = t^2 - 1$

2. $z = x^2 + y^2 + xy, \quad x = \sin t, \quad y = e^t$

3. $z = \dfrac{x - y}{x + 2y}, \quad x = e^{\pi t}, \quad y = e^{-\pi t}$

4. $z = \sin x \cos y, \quad x = \sqrt{t}, \quad y = \dfrac{1}{t}$

5. $z = \sqrt{1 + x^2 + y^2}, \quad x = \ln t, \quad y = \cos t$

6. $z = \tan^{-1}\dfrac{y}{x}, \quad x = e^t, \quad y = 1 - e^{-t}$

7. $w = xe^{y/z}, \quad x = t^2, \quad y = 1 - t \quad z = 1 + 2t$

8. $w = \ln\sqrt{x^2 + y^2 + z^2}, \quad x = \sin t, \quad y = \cos t, \quad z = \tan t$

Use the Chain Rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

9. $z = (x - y)^5, \quad x = s^2t, \quad y = st^2$

10. $z = x^2y^3, \quad x = s\cos t, \quad y = s\sin t$

11. $z = \sin^{-1}(x - y), \quad x = s^2 + t^2, \quad y = 1 - 2st$

12. $z = \tan^{-1}(x^2 + y^2), \quad x = s\ln t, \quad y = te^s$

13. $z = \sin\theta\cos\phi, \quad \theta = st^2, \quad \phi = s^2t$

14. $z = e^{x+2y}, \quad x = \dfrac{s}{t}, \quad y = \dfrac{t}{s}$

15. $z = \sqrt{x}\,e^{xy}, \quad x = 1 + st, \quad y = s^2 - t^2$

16. $z = e^r\cos\theta, \quad r = st, \quad \theta = \sqrt{s^2 + t^2}$

17. $z = \tan\dfrac{u}{v}, \quad u = 2s + 3t, \quad v = 3s - 2t$

18. Suppose f is a differentiable function of x and y, and $z = f(x, y)$, $x = g(t)$, $g(3) = 2$, $g'(3) = 5$, $y = h(t)$, $h(3) = 7$, $h'(3) = -4$, $f_x(2, 7) = 6$, $f_y(2, 7) = -8$. Find $\dfrac{dz}{dt}$ when $t = 3$.

19. Suppose f is a differentiable function of x and y, and $p(t) = f(g(t), h(t))$, $g(20) = 4$, $g'(2) = -3$, $h(2) = 5$, $h'(2) = 6$, $f_x(4, 5) = 2$, $f_y(4, 5) = 8$. Find $p'(2)$.

20. Let $W(s, t) = F(u(s, t), v(s, t))$, where F, u, and v are differentiable, $u(1, 0) = 2$, $u_s(1, 0) = -2$, $u_t(1, 0) = 6$, $v(1, 0) = 3$, $v_s(1, 0) = 5$, $v_t(1, 0) = 4$, $F_u(2, 3) = -1$, $F_v(2, 3) = 10$. Find $W_s(1, 0)$ and $W_t(1, 0)$.

21. Let $R(s, t) = G(u(s, t), v(s, t))$, where G, u, and v are differentiable, $u(1, 2) = 5$, $u_s(1, 2) = 4$, $u_t(1, 2) = -3$, $v(1, 2) = 7$, $v_s(1, 2) = 2$, $v_t(1, 2) = 6$, $G_u(5, 7) = 9$, $G_v(5, 7) = -2$. Find $R_s(1, 2)$ and $R_t(1, 2)$.

22. Suppose f is a differentiable function of x and y, and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the table of values to find the values of $g_u(0, 0)$ and $g_v(0, 0)$.

	f	g	f_x	f_y
(0, 0)	3	6	4	8
(1, 2)	6	3	2	5

23. Suppose f is a differentiable function of x and y, and $g(r, s) = f(2r - s, s^2 - 4r)$. Use the table of values in Exercise 22 to find the values of $g_r(1, 2)$ and $g_s(1, 2)$.

Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

24. $u = f(x, y)$, where $x = x(r, s, t)$, $y = y(r, s, t)$

25. $w = f(r, s, t)$, where $r = r(x, y)$, $s = s(x, y)$, $t = t(x, y)$

26. $R = F(t, u)$, where $t = t(w, x, y, z)$, $u = u(w, x, y, z)$

27. $t = f(u, v, w)$, where $u = u(p, q, r, s)$, $v = v(p, q, r, s)$, $w = w(p, q, r, s)$

Use the Chain Rule to find the indicated partial derivatives.

28. $z = x^2 + xy^3, \quad x = uv^2 + w^3, \quad y = u + ve^w;$

$\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}, \dfrac{\partial z}{\partial w}$, where $u = 2$, $v = 1$, $w = 0$

29. $u = \sqrt{r^2 + s^2}, \quad r = y + x\cos t, \quad s = x + y\sin t;$

$\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial t}$, where $x = 1$, $y = 2$, $t = 0$

30. $T = \dfrac{v}{2u + v}, \quad u = pq\sqrt{r}, \quad v = p\sqrt{qr};$

$\dfrac{\partial T}{\partial p}, \dfrac{\partial T}{\partial q}, \dfrac{\partial T}{\partial r}$, where $p = 2$, $q = 1$, $r = 4$

31. $R = \ln(u^2 + v^2 + w^2)$,

$u = x + 2y, \quad v = 2x - y, \quad w = 2xy;$

$\dfrac{\partial R}{\partial x}, \dfrac{\partial R}{\partial y}$, where $x = y = 1$

32. $M = xe^{y - z^2}, \quad x = 2uv, \quad y = u - v, \quad z = u + v;$

$\dfrac{\partial M}{\partial u}, \dfrac{\partial M}{\partial v}$, where $u = 3$, $v = -1$

33. $u = x^2 + yz, \quad x = pr\cos\theta, \quad y = pr\sin\theta, \quad z = p + r;$

$\dfrac{\partial u}{\partial p}, \dfrac{\partial u}{\partial r}, \dfrac{\partial u}{\partial \theta}$ where $p = 2$, $r = 3$, $\theta = 0$

Use Equation 6 to find $\dfrac{dy}{dx}$.

34. $y^5 + x^2y^3 = 1 + ye^{x^2}$

35. $\sin x + \cos y = \sin x \cos y$

36. $\tan^{-1}(x^2y) = x + xy^2$

37. $e^y \sin x = x + xy$

Use the expressions given in Equation 7 to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

38. $x^2 + y^2 + z^2 = 3xyz$

39. $xyz = \cos(x + y + z)$

40. $e^z = xyz$

41. $yz + x\ln y = z^2$

42. $x - z = \arctan(yz)$

43. $yz = \ln(x + z)$

44. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

45. Wheat production W in a given year depends on the average temperature T and the annual rainfall R. Scientists estimate that the average temperature is rising at a rate of 0.15°C/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that, at current production levels, $\dfrac{\partial W}{\partial T} = -2$ and $\dfrac{\partial W}{\partial R} = 8$.

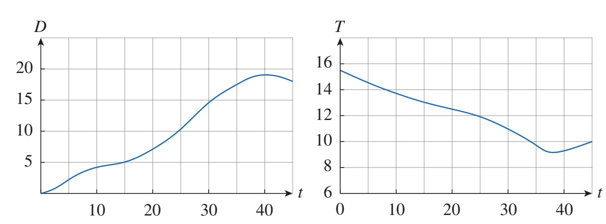
(a) Explain the significance of the signs of these partial derivatives in the context of the problem.

(b) Estimate the current rate of change of wheat production, $\dfrac{dW}{dt}$.

46. The speed of sound traveling through ocean water with salinity 35 parts per thousand has been modeled by the equation

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$$

where C is the speed of sound (in meters per second), T is the temperature (in degrees Celsius), and D is the depth below the ocean surface (in meters). A scuba diver began a leisurely dive into the ocean water; the diver's depth and the surrounding water temperature over time (in minutes) are recorded in the following graphs. Using correct units, estimate the rate of change (with respect to time) of the speed of sound through the ocean water experienced by the diver 20 minutes into the dive.



47. The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in.?

48. The length ℓ, width w, and height h of a box change with time. At a certain instant the dimensions are $\ell = 1$ m and $w = h = 2$ m, and ℓ and w are increasing at a rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that instant, find the rates at which the following quantities are changing.
(a) The volume
(b) The surface area
(c) The length of a diagonal

49. The voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance is slowly increasing as the resistor heats up. Use Ohm's Law, $V = IR$, to find how the current I is changing at the moment when $R = 400\ \Omega$, $I = 0.08$ A, $\dfrac{dV}{dt} = -0.01$ V/s, and $\dfrac{dR}{dt} = 0.03\ \Omega/\text{s}$.

50. The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is increasing at a rate of 0.15 K/s. Use the equation in Example 2 to find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K.

51. A manufacturer has modeled its yearly production function P (the value of its entire production in millions of dollars) as a Cobb–Douglas function

$$P(L, K) = 1.47L^{0.65}K^{0.35}$$

where L is the number of labor hours (in thousands) and K is the invested capital (in millions of dollars). Suppose that when $L = 30$ and $K = 8$, the labor force is decreasing at a rate of 2000 labor hours per year and capital is increasing at a rate of \$500,000 per year. Find the rate of change of production.

52. One side of a triangle is increasing at a rate of 3 cm/s and a second side is decreasing at a rate of 2 cm/s. If the area of the triangle remains constant, at what rate does the angle between the sides change when the first side is 20 cm long, the second side is 30 cm, and the angle is $\pi/6$?

53. If a sound with frequency f_s is produced by a source traveling along a line with speed v_s and an observer is traveling with speed v_o along the same line from the opposite direction toward the source, then the frequency of the sound heard by the observer is

$$f_o = \left(\frac{c + v_o}{c - v_s}\right) f_s$$

where c is the speed of sound, about 332 m/s. This is called the **Doppler effect**. Suppose that, at a particular moment, you are in a train traveling at 34 m/s and accelerating at 1.2 m/s^2. A train is approaching you from the opposite direction on the other track at 40 m/s, accelerating at 1.4 m/s^2, and sounds its whistle, which has a frequency of 460 Hz. At that instant, what is the perceived frequency that you hear and how fast is it changing?

Assume that all the given functions are differentiable.

54. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, (a) find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$ and (b) show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

55. If $u = f(x, y)$, where $x = e^s\cos t$ and $y = e^s\sin t$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]$$

56. If $z = f(x - y)$, show that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$

57. If $z = f(x, y)$, where $x = s + t$ and $y = s - t$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial s}\frac{\partial z}{\partial t}$$

Assume that all the given functions have continuous second-order partial derivatives.

58. Show that any function of the form

$$z = f(x + at) + g(x - at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$$

Hint: Let $u = x + at$, $v = x - at$

59. If $u = f(x, y)$, where $x = e^s\cos t$ and $y = e^s\sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right]$$

60. If $z = f(x, y)$, where $x = r^2 + s^2$ and $y = 2rs$, find $\dfrac{\partial^2 z}{\partial r\partial s}$. Compare your answer with Example 7.

61. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, find (a) $\dfrac{\partial z}{\partial r}$, (b) $\dfrac{\partial z}{\partial \theta}$, and (c) $\dfrac{\partial^2 z}{\partial r\,\partial \theta}$.

62. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

63. Suppose $z = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$.

(a) Show that

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

(b) Find a similar formula for $\dfrac{\partial^2 z}{\partial s\,\partial t}$.

A function f is called *homogeneous of degree n* if it satisfies the equation

$$f(tx, ty) = t^n f(x, y)$$

for all t, where n is a positive integer and f has continuous second-order partial derivatives.

64. Verify that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.

65. Show that if f is homogeneous of degree n, then

(a) $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x, y)$

Hint: Use the Chain Rule to differentiate $f(tx, ty)$ with respect to t.

(b) $x^2\dfrac{\partial^2 f}{\partial x^2} + 2xy\dfrac{\partial^2 f}{\partial x\,\partial y} + y^2\dfrac{\partial^2 f}{\partial y^2} = n(n-1)f(x, y)$

66. If f is homogeneous of degree n, show that

$$f_x(tx, ty) = t^{n-1}f_x(x, y)$$

67. Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables x, y, and z as functions of the other two: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. If F is differentiable and F_x, F_y, and F_z are all nonzero, show that

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1$$

68. Equation 6 is a formula for the derivative $\dfrac{dy}{dx}$ of a function defined implicitly by an equation $F(x, y) = 0$, provided that F is differentiable and $F_y \neq 0$. Prove that if F has continuous second derivatives, then a formula for the second derivative of y is

$$\frac{d^2y}{dx^2} = -\frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{F_y^3}$$

11.6 Directional Derivatives and the Gradient Vector

The weather map in Figure 11.46 shows a contour map of the temperature function $T(x, y)$ (in °F) for the New England region of the United States at 12:00 PM on a day in October. The level curves, or isothermals, join locations with the same temperature. The partial derivative T_x, at a location such as Boston, is the rate of change of temperature with respect to distance if we travel east from Boston (toward the bay); T_y is the rate of change of temperature if we travel north (toward New Hampshire). But what if we want to know the rate of change of temperature when we travel southeast (toward Hartford), or in some other direction? In this section, we will investigate a type of derivative, called a *directional derivative*, that enables us to find the rate of change of a function of two or more variables in any direction.

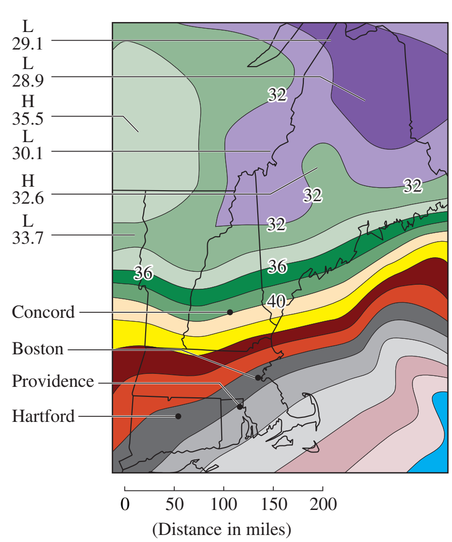


Figure 11.46
Contour map of the temperature.
(Plymouth State Weather Center)

Directional Derivatives

Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$
$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \qquad (1)$$

and represent the rates of change of z in the x- and y-directions, that is, in the direction of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

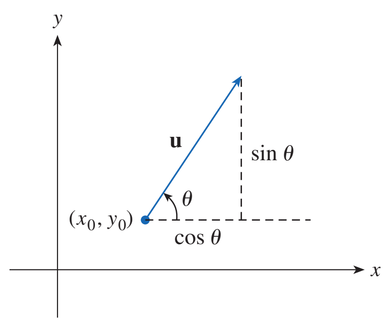


Figure 11.47
A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$.

Suppose that we would like to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. See Figure 11.47. To compute this value, consider the surface S with the equation $z = f(x, y)$ (the graph of f) and let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction of $\mathbf{u}$ intersects S in a curve C, as shown in Figure 11.48. The slope of the tangent line T to C at the point P is the rate of change of z in the direction of $\mathbf{u}$.

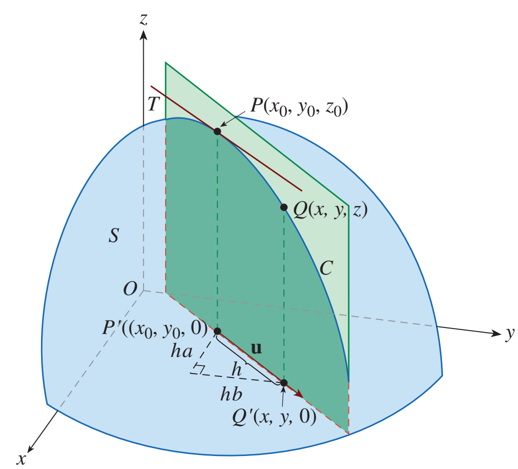


Figure 11.48
The vertical plane through P in the direction of $\mathbf{u}$ intersects S in a curve C.

If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to $\mathbf{u}$ and so

$$\overrightarrow{P'Q'} = h\,\mathbf{u} = \langle ha, hb \rangle$$

for some scalar h. Therefore, $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of **u**, which is called the directional derivative of f in the direction of **u**.

Definition • Directional Derivative

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

If we compare this definition with the Equation 1, we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}} f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}} f = f_y$. In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

Example 1 Estimate a Directional Derivative

Use the weather map in Figure 11.46 to estimate the value of the directional derivative of the temperature function at Boston in the northwesterly direction.

Solution

The unit vector directed toward the northwest is $\mathbf{u} = \dfrac{\mathbf{j} - \mathbf{i}}{\sqrt{2}}$.

However, we will not need to use this expression.

Start by drawing a line through Boston toward the northwest; see Figure 11.49.

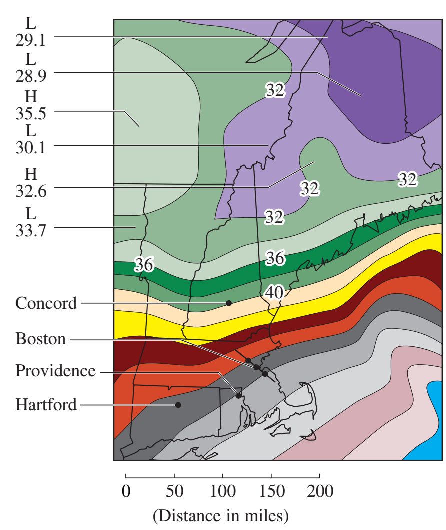


Figure 11.49
Contour map of temperature with northwest line segment.

Approximate the directional derivative $D_{\mathbf{u}}T$ by the average rate of change of the temperature between the points where this line intersects the isothermals $T = 48$ and $T = 50$.

The temperature at the point northwest of Boston is $T = 48°\text{F}$ and the temperature at the point southeast of Boston is $T = 50°\text{F}$.

The distance between these two points looks like about 25 miles.

Therefore, the rate of change of the temperature in the northwesterly direction is

$$D_{\mathbf{u}}T \approx \frac{48 - 50}{25} = -0.08°\text{F/mi}.$$ ■

The next theorem can be used to compute the directional derivative of a function defined by a formula.

Theorem • Directional Derivative Formula

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)\, b$$

Proof

Define a function g of the single variable h by $g(h) = f(x_0 + ha, y_0 + hb)$.

By the definition of the derivative:

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \tag{2}$$

$$= D_{\mathbf{u}} f(x_0, y_0)$$

We can also write $g(h) = f(x, y)$, where $x = x_0 + ha$ and $y = y_0 + hb$.

The Chain Rule gives

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)a + f_y(x, y)\, b.$$

Let $h = 0$, then $x = x_0$, $y = y_0$, and

$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)\, b. \tag{3}$$

Compare Equations 2 and 3:

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0) b$$ ■

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis (as in Figure 11.47), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta\rangle$ and the expression for the directional derivative becomes

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \tag{4}$$

Example 2 Find a Directional Derivative

Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector given by the angle $\theta = \dfrac{\pi}{6}$. Find the value of $D_{\mathbf{u}} f(1, 2)$.

Solution

Use Equation 4:

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos \frac{\pi}{6} + f_y(x, y) \sin \frac{\pi}{6}$$ Equation 4 with $\theta = \frac{\pi}{6}$.

$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2}$$ Partial derivatives; evaluate trigonometric functions.

$$= \frac{1}{2}[3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y]$$ Simplify.

Therefore,

$$D_{\mathbf{u}} f(1, 2) = \frac{1}{2}[3\sqrt{3}(1)^2 - 3(1) + (8 - 3\sqrt{3})(2)] = \frac{13 - 3\sqrt{3}}{2}.$$

Figure 11.50 illustrates this directional derivative. ■

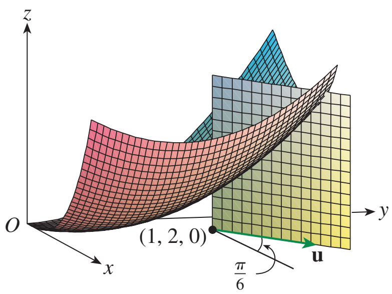


Figure 11.50
The directional derivative $D_{\mathbf{u}} f(1, 2)$ represents the rate of change of z in the direction of **u**. This is the slope of the tangent line to the curve of intersection of the surface $z = x^3 - 3xy + 4y^2$ and the vertical plane through (1, 2, 0) in the direction of **u**.

The Gradient Vector

Notice that the directional derivative of a differentiable function given above can be written as the dot product of two vectors.

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y)a + f_y(x, y)b \\ &= \langle f_x(x, y), f_y(x, y)\rangle \cdot \langle a, b\rangle \\ &= \langle f_x(x, y), f_y(x, y)\rangle \cdot \mathbf{u} \end{aligned} \tag{5}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well. Therefore, it has a special name, the *gradient* of f, and special notation, **grad** f of ∇f (read as "del f").

Definition • Gradient

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example 3 Evaluate a Gradient Vector

If $f(x, y) = \sin x + e^{xy}$, then

$\nabla f(x, y) = \langle f_x, f_y\rangle = \langle \cos x + ye^{xy}, xe^{xy}\rangle$ and

$\nabla f(0, 1) = \langle \cos 0 + 1 \cdot e^{0\cdot 1}, 0 \cdot e^{0\cdot 1}\rangle = \langle 2, 0\rangle$ ■

Using this notation for the gradient vector, we can rewrite the expression for the directional derivative of a differentiable function given in Equation 5 as

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} \tag{6}$$

This is interpreted as the directional derivative in the direction of **u** is the scalar projection of the gradient vector onto **u**.

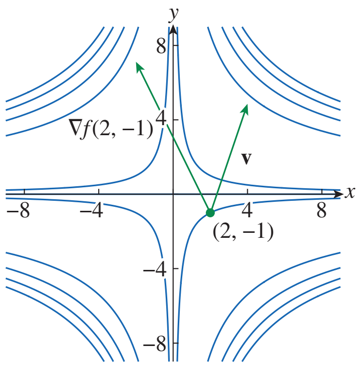


Figure 11.51
The gradient vector $\nabla f(2, -1)$ is shown with initial point $(2, -1)$. The vector $\mathbf{v}$ indicates the direction of the directional derivative. Both vectors are superimposed on a contour plot of the graph of f.

Example 4 Use a Gradient Vector to Find a Directional Derivative

Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\,\mathbf{i} + 5\,\mathbf{j}$.

Solution

Find the gradient vector at $(2, -1)$.

$\nabla f(x, y) = f_x\,\mathbf{i} + f_y\,\mathbf{j} = 2xy^3\,\mathbf{i} + (3x^2y^2 - 4)\,\mathbf{j}$

$\nabla f(2, -1) = 2(2)(-1)^3\,\mathbf{i} + (3(2)^2(-1)^2 - 4)\,\mathbf{j} = -4\,\mathbf{i} + 8\,\mathbf{j}$

Find the unit vector in the direction of $\mathbf{v}$.

$$|\mathbf{v}| = \sqrt{2^2 + 5^2} = \sqrt{29} \quad \Rightarrow \quad \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}$$

Use Equation 6:

$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = (-4\,\mathbf{i} + 8\,\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}\right)$$
$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

Figure 11.51 illustrates this result geometrically. ■

Functions of Three Variables

For functions of three variables we can define directional derivatives in a similar manner. Again $D_{\mathbf{u}} f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector $\mathbf{u}$.

Definition • Directional Derivative for a Function of Three Variables

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

Using vector notation, we can write the definitions of the directional derivative in a more compact form:

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h\to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h} \tag{7}$$

where $\mathbf{x}_0 = \langle x_0, y_0\rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0\rangle$ if $n = 3$. This notation is reasonable because the vector equation of the line through $\mathbf{x}_0$ in the direction of the vector $\mathbf{u}$ is given by $\mathbf{x} = \mathbf{x}_0 + t\mathbf{u}$ (Equation 9.5.1). Therefore, $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c\rangle$, then the same method that was used to prove the directional derivative formula can be used to show that

$$D_{\mathbf{u}} f(x, y, z) = f_x(x, y, z)\,a + f_y(x, y, z)\,b + f_z(x, y, z)\,c \tag{8}$$

For a function f of three variables, the **gradient vector**, denoted ∇f or **grad** f, is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$$

or, simply,

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{9}$$

And, just as with functions of two variables, Equation 8 for the directional derivative can be written as

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u} \tag{10}$$

Example 5 Find a Gradient and Directional Derivative

If $f(x, y, z) = x \sin(yz)$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution

(a) The gradient of f is

$$\begin{aligned}\nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= \langle \sin(yz), xz\cos(yz), xy\cos(yz) \rangle\end{aligned}$$

(b) At the point $(1, 3, 0)$:

$$\nabla f(1, 3, 0) = \langle \sin(3 \cdot 0), 1 \cdot 0 \cdot \cos(3 \cdot 0), 1 \cdot 3\cos(3 \cdot 0) \rangle = \langle 0, 0, 3 \rangle.$$

Find the magnitude of $\mathbf{v}$: $|\mathbf{v}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$.

Therefore, the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}.$$

Use Equation 10 to find the directional derivative.

$$\begin{aligned}D_{\mathbf{u}} f(1, 3, 0) &= \nabla f(1, 3, 0) \cdot \mathbf{u} \\ &= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} \right) \\ &= 0 \cdot \frac{1}{\sqrt{6}} + 0 \cdot \frac{2}{\sqrt{6}} + 3\left(-\frac{1}{\sqrt{6}}\right) = -\sqrt{\frac{3}{2}}\end{aligned}$$

Maximizing the Directional Derivative

Suppose f is a function of two or three variables and we consider all possible directional derivatives of f at a given point. These values are the rates of change of f in all possible directions. It seems reasonable to ask in which of these directions does f change fastest and what is the maximum rate of change? The next theorem provides these answers.

Theorem • Maximum Value of the Directional Derivative

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Proof

Using Equation 9 or 14:

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta,$$

where θ is the angle between ∇f and $\mathbf{u}$.

The maximum value of $\cos\theta$ is 1, and this occurs when $\theta = 0$.

Therefore, the maximum value of $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\mathbf{u}$ has the same direction as ∇f. ■

Example 6 Determine a Maximum Rate of Change

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Solution

(a) Compute the gradient vector.

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle \quad \Rightarrow \quad \nabla f(2, 0) = \langle e^0, 2e^0 \rangle = \langle 1, 2 \rangle$$

Find the unit vector in the direction of $\overrightarrow{PQ}$.

$$\overrightarrow{PQ} = \left\langle \frac{1}{2} - 2, 2 - 0 \right\rangle = \left\langle -\frac{3}{2}, 2 \right\rangle \quad \Rightarrow \quad |\overrightarrow{PQ}| = \sqrt{\left(-\frac{3}{2}\right)^2 + 2^2} = \frac{5}{2}$$

$$\mathbf{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{\left\langle -\frac{3}{2}, 2 \right\rangle}{\frac{5}{2}} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$$

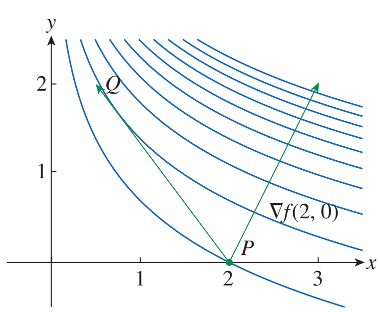


Figure 11.52
Graph of the gradient vector $\nabla f(2, 0)$ and the vector $\overrightarrow{PQ}$ superimposed on a contour map of f.

The rate of change of f in the direction from P to Q is

$$D_{\mathbf{u}}f(2, 0) = \nabla f(2, 0) \cdot \mathbf{u} = \langle 1, 2 \rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$$

$$= 1\left(-\frac{3}{5}\right) + 2\left(\frac{4}{5}\right) = 1$$

(b) f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$.

The maximum rate of change is $|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$.

Figure 11.52 shows the gradient vector $\nabla f(2, 0)$ superimposed on a contour map of f. Notice that this vector appears to be perpendicular to the level curve through $(2, 0)$. Figure 11.53 shows the graph of f and the gradient vector. ■

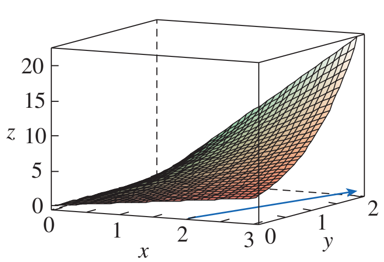


Figure 11.53
Graph of f and the gradient vector.

Example 7 Find the Direction of Greatest Increase in Temperature

Suppose that the temperature at a point (x, y, z) in space is given by

$$T(x, y, z) = \frac{80}{1 + x^2 + 2y^2 + 3z^2}$$

where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

Solution

Find the gradient of T.

$$\begin{aligned}\nabla T &= \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} \\ &= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{k} \\ &= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-x\,\mathbf{i} - 2y\,\mathbf{j} - 3z\,\mathbf{k})\end{aligned}$$

At the point $(1, 1, -2)$, the gradient vector is

$$\begin{aligned}\nabla T(1, 1, -2) &= \frac{160}{(1 + 1^2 + 2(1)^2 + 3(-2)^2)^2}(-1\,\mathbf{i} - 2(1)\,\mathbf{j} - 3(-2)\,\mathbf{k}) \\ &= \frac{160}{256}(-\mathbf{i} - 2\,\mathbf{j} + 6\,\mathbf{k}) = \frac{5}{8}(-\mathbf{i} - 2\,\mathbf{j} + 6\mathbf{k})\end{aligned}$$

The temperature increases fastest in the direction of the gradient vector $\nabla T(1, 1, -2) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $\frac{1}{\sqrt{41}}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$.

The maximum rate of increase is the length of the gradient vector:

$$\begin{aligned}|\nabla T(1, 1, -2)| &= \frac{5}{8}|-\mathbf{i} - 2\,\mathbf{j} + 6\,\mathbf{k}| \\ &= \frac{5}{8}\sqrt{(-1)^2 + (-2)^2 + 6^2} = \frac{5}{8}\sqrt{41}\end{aligned}$$

Therefore, the maximum rate of increase of temperature is $\frac{5}{8}\sqrt{41} \approx 4°\mathrm{C/m}$. ■

Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through the point P. Recall from Section 10.1 that the curve C is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Let t_0 be the parameter value corresponding to P; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. Since C lies on S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is,

$$F(x(t), y(t), z(t)) = k \tag{11}$$

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 11 as follows:

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0 \tag{12}$$

But, since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation 12 can be written in terms of a dot product as

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

In particular, when $t = t_0$ then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$ and

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0 \qquad (13)$$

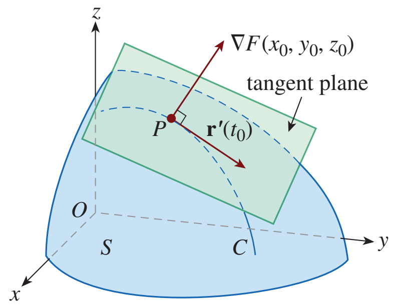


Figure 11.54
The gradient vector is perpendicular to the tangent vector.

Equation 13 says that *the gradient vector at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P*. Figure 11.54 illustrates this concept. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is reasonable to define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. Using the standard equation of a plane (Equation 9.5.7), we can write the equation of this tangent plane as

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \qquad (14)$$

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, by Equation 9.5.3, its symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)} \qquad (15)$$

In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function f of two variables), we can rewrite the equation for the surface as

$$F(x, y, z) = f(x, y) - z = 0$$

and consider S as a level surface (with $k = 0$) of F. Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0) \qquad F_y(x_0, y_0, z_0) = f_y(x_0, y_0) \qquad F_z(x_0, y_0, z_0) = -1$$

and Equation 14 can be written as

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

which is equivalent to Equation 11.4.2. Therefore, this new, more general, definition of a tangent plane is consistent with the definition that was given in the special case of Section 11.4.

Example 8 Find a Tangent Plane and Normal Line

Find equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Solution

The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}.$$

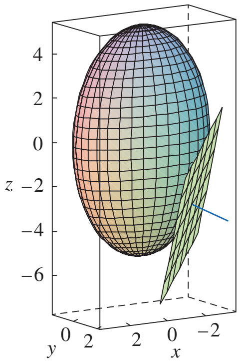


Figure 11.55
Graph of the ellipsoid, tangent plane, and normal line.

Find the partial derivatives and evaluate each at $(-2, 1, -3)$.

$$F_x(x, y, z) = \frac{x}{2} \quad \Rightarrow \quad F_x(-2, 1, -3) = \frac{-2}{2} = -1$$
$$F_y(x, y, z) = 2y \quad \Rightarrow \quad F_y(-2, 1, -3) = 2(1) = 2$$
$$F_z(x, y, z) = \frac{2z}{9} \quad \Rightarrow \quad F_z(-2, 1, -3) = \frac{2(-3)}{9} = -\frac{2}{3}$$

Use Equation 14 to write an equation of the tangent plane at $(-2, 1, -3)$.

$$-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0 \quad \Rightarrow \quad 3x - 6y + 2z + 18 = 0$$

Use Equation 15 to write the symmetric equations of the normal line.

$$\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-\frac{2}{3}}$$

Figure 11.55 shows the ellipsoid, tangent plane, and the normal line. ■

Significance of the Gradient Vector

Let's summarize the ways in which the gradient vector is significant. First, consider a function f of three variables and a point $P(x_0, y_0, z_0)$ in its domain. We know that the gradient vector $\nabla f(x_0, y_0, z_0)$ represents the direction of fastest increase of f. In addition, we know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface S of f through P, as illustrated in Figure 11.54. These two properties are consistent intuitively because as we move away from P on the level surface S, the value of f does not change at all. So, it seems reasonable that if we move in the perpendicular direction, the result would be the maximum increase.

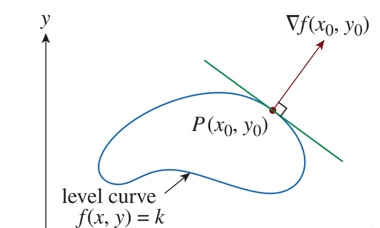


Figure 11.56
$\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P.

Similarly, consider a function f of two variables and a point $P(x_0, y_0)$ in its domain. Again, the gradient vector $\nabla f(x_0, y_0)$ represents the direction of fastest increase of f. And, similar to the discussion of tangent planes, it can be shown that $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P. Once more, this is intuitively plausible because the values of f remain constant as we move along the curve. See Figure 11.56.

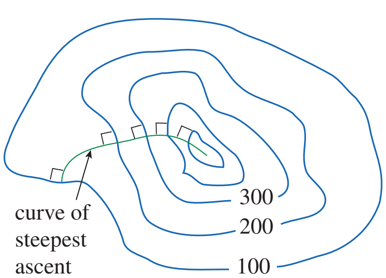


Figure 11.57
The curve of steepest ascent is perpendicular to all of the contour lines.

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y), then a curve of steepest ascent can be drawn as in Figure 11.57 by sketching it perpendicular to all of the contour lines. This phenomenon is evident in Figure 11.5 in Section 11.1, where Lonesome Creek follows a curve of steepest descent.

Many computer algebra systems include commands that will plot sample gradient vectors. Each gradient vector $\nabla f(a, b)$ is plotted starting at the point (a, b). Figure 11.58 shows a plot of a *gradient vector field* for the function $f(x, y) = x^2 - y^2$ superimposed on a contour map of f. As expected, the gradient vectors point *uphill* and are perpendicular to the level curves.

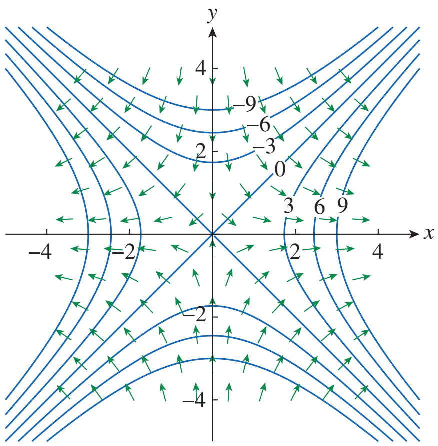
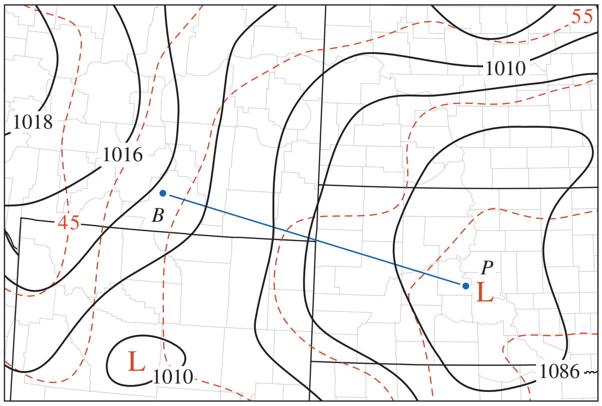


Figure 11.58
Plot of a gradient vector field.

11.6 Exercises

1. Level curves for barometric pressure (in millibars) (and for temperature in dashed lines) are shown for early morning on August 24, 2021. A low-pressure system is moving over Central South Dakota. The distance along the line from P (Pierre, South Dakota) to B (Billings, Montana) is 409 miles. Estimate the value of the directional derivative of the pressure function at Pierre in the direction of Billings. What are the units of the directional derivative?



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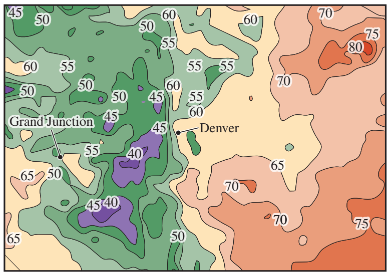
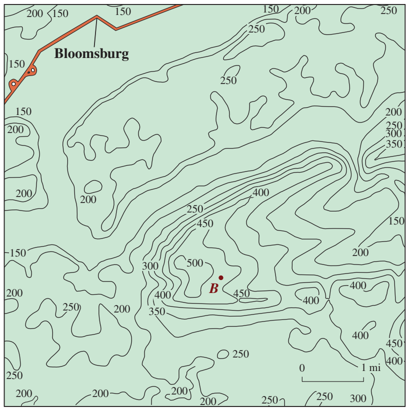
2. The contour map shows the temperature (in °F) early in the morning on August 24, 2021. The distance from Grand Junction to Denver is 196 miles. Using correct units, estimate the value of the directional derivative of this temperature function at Grand Junction, in the direction of Denver.



(Weather Underground)

3. The topographical map shows the elevations (in feet) for a region near Bloomsburg, PA. Using correct units, estimate the value of the directional derivative of this elevation function at the point B in an easterly direction.



4. A table of values for the wind-chill index $W = f(v, T)$ is given in Exercise 29 on page 28. Use the table to estimate the value of $D_{\mathbf{u}} f(15, -10)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

Find the directional derivative of f at the given point in the direction indicated by the angle θ.

5. $f(x, y) = x^2y^3 - y^4$, $(2, 1)$, $\theta = \dfrac{\pi}{4}$

6. $f(x, y) = ye^{-x}$, $(0, 4)$, $\theta = \dfrac{2\pi}{3}$

7. $f(x, y) = x \sin(xy)$, $(2, 0)$, $\theta = \dfrac{\pi}{3}$

8. $f(x, y) = \sqrt{2x + 3y}$, $(3, 1)$, $\theta = -\dfrac{\pi}{6}$

(a) Find the gradient of f.
(b) Evaluate the gradient at the point P.
(c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

9. $f(x, y) = \sin(2x + 3y)$, $P(-6, 4)$, $\mathbf{u} = \frac{1}{2}(\sqrt{3}\,\mathbf{i} - \mathbf{j})$

10. $f(x, y) = \dfrac{y^2}{x}$, $P(1, 2)$, $\mathbf{u} = \frac{1}{3}(2\mathbf{i} + \sqrt{5}\,\mathbf{j})$

11. $f(x, y) = x^2 \ln y$, $P(3, 1)$, $\mathbf{u} = -\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}$

12. $f(x, y, z) = x^2yz - xyz^3$, $P(2, -1, 1)$, $\mathbf{u} = \left\langle 0, \frac{4}{5}, -\frac{3}{5} \right\rangle$

13. $f(x, y, z) = xe^{2yz}$, $P(3, 0, 2)$, $\mathbf{u} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$

14. $f(x, y, z) = \sqrt{x + yz}$, $P(1, 3, 1)$, $\mathbf{u} = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$

Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

15. $f(x, y) = 1 + 2x\sqrt{y}$, $(3, 4)$, $\mathbf{v} = \langle 4, -3 \rangle$

16. $f(x, y) = \dfrac{x}{x^2 + y^2}$, $(1, 2)$, $\mathbf{v} = \langle 3, 5 \rangle$

17. $f(x, y) = x^4 - x^2y^3$, $(2, 1)$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

18. $f(x, y) = \tan^{-1}(xy)$, $(1, 2)$, $\mathbf{v} = 5\mathbf{i} + 10\mathbf{j}$

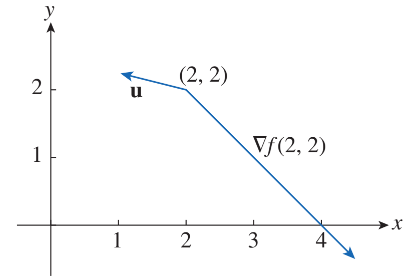
19. $f(x, y, z) = xe^y + ye^z + ze^x$, $(0, 0, 0)$, $\mathbf{v} = \langle 5, 1, -2 \rangle$

20. $f(x, y, z) = \sqrt{xyz}$, $(3, 2, 6)$, $\mathbf{v} = \langle -1, -2, 2 \rangle$

21. $g(x, y, z) = (x + 2y + 3z)^{3/2}$, $(1, 1, 2)$, $\mathbf{v} = 2\mathbf{j} - \mathbf{k}$

22. $h(x, y, z) = \ln(3x + 6y + 9z)$, $(1, 1, 1)$, $\mathbf{v} = \langle 4, 12, 6 \rangle$

23. Use the figure to estimate $D_{\mathbf{u}} f(2, 2)$.



24. Find the directional derivative of $f(x, y) = \sqrt{xy}$ at $P(2, 8)$ in the direction of $Q(5, 4)$.

25. Find the directional derivative of $f(x, y, z) = xy + yz + zx$ at $P(1, -1, 3)$ in the direction of $Q(2, 4, 5)$.

Find the maximum rate of change of f at the given point and the direction in which it occurs.

26. $f(x, y) = 4y\sqrt{x}$, $(4, 1)$

27. $f(x, y) = ye^{xy}$, $(0, 2)$

28. $f(x, y) = \sin(xy)$, $(1, 0)$

29. $f(x, y) = ye^{-x} + xe^{-y}$, $(0, 0)$

30. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $(3, 6, -2)$

31. $f(x, y, z) = \dfrac{x + y}{z}$, $(1, 1, -1)$

32. $f(x, y, z) = \arctan(xyz)$, $(1, 2, 1)$

33. (a) Show that a differentiable function f decreases most rapidly at $\mathbf{x}$ in the direction opposite to the gradient vector, that is, in the direction $-\nabla f(\mathbf{x})$.
(b) Use the result of part (a) to find the direction in which the function $f(x, y) = x^4y - x^2y^3$ decreases fastest at the point $(2, -3)$.

34. Find the directions in which the directional derivative of $f(x, y) = ye^{-xy}$ at the point $(0, 2)$ has the value 1.

35. Find all points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

36. Near a buoy, the depth of a lake at the point with coordinates (x, y) is $z = 200 + 0.02x^2 - 0.001y^3$, where x, y, and z are measured in meters. A person in a small boat starts at the point $(80, 60)$ and moves toward the buoy, which is located at $(0, 0)$. Is the water under the boat getting deeper or shallower when they depart? Explain your reasoning.

37. The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is $120°$.

(a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.
(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin.

38. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$

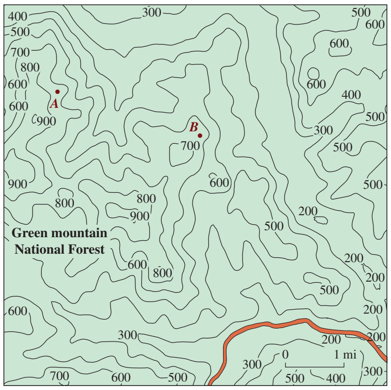
where T is measured in °C and x, y, z in meters.
(a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.
(b) In which direction does the temperature increase fastest at P?
(c) Find the maximum rate of increase at P.

39. Suppose that over a certain region of space the electrical potential V is given by $V(x, y, z) = 5x^2 - 3xy + xyz$.
(a) Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.
(b) In which direction does V change most rapidly at P?
(c) What is the maximum rate of change at P?

40. Suppose a person is climbing a hill whose shape is given by the equation $z = 1000 - 0.005x^2 - 0.01y^2$, where x, y, and z are measured in meters, and they are standing at a point with coordinates $(60, 40, 966)$. The positive x-axis points east and the positive y-axis points north.
(a) If they walk due south, will they start to ascend or descend? At what rate?
(b) If they walk northwest, will they start to ascend or descend? At what rate?
(c) In which direction is the slope largest? What is the rate of ascent in that direction? At what angle above the horizontal does the path in that direction begin?

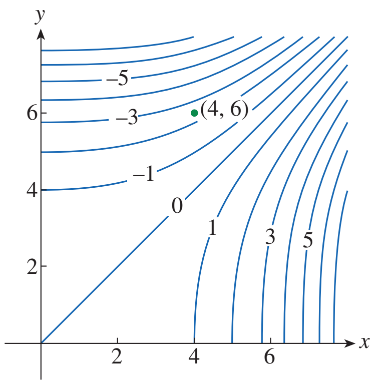
41. Let f be a function of two variables that has continuous partial derivatives and consider the points $A(1, 3)$, $B(3, 3)$, $C(1, 7)$, and $D(6, 15)$. The directional derivative of f at A in the direction of the vector $\overrightarrow{AB}$ is 3 and the directional derivative at A in the direction of $\overrightarrow{AC}$ is 26. Find the directional derivative of f at A in the direction of the vector $\overrightarrow{AD}$.

42. The topographic map shown is of an area in the Green Mountain National Forest in Vermont. Draw curves of steepest descent from Point A and from Point B.



43. Assume that u and v are differentiable functions of x and y and that a, b are constants. Show that the operation of taking the gradient of a function has the given property.
(a) $\nabla(au + bv) = a\nabla u + b\nabla v$ (b) $\nabla(uv) = u\nabla v + v\nabla u$
(c) $\nabla\left(\dfrac{u}{v}\right) = \dfrac{v\nabla u - u\nabla v}{v^2}$ (d) $\nabla u^n = nu^{n-1}\nabla u$

44. Sketch the gradient vector $\nabla f(4, 6)$ for the function f whose level curves are shown. Explain how you chose the direction and length of this vector.



45. The **second directional derivative** of $f(x, y)$ is

$$D_{\mathbf{u}}^2 f(x, y) = D_{\mathbf{u}}[D_{\mathbf{u}} f(x, y)]$$

If $f(x, y) = x^3 + 5x^2y + y^3$ and $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$ find $D_{\mathbf{u}}^2 f(2, 1)$.

46. (a) If $\mathbf{u} = \langle a, b\rangle$ is a unit vector and f has continuous second partial derivatives, show that

$$D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$$

(b) Find the second directional derivative of $f(x, y) = xe^{2y}$ in the direction of $\mathbf{v} = \langle 4, 6\rangle$.

Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

47. $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$, $(3, 3, 5)$

48. $y = x^2 - z^2$, $(4, 7, 3)$

49. $x^2 - 2y^2 + z^2 + yz = 2$, $(2, 1, -1)$

50. $x - z = 4\arctan(yz)$, $(1 + \pi, 1, 1)$

51. $z + 1 = xe^y \cos z$, $(1, 0, 0)$

52. $yz = \ln(x + z)$, $(0, 0, 1)$

53. $x^4 + y^4 + z^4 = 3x^2y^2z^2$, $(1, 1, 1)$

Use technology to graph the surface, the tangent plane, and the normal line on the same coordinate axes. Choose the domain carefully to avoid extraneous vertical planes. Select a viewpoint so that all three objects are visible.

54. $4xy + yz + zx = 3, \quad (1, 1, 1)$

55. $xyz = 6, \quad (1, 2, 3)$

56. If $f(x, y) = xy$, find the gradient vector $\nabla f(3, 2)$ and use it to find the tangent line to the level curve $f(x, y) = 6$ at the point $(3, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

57. If $g(x, y) = x^2 + y^2 - 4x$, find the gradient vector $\nabla g(1, 2)$ and use it to find the tangent line to the level curve $g(x, y) = 1$ at the point $(1, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

58. Show that the equation of the tangent plane to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

59. Find an equation of the tangent plane to the hyperboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) and express it in a form similar to the one in Exercise 58.

60. Show that the equation of the tangent plane to the elliptic paraboloid $\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ at the point (x_0, y_0, z_0) can be written as

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} = \frac{z + z_0}{c}$$

61. At what point on the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is the tangent plane parallel to the plane $x + 2y + z = 1$?

62. At what point on the paraboloid $y = x^2 + z^2$ is the tangent plane parallel to the plane $x + 2y + 3z = 1$?

63. Are there any points on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?

64. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$. (This means that they have a common tangent plane at this point.)

65. Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

66. Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.

67. Find the point where the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1, 1, 2)$ intersects the paraboloid a second time.

68. Find the point where the normal line through the point $(1, 2, 1)$ on the ellipsoid $4x^2 + y^2 + 4z^2 = 12$ intersects the sphere $x^2 + y^2 + z^2 = 102$.

69. Show that the sum of the x-, y-, and z-intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

70. Show that the pyramids cut off from the first octant by any tangent planes to the surface $xyz = 1$ at points in the first octant must all have the same volume.

71. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

72. (a) The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 1)$.
(b) Graph the cylinder, the plane, and the tangent line on the same coordinate axes.

73. Two surfaces are called **orthogonal** at a point of intersection if their normal lines are perpendicular at that point.
(a) Show that surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at a point P where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$ if and only if

$$F_x G_x + F_y G_y + F_z G_z = 0 \quad \text{at } P$$

(b) Use part (a) to show that the surfaces $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = r^2$ are orthogonal at every point of intersection. Explain this result without using calculus.

74. (a) Show that the function $f(x, y) = \sqrt[3]{xy}$ is continuous and the partial derivatives f_x and f_y exist at the origin but the directional derivatives in all other directions do not exist.
(b) Graph f near the origin and explain how the graph confirms the result in part (a).

75. Suppose the directional derivatives of $f(x, y)$ are known at a given point in two nonparallel directions given by the unit vectors $\mathbf{u}$ and $\mathbf{v}$. Determine whether it is possible to find ∇f at this point and if so, how to find it.

76. Show that if $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = \langle x_0, y_0 \rangle$, then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$$

Hint: Use Definition 11.4.7 directly.

11.7 Maximum and Minimum Values

As we discussed in Chapter 4, one of the main uses of ordinary derivatives is in finding maximum and minimum values. In this section, we will discover how to use partial derivatives to locate maxima and minima of functions of two variables. We will also explore some practical applications of this concept.

Local Maximum and Minimum Values

Consider the *hills* and *valleys* in the graph of f shown in Figure 11.59. It appears that there are two points (a, b) where f has a *local maximum*, that is, where $f(a, b)$ is the largest value in a *neighborhood* of, or nearby, (a, b). The larger of these two values is the *absolute maximum*. Similarly, f has two *local minima*, where $f(a, b)$ is the smallest value in a neighborhood. The smaller of these two values is the *absolute minimum*.

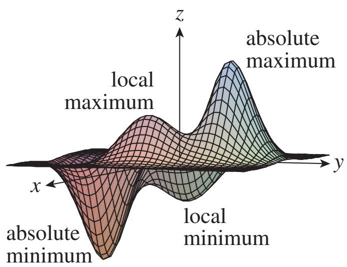


Figure 11.59
Graph of f with local and absolute extreme values.

Definition • Local Extrema

A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

If the inequalities in this definition hold for *all* points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

For functions of a single variable, Fermat's Theorem (Section 4.2) states that if f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$. The next theorem provides a similar result for functions of two variables.

Note that the conclusion of this theorem can be stated using the gradient vector: $\nabla f(a, b) = \mathbf{0}$.

Fermat's Theorem for Functions of Two Variables

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Proof

Let $g(x) = f(x, b)$.

If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a.

Therefore, $g'(a) = 0$ by Fermat's Theorem for functions of one variable.

But $g'(a) = f_x(a, b) \quad \Rightarrow \quad f_x(a, b) = 0$.

Similarly, apply Fermat's Theorem to the function $G(y) = f(a, y)$ and $f_y(a, b) = 0$. ■

A Closer Look

1. If we let $f_x(a, b) = 0$ and $f_y(a, b) = 0$ in the equation of a tangent plane, then $z = z_0$. Therefore, the geometric interpretation of Fermat's theorem for functions of two variables is that if the graph of f has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.

2. A point (a, b) is called a **critical point** (or **stationary point**) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Fermat's theorem for functions of two variables says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f. However, just as in single-variable calculus, not all critical points result in a local or absolute extreme value. At a critical point, a function could have a local maximum, a local minimum, or neither a local maximum nor a local minimum.

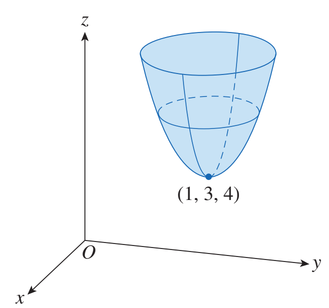


Figure 11.60
Graph of the elliptic paraboloid.

Example 1 A Function with an Absolute Minimum

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Find the partial derivatives and set each expression equal to 0.

$$f_x(x, y) = 2x - 2 = 0 \quad \Rightarrow \quad 2(x - 1) = 0 \quad \Rightarrow \quad x = 1$$

$$f_y(x, y) = 2y - 6 = 0 \quad \Rightarrow \quad 2(y - 3) = 0 \quad \Rightarrow \quad y = 3$$

The only critical point is $(1, 3)$.

Rewrite the expression for f by completing the square.

$$f(x, y) = (x^2 - 2x + 1) + (y^2 - 6y + 9) + 14 - 1 - 9$$
$$= (x - 1)^2 + (y - 3)^3 + 4$$

Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, then $f(x, y) \geq 4$ for all values of x and y.

Therefore, $f(1, 3) = 4$ is a local minimum, and in fact, is the absolute minimum of f.

We can confirm this geometrically from the graph of f, which is an elliptic paraboloid with vertex at $(1, 3, 4)$, shown in Figure 11.60.

Example 2 A Function with No Extreme Values

Find the extreme values of $f(x, y) = y^2 - x^2$.

Solution

Find the partial derivatives and set each expression equal to 0.

$$f_x = -2x = 0 \quad \Rightarrow \quad x = 0 \qquad f_y = 2y = 0 \quad \Rightarrow \quad y = 0$$

The only critical point is $(0, 0)$.

For points on the x-axis, $y = 0$, so $f(x, y) = -x^2 < 0$ (if $x \neq 0$).

For points on the y-axis, $x = 0$, so $f(x, y) = y^2 > 0$ (if $y \neq 0$).

Therefore, every disk with center $(0, 0)$ contains points where f takes on positive values as well as points where f takes on negative values.

Therefore, $f(0, 0)$ cannot be an extreme value for f.

So, f has no extreme values.

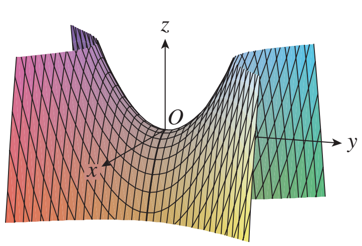


Figure 11.61
Graph of the hyperbolic paraboloid $z = y^2 - x^2$.

Example 2 shows that a critical point is a candidate for a maximum or minimum; that is, a function need not have a maximum or minimum value at a critical point. Figure 11.61 illustrates how this might happen. The graph of f is the hyperbolic paraboloid $z = y^2 - x^2$, which has horizontal tangent plane $(z = 0)$ at the origin. The graph suggests that $f(0, 0) = 0$ is a maximum in the direction of the x-axis but a minimum in the direction of the y-axis. Near the origin, the graph has the shape of a saddle; therefore, $(0, 0)$ is called a *saddle point* of f.

We need a method to determine whether or not a function has an extreme value at a critical point. The following test, proved in Appendix E, is analogous to the Second Derivative Test for functions of one variable.

Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$; that is, (a, b) is a critical point of f. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is neither a local maximum nor a local minimum.

A Closer Look

1. In case (c), the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).
2. If $D = 0$, the test is inconclusive: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.
3. To help remember the formula, the expression for D is often written as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$ ■

Example 3 Classify Critical Points

Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

Solution

Find the critical points; compute the partial derivatives and set each expression equal to 0.

$$f_x = 4x^3 - 4y = 0 \quad \Rightarrow \quad x^3 - y = 0$$
$$f_y = 4y^3 - 4x = 0 \quad \Rightarrow \quad y^3 - x = 0$$

Solve these equations simultaneously.

$y = x^3$ — Solve the first equation for y.

$0 = (x^3)^3 - x = x^9 - x = x(x^8 - 1)$ — Substitute into the second equation.

$\quad = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$

$\quad = x(x - 1)(x + 1)(x^2 + 1)(x^4 + 1)$ — Factor completely.

There are three real roots: $x = 0, 1, -1$.

The three critical points are: $(0, 0), (1, 1), (-1, -1)$. — Use $y = x^3$.

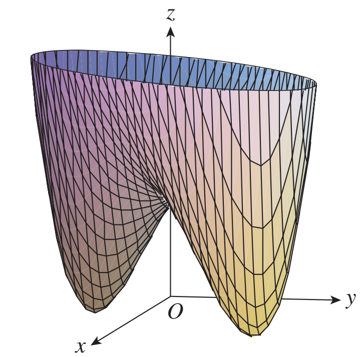


Figure 11.62
Graph of $z = x^4 + y^4 - 4xy + 1$.

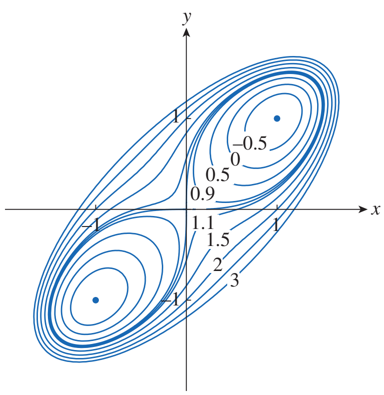


Figure 11.63
A contour map of the function f.

Calculate the second partial derivatives and $D(x, y)$.

$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$

$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$

$D(0, 0) = 144(0)^2(0)^2 - 16 = -16 < 0 \quad \Rightarrow \quad (0, 0)$ is a saddle point. f has neither a local maximum nor local minimum at $(0, 0)$.

$D(1, 1) = 144(1)^2(1)^2 - 16 = 128 > 0$ and $f_{xx}(1, 1) = 12(1)^2 = 12 > 0$
Therefore, $f(1, 1) = -1$ is a local minimum.

$D(-1, -1) = 144(-1)^2(-1)^2 - 16 = 128 > 0$ and $f_{xx} = 12(-1)^2 = 12 > 0$
Therefore, $f(-1, -1) = -1$ is also a local minimum.

The graph of f is shown in Figure 11.62.

A contour map of the function f is shown in Figure 11.63. The level curves near $(1, 1)$ and $(-1, -1)$ are oval in shape and indicate that as we move away from $(1, 1)$ and $(-1, -1)$ in any direction, the values of f are increasing.

The level curves near $(0, 0)$ look like hyperbolas. They suggest that as we move away from the origin, where the value of f is 1, the values of f decrease in some directions but increase in other directions.

Therefore, the contour map suggests the presence of the minima and saddle point as determined analytically. ■

Example 4 Estimate Critical Points Numerically

Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Find the highest point on the graph of f.

Solution

Find the partial derivatives and set each expression equal to 0.

$f_x = 20xy - 10x - 4x^3 \qquad f_y = 10x^2 - 8y - 8y^3$

To find the critical points, solve the system of equations.

$$2x(10y - 5 - 2x^2) = 0 \tag{1}$$
$$5x^2 - 4y - 4y^3 = 0 \tag{2}$$

Use the Principle of Zero Products in Equation 1.

$x = 0 \quad$ or $\quad 10y - 5 - 2x^2 = 0$

If $x = 0$, then Equation 2 becomes $-4y(1 + y^2) = 0 \quad \Rightarrow \quad y = 0$.

Therefore, $(0, 0)$ is a critical point.

If $10y - 5 - 2x^2 = 0$, then $x^2 = 5y - 2.5$

Using this expression in Equation 2, we have $25y - 12.5 - 4y - 4y^3 = 0$.

Use technology to solve this cubic equation: $y \approx -2.5452, 0.6468, 1.8984$.

The corresponding x-values are given by $x = \pm\sqrt{5y - 2.5}$.

If $y \approx -2.5452$, then x has no corresponding real values.

If $y \approx 0.6468$, then $x \approx \pm 0.8567$.

If $y \approx 1.8984$, then $x \approx \pm 2.6442$.

Therefore, there are five critical points. Each point is analyzed in the following chart.

Critical point	Value of f	f_{xx}	D	Conclusion
(0.0)	0.00	-10.000	80.000	local maximum
$(\pm 2.6442,\ 1.8984)$	8.496	-55.936	2488.717	local maximum
$(\pm 0.8567,\ 0.6468)$	-1.485	-5.871	-187.636	saddle point

Figures 11.64 and 11.65 show two views of the graph of f; the surface opens downward.

This is confirmed by analyzing the expression for $f(x,\ y)$: the dominant terms are $-x^4 - 2y^4$ when $|x|$ and $|y|$ are large.

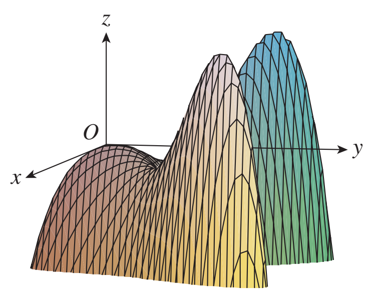


Figure 11.64
Graph of $z = f(x,\ y)$.

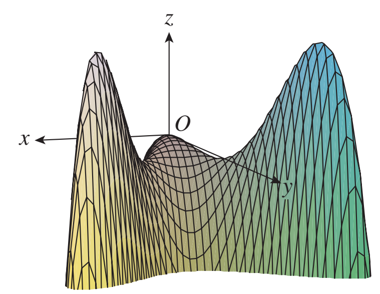


Figure 11.65
Graph of $z = f(x,\ y)$; a different viewpoint.

Compare the values of f at its local maximum points.
The absolute maximum value of f is $f(\pm 2.6442,\ 1.8984) = 8.496$.
The highest points on the graph of f are $(\pm 2.6442,\ 1.8984,\ 8.496)$.

Figure 11.66 shows a contour map of the function f with the five critical points. This graph confirms our analytical results.

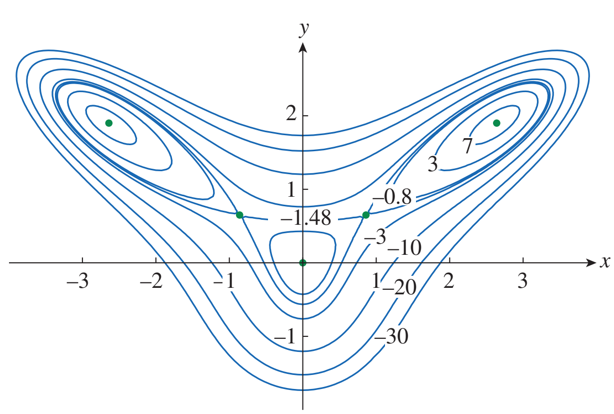


Figure 11.66
Contour map of the function f with five critical points.

Example 5 Shortest Distance

Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

Solution

The distance from any point (x, y, z) to the point $(1, 0, -2)$ is

$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$.

If (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$ and the expression for d is $d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$.

We can minimize d by minimizing the simpler expression

$d^2 = f(x, y) = (x-1)^2 + y^2 + (6-x-2y)^2$.

Find the partial derivatives and set each expression equal to 0.

$f_x = 2(x-1) - 2(6-x-2y) = 4x + 4y - 14 = 0$

$f_y = 2y - 4(6-x-2y) = 4x + 10y - 24 = 0$

Solve these equations simultaneously; the result is one critical point: $\left(\frac{11}{6}, \frac{5}{3}\right)$.

Find the second-order partial derivatives and evaluate the function D.

$f_{xx} = 4, \quad f_{xy} = 4, \quad f_{yy} = 10$

$D\left(\frac{11}{6}, \frac{5}{3}\right) = f_{xx}f_{yy} - (f_{xy})^2 = (4)(10) - (4)^2 = 24 > 0$

By the Second Derivatives Test, f has a local minimum at $\left(\frac{11}{6}, \frac{5}{3}\right)$.

Intuitively, this local minimum must be an absolute minimum because there must be a point on the given plane that is closest to $(1, 0, -2)$.

If $x = \frac{11}{6}$ and $y = \frac{5}{3}$, then $z = -\frac{7}{6}$ and the distance to the plane is

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2} = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2} = \frac{5}{6}\sqrt{6}.$$

The shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$.

Figure 11.67 shows a graph of the point on the plane closest to $(1, 0, -2)$. ■

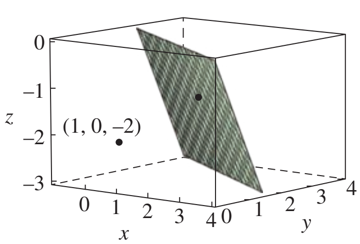


Figure 11.67
Graph of the point $(1, 0, -2)$, the plane, and the point on the plane closest to $(1, 0, -2)$.

Example 6 Maximum Volume

A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Solution

Let the length, width, and height of the box (in meters) be x, y, and z, as shown in Figure 11.68.

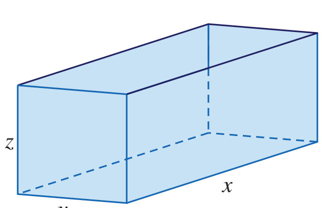


Figure 11.68
Rectangular box with dimensions x, y, and z.

The volume of the box is $V = xyz$.

Write an expression for the area of the four sides and the bottom of the box.
$2xz + 2yz + xy = 12$

Solve this equation for z, and substitute into the formula for volume.

$$z = \frac{12 - xy}{2(x + y)} \quad \Rightarrow \quad V = xy\frac{12 - xy}{2(x + y)} = \frac{12xy - x^2y^2}{2(x + y)}$$

Find the partial derivatives and set each expression equal to 0.

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x + y)^2} = 0 \qquad \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x + y)^2} = 0$$

If $x = 0$ or $y = 0$ then the volume $V = 0$.

Therefore, we need to solve the following equations.

$$12 - 2xy - x^2 = 0 \qquad 12 - 2xy - y^2 = 0$$

Note that x and y must both be positive in this problem.

Solving these simultaneously: $x^2 = y^2 \quad \Rightarrow \quad x = y$.

Let $x = y$ in either equation.

$$12 - 3x^2 = 0 \quad \Rightarrow \quad x = 2, y = 2, \quad \text{and} \quad z = \frac{12 - 2 \cdot 2}{2(2 + 2)} = 1$$

We can use the Second Derivatives Test to confirm that this solution yields a local maximum, or argue from the physical nature of this problem that there must be an absolute maximum volume, which has to occur at a critical point of V. Figure 11.69 shows a graph of the volume function and provides additional graphical confirmation of this conclusion.

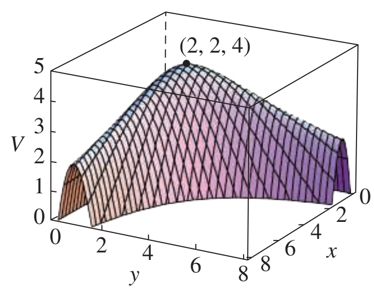


Figure 11.69
Graph of the volume function.

Therefore, the maximum volume occurs when $x = 2$, $y = 2$, and $z = 1$.

The maximum volume is $V = 2 \cdot 2 \cdot 1 = 4\ \text{m}^3$. ■

Absolute Maximum and Minimum Values

For a function f of one variable, the Extreme Value Theorem says that if f is continuous on a closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value. Using the Closed Interval Method from Section 4.2, we found these extreme values by evaluating f at the critical points and endpoints of the interval, a and b.

There is a similar method for functions of two variables. Just as a closed interval contains both endpoints, a **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points. A boundary point in a set D is exactly what you would expect, and is defined as a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D. For example, the disk

$$D = \{(x, y) \mid x^2 + y^2 \le 1\}$$

which consists of all points on and inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points, which are the points on the circle $x^2 + y^2 = 1$. Note that if even one point on the boundary curve were omitted, the set would not be closed. See Figure 11.70

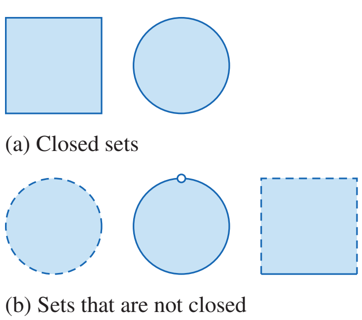


Figure 11.70
Examples of sets in $\mathbb{R}^2$.

A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk; that is, it is finite in extent. Now, the Extreme Value Theorem in two dimensions can be stated in terms of closed and bounded sets.

Can you think of an example of a closed, unbounded set in $\mathbb{R}^2$?

Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the extreme values guaranteed to exist by this theorem, we can use Fermat's theorem for functions of two variables: If f has an extreme value at (x_1, y_1), then (x_1, y_1) is either a critical point of f or a boundary point of D. This leads to the following extension of the Closed Interval Method.

The Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 7 Test for Absolute Extreme Values on the Boundary

Find the absolute maximum and minimum values of the function
$f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

Solution

Since f is a polynomial, it is continuous on the closed, bounded rectangle D.
Theorem 6 guarantees there is both an absolute maximum and an absolute minimum.

Find the critical points: set the partial derivatives equal to 0.

$$f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0$$

Solve these equations simultaneously:
the only critical point is $(1, 1)$ and $f(1, 1) = 1$.

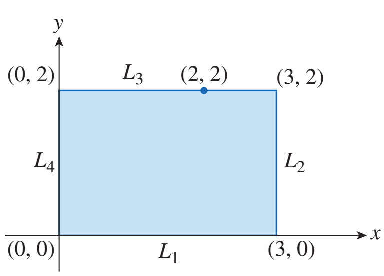


Figure 11.71
The boundary of D consists of four line segments.

Next, consider the values of f on the boundary of D.
The boundary consists of four line segments L_1, L_2, L_3, L_4 as shown in Figure 11.71.

On L_1: $y = 0$ and $f(x, 0) = x^2$, $0 \le x \le 3$.

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$.

On L_2: $x = 3$ and $f(3, y) = 9 - 4y$, $0 \le y \le 2$.

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$.

On L_3: $y = 2$ and $f(x, 2) = x^2 - 4x + 4 = (x - 2)^2$, $0 \le x \le 3$.

Using the methods presented in Chapter 4, the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$.

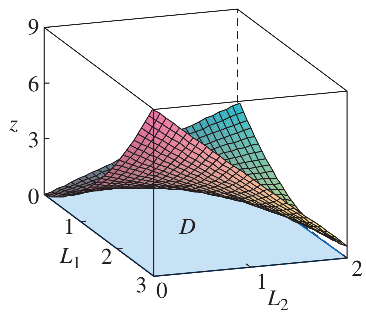


Figure 11.72
Graph of f on the domain D.

On L_4: $x = 0$ and $f(0, y) = 2y$, $0 \le y \le 2$.

This is an increasing function of y, so its maximum value is $f(0, 2) = 4$ and its minimum value is $f(0, 0) = 0$.

Therefore, on the boundary of D, the minimum value of f is 0 and the maximum value is 9.

Compare these values on the boundary with the value $f(1, 1) = 1$ at the critical point.

Finally, we conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$.

Figure 11.72 shows the graph of f and provides some graphical confirmation of our results. ■

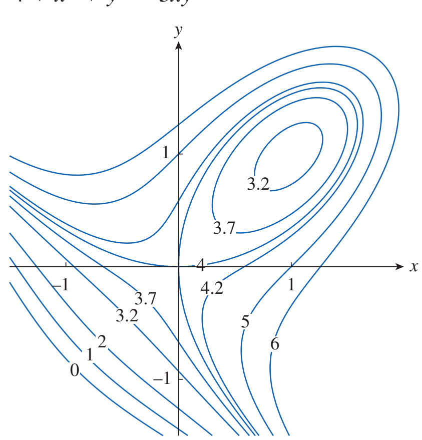
11.7 Exercises

1. Suppose (1, 1) is a critical point of a function f with continuous second derivatives. In each case, what can you say about f?
(a) $f_{xx}(1, 1) = 4$, $f_{xy}(1, 1) = 1$, $f_{yy}(1, 1) = 2$
(b) $f_{xx}(1, 1) = 4$, $f_{xy}(1, 1) = 3$, $f_{yy}(1, 1) = 2$

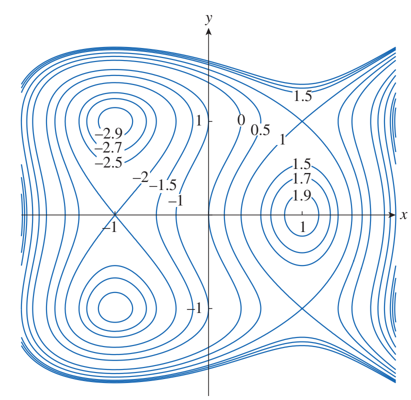
2. Suppose (0, 2) is a critical point of a function g with continuous second derivatives. In each case, what can you say about g?
(a) $g_{xx}(0, 2) = -1$, $g_{xy}(0, 2) = 6$, $g_{yy}(0, 2) = 1$
(b) $g_{xx}(0, 2) = -1$, $g_{xy}(0, 2) = 2$, $g_{yy}(0, 2) = -8$
(c) $g_{xx}(0, 2) = 4$, $g_{xy}(0, 2) = 6$, $g_{yy}(0, 2) = 9$

Use the level curves in the figure to predict the location of the critical points of f and whether f has a saddle point or a local maximum or minimum at each critical point. Explain your reasoning. Then use the Second Derivatives Test to confirm your predictions.

3. $f(x, y) = 4 + x^3 + y^3 - 3xy$



4. $f(x, y) = 3x - x^3 - 2y^2 + y^4$



Find the local maximum and minimum values and saddle point(s) of the function. Use technology to graph the function with a domain and viewpoint that reveal all the important aspects of the function.

5. $f(x, y) = x^2 + xy + y^2 + y$

6. $f(x, y) = x^3y + 12x^2 - 8y$

7. $f(x, y) = x^4 + y^4 - 4xy + 2$

8. $f(x, y) = x^3 - 12xy + 8y^3$

9. $f(x, y) = xy - 2x - 2y - x^2 - y^2$

10. $f(x, y) = (x - y)(1 - xy)$

11. $f(x, y) = 2 - x^4 + 2x^2 - y^2$

12. $f(x, y) = x^3 - 3x + 3xy^2$

13. $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$

14. $f(x, y) = x^4 - 2x^2 + y^3 - 3y$

15. $f(x, y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$

16. $f(x, y) = (x^2 + y^2)e^{-x}$

17. $f(x, y) = xye^{-(x^2+y^2)/2}$

18. $f(x, y) = (x^2 + y^2)e^{y^2-x^2}$

19. $f(x, y) = e^y(y^2 - x^2)$

20. $f(x, y) = e^x \cos y$

21. $f(x, y) = y \cos x$

22. $f(x, y) = y^2 - 2y \cos x, \quad -1 \le x \le 7$

23. $f(x, y) = \sin x \sin y, \quad -\pi < x < \pi, \quad -\pi < y < \pi$

24. Show that $f(x, y) = x^2 + 4y^2 - 4xy + 2$ has an infinite number of critical points and that $D = 0$ at each one. Then show that f has a local (and absolute) minimum at each critical point.

25. Show that $f(x, y) = x^2ye^{-x^2-y^2}$ has maximum values at $\left(\pm 1, \frac{1}{\sqrt{2}}\right)$ and minimum values at $\left(\pm 1, -\frac{1}{\sqrt{2}}\right)$. Also show that f has infinitely many other critical points and $D = 0$ at each of them. Which of these yield maximum values? Minimum values? Saddle points?

Use a graph and/or level curves to estimate the local maximum and minimum values and saddle point(s) of the function. Use calculus to confirm your estimates and to find these values precisely.

26. $f(x, y) = x^2 + y^2 + x^{-2}y^{-2}$

27. $f(x, y) = xye^{-x^2-y^2}$

28. $f(x, y) = \sin x + \sin y + \sin(x + y)$
$0 \le x \le 2\pi, \quad 0 \le y \le 2\pi$

29. $f(x, y) = \sin x + \sin y + \cos(x + y)$
$0 \le x \le \dfrac{\pi}{4}, \quad 0 \le y \le \dfrac{\pi}{4}$

Use technology to find the critical points of f. Classify each critical point and find the highest or lowest points on the graph, if any.

30. $f(x, y) = x^4 - 5x^2 + y^2 + 3x + 2$

31. $f(x, y) = x^4 + y^4 - 4x^2y + 2y$

32. $f(x, y) = 5 - 10xy - 4x^2 + 3y - y^4$

33. $f(x, y) = 2x + 4x^2 - y^2 + 2xy^2 - x^4 - y^4$

34. $f(x, y) = x^4 + y^3 - 3x^2 + y^2 + x - 2y + 1$

35. $f(x, y) = e^x + y^4 - x^3 + 4 \cos y$

36. $f(x, y) = 20e^{-x^2-y^2} \sin 3x \cos 3y, \ |x| \le 1, \ |y| \le 1$

Find the absolute maximum and minimum values of f on the set D.

37. $f(x, y) = 1 + 4x - 5y$, D is the closed triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 3)$.

38. $f(x, y) = x + y - xy$, D is the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$.

39. $f(x, y) = x^2 + y^2 + x^2y + 4$,
$D = \{(x, y) \mid |x| \le 1, |y| \le 1\}$

40. $f(x, y) = 4x + 6y - x^2 - y^2$,
$D = \{(x, y) \mid 0 \le x \le 4, \ 0 \le y \le 5\}$

41. $f(x, y) = x^2 + xy + y^2 - 6y$,
$D = \{(x, y) \mid -3 \le x \le 3, \ 0 \le y \le 5\}$

42. $f(x, y) = 2x^3 + y^4, \quad D = \{(x, y) \mid x^2 + y^2 \le 1\}$

43. $f(x, y) = x^3 - 3x - y^3 + 12y$, D is the quadrilateral whose vertices are $(-2, 3)$, $(2, 3)$, $(2, 2)$, and $(-2, -2)$.

44. For functions of one variable, it is impossible for a continuous function to have two local maximum and no local minimum. However, these characteristics are possible for a function of two variables. Show that the function

$$f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2$$

has only two critical points, but has a local maximum at both of them. Use technology to graph f with a carefully chosen domain and viewpoint to see how this is possible.

45. If a function of one variable is continuous on an interval and has only one critical number, then a local maximum must also be an absolute maximum. However, this is not true for functions of two variables. Show that the function

$$f(x, y) = 3xe^y - x^3 - e^{3y}$$

has exactly one critical point, and that f has a local maximum there that is not an absolute maximum. Use technology to graph f with a carefully chosen domain and viewpoint to see how this is possible.

46. Find the shortest distance from the point $(2, 1, -1)$ to the plane $x + y - z = 1$.

47. Find the point on the plane $x - 2y + 3z = 6$ that is closest to the point $(0, 1, 1)$.

48. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

49. Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

50. Find three positive numbers whose sum is 100 and whose product is a maximum.

51. Find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible.

52. Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r.

53. Find the dimensions of the box with volume $1000\ \text{cm}^3$ that has minimal surface area.

54. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

55. Find the dimensions of the rectangular box with largest volume if the total surface area is given as $64\ \text{cm}^2$.

56. Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant c.

57. The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.

58. A cardboard box without a lid is to have a volume of $32{,}000\ \text{cm}^3$. Find the dimensions that minimize the amount of cardboard used.

59. A rectangular building is being designed to minimize heat loss. The east and west walls lose heat at a rate of $10\ \text{units/m}^2$ per day, the north and south walls at a rate of $8\ \text{units/m}^2$ per day, the floor at a rate of $1\ \text{unit/m}^2$ per day, and the roof at a rate of $5\ \text{units/m}^2$ per day. Each wall must be at least 30 m long, the height must be at least 4 m, and the volume must be exactly $4000\ \text{m}^3$.
(a) Find and sketch the domain of the heat loss as a function of the lengths of the sides.
(b) Find the dimensions that minimize heat loss. Check both the critical points and the points on the boundary of the domain.
(c) Is it possible to design a building with even less heat loss if the restrictions on the lengths of the walls were removed? Explain your reasoning.

60. If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

61. A model for the yield Y of an agricultural crop as a function of the nitrogen level N and phosphorus level P in the soil (measured in appropriate units) is

$$Y(N,\ P) = kNPe^{-N-P}$$

where k is a positive constant. Find the levels of nitrogen and phosphorus that result in the best yield.

62. The Shannon index (sometimes called the Shannon–Wiener index or Shannon–Weaver index) is a measure of diversity in an ecosystem. For the case of three species, it is defined as

$$H = -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3$$

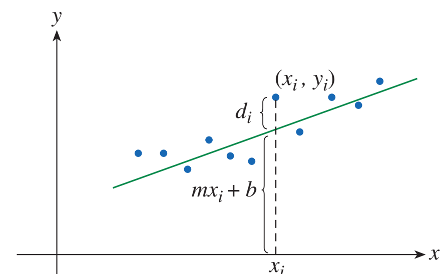
where p_i is the proportion of species i in the ecosystem.
(a) Express H as a function of two variables using the fact that $p_1 + p_2 + p_3 = 1$.
(b) Find the domain of H.
(c) Find the maximum value of H. For what values of p_1, p_2, p_3 does this occur?

63. Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO), and AB. The Hardy–Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

$$P = 2pq + 2pr + 2rq$$

where p, q, and r are the proportions of A, B, and O in the population. Use the fact that $p + q + r = 1$ to show that P is at most $\frac{2}{3}$.

64. Suppose that a researcher has reason to believe that two quantities x and y are related linearly, that is $y = mx + b$, at least approximately, for some values of m and b. The researcher conducts an experiment and collects data in the form of ordered pairs $(x_1,\ y_1)$, $(x_2,\ y_2)$, . . . , $(x_n,\ y_n)$, and then plots these points. The points do not lie exactly on a straight line. However, the researcher would like to find constants m and b so that the line $y = mx + b$ fits the points as well as possible. The figure shows a scatter plot of the points and a possible line of best fit.



Let $d_i = y_i - (mx_i + b)$ be the vertical deviation of the point $(x_i,\ y_i)$ from the line. The **method of least squares** produces values for m and b such that the sum of squares of the deviations, $\sum_{i=1}^{n} d_i^2$, is a minimum.

Show that, using this method to find m and b, the line of best fit is obtained when

$$m\sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i$$

$$m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

Therefore, the line is determined by solving these two equations in the two unknown variables m and b.

65. Find an equation of the plane that passes through the point (1, 2, 3) and cuts off the smallest volume in the first octant.

Applied Project | Designing a Dumpster

For this project, you will consider the shape and construction of a rectangular trash dumpster. You will then attempt to determine the dimensions of a container of similar design that minimize construction cost.

1. Consider a typical rectangular trash dumpster. Carefully study and describe all details of its construction, and determine its volume. Include a sketch of the container.
2. While maintaining the general shape and method of construction, determine the dimensions such a container of the same volume should have in order to minimize the cost of construction. Use the following assumptions in your analysis:
   - The sides, back, and front are to be made from 12-gauge (0.1046 inch thick) steel sheets, which cost $12.00 per square foot (including any required cuts or bends).
   - The base is to be made from a 10-gauge (0.1345 inch thick) steel sheet, which costs $15.00 per square foot.
   - Lids cost approximately $200.00 each, regardless of dimensions.
   - Welding costs approximately $2.00 per foot for material and labor combined.

   Give justification of any further assumptions or simplifications made of the details of construction.
3. Describe how any of your assumptions or simplifications may affect the final result.
4. If you were hired as a consultant on this investigation, what would your conclusions be? Would you recommend altering the design of the dumpster? If so, describe the savings that would result.

Laboratory Project | Quadratic Approximations and Critical Points

The Taylor polynomial approximation to functions of one variable discussed in Chapter 8 can be extended to functions of two or more variables. The purpose of this project is to investigate quadratic approximations to functions of two variables and use them to give insight into the Second Derivatives Test for classifying critical points.

In Section 11.4, we discussed the linearization of a function f of two variables at a point (a, b):

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Recall that the graph of L is the tangent plane to the surface $z = f(x, y)$ at $(a, b, f(a, b))$ and the corresponding linear approximation is $f(x, y) \approx L(x, y)$. The linearization L is also called the **first-degree Taylor polynomial** of f at (a, b).

1. If f has continuous second-order partial derivatives at (a, b), then the **second-degree Taylor polynomial** of f at (a, b) is

   $$\begin{aligned} Q(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &\quad + \frac{1}{2} f_{xx}(a, b)(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2} f_{yy}(y - b)^2 \end{aligned}$$

   and the approximation $f(x, y) \approx Q(x, y)$ is called the **quadratic approximation** to f at (a, b). Verify that Q has the same first- and second-order partial derivatives as f at (a, b).

2. (a) Find the first- and second-degree Taylor polynomials L and Q of $f(x, y) = e^{-x^2-y^2}$ at $(0, 0)$.

(b) Graph f, L, and Q. Use these graphs to explain how well L and Q approximate f.

3. (a) Find the first- and second-degree Taylor polynomials L and Q for $f(x, y) = xe^y$ at $(1, 0)$.

(b) Compare the values of L, Q, and f at $(0.9, 0.1)$.

(c) Graph f, L, and Q. Use these graphs to explain how well L and Q approximate f.

4. In this problem, we will analyze the behavior of the polynomial $f(x, y) = ax^2 + bxy + cy^2$ (without using the Second Derivatives Test) by identifying the graph as a paraboloid.

(a) By completing the square, show that if $a \neq 0$, then

$$f(x, y) = ax^2 + bxy + cy^2 = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)y^2\right]$$

(b) Let $D = 4ac - b^2$. Show that if $D > 0$ and $a > 0$, then f has a local minimum at $(0, 0)$.

(c) Show that if $D > 0$ and $a < 0$, the f has a local maximum at $(0, 0)$.

(d) Show that if $D < 0$, then $(0, 0)$ is a saddle point.

5. (a) Suppose f is any function with continuous second-order partial derivatives such that $f(0, 0) = 0$ and $(0, 0)$ is a critical point of f. Write an expression for the second-degree Taylor polynomial, Q, of f at $(0, 0)$.

(b) What can you conclude about Q from Problem 4?

(c) In view of the quadratic approximation $f(x, y) \approx Q(x, y)$, what does part (b) suggest about f?

11.8 Lagrange Multipliers

In Example 11.7.6, we found the maximum value of the volume function $V = xyz$ subject to the constraint $2xz + 2yz + xy = 12$, an expression for the extra condition that the surface area was $12\ \text{m}^2$. In this section, we consider Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

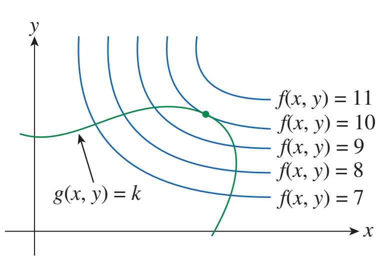


Figure 11.73
Graphs of level curves of f and the graph of the level curve $g(x, y) = k$.

Lagrange Multipliers: One Constraint

First, let's consider the geometric basis of Lagrange's methods for functions of two variables. Suppose we would like to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$. Here's a way to think about this problem geometrically. We want the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$. Figure 11.73 shows this curve together with several level curves of f. These level curves have the equations $f(x, y) = c$, where $c = 7, 8, 9, 10, 11$.

To maximize $f(x, y)$ subject to $g(x, y) = k$ means we need to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$. It appears from Figure 11.73 that this happens when these curves just touch each other, that is, when they have a common tangent line. Otherwise we could find a larger value of c. This means that the normal lines at the point (x_0, y_0) where they touch are identical. So, the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ.

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. Therefore, the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$. Instead of the level curves in Figure 11.73, we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.

Let's make this geometric argument more precise. Suppose that a function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S and let C be a curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ that lies on S and passes through P. If t_0 is the parameter value corresponding to P, then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. The composite function $h(t) = f(x(t), y(t), g(t))$ represents the values that f takes on the curve C. Since f has an extreme value at (x_0, y_0, z_0), it follows that h has an extreme value at t_0, so $h'(t_0) = 0$. But if f is differentiable, we can use the Chain Rule to write

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

Since this dot product is equal to 0, the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C. From Section 11.6, we know that the gradient vector of g, $\nabla g(x_0, y_0, z_0)$, is also orthogonal to $\mathbf{r}'(t_0)$ for every such curve. (See Equation 11.6.18). This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, there is a number λ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \tag{1}$$

Lagrange multipliers are named after the French-Italian mathematician Joseph-Louis Lagrange (1736–1813).

The number λ in Equation 1 is called a **Lagrange multiplier**. The procedure for finding maximum and minimum values based on Equation 1 is described next.

In deriving Lagrange's method, we assumed that $\nabla g \neq \mathbf{0}$. In each of our examples, you can check that $\nabla g \neq \mathbf{0}$ at all points where $g(x, y, z) = k$. See Exercise 27 for what can go wrong if $\nabla g = \mathbf{0}$.

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$):

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

If we write the vector equation $\nabla f = \lambda \nabla g$ in terms of components, then the equations in step (a) become

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x, y, z) = k$$

This is a system of four equations in the four unknowns x, y, z, and λ. We need to find all possible solutions to this system, but not necessarily the explicit values of λ.

For functions of two variables, the method of Lagrange multipliers is similar to the procedure just described. To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x, y, and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = k$$

We need to solve three equations in three unknowns:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

Let's start with an example illustrating Lagrange's method in which we reconsider the problem in Example 6, Section 11.7.

Example 1 Maximize a Volume Using Lagrange Multipliers

A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Solution

As in Example 6 in Section 11.7, let x, y, and z be the length, width, and height, respectively, of the box in meters.

We need to maximize $V = xyz$ subject to the constraint $g(x, y, z) = 2xz + 2yz + xy = 12$.

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$.

This leads to four equations.

$$V_x = \lambda g_x \qquad V_y = \lambda g_y \qquad V_z = \lambda g_z \qquad 2xz + 2yz + xy = 12$$

Find the partial derivatives; the equations become:

$$yz = \lambda(2z + y) \tag{2}$$

$$xz = \lambda(2z + x) \tag{3}$$

$$xy = \lambda(2x + 2y) \tag{4}$$

$$2xz + 2yz + xy = 12 \tag{5}$$

There are no general rules for solving a system of equations. In this case, we could solve each of Equations 2, 3, and 4 for λ, and then equate the resulting expressions.

Another, perhaps easier way, is to multiply Equation 2 by x, Equation 3 by y, and Equation 4 by z, then the left sides of these three resulting equations will be the same.

Here are the three equations after these operations.

$$xyz = \lambda(2xz + xy) \tag{6}$$

$$xyz = \lambda(2yz + xy) \tag{7}$$

$$xyz = \lambda(2xz + 2yz) \tag{8}$$

Note that $\lambda \neq 0$. If $\lambda = 0$, then $yz = xz = xy = 0$ from Equations 2, 3, and 4. This would contradict Equation 5.

Solve for λ in Equations 6 and 7, equate the resulting expressions, and we are left with:

$2xz + xy = 2yz + xy \quad \Rightarrow \quad xz = yz$

But $z \neq 0$ since $z = 0$ would result in $V = 0$. Therefore, $x = y$.

Using the same strategy in Equations 7 and 8, we have

$2yz + xy = 2xz + 2yz \quad \Rightarrow \quad xy = 2xz \quad \Rightarrow \quad y = 2z$ (since $x \neq 0$).

Let $x = y = 2z$ in Equation 5.

$4z^2 + 4z^2 + 4z^2 = 12 \quad \Rightarrow \quad 12z^2 = 12 \quad \Rightarrow \quad z^2 = 1 \quad \Rightarrow \quad z = 1$

(x, y, and z are all positive.)

Therefore, $z = 1 \quad \Rightarrow \quad x = 2,\ y = 2.$

And this is the same answer as in Section 11.7. ■

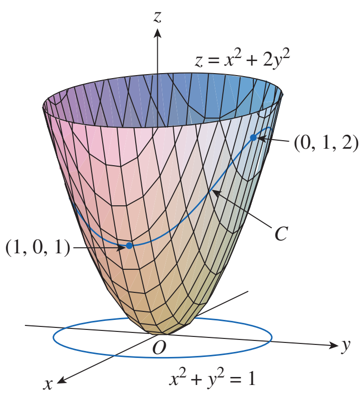


Figure 11.74
Geometrically, we are finding the highest and lowest points on the curve C that lie on the paraboloid $z = x^2 + 2y^2$ and directly above the constraint circle $x^2 + y^2 = 1$.

Example 2 Extreme Values on a Curve C

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution

We need the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$.

Using the method of Lagrange multipliers, we need to solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$.

This leads to the three equations: $f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = 1$

Find the partial derivatives; the equations become:

$$2x = 2x\lambda \tag{9}$$

$$4y = 2y\lambda \tag{10}$$

$$x^2 + y^2 = 1 \tag{11}$$

Using Equation 9: $x = 0$ or $\lambda = 1$.

If $x = 0$, then from Equation 11: $y = \pm 1$.

If $\lambda = 1$, then from Equation 10: $y = 0$ and from Equation 11: $x = \pm 1$.

Therefore, f has possible extreme values at the points $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$.

Evaluate f at these four points.

$f(0, 1) = 2 \qquad f(0, -1) = 2 \qquad f(1, 0) = 1 \qquad f(-1, 0) = 1$

Therefore, the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$.

Figure 11.74 shows the graph of the paraboloid, the circle, and the curve C on the paraboloid. The figure suggests that these extreme values are reasonable.

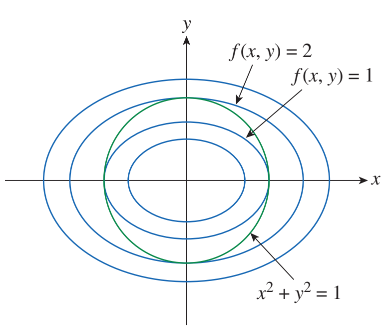


Figure 11.75
Graph of the level curves of f and the circle $x^2 + y^2 = 1$.

Figure 11.75 shows a contour map of f. The extreme values of f correspond to the level curves of f that just touch the circle $x^2 + y^2 = 1$. ■

Example 3 Extreme Values on a Disk

Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \le 1$.

Solution

Using the procedure described in Section 11.7, we need to compare the values of f at the critical point with values at the points on the boundary.

Find the critical points:

$$f_x = 2x = 0 \quad \Rightarrow \quad x = 0 \qquad f_y = 4y = 0 \quad \Rightarrow \quad y = 0$$

Therefore, the only critical point is (0, 0).

Compare the value of f at this point with the extreme values on the boundary from Example 2:

$$f(0, 0) = 0 \qquad f(\pm 1, 0) = 1 \qquad f(0, \pm 1) = 2$$

Therefore, the maximum value of f on the disk $x^2 + y^2 \le 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(0, 0) = 0$. ■

Example 4 Distance to a Sphere

Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

Solution

The distance from a point (x, y, z) to the point $(3, 1, -1)$ is $d = \sqrt{(x - 3)^2 + (y - 1)^2 + (z + 1)^2}$.

The algebra is a little easier if we instead maximize and minimize the square of the distance:

Can you explain why the points that produce a maximum and minimum of d^2 also produce a maximum and minimum of d?

$$d^2 = f(x, y, z) = (x - 3)^2 + (y - 1)^2 + (z + 1)^2.$$

The constraint is that the point (x, y, z) lies on the sphere, that is, $g(x, y, z) = x^2 + y^2 + z^2 = 4$.

Using the method of Lagrange multipliers, we need to solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y, z) = 4$. This leads to the equations:

$$2(x - 3) = 2x\lambda \tag{12}$$

$$2(y - 1) = 2y\lambda \tag{13}$$

$$2(z + 1) = 2z\lambda \tag{14}$$

$$x^2 + y^2 + z^2 = 4 \tag{15}$$

In Equations 12, 13, and 14, solve for x, y, and z in terms of λ.

From Equation 12:

$$x - 3 = x\lambda \quad \Rightarrow \quad x(1 - \lambda) = 3 \quad \Rightarrow \quad x = \frac{3}{1 - \lambda}$$

Note that $1 - \lambda \ne 0$ because $\lambda = 1$ is not possible from Equation 12. Similarly, from Equations 13 and 14:

$$y = \frac{1}{1 - \lambda} \qquad z = -\frac{1}{1 - \lambda}$$

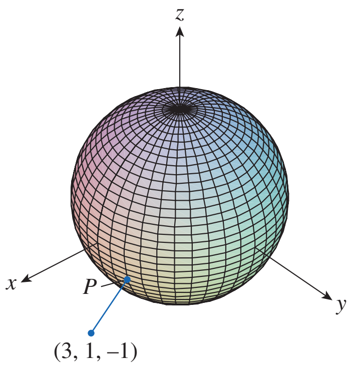


Figure 11.76
The sphere and the nearest point P.

Substitute these values into Equation 15.

$$\frac{3^2}{(1-\lambda)^2} + \frac{1^2}{(1-\lambda)^2} + \frac{(-1)^2}{(1-\lambda)^2} = 4$$

Solve this equation for λ:

$$(1-\lambda)^2 = \frac{11}{4} \quad \Rightarrow \quad 1 - \lambda = \pm\frac{\sqrt{11}}{2} \quad \Rightarrow \quad \lambda = 1 \pm \frac{\sqrt{11}}{2}$$

Use these values of λ in Equations 12, 13, and 14 to find the corresponding points (x, y, z):

$$P\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \quad \text{and} \quad Q\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

The function f has a smaller value at P, so the closest point is P and the farthest point is Q. Figure 11.76 illustrates this result. ■

Lagrange Multipliers: Two Constraints

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (or side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$, as illustrated in Figure 11.77.

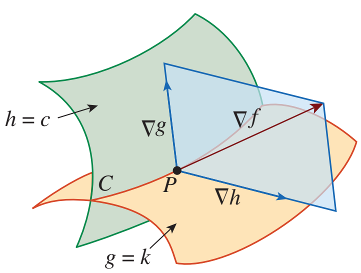


Figure 11.77
The geometric interpretation of two constraints.

Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$. We know from the beginning of this section that ∇f is orthogonal to C at P. But we also know that ∇g is orthogonal to $g(x, y, z) = k$ and ∇h is orthogonal to $h(x, y, z) = c$, so ∇g and ∇h are both orthogonal to C. This means that the gradient vector $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. (We assume that these gradient vectors are not zero and not parallel.) So, there are numbers λ and μ (called the Lagrange multipliers) such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0) \tag{16}$$

In this case, using Lagrange's method to find extreme values means we need to solve a system of five equations in the five unknowns x, y, z, λ, and μ. These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$\begin{aligned} f_x &= \lambda g_x + \mu h_x \\ f_y &= \lambda g_y + \mu h_y \\ f_z &= \lambda g_z + \mu h_z \\ g(x, y, z) &= k \\ h(x, y, z) &= c \end{aligned}$$

Example 5 Extreme Value: Two Constraints

Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

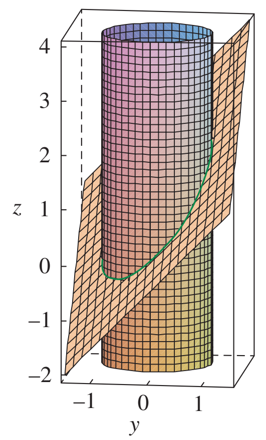


Figure 11.78
The cylinder $x^2 + y^2 = 1$ intersects the plane $x - y + z = 1$ in an ellipse. We need to find the maximum value of f when (x, y, z) is restricted to lie on the ellipse.

Solution

We need to maximize the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$. Figure 11.78 illustrates this problem geometrically.

Using the method of Lagrange multipliers, we need to solve the equation $\nabla f = \lambda \nabla g + \mu \nabla h$.

The five component equations are

$$1 = \lambda + 2x\mu \tag{17}$$
$$2 = -\lambda + 2y\mu \tag{18}$$
$$3 = \lambda \tag{19}$$
$$x - y + z = 1 \tag{20}$$
$$x^2 + y^2 = 1 \tag{21}$$

Let $\lambda = 3$ (from Equation 19) in Equation 17: $2x\mu = -2 \quad \Rightarrow \quad x = -\dfrac{1}{\mu}$.

Similarly, let $\lambda = 3$ in Equation 18: $5 = 2y\mu \quad \Rightarrow \quad y = \dfrac{5}{2\mu}$.

Substitute these values for x and y into Equation 21:

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1 \quad \Rightarrow \quad \mu^2 = \frac{29}{4} \quad \Rightarrow \quad \mu = \pm\frac{\sqrt{29}}{2}.$$

Then $x = -\dfrac{1}{\mu} = \mp\dfrac{2}{\sqrt{29}}, \quad y = \dfrac{5}{2\mu} = \pm\dfrac{5}{\sqrt{29}}$ and

$$z = 1 - x + y = 1 \pm \frac{7}{\sqrt{29}}.$$

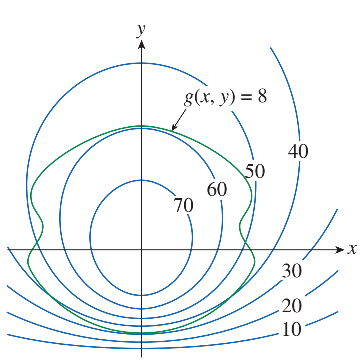
The corresponding values of f are

$$\mp\frac{2}{\sqrt{29}} + 2\left(\pm\frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}.$$

Therefore, the maximum value of f on the given curve is $3 + \sqrt{29}$. ■

11.8 Exercises

1. A contour map of f and a curve with equation $g(x, y) = 8$ are shown in the figure.



Estimate the maximum and minimum values of f subject to the constraint $g(x, y) = 8$. Explain your reasoning.

2. (a) Use technology to graph the circle $x^2 + y^2 = 1$. In the same viewing window, graph several curves of the form $x^2 + y = c$ until you find two that just touch the circle. What is the significance of the values of c for these two curves?

(b) Use Lagrange multipliers to find the extreme values of $f(x, y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$. Compare your answers with those in part (a).

Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

3. $f(x, y) = x^2 + y^2; \quad xy = 1$

4. $f(x, y) = x^2 - y^2; \quad x^2 + y^2 = 1$

5. $f(x, y) = 4x + 6y; \quad x^2 + y^2 = 13$

6. $f(x, y) = x^2y; \quad x^2 + 2y^2 = 6$

7. $f(x, y) = xy; \quad 4x^2 + y^2 = 8$

8. $f(x, y) = xe^y; \quad x^2 + y^2 = 2$

9. $f(x, y) = e^{xy}; \quad x^3 + y^3 = 16$

10. $f(x, y, z) = 2x + 2y + z; \quad x^2 + y^2 + z^2 = 9$

11. $f(x, y, z) = 8x - 4z; \quad x^2 + 10y^2 + z^2 = 5$

12. $f(x, y, z) = e^{xyz}; \quad 2x^2 + y^2 + z^2 = 24$

13. $f(x, y, z) = xyz; \quad x^2 + 2y^2 + 3z^2 = 6$

14. $f(x, y, z) = x^2y^2z^2; \quad x^2 + y^2 + z^2 = 1$

15. $f(x, y, z) = x^2 + y^2 + z^2; \quad x^4 + y^4 + z^4 = 1$

16. $f(x, y, z) = x^4 + y^4 + z^4; \quad x^2 + y^2 + z^2 = 1$

17. $f(x, y, z) = \ln(x^2 + 1) + \ln(y^2 + 1) + \ln(z^2 + 1);$
$x^2 + y^2 + z^2 = 12$

18. $f(x, y, z, t) = x + y + z + t; \quad x^2 + y^2 + z^2 + t^2 = 1$

19. $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n;$
$x_1^2 + x_2^2 + \cdots + x_n^2 = 1$

20. $f(x, y, z) = x + 2y; \quad x + y + z = 1, \quad y^2 + z^2 = 4$

21. $f(x, y, z) = 3x - y - 3z; \quad x + y - z = 0, \quad x^2 + 2z^2 = 1$

22. $f(x, y, z) = yz + xy; \quad xy = 1, \quad y^2 + z^2 = 1$

Find the extreme values of f on the region described by the inequality.

23. $f(x, y) = x^2 + y^2 + 4x - 4y; \quad x^2 + y^2 \le 9$

24. $f(x, y) = 2x^2 + 3y^2 - 4x - 5; \quad x^2 + y^2 \le 16$

25. $f(x, y) = e^{-xy}; \quad x^2 + 4y^2 \le 1$

26. Consider the problem of maximizing the function $f(x, y) = 2x + 3y$ subject to the constraint $\sqrt{x} + \sqrt{y} = 5$.
(a) Try using Lagrange multipliers to solve this problem.
(b) Does $f(25, 0)$ result in a larger value than the answer found in part (a)?
(c) Use technology to solve the problem by graphing the constraint equation and several level curves of f.
(d) Explain why the method of Lagrange multipliers fails in this problem.
(e) What is the significance of $f(9, 4)$?

27. Consider the problem of minimizing the function $f(x, y) = x$ on the curve $y^2 + x^4 - x^3 = 0$ (a piriform).
(a) Try using Lagrange multipliers to solve this problem.
(b) Show that the minimum value is $f(0, 0) = 0$ but the Lagrange condition $\nabla f(0, 0) = \lambda \nabla g(0, 0)$ is not satisfied for any value of λ.
(c) Explain why the method of Lagrange multipliers fails in this problem.

28. (a) Use technology to plot the appropriate curves and use your sketch to estimate the minimum and maximum values of $f(x, y) = x^3 + y^3 + 3xy$ subject to the constraint $(x - 3)^2 + (y - 3)^2 = 9$.
(b) Solve the problem in part (a) using the method of Lagrange multipliers. Use technology to solve the system of equations numerically. Compare your answers with those in part (a).

29. The total production P of a certain product depends on the amount L of labor used and the amount K of capital investment. In Sections 11.1 and 11.3, we discussed how the Cobb–Douglas model $P = bL^{\alpha}K^{1-\alpha}$ follows from certain economic assumptions, where b and α are positive constants and $\alpha < 1$. If the cost of a unit of labor is m and the cost of a unit of capital is n, and the company can spend only p dollars as its total budget, then maximizing the production P is subject to the constraint $mL + nK = p$. Show that the maximum production occurs when

$$L = \frac{\alpha p}{m} \quad \text{and} \quad K = \frac{(1 - \alpha)p}{n}$$

30. Referring to Exercise 29, suppose that the production is fixed at $bL^{\alpha}K^{1-\alpha} = Q$, where Q is a constant. What values of L and K minimize the cost function $C(L, K) = mL + nK$?

31. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.

32. Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral.
Hint: Use Heron's formula for the area:

$$A = \sqrt{s(s - x)(s - y)(s - z)}$$

where $s = p/2$ and x, y, z are the lengths of the sides.

Use Lagrange multipliers to provide an alternate solution to the indicated exercise in Section 11.7.

33. Exercise 46
34. Exercise 48
35. Exercise 49
36. Exercise 50
37. Exercise 51
38. Exercise 52
39. Exercise 53
40. Exercise 54
41. Exercise 55
42. Exercise 56
43. Exercise 57
44. Exercise 60

45. Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm^2 and whose total edge length is 200 cm.

46. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

47. The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse.
(a) Use technology to graph the cone and the plane and observe the elliptical intersection.
(b) Use Lagrange multipliers to find the highest and lowest points on the ellipse.

Find the maximum and minimum values of f subject to the given constraints. Use technology to solve the system of equations that arises in using Lagrange multipliers.

48. $f(x, y, z) = ye^{x-z}$; $9x^2 + 4y^2 + 36z^2 = 36$, $xy + yz = 1$

49. $f(x, y, z) = x + y + z$; $x^2 - y^2 = z$, $x^2 + z^2 = 4$

---

50. (a) Find the maximum value of

$$f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

given that $x_1, x_2, \ldots, x_n$ are positive numbers and $x_1 + x_2 + \cdots + x_n = c$, where c is a constant.
(b) Use the result in part (a) to conclude that if $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \le \frac{x_1 + x_2 + \cdots + x_n}{n}$$

This inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal?

51. (a) Maximize $\sum_{i=1}^{n} x_i y_i$ subject to the constraints $\sum_{i=1}^{n} x_i^2 = 1$ and $\sum_{i=1}^{n} y_i^2 = 1$.
(b) Let

$$x_i = \frac{a_i}{\sqrt{\sum_{j=1}^{n} a_j^2}} \quad \text{and} \quad y_i = \frac{b_i}{\sqrt{\sum_{j=1}^{n} b_j^2}}$$

to show that

$$\sum_{i=1}^{n} a_i b_i \le \sqrt{\sum_{j=1}^{n} a_j^2}\sqrt{\sum_{j=1}^{n} b_j^2}$$

for any numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$. This expression is called the Cauchy–Schwarz Inequality.

Applied Project | Rocket Science

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Construction of a Saturn V rocket.

Sharon Wildie/Shutterstock.com

The Ariane 5 rocket ready for launch.

Many rockets, such as the *Ariane 1* and the *Saturn V*, are designed to use three stages in their ascent into space. A large first stage initially propels the rocket until its fuel is consumed, at which point the stage is jettisoned to reduce the mass of the rocket. The smaller second and third stages function similarly in order to place the rocket's payload into orbit about Earth. (With this design, at least two stages are required in order to reach the necessary velocities, and using three stages has proven to be a good compromise between cost and performance.) The goal of this project is to determine the individual masses of the three stages, which are to be designed in such a way as to minimize the total mass of the rocket while enabling it to reach a desired velocity.

For a single-stage rocket consuming fuel at a constant rate, the change in velocity resulting from the acceleration of the rocket vehicle has been modeled by

$$\Delta V = -c \ln\left(1 - \frac{(1 - S)M_r}{P + M_r}\right)$$

where M_r is the mass of the rocket engine including initial fuel, P is the mass of the payload, S is a *structural factor* determined by the design of the rocket (specifically, it is the ratio of the mass of the rocket vehicle without fuel to the total mass of the rocket with payload), and c is the (constant) speed of exhaust relative to the rocket.

Now consider a rocket with three stages and a payload of mass A. Assume that outside forces are negligible and that c and S remain constant for each stage. If M_i is the mass of the ith stage, we can initially consider the rocket engine to have mass M_1 and its payload to have mass $M_2 + M_3 + A$; the second and third stages can be treated similarly.

1. Show that the velocity attained after all three stages have been jettisoned is given by

$$v_f = c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A}\right) + \ln\left(\frac{M_3 + A}{SM_3 + A}\right)\right].$$

2. The goal is to minimize the total mass $M = M_1 + M_2 + M_3$ of the rocket engine subject to the constraint that the desired velocity v_f from Problem 1 is attained. The method of Lagrange multipliers is appropriate here, but difficult to implement using the current expressions.

To simplify, define variables N_i so that the constraint equation may be expressed as $v_f = c(\ln N_1 + \ln N_2 + \ln N_3)$. Since M is now difficult to express in terms of the N_i's, we need a simpler function that will be minimized at the same value as M.

Show that

$$\frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} = \frac{(1 - S)N_1}{1 - SN_1}$$

$$\frac{M_2 + M_3 + A}{M_3 + A} = \frac{(1 - S)N_2}{1 - SN_2}$$

$$\frac{M_3 + A}{A} = \frac{(1 - S)N_3}{1 - SN_3}$$

and conclude that

$$\frac{M + A}{A} = \frac{(1 - S)^3 N_1 N_2 N_3}{(1 - SN_1)(1 - SN_2)(1 - SN_3)}$$

3. Verify that $\ln \dfrac{M + A}{A}$ is minimized at the same value as M; use Lagrange multipliers and the results of Problem 2 to find expressions for the values of N_i where the minimum occurs subject to the constraint $v_f = c(\ln N_1 + \ln N_2 + \ln N_3)$.

Hint: Use properties of logarithms to help simplify the expressions.

4. Find an expression for the minimum value of M as a function of v_f.

5. If a space agency would like to place a three-stage rocket into orbit 100 miles above Earth's surface, a final velocity of approximately 17,500 mi/h is required. Suppose that each stage is built with a structural factor $S = 0.2$ and exhaust speed of $c = 6000$ mi/h.

(a) Find the minimum total mass M of the rocket engines as a function of A.

(b) Find the mass of each individual stage as a function of A. (They are not all of equal size!)

6. The same rocket would require a final velocity of approximately 24,700 mi/h in order to escape Earth's gravity. Find the mass of each individual stage that would minimize the total mass of the rocket engines and allow the rocket to propel a 500-pound probe into deep space.

Applied Project | Hydro-Turbine Optimization

The James W. Broderick Hydroelectric Power Facility at Pueblo Reservoir began operation during the summer of 2019. Water flows through a dam on the reservoir to the power station, and then into the Arkansas River. The rate at which the water flows through the pipes varies, depending on external conditions.

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A hydroelectric turbine at the James W. Broderick Hydroelectric Power Facility at Pueblo Dam.

The power station has three different hydro electric turbines. Suppose each has a known (and unique) power function that gives the amount of electric power generated as a function of the water flow arriving at the turbine. The incoming water can be apportioned in different volumes to each turbine, so the goal is to determine how to distribute water among the turbines to give the maximum total energy production for any rate of flow.

Using experimental evidence and *Bernoulli's equation*, suppose the following quadratic models were determined for the power output of each turbine, along with the allowable flows of operation:

$$KW_1 = (-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5}Q_1^2)(170 - 1.6 \cdot 10^{-6}Q_T^2)$$

$$KW_2 = (-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5}Q_2^2)(170 - 1.6 \cdot 10^{-6}\ Q_T^2)$$

$$KW_3 = (-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5}\ Q_3^2)(170 - 1.6 \cdot 10^{-6}\ Q_T^2)$$

$$250 \le Q_1 \le 1110, \quad 250 \le Q_2 \le 1110, \quad 250 \le Q_3 \le 1225$$

where

$Q_i =$ flow through turbine i in cubic feet per second

$KW_i =$ power generated by turbine i in kilowatts

$Q_T =$ total flow through the station in cubic feet per second

1. If all three turbines are being used, we would like to determine the flow Q_i to each turbine that will yield the maximum total energy production. The limitations are that the flows must sum to the total incoming flow and the given domain restrictions must be maintained.

Use Lagrange multipliers to find the values for the individual flows (as functions of Q_T) that maximize the total energy production $KW_1 + KW_2 + KW_3$ subject to the constraints $Q_1 + Q_2 + Q_3 = Q_T$ and the domain restrictions on each Q_i.

2. For which values of Q_T is your result valid?

3. For an incoming flow of 2500 ft^3/s, determine the distribution to the turbines and verify (by trying some other, close values for Q_i) that your result is indeed a maximum.

4. Until now we have assumed that all three turbines are operating. However, it may be possible that in some situations more power could be produced by using only one turbine. Construct a graph of the three power functions and use it to help decide if an incoming flow of 1000 ft^3/s should be distributed to all three turbines or routed to just one. If you determine that only one turbine should be used, which one should it be? What if the flow is only 600 ft^3/s?

5. For some flow levels, it might be advantageous to use just two turbines. If the incoming flow is 1500 ft^3/s, which two turbines would you recommend using? Use Lagrange multipliers to determine how the flow should be distributed between the two turbines to maximize the energy produced. For this flow, is using two turbines more efficient than using all three?

6. If the incoming flow is 3400 ft^3/s, what would you recommend to the company, that is, what should the flow rate be to each turbine?

11 Review

Concepts and Vocabulary

1. (a) What is a function of two variables?
   (b) Describe two methods for visualizing a function of two variables. Explain the connection between these methods.

2. What is a function of three variables? Explain how to visualize a function of three variables.

3. What does

$$\lim_{(x,\,y)\to(a,\,b)} f(x,\,y) = L$$

mean? How can you show that such a limit does not exist?

4. (a) What does it mean to say that f is continuous at (a, b)?
   (b) If f is continuous on $\mathbb{R}^2$, what can you say about its graph?

5. (a) Write expressions involving limits for the partial derivatives $f_x(a, b)$ and $f_y(a, b)$.
   (b) How do you interpret $f_x(a, b)$ and $f_y(a, b)$ geometrically? How do you interpret these values as rates of change?
   (c) If $f(x, y)$ is given by a formula, explain how to calculate f_x and f_y.

6. Explain Clairaut's Theorem in your own words.

7. How do you find a tangent plane to each of the following types of surfaces?
   (a) A graph of a function of two variables, $z = f(x, y)$
   (b) A level surface of a function of three variables, $F(x, y, z) = k$
   (c) A parametric surface given by a vector function $\mathbf{r}(u, v)$

8. Define the linearization of f at (a, b). What is the corresponding linear approximation? What is the geometric interpretation of the linear approximation?

9. (a) What does it mean to say that f is differentiable at (a, b)?
   (b) How do you usually verify that f is differentiable?

10. If $z = f(x, y)$, what are the differentials dx, dy, and dz?

11. State the Chain Rule for the case where $z = f(x, y)$ and x and y are functions of one variable. What if x and y are functions of two variables?

12. If z is defined implicitly as a function of x and y by an equation of the form $F(x, y, z) = 0$, how do you find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$?

13. (a) Write an expression as a limit for the directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$. How do you interpret the directional derivative as a rate? How do you interpret it geometrically?
    (b) If f is differentiable, write an expression for $D_{\mathbf{u}} f(x_0, y_0)$ in terms of f_x and f_y.

14. (a) Define the gradient vector ∇f for a function f of two or three variables.
    (b) Express $D_{\mathbf{u}} f$ in terms of ∇f.
    (c) Explain the geometric significance of the gradient.

15. What do the following statements mean?
    (a) f has a local maximum at (a, b).
    (b) f has an absolute maximum at (a, b).
    (c) f has a local minimum at (a, b).
    (d) f has an absolute minimum at (a, b).
    (e) f has a saddle point at (a, b).

16. (a) If f has a local maximum at (a, b), what can you say about its partial derivatives at (a, b)?
    (b) What is a critical point of f?

17. State the Second Derivatives Test for functions of two variables.

18. (a) What is a closed set in $\mathbb{R}^2$? What is a bounded set?
    (b) State the Extreme Value Theorem for functions of two variables.
    (c) How do you find the values that the Extreme Value Theorem guarantees must exist?

19. Explain how the method of Lagrange multipliers works in finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. What if there is a second constraint $h(x, y, z) = c$?

True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. $f_y(a, b) = \displaystyle\lim_{y\to b} \frac{f(a, y) - f(a, b)}{y - b}$

2. There exists a function f with continuous second-order partial derivatives such that $f(x, y) = x + y^2$ and $f_y(x, y) = x - y^2$.

3. $f_{xy} = \dfrac{\partial^2 f}{\partial x\, \partial y}$

4. $D_{\mathbf{k}} f(x, y, z) = f_z(x, y, z)$

5. If $f(x, y) \to L$ as $(x, y) \to (a, b)$ along every straight line through (a, b), then $\displaystyle\lim_{(x,\,y)\to(a,\,b)} f(x, y) = L$.

6. If $f_x(a, b)$ and $f_y(a, b)$ both exist, then f is differentiable at (a, b).

7. If f has a local minimum at (a, b) and f is differentiable at (a, b), then $\nabla f(a, b) = \mathbf{0}$.

8. If f is a function, then

$$\lim_{(x, y) \to (2, 5)} f(x, y) = f(2, 5)$$

9. If $f(x, y) = \ln y$, then $\nabla f(x, y) = \dfrac{1}{y}$.

10. If $(2, 1)$ is a critical point of f and

$$f_{xx}(2, 1) f_{yy}(2, 1) < [f_{xy}(2, 1)]^2$$

then f has a saddle point at $(2, 1)$.

11. If $f(x, y) = \sin x + \sin y$, then $-\sqrt{2} \le D_{\mathbf{u}} f(x, y) \le \sqrt{2}$.

12. If $f(x, y)$ has two local maxima, then f must have a local minimum.

Exercises

Find and sketch the domain of the function.

1. $f(x, y) = \ln(x + y + 1)$

2. $f(x, y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 - x^2}$

Sketch the graph of the function.

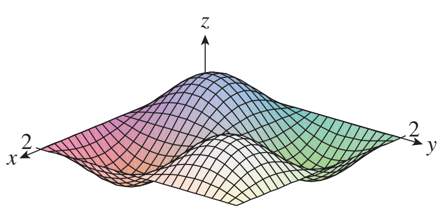
3. $f(x, y) = 1 - y^2$

4. $f(x, y) = x^2 + (y - 2)^2$

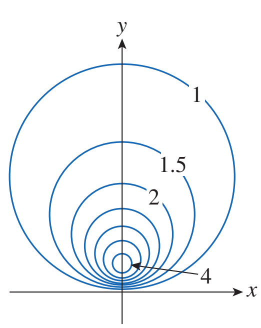
Sketch several level curves of the function.

5. $f(x, y) = \sqrt{4x^2 + y^2}$ **6.** $f(x, y) = e^x + y$

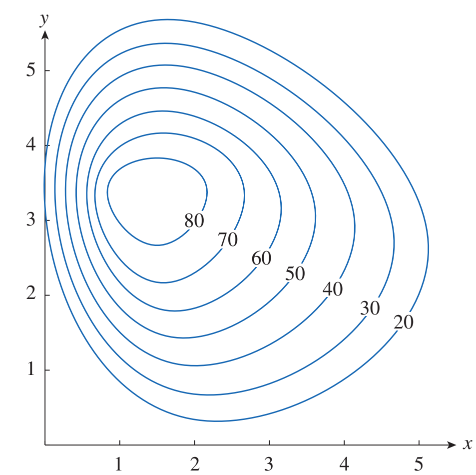
7. Make a rough sketch of a contour map for the function whose graph is shown.



8. A contour map of a function f is shown. Use it to make a rough sketch of the graph of f.



9. The contour map of a function f is shown.
(a) Estimate the value of $f(3, 2)$.
(b) Is $f_x(3, 2)$ positive or negative? Explain your reasoning.
(c) Which is greater, $f_y(3, 1)$ or $f_y(1, 3)$? Explain your reasoning.



Evaluate the limit or show that it does not exist.

10. $\displaystyle\lim_{(x, y) \to (1, 1)} \frac{2xy}{x^2 + 2y^2}$ **11.** $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{2xy}{x^2 + 2y^2}$

12. A metal plate is situated in the xy-plane and occupies the rectangle $0 \le x \le 10$, $0 \le y \le 8$, where x and y are measured in meters. The temperature at the point (x, y) in the plate is $T(x, y)$, where T is measured in degrees Celsius. Temperatures at equally spaced points were measured and are reported in the table.
(a) Using correct units, estimate the values of the partial derivatives $T_x(6, 4)$ and $T_y(6, 4)$.
(b) Estimate the value of $D_{\mathbf{u}} T(6, 4)$ where $\mathbf{u} = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$.
(c) Estimate the value of $T_{xy}(6, 4)$.

$x \backslash y$	0	2	4	6	8
0	30	38	45	51	55
2	52	56	60	62	61
4	78	74	72	68	66
6	98	87	80	75	71
8	96	90	86	80	75
10	92	92	91	87	78

13. Find a linear approximation to the temperature function $T(x, y)$ in Exercise 12 near the point (6, 4). Then use it to estimate the temperature at the point (5, 3.8).

Find the first partial derivatives.

14. $f(x, y) = \sqrt{2x + y^2}$

15. $u = e^{-r} \sin 2\theta$

16. $g(u, v) = u \tan^{-1} v$

17. $w = \dfrac{x}{y - z}$

18. $T(p, q, r) = p \ln(q + e^r)$

19. $u = e^{-xy} \sin \dfrac{y}{z}$

20. The speed of sound traveling through ocean water is a function of temperature, salinity, and pressure. It has been modeled by the function

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D$$

where C is the speed of sound (in meters per second), T is the temperature (in degrees Celsius), S is the salinity (the concentration of salts in parts per thousand, which means the number of grams of dissolved solids per 1000 g of water), and D is the depth below the ocean surface (in meters). Compute $\dfrac{\partial C}{\partial T}$, $\dfrac{\partial C}{\partial S}$, and $\dfrac{\partial C}{\partial D}$ when $T = 10°\text{C}$, $S = 35$ parts per thousand, and $D = 100$ m. Explain the physical significance of these partial derivatives.

Find the second partial derivatives of f.

21. $f(x, y) = 4x^3 - xy^2$

22. $f(x, y) = xe^{-2y}$

23. $f(x, y) = \ln(x^2 + y^2)$

24. $f(x, y) = x \tan(xy^2)$

25. $f(x, y, z) = x^k y^l z^m$

26. $f(x, y, z) = x \cos(y + 2z)$

27. If $z = xy + xe^{y/x}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = xy + z$.

28. If $z = \sin(x + \sin t)$, show that

$$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\, \partial t} = \frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2}$$

Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

29. $z = 3x^2 - y^2 + 2x$, $(1, -2, 1)$

30. $z = e^x \cos y$, $(0, 0, 1)$

31. $x^2 + 2y^2 - 3z^2 = 3$, $(2, -1, 1)$

32. $xy + yz + zx = 3$, $(1, 1, 1)$

33. $\mathbf{r}(u, v) = (u + v)\mathbf{i} + u^2\mathbf{j} + v^2\mathbf{k}$, $(3, 4, 1)$

34. Use technology to graph the surface $z = x^2 + y^4$ and the tangent plane and normal line at (1, 1, 2) on the same coordinate axes. Choose the domain and viewpoint so that all three objects are visible and discernible.

35. Find the points on the hyperboloid

$$x^2 + 4y^2 - z^2 = 4$$

where the tangent plane is parallel to the plane

$$2x + 2y + z = 5$$

36. Find du if $u = \ln(1 + se^{2t})$.

37. Find the linear approximation of the function $f(x, y, z) = x^3\sqrt{y^2 + z^2}$ at the point (2, 3, 4) and use it to estimate the number $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2}$.

38. The two legs of a right triangle are measured as 5 m and 12 m with a possible error in measurement of at most 0.2 cm in each. Use differentials to estimate the maximum error in the calculated value of (a) the area of the triangle and (b) the length of the hypotenuse.

39. If $u = x^2y^3 + z^4$, where $x = p + 3p^2$, $y = pe^p$, and $z = p \sin p$, use the Chain Rule to find $\dfrac{du}{dp}$.

40. If $v = x^2 \sin y + ye^{xy}$, where $x = s + 2t$ and $y = st$, use the Chain Rule to find $\dfrac{\partial v}{\partial s}$ and $\dfrac{\partial v}{\partial t}$ when $s = 0$ and $t = 1$.

41. Suppose $z = f(x, y)$, where $x = g(s, t)$, $y = h(s, t)$, $g(1, 2) = 3$, $g_s(1, 2) = -1$, $g_t(1, 2) = 4$, $h(1, 2) = 6$, $h_s(1, 2) = -5$, $h_t(1, 2) = 10$, $f_x(3, 6) = 7$, and $f_y(3, 6) = 8$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ when $s = 1$ and $t = 2$.

42. Use a tree diagram to write out the Chain Rule for the case where $w = f(t, u, v)$, $t = t(p, q, r, s)$, $u = u(p, q, r, s)$, and $v = v(p, q, r, s)$ are all differentiable functions.

43. If $z = y + f(x^2 - y^2)$, where f is differentiable, show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x$$

44. The length x of a side of a triangle is increasing at a rate of 3 in/s, the length y of another side is decreasing at a rate of 2 in/s, and the contained angle θ is increasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when $x = 40$ in., $y = 50$ in., and $\theta = \pi/6$?

45. If $z = f(u, v)$, where $u = xy$, $v = y/x$, and f has continuous second partial derivatives, show that

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = -4uv\frac{\partial^2 z}{\partial u\,\partial v} + 2v\frac{\partial z}{\partial v}$$

46. If $\cos(xyz) = 1 + x^2y^2 + z^2$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

47. Find the gradient of the function $f(x, y, z) = x^2e^{yz^2}$.

48. (a) When is the directional derivative of f a maximum?
(b) When is it a minimum?
(c) When is it 0?
(d) When is it half of its maximum value?

Find the directional derivative of f at the given point in the indicated direction.

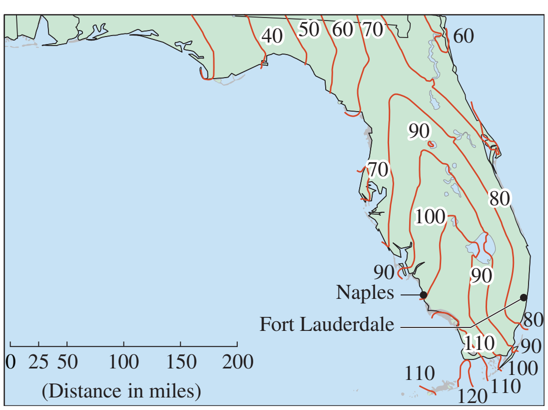
49. $f(x, y) = x^2e^{-y}$, $(-2, 0)$, in the direction toward the point $(2, -3)$

50. $f(x, y, z) = x^2y + x\sqrt{1 + z}$, $(1, 2, 3)$, in the direction of $\mathbf{v} = 2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$

51. Find the maximum rate of change of $f(x, y) = x^2y + \sqrt{y}$ at the point (2, 1). In which direction does it occur?

52. Find the direction in which $f(x, y, z) = ze^{xy}$ increases most rapidly at the point (0, 1, 2). What is the maximum rate of increase?

53. The contour map shows the 3-second gust wind speed in mi/h during Hurricane Irma on September 12, 2017. Use it to estimate the value of the directional derivative of the wind speed at Naples, Florida, in the direction of Fort Lauderdale.



54. Find parametric equations of the tangent line at the point $(-2, 2, 4)$ to the curve of intersection of the surface $z = 2x^2 - y^2$ and the plane $z = 4$.

Find the local maximum and minimum values and saddle points of the function. Use technology to graph the function with a domain and viewpoint that reveal all the important aspects of the function.

55. $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10$

56. $f(x, y) = x^3 - 6xy + 8y^3$

57. $f(x, y) = 3xy - x^2y - xy^2$

58. $f(x, y) = (x^2 + y)e^{y/2}$

Find the absolute maximum and minimum values of f on the set D.

59. $f(x, y) = 4xy^2 - x^2y^2 - xy^3$; D is the closed triangular region in the xy-plane with vertices (0, 0), (0, 6), and (6, 0).

60. $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$; D is the disk $x^2 + y^2 \le 4$.

61. Use a graph or level curve or both to estimate the local maximum and minimum values and saddle points of $f(x, y) = x^3 - 3x + y^4 - 2y^2$. Then use calculus to find these values precisely.

62. Use technology to find the critical points of $f(x, y) = 12 + 10y - 2x^2 - 8xy - y^4$. Classify each critical point and find the highest point on the graph.

Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint(s).

63. $f(x, y) = x^2y$; $x^2 + y^2 = 1$

64. $f(x, y) = \dfrac{1}{x} + \dfrac{1}{y}$; $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 1$

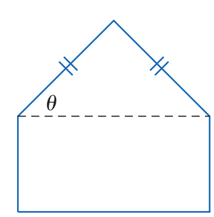
65. $f(x, y, z) = xyz$; $x^2 + y^2 + z^3 = 3$

66. $f(x, y, z) = x^2 + 2y^2 + 3z^2$; $x + y + z = 1$, $x - y + 2z = 2$

67. Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.

68. A package in the shape of a rectangular box can be mailed by the US Postal Service if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108 in. Find the dimensions of the package with largest volume that can be mailed.

69. A pentagon is formed by placing an isosceles triangle on a rectangle, as shown in the figure. If the pentagon has fixed perimeter P, find the lengths of the sides of the pentagon that maximize the area of the pentagon.



70. A particle of mass m moves on the surface $z = f(x, y)$. Let $x = x(t)$ and $y = y(t)$ be the x- and y-coordinates of the particle at time t.
(a) Find the velocity vector $\mathbf{v}$ and the kinetic energy $K = \dfrac{1}{2}m|\mathbf{v}|^2$ of the particle.
(b) Determine the acceleration vector $\mathbf{a}$.
(c) Let $z = x^2 + y^2$ and $x(t) = t\cos t$, $y(t) = t\sin t$. Find the velocity vector, the kinetic energy, and the acceleration vector.

Focus on Problem Solving

1. A rectangle with length L and width W is cut into four smaller rectangles by two lines parallel to the sides. Find the maximum and minimum values of the sum of the squares of the areas of the smaller rectangles.

2. Marine biologists have determined that when a shark detects the presence of blood in the water, it will swim in the direction in which the concentration of the blood increases most rapidly. Based on certain tests, the concentration of blood (in parts per million) at a point $P(x, y)$ on the surface of seawater is approximated by

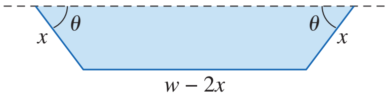
$$C(x, y) = e^{-(x^2+2y^2)/10^4}$$

where x and y are measured in meters in a rectangular coordinate system with the blood source at the origin.

(a) Identify the level curves of the concentration function and sketch several members of this family together with a path that a shark will follow to the source.

(b) Suppose a shark is at the point (x_0, y_0) when it first detects the presence of blood in the water. Find an equation of the shark's path by setting up and solving a differential equation.

3. A long piece of galvanized sheet metal with width w is to be bent into a symmetric form with three straight sides to make a rain gutter. A cross-section is shown in the figure.



(a) Determine the dimensions that allow the maximum possible flow; that is, find the dimensions that give the maximum possible cross-sectional area.

(b) Would it be better to bend the metal into a gutter with a semicircular cross-section?

4. For what values of the number r is the function

$$f(x, y, z) = \begin{cases} \dfrac{(x+y+z)^r}{x^2+y^2+z^2} & \text{if } (x, y, z) \neq 0 \\ 0 & \text{if } (x, y, z) = 0 \end{cases}$$

continuous on $\mathbb{R}^3$?

5. Suppose f is a differentiable function of one variable. Show that all tangent planes to the surface $z = xf(y/x)$ intersect in a common point.

6. (a) Newton's method for approximating a root of an equation $f(x) = 0$ (see Section 4.7) can be adapted to approximating a solution of a system of equations $f(x, y) = 0$ and $g(x, y) = 0$. The surfaces $z = f(x, y)$ and $z = g(x, y)$ intersect in a curve that intersects the xy-plane at the point (r, s), which is the solution of the system. If an initial approximation (x_1, y_1) is close to this point, then the tangent planes to the surfaces at (x_1, y_1) intersect in a straight line that intersects the xy-plane in a point (x_2, y_2), which should be closer to (r, s). (Compare with Figure 4.81 in Section 4.7.) Show that

$$x_2 = x_1 - \frac{fg_y - f_y g}{f_x g_y - f_y g_x} \quad \text{and} \quad y_2 = y_1 - \frac{f_x g - fg_x}{f_x g_y - f_y g_x}$$

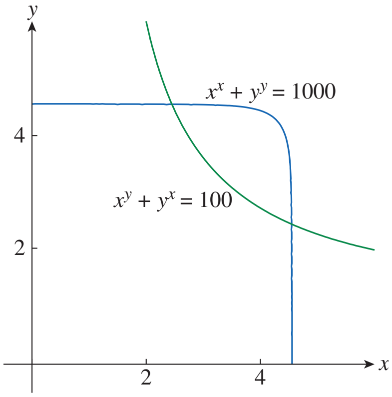
where f, g, and their partial derivatives are evaluated at (x_1, y_1). If we continue this procedure, we obtain successive approximations (x_n, y_n).

Hint: Solve 2 linear equations in 2 unknowns.

(b) It was Thomas Simpson (1710–1761) who formulated Newton's method as we know it today and who extended it to functions of two variables as in part (a). The example that he gave to illustrate the method was to solve the system of equations

$$x^x + y^y = 1000 \qquad x^y + y^x = 100$$

In other words, he found the points of intersection of the curves in the figure.



Use the method of part (a) to find the coordinates of the points of intersection correct to six decimal places.

7. (a) Show that when Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is written in cylindrical coordinates, it becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

(b) Show that when Laplace's equation is written in spherical coordinates, it becomes

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial \phi} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} = 0$$

8. Among all planes that are tangent to the surface $xy^2z^2 = 1$, find the ones that are farthest from the origin.

9. If the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is to enclose the circle $x^2 + y^2 = 2y$, what values of a and b minimize the area of the ellipse?

Nate Hovee/Shutterstock.com

Mercedes-Benz Stadium is home to the Atlanta Falcons of the National Football League and Atlanta United FC of Major League Soccer. The stadium roof, which resembles a camera eye, has eight moving panels, petals that weigh over 4000 tons, and Falcon owner Arthur Blank has called it "the most complicated roof design in the history of the world." Using a function to model the height of the roof, we can use calculus to estimate the total volume of the stadium.

12 Multiple Integrals

Contents

In this chapter, we extend the idea of a definite integral to double and triple integrals of functions of two or three variables. These concepts are then used to compute volumes, surface areas, masses, and centroids of more general regions than we were able to consider in Chapter 6. We also use double integrals to compute probabilities associated with two jointly distributed random variables.

We will also see that using polar coordinates can be advantageous in computing double integrals over some types of regions. And we will also use cylindrical and spherical coordinates to simplify certain computations associated with triple integrals over commonly occurring solid regions.

12.1 Double Integrals over Rectangles

One of the most important unifying problems in calculus is the area problem, and solving this problem led to the definition of a definite integral. Now, similarly, we will try to find the volume of a solid and the solution will help establish the definition of a double integral.

Review of the Definite Integral

First let's recall some basic facts concerning definite integrals of functions of a single variable. Suppose $f(x)$ is defined for $a \le x \le b$. Divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \dfrac{b-a}{n}$ and choose sample points x_i^* in each subinterval. A Riemann sum is of the form

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x \tag{1}$$

and if we consider the limit as $n \to \infty$, we obtain the definition of the definite integral of f from a to b:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \tag{2}$$

In the special case where $f(x) \ge 0$, the Riemann sum can be interpreted as the sum of areas of the approximating rectangles as illustrated in Figure 12.1, and $\int_a^b f(x)\,dx$ represents the area under the graph of $y = f(x)$ from a to b.

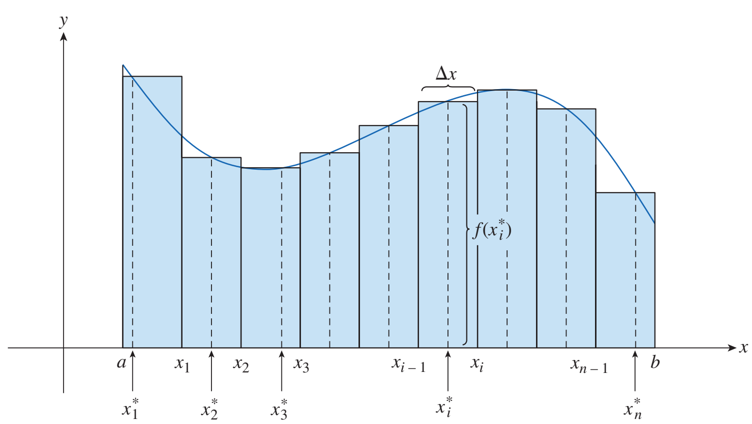


Figure 12.1
Approximating rectangles using an arbitrary value x_i^* in each subinterval.

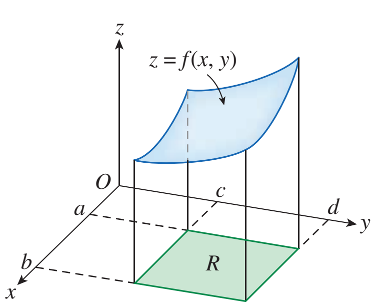


Figure 12.2
Graph of the function f on the closed rectangle R.

Volumes and Double Integrals

In a similar manner, let's consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \,|\, a \le x \le b, c \le y \le d\}$$

and suppose that $f(x, y) \ge 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 \,|\, 0 \le z \le f(x, y), (x, y) \in R\}$$

See Figure 12.2. Our goal is to find the volume of S.

Divide the rectangle R into subrectangles; divide the interval $[a, b]$ in m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \dfrac{b-a}{m}$ and divide the interval $[c, d]$ into n subintervals

$[y_{j-1}, y_j]$ of equal width $\Delta y = \dfrac{d-c}{n}$. Draw parallel lines to the coordinate axes through the endpoints of these subintervals, as in Figure 12.3, to form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j\}$$

each with area $\Delta A = \Delta x\, \Delta y$.

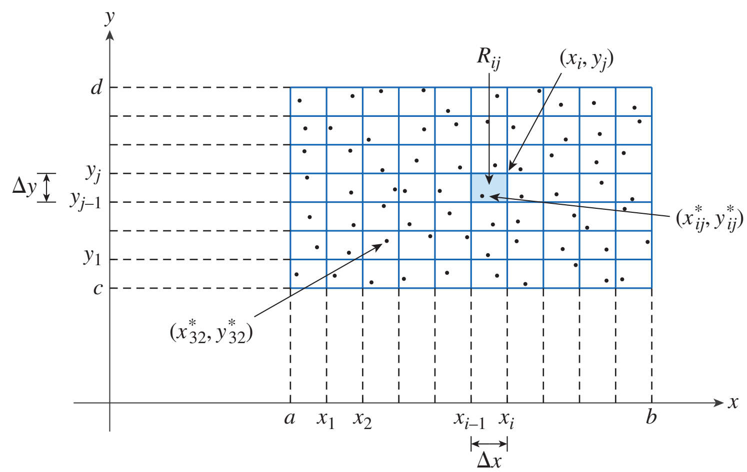


Figure 12.3
The region R divided into subrectangles.

Choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}. Then we can approximate the part of the solid S that lies above each R_{ij} by a thin rectangular box (or column) with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in Figure 12.4. (Compare this with Figure 12.1.) The volume of the box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

Use this same procedure for all the rectangles and add the volumes of the corresponding boxes; an approximation to the total volume of S is:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A \tag{3}$$

(See Figure 12.5.) This double sum means that for each subrectangle, we evaluate f at the chosen sample point and multiply by the area of the subrectangle, and then add the results.

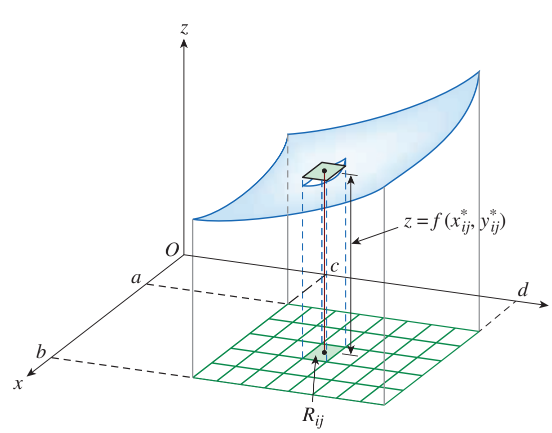


Figure 12.4
The rectangular box with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$.

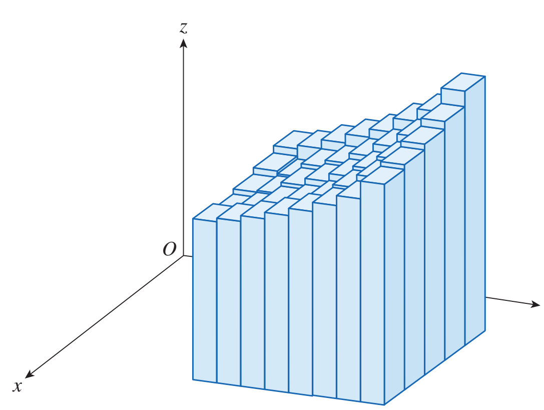


Figure 12.5
The sum of the volumes of all the rectangular boxes is an approximation to the total volume of S.

The double limit in Equation 4 means that we can make the double sum as close as we like to the number V [for any choice of (x_{ij}^*, y_{ij}^*) in R_{ij}] by taking m and n sufficiently large.

It seems reasonable that the approximation given in Equation 3 becomes better and better as m and n become larger; so we would expect that

$$V = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A \tag{4}$$

We use the expression in Equation 4 to define the **volume** of the solid S that lies under the graph of f and above the rectangle R. (It can be shown that this definition is consistent with the formula for volume in Section 6.2.)

Limits of the type that appear in Equation 4 occur frequently, not just in finding volumes but in a variety of other applications as well, as we will see in Section 12.5, even when f is not a positive function. We will use the next definition to represent this type of limit.

Definition • Double Integral

The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

A function f is called **integrable** if the limit in this definition exists. In most courses on advanced calculus, it is shown that all continuous functions are integrable. In fact, the double integral of f exists provided that f is "not too discontinuous." In particular, if f is bounded [that is, there is a constant M such that $|f(x, y)| \leq M$ for all (x, y) in R], and f is continuous there, except on a finite number of smooth curves, then f is integrable over R.

The sample point (x_{ij}^*, y_{ij}^*) can be chosen to be any point in the subrectangle R_{ij}; if we select the upper right corner of R_{ij}, (namely (x_i, y_j), as identified in Figure 12.3), then the expression for the double integral looks a little simpler:

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A \tag{5}$$

By comparing Equations 4 and 5, we see that a volume can be written as a double integral of a positive function.

If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

The sum in the definition of a double integral,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. Notice how similar it is to the Riemann sum in Equation 1 for a function of a single variable. If f happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 12.5, and is an approximation to the volume under the graph of f and above the rectangle R.

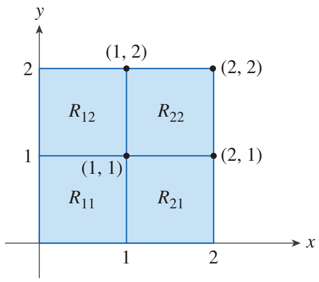


Figure 12.6
The squares R_{ij} and the sample points.

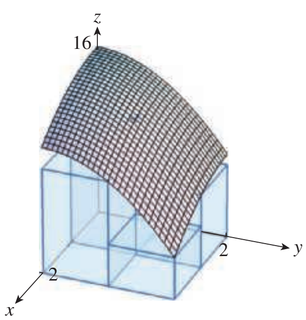


Figure 12.7
The approximating rectangular boxes.

Example 1 Use a Double Riemann Sum to Estimate a Volume

Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

Solution

The squares R_{ij} and the sample points are shown in Figure 12.6.

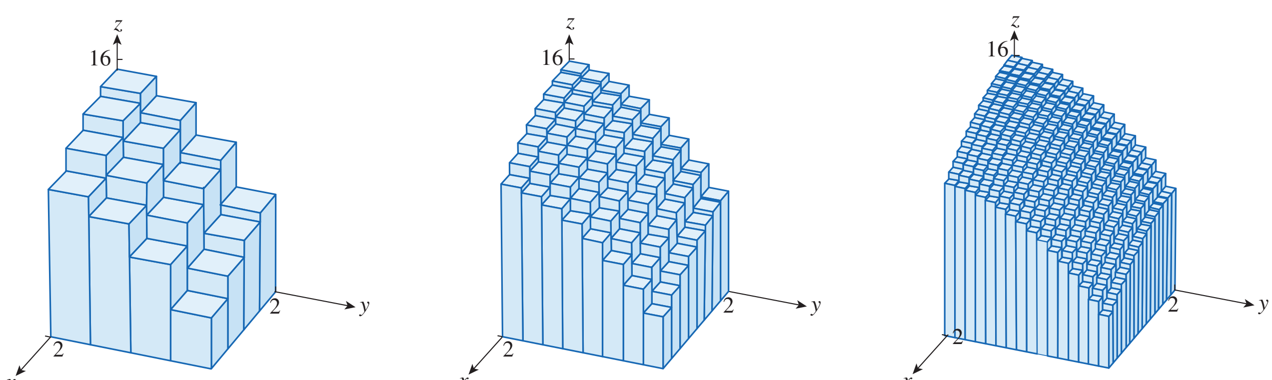
The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$.

Approximate the volume using the Riemann sum with $m = n = 2$.

$$
\begin{aligned}
V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_i)\,\Delta A \\
&= f(1, 1)\,\Delta A + f(1, 2)\,\Delta A + f(2, 1)\,\Delta A + f(2, 2)\,\Delta A \\
&= (13)(1) + (7)(1) + (10)(1) + (4)(1) = 34
\end{aligned}
$$

The volume of the approximating rectangular boxes, shown in Figure 12.7, is 34. ■

It seems reasonable that the approximations to the volume in Example 1 get better as we increase the number of squares (and rectangular boxes). Figure 12.8 shows how the rectangular boxes start to look more like the actual solid and how the corresponding approximations become more accurate when we use 16, 64, and 256 squares. In the next section, we will be able to show that the exact volume is 48.



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Figure 12.8
The Riemann sum approximations to the volume under $z = 16 - x^2 - 2y^2$ become more accurate as m and n increase.

Example 2 Evaluate a Double Integral by Interpreting It Geometrically

If $R = \{(x, y) \mid -1 \le x \le 1, -2 \le y \le 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$

Solution

It would be difficult to evaluate this integral directly using the definition of a double integral.

However, because $\sqrt{1 - x^2} \ge 0$, we can compute the integral by interpreting it as the volume of a solid.

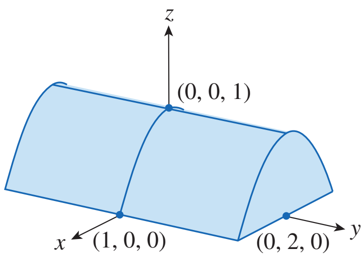


Figure 12.9
Geometric interpretation of the double integral.

If $z = \sqrt{1 - x^2}$, then $x^2 + z^2 = 1$ and $z \geq 0$.

Therefore, the given double integral represents the volume of the solid S that lies below the circular cylinder $x^2 + z^2 = 1$ and above the rectangle R, as illustrated in Figure 12.9.

The volume of S is the area of a semicircle with radius 1 times the length of the cylinder.

$$\iint_R \sqrt{1 - x^2}\, dA = \frac{1}{2}\pi(1)^2 \cdot 4 = 2\pi$$

The Midpoint Rule

The methods that we used for approximating single definite integrals (the Midpoint Rule, the Trapezoidal Rule, Simpson's Rule) all have counterparts for double integrals. Here we consider only the Midpoint Rule for double integrals. This means that we use a double Riemann sum to approximate the double integral, where the sample point (x_{ij}^*, y_{ij}^*) in R_{ij} is chosen to be the center $(\bar{x}_i, \bar{y}_i)$ of R_{ij}, that is, $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Midpoint Rule for Double Integrals

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example 3 Midpoint Rule Approximation

Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral

$$\iint_R (x - 3y^2)\, dA, \text{ where } R = \{(x, y) \mid 0 \leq x \leq 2,\ 1 \leq y \leq 2\}.$$

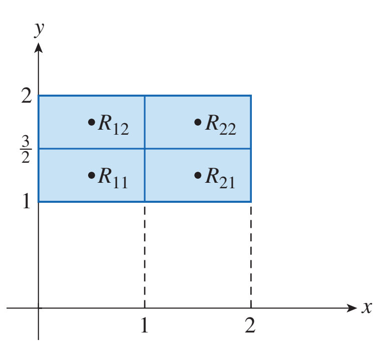


Figure 12.10
The four subrectangles and midpoints.

Solution

Use the Midpoint Rule with $m = n = 2$.

Evaluate $f(x, y) = x - 3y^2$ at the centers of the four subrectangles shown in Figure 12.10.

The midpoints are $\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{3}{2}$, $\bar{y}_1 = \frac{5}{4}$, $\bar{y}_2 = \frac{7}{4}$.

The area of each rectangle is $\Delta A = 1 \cdot \frac{1}{2} = \frac{1}{2}$.

The Midpoint Rule approximation is

$$\begin{aligned}\iint_R (x - 3y^2)\, dA &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\, \Delta A \\ &= f(\bar{x}_1, \bar{y}_1)\, \Delta A + f(\bar{x}_1, \bar{y}_2)\, \Delta A + f(\bar{x}_2, \bar{y}_1)\, \Delta A + f(\bar{x}_2, \bar{y}_2)\, \Delta A \\ &= f\left(\tfrac{1}{2}, \tfrac{5}{4}\right) \Delta A + f\left(\tfrac{1}{2}, \tfrac{7}{4}\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{5}{4}\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{7}{4}\right) \Delta A \\ &= \left(-\frac{67}{16}\right)\frac{1}{2} + \left(-\frac{139}{16}\right)\frac{1}{2} + \left(-\frac{51}{16}\right)\frac{1}{2} + \left(-\frac{123}{16}\right)\frac{1}{2} \\ &= -\frac{95}{8} = -11.875\end{aligned}$$

Therefore, using the Midpoint Rule, $\displaystyle\iint_R (x - 3y^2)\, dA \approx -11.875.$ ■

Number of subrectangles	Midpoint Rule approximation
1	-11.5000
4	-11.8750
16	-11.9687
64	-11.9922
256	-11.9980
1024	-11.9995

Table 12.1
Midpoint Rule approximations as the number of subrectangles increases.

Note: In the next section, we will develop an efficient method for computing double integrals and we will see that the exact value of the double integral in Example 3 is -12. (Remember that the geometric interpretation of a double integral as the volume of a solid is valid only when the integrand is a *positive* function.) The integrand in Example 3 is not a positive function, so its integral does not represent a volume. In Examples 2 and 3 in Section 12.2, we will discuss how to interpret integrals of functions that are not always positive in terms of volumes. If we continue dividing each subrectangle in Figure 12.10 into four smaller ones with similar shape, we get the Midpoint Rule approximations as shown in Table 12.1. Notice how these approximations approach the exact value of the double integral, -12.

Average Value

Recall from Section 6.5 that the average value of a function f of one variable defined on an interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\, dx$$

In a similar manner, we define the **average value** of a function f of two variables defined on a rectangle R to be

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x, y)\, dA$$

where $A(R)$ is the area of R.

If $f(x, y) \geq 0$, the equation

$$A(R) \times f_{\text{ave}} = \iint_R f(x, y)\, dA$$

is interpreted as the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f. If $z = f(x, y)$ describes a mountainous region and you chop off the tops of the mountains at height f_{ave}, then you can use them to fill in the valleys so that the region becomes completely flat.

Example 4 Use the Midpoint Rule to Estimate Average Snowfall

The contour map in Figure 12.11 shows the snowfall, in inches, that fell on the state of Colorado from September 2020 to May 2021. The state is in the shape of a rectangle that measures 388 mi west to east and 276 mi south to north. Use the contour map to estimate the average snowfall for the entire state of Colorado during this winter season.

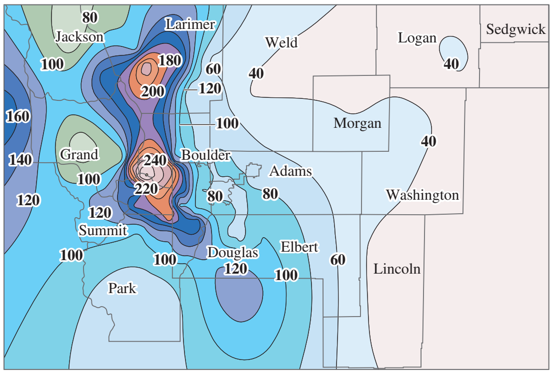


Figure 12.11
Snowfall totals for Colorado September 2020–May 2021. (National Weather Service)

Solution

Place the origin at the southwest corner of the state.
Then $0 \le x \le 388$, $0 \le y \le 276$, and $f(x, y)$ is the snowfall, in inches, at a location x miles to the east and y miles to the north of the origin.

If R is the rectangle that represents Colorado, then the average snowfall for the state during the winter season was

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA \text{ where } A(R) = 388 \cdot 276.$$

Use the Midpoint Rule with $m = n = 4$ to estimate the value of this double integral. Divide R into 16 subrectangles of equal area, as shown in Figure 12.12.

The area of each subrectangle is $\Delta A = \frac{1}{16}(388)(276) = 6693 \text{ mi}^2$.

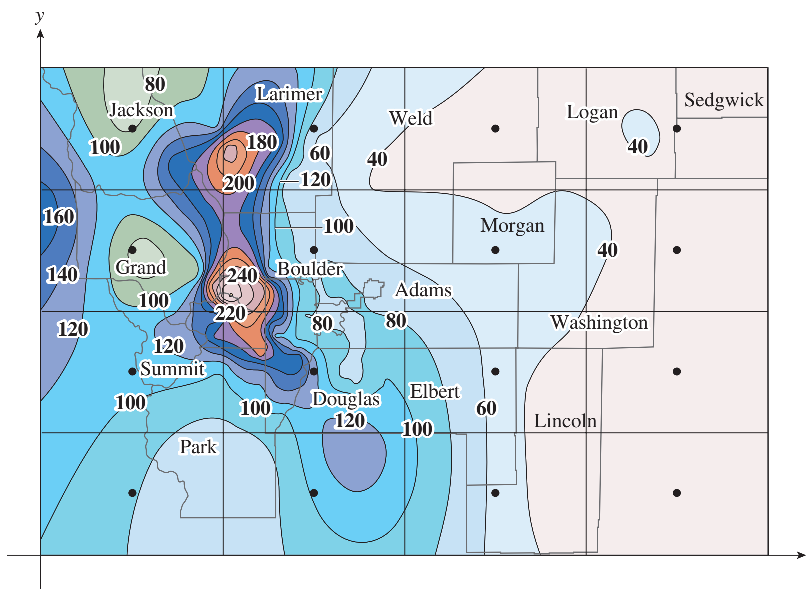


Figure 12.12
The subrectangles and midpoints.

Use the contour map to estimate the value of f at the center of each subrectangle.

$$\begin{aligned}\iint_R f(x, y)\, dA &\approx \sum_{i=1}^{4}\sum_{j=1}^{4} f(\bar{x}_i, \bar{y}_j)\, \Delta A \\ &\approx \Delta A[82 + 102 + 58 + 30 + 110 + 130 + 55 + 30 \\ &\qquad + 78 + 70 + 50 + 30 + 80 + 75 + 30 + 30] \\ &= (6693)(1040)\end{aligned}$$

Therefore, $f_{\text{ave}} \approx \frac{(6693)(1040)}{(388)(276)} = 65.$

From September 2020 to May 2021, Colorado received an average of approximately 65 inches of snow. ■

Properties of Double Integrals

Here are three properties of double integrals that can be proved in the same manner as in Section 5.2. We assume that all of the integrals exist. Properties 7 and 8 are referred to

These properties are proved using the definition of a double integral and properties of double sums.

as the *linearity* of the integral.

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA \tag{6}$$

$$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \quad \text{where } c \text{ is a constant} \tag{7}$$

If $f(x, y) \geq g(x, y)$ for all (x, y) in R, then

$$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA \tag{8}$$

12.1 Exercises

1. (a) Estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle

$$R = \{(x, y) \mid 0 \leq x \leq 6, 0 \leq y \leq 4\}$$

Use a Riemann sum with $m = 3$, $n = 2$, and take the sample point to be the upper right corner of each square.

(b) Use the Midpoint Rule to estimate the volume of the solid in part (a).

2. If $R = [0, 4] \times [-1, 2]$, use a Riemann sum with $m = 2$, $n = 3$ to estimate the value of $\iint_R (1 - xy^2)\, dA$. Choose sample points in (a) the lower right corners and (b) the upper left corners of the rectangles.

3. (a) Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R \sin(x + y)\, dA$, where $R = [0, \pi] \times [0, \pi]$. Choose samples points in the lower left corner of each subrectangle.

(b) Use the Midpoint Rule to estimate the integral in part (a).

4. (a) Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R xe^{-xy}\, dA$, where $R = [0, 2] \times [0, 1]$. Take the sample points to upper right corners.

(b) Use the Midpoint Rule to estimate the integral in part (a).

5. (a) Estimate the volume of the solid that lies below the surface $z = x + 2y^2$ and above the rectangle $R = [0, 2] \times [0, 4]$. Use a Riemann sum with $m = n = 2$ and choose the sample points to be lower right corners.

(b) Use the Midpoint Rule to estimate the integral in part (a).

6. (a) Estimate the volume of the solid that lies below the surface $z = 1 + x^2 + 3y$ and above the rectangle $R = [1, 2] \times [0, 3]$. Use a Riemann sum with $m = n = 2$ and choose the sample points to be lower left corners.

(b) Use the Midpoint Rule to estimate the integral in part (a).

7. A table of values is given for a function $f(x, y)$ defined on $R = [0, 4] \times [2, 4]$.

(a) Estimate $\iint_R f(x, y)\, dA$ using the Midpoint Rule with $m = n = 2$.

(b) Estimate the double integral with $m = n = 4$ by choosing the sample points to be the points closest to the origin.

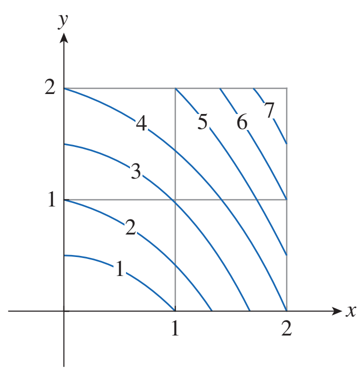
x \ y	2.0	2.5	3.0	3.5	4.0
0	-3	-5	-6	-4	-1
1	-1	-2	-3	-1	1
2	1	0	-1	1	4
3	2	2	1	3	7
4	3	4	2	5	9

8. A 20-ft-by-30-ft swimming pool is filled with water. The depth is measured at 5-ft intervals, starting at one corner of the pool, and the values are given in the table. Estimate the volume of water in the pool.

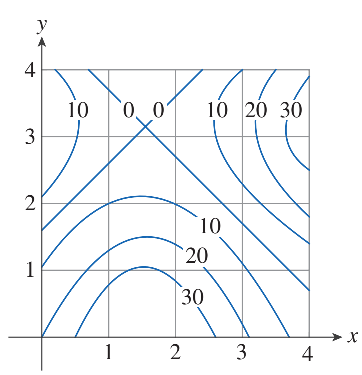
	0	5	10	15	20	25	30
0	2	3	4	6	7	8	8
5	2	3	4	7	8	10	8
10	2	4	6	8	10	12	10
15	2	3	4	5	6	8	7
20	2	2	2	2	3	4	4

9. Let V be the volume of the solid that lies under the graph of $f(x, y) = \sqrt{52 - x^2 - y^2}$ and above the rectangle R given by $2 \leq x \leq 4, 2 \leq y \leq 6$. Use the lines $x = 3$ and $y = 4$ to divide R into subrectangles. Let L and U be the Riemann sums computed using lower left corners and upper right corners, respectively. Without calculating the numbers V, L, and U, arrange them in increasing order and explain your reasoning.

10. Level curves of a function f in the square $R = [0, 2] \times [0, 2]$ are shown in the figure. Use the Midpoint Rule with $m = n = 2$ to estimate $\iint_R f(x, y)\, dA$. Explain how to improve your estimate.



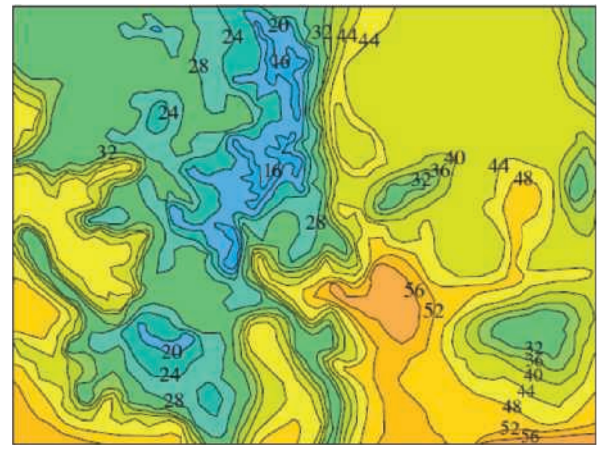
11. A contour map for a function f on the square $R = [0, 4] \times [0, 4]$ is shown in the figure.



(a) Use the Midpoint Rule with $m = n = 2$ to estimate the value of $\iint_R f(x, y)\, dA$.

(b) Estimate the average value of f.

12. The contour map shows the temperature, in degrees Fahrenheit, at 4:00 PM on a day in February in Colorado. The state measures 388 mi west to east and 276 mi south to north. Use the Midpoint Rule with $m = n = 4$ to estimate the average temperature in Colorado at that time.



Evaluate the double integral by identifying it as the volume of a solid.

13. $\iint_R 3\, dA, \quad R = \{(x, y) \mid -2 \leq x \leq 2, 1 \leq y \leq 6\}$

14. $\iint_R (5 - x)\, dA, \quad R = \{(x, y) \mid 0 \leq x \leq 5, 0 \leq y \leq 3\}$

15. $\iint_R (4 - 2y)\, dA, \quad R = [0, 1] \times [0, 1]$

16. The integral $\iint_R \sqrt{9 - y^2}\, dA$, where $R = [0, 4] \times [0, 2]$, represents the volume of a solid. Sketch the solid.

Use technology to estimate each double integral. Use the Midpoint Rule with the following numbers of squares of equal size: 1, 4, 16, 64, 256, and 1024.

17. $\iint_R \sqrt{1 + xe^{-y}}\, dA, \quad R = [0, 1] \times [0, 1]$

18. $\iint_R \sin(x + \sqrt{y})\, dA, \quad R = [0, 1] \times [0, 1]$

19. $\iint_R e^{-x^2 - y^2} dA, \quad R = [-3, 3] \times [-3, 3]$

20. If f is a constant function, $f(x, y) = k$, and $R = [a, b] \times [c, d]$, show that

$$\iint_R k\, dA = k(b - a)(d - c)$$

21. Use the result in Exercise 20 to show that

$$0 \leq \iint_R \sin \pi x \, \cos \pi y\, dA \leq \frac{1}{32}$$

where $R = \left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, \frac{1}{2}\right]$.

12.2 Iterated Integrals

We have seen that it can be difficult to evaluate the definite integral of a function of a single variable by definition. However, the Fundamental Theorem of Calculus provides an easier method. The evaluation of double integrals from definition is often even more difficult, but in this section, we will see how to express a double integral as an iterated integral, which can be evaluated by calculating two single integrals.

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$. The notation $\int_c^d f(x, y)\,dy$ means that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y*. (Notice how this is similar to partial differentiation.)

The expression $\int_c^d f(x, y)\,dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)\,dy$$

If we integrate the function A with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx \tag{1}$$

The integral on the right side of Equation 1 is called an **iterated integral**. Usually the brackets are omitted. Therefore,

$$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx \tag{2}$$

means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, the iterated integral

$$\int_c^d \int_a^b f(x, y)\,dx\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx \right] dy \tag{3}$$

means that we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate with respect to y from $y = c$ to $y = d$. Notice that in both Equations 2 and 3, we work *from the inside out*.

Example 1 Integrate in Both Orders

Evaluate the iterated integrals.

(a) $\displaystyle\int_0^3 \int_1^2 x^2 y\,dy\,dx$ (b) $\displaystyle\int_1^2 \int_0^3 x^2 y\,dx\,dy$

Solution

(a) Treat x as a constant and integrate with respect to y.

$$\int_1^2 x^2 y\,dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \frac{3}{2} x^2$$

Integrate this function with respect to x.

$$\int_0^3 \int_1^2 x^2 y\, dy\, dx = \int_0^3 \left[\int_1^2 x^2 y\, dy \right] dx$$

$$= \int_0^3 \frac{3}{2} x^2\, dx = \left[\frac{x^3}{2} \right]_0^3 = \frac{27}{2}$$

(b) In this case, first treat y as a constant and integrate with respect to x.

$$\int_1^2 \int_0^3 x^2 y\, dx\, dy = \int_1^2 \left[\int_0^3 x^2 y\, dx \right] dy$$

$$= \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy \qquad \text{Integrate with respect to } x.$$

$$= \int_1^2 \left[\frac{3^3}{3} y - \frac{0^3}{3} y \right] dy \qquad \text{FTC.}$$

$$= \int_1^2 9y\, dy = \left[9\frac{y^2}{2} \right]_1^2 = \frac{27}{2} \qquad \text{Integrate with respect to } y\text{; FTC; simplify.}$$

Notice that in Example 1, we obtained the same answer whether we integrated with respect to y or x first. In general, it turns out (see Fubini's Theorem below) that the two iterated integrals in Equations 2 and 3 are always equal; that is, the order of integration does not matter. This result is similar to Clairaut's Theorem and the equality of mixed partial derivatives.

The next theorem provides a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x, y) \mid a \le x \le b,\ c \le y \le d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy \tag{4}$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Although we will not present a proof of Fubini's Theorem, let's consider an intuitive approach to show why this result is true for the case where $f(x, y) \ge 0$. Recall that if f is positive, then we can interpret the double integral $\iint_R f(x, y)\, dA$ as the volume V of the solid S that lies above R and under the surface $z = f(x, y)$. But we have another formula that we used for volume in Chapter 6, namely,

$$V = \int_a^b A(x)\, dx$$

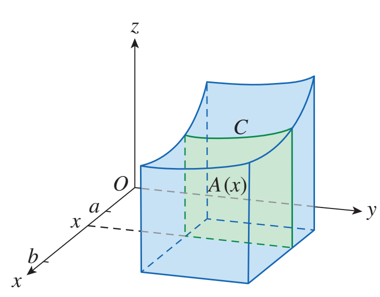


Figure 12.13
$A(x)$ is the area of the cross-section S in the plane through x perpendicular to the x-axis.

where $A(x)$ is the area of a cross-section of S in the plane through x perpendicular to the x-axis. From Figure 12.13, we can see that $A(x)$, in this case, is the area under the curve C whose equation is $z = f(x, y)$, where x is held constant and $c \le y \le d$. Therefore,

$$A(x) = \int_c^d f(x, y)\, dy$$

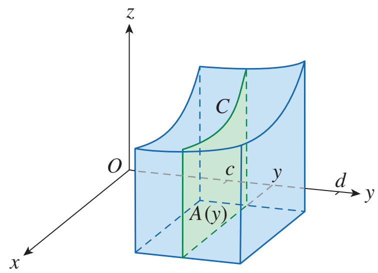


Figure 12.14
$A(y)$ is the area of the cross-section S in the plane through y perpendicular to the y-axis.

and we can write

$$\iint_R f(x, y)\, dA = V = \int_a^b A(x)\, dx = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

A similar argument, using cross-sections perpendicular to the y-axis as in Figure 12.14, shows that

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Example 2 Use Fubini's Theorem to Calculate a Double Integral

Evaluate the double integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$. (Compare this answer with the work in Example 3, Section 12.1.)

Solution 1

Use Fubini's Theorem to write the double integral as an iterated integral, and evaluate.

$$\iint_R (x - 3y^2)\, dA = \int_0^2 \int_1^2 (x - 3y^2)\, dy\, dx \qquad \text{Fubini's Theorem.}$$

$$= \int_0^2 \Big[xy - y^3 \Big]_{y=1}^{y=2} dx \qquad \text{Integrate with respect to } y.$$

$$= \int_0^2 (x - 7)\, dx = \left[\frac{x^2}{2} - 7x \right]_0^2 \qquad \text{FTC; integrate with respect to } x.$$

$$= \frac{2^2}{2} - (7)(2) = -12 \qquad \text{FTC; simplify.}$$

Solution 2

Use Fubini's Theorem again, but this time integrate with respect to x first.

$$\iint_R (x - 3y^2)\, dA = \int_1^2 \int_0^2 (x - 3y^2)\, dx\, dy \qquad \text{Fubini's Theorem.}$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2 \right]_{x=0}^{x=2} dy \qquad \text{Integrate with respect to } x.$$

$$= \int_1^2 (2 - 6y^2)\, dy = \Big[2y - 2y^3 \Big]_1^2 \qquad \text{FTC; integrate with respect to } y.$$

$$= (2 \cdot 2 - 2 \cdot 2^3) - (2 \cdot 1 - 2 \cdot 1^3) = -12 \qquad \text{FTC; simplify.}$$

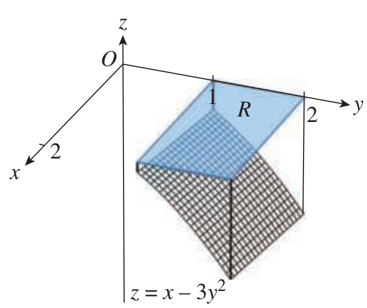


Figure 12.15
Graph of f and R.

Note: The negative answer here might seem wrong, but the function f is not positive over R. Therefore, the integral does not represent a volume. Figure 12.15 shows that f is negative on R, so the value of the integral is the negative of the volume that lies above the graph of f and below R. ■

Example 3 Compare the Difficulty in Different Orders of Integration

Evaluate $\iint_R y \sin(xy)\, dA$, where $R = [1, 2] \times [0, \pi]$.

Solution 1

Integrate first with respect to x.

$$\begin{aligned}
\iint_R y\,\sin(xy)\,dA &= \int_0^{\pi}\int_1^2 y\,\sin(xy)\,dx\,dy && \text{Fubini's Theorem.}\\
&= \int_0^{\pi}\Big[-\cos(xy)\Big]_{x=1}^{x=2}\,dy && \text{Integrate with respect to } x.\\
&= \int_0^{\pi}(-\cos 2y + \cos y)\,dy && \text{FTC.}\\
&= \left[-\frac{1}{2}\sin 2y + \sin y\right]_0^{\pi} = 0 && \text{Antiderivative; FTC; simplify.}
\end{aligned}$$

Solution 2

Suppose we reverse the order of integration and integrate first with respect to y.

$$\iint_R y\,\sin(xy)\,dA = \int_1^2\int_0^{\pi} y\,\sin(xy)\,dy\,dx$$

Use integration by parts to evaluate the inner integral.

$$\begin{aligned}
u &= y & dv &= \sin(xy)\,dy\\
du &= dy & v &= \int \sin(xy)\,dy = -\frac{\cos(xy)}{x}
\end{aligned}$$

$$\begin{aligned}
\int_0^{\pi} y\,\sin(xy)\,dy &= -\frac{y\,\cos(xy)}{x}\Bigg]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^{\pi}\cos(xy)\,dy && \text{Integration by parts.}\\
&= -\frac{\pi\,\cos\pi x}{x} + \frac{1}{x^2}\Big[\sin(xy)\Big]_{y=0}^{y=\pi} && \text{FTC; antiderivative.}\\
&= -\frac{\pi\,\cos\,\pi x}{x} + \frac{\sin\,\pi x}{x^2} && \text{FTC.}
\end{aligned}$$

Integrate this expression with respect to x.
Use integration by parts for the first term; leave the second term alone.

$$\begin{aligned}
u &= -\frac{1}{x} & dv &= \pi\cos\pi x\,dx\\
du &= \frac{1}{x^2}\,dx & v &= \int \pi\cos\pi x\,dx = \sin\pi x
\end{aligned}$$

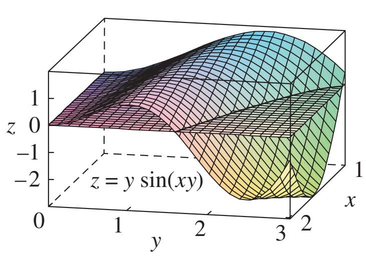


Figure 12.16
The volume V_1 above R and below the graph of f is equal to the volume V_2 below R and above the graph of f.

$$\begin{aligned}
&\int\left(-\frac{\pi\,\cos\,\pi x}{x}\right)dx + \int\frac{\sin\,\pi x}{x^2}\,dx\\
&= \left[-\frac{\sin\,\pi x}{x} - \int\frac{\sin\,\pi x}{x^2}\,dx\right] + \int\frac{\sin\,\pi x}{x^2}\,dx && \text{Integration by parts.}\\
&= -\frac{\sin\,\pi x}{x} && \text{Simplify.}
\end{aligned}$$

Evaluate the iterated integral.

$$\begin{aligned}
\int_1^2\int_0^{\pi} y\,\sin(xy)\,dy\,dx &= \left[-\frac{\sin\,\pi x}{x}\right]_1^2 && \text{Antiderivative.}\\
&= -\frac{\sin\,2\pi}{2} + \sin\,\pi = 0 && \text{FTC; simplify.}
\end{aligned}$$

The first solution is a little easier than the second. Therefore, when we evaluate double integrals, a judicious choice of the order of integration can lead to simpler integrals.

The function $f(x, y) = y \sin(xy)$ takes on both positive and negative values over R. Therefore, $\iint_R f(x, y)\, dA$ represents a difference in volumes: $V_1 - V_2$, where V_1 is the volume above R and below the graph of f, and V_2 is the volume below R and above the graph. Since the integral is 0, then the two volumes V_1 and V_2 are equal. Figure 12.16 illustrates this result. ■

Example 4 Use a Double Integral to Compute a Volume

Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

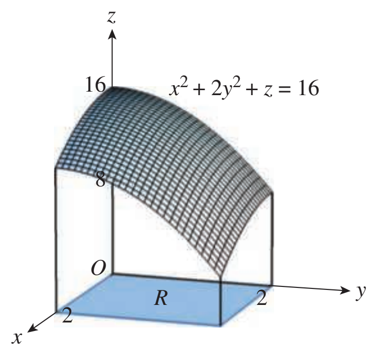


Figure 12.17
The solid S lies below the surface $z = 16 - x^2 - 2y^2$ and above R.

Solution

The solid S lies under the surface $z = 16 - x^2 - 2y^2$ and above the square $R = [0, 2] \times [0, 2]$, as illustrated in Figure 12.17.

We considered this solid in Section 12.1 Example 1, but now we have the calculus tools to evaluate the double integral using Fubini's Theorem.

$$V = \iint_R (16 - x^2 - 2y^2)\, dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy \qquad \text{Fubini's Theorem.}$$

$$= \int_0^2 \left[16x - \frac{1}{3}x^3 - 2y^2 x \right]_{x=0}^{x=2} dy \qquad \text{Integrate with respect to } x.$$

$$= \int_0^2 \left(\frac{88}{3} - 4y^2 \right) dy = \left[\frac{88}{3}y - \frac{4}{3}y^3 \right]_0^2 = 48 \qquad \text{FTC; antiderivative; simplify.}$$

■

In the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a simpler form.

Suppose $f(x, y) = g(x)h(y)$ and $R = [a, b] \times [c, d]$. Using Fubini's Theorem, we have

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy$$

In the inner integral, y is a constant, so $h(y)$ is a constant and passes freely through the inner integral symbol. And, since $\int_a^b g(x)\, dx$ is constant with respect to y, we can write

$$\int_c^d \left[\int_a^b g(x)h(y) dx \right] dy = \int_c^d \left[h(y) \left(\int_a^b g(x)\, dx \right) \right] dy = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$$

Therefore, in this case, the double integral of f can be written as the product of two single integrals:

$$\iint_R g(x)h(y)\, dA = \int_a^b g(x)\, dx \int_c^d h(y)\, dy \quad \text{where } R = [a, b] \times [c, d] \qquad (5)$$

Example 5 Integrate a Function of *x* Times a Function of *y*

Evaluate $\iint_R f(x, y)\, dA$, where $f(x, y) = \sin x \cos y$ and $R = [0, \pi/2] \times [0, \pi/2]$.

Solution

Since f can be written as the product of a function in x times a function of y, use Equation 5.

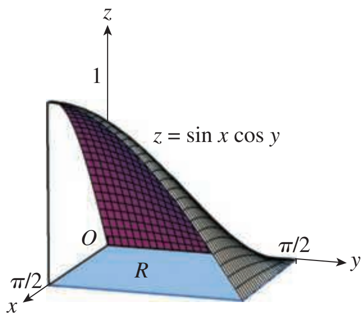


Figure 12.18
Graph of f and the region R.

$$\iint_R \sin x \cos y \, dA = \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy \qquad \text{Equation 5.}$$

$$= \Big[-\cos x\Big]_0^{\pi/2} \Big[\sin y\Big]_0^{\pi/2} \qquad \text{Antiderivatives.}$$

$$= \left[-\cos\frac{\pi}{2} + \cos 0\right] \cdot \left[\sin\frac{\pi}{2} - \sin 0\right] = 1 \cdot 1 = 1 \qquad \text{FTC; simplify.}$$

The function $f(x, y) = \sin x \cos y$ is positive on R. Therefore, the integral represents the volume of the solid that lies above R and below the graph of f as shown in Figure 12.18.

12.2 Exercises

Find $\int_0^5 f(x, y)\, dx$ and $\int_0^1 f(x, y)\, dy$.

1. $f(x, y) = 12x^2y^3$.

2. $f(x, y) = y + xe^y$

3. $f(x, y) = y^2 \sin(2\pi x)$

4. $f(x, y) = \dfrac{e^x}{y^2 + 2}$

Calculate the iterated integral.

5. $\int_1^3 \int_0^1 (1 + 4xy)\, dx\, dy$

6. $\int_0^1 \int_1^2 (4x^3 - 9x^2y^2)\, dy\, dx$

7. $\int_0^2 \int_0^{\pi/2} x \sin y\, dy\, dx$

8. $\int_{\pi/6}^{\pi/2} \int_{-1}^5 \cos y\, dx\, dy$

9. $\int_0^2 \int_0^1 (2x + y)^8\, dx\, dy$

10. $\int_0^1 \int_1^2 \dfrac{xe^x}{y}\, dy\, dx$

11. $\int_1^4 \int_1^2 \left(\dfrac{x}{y} + \dfrac{y}{x}\right) dy\, dx$

12. $\int_0^1 \int_0^3 e^{x+3y}\, dx\, dy$

13. $\int_0^1 \int_0^1 (u - v)^5\, du\, dv$

14. $\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2}\, dy\, dx$

15. $\int_0^2 \int_0^{\pi} r \sin^2\theta\, d\theta\, dr$

16. $\int_0^1 \int_0^1 \sqrt{s + t}\, ds\, dt$

17. $\int_0^1 \int_0^1 \dfrac{y}{x^2 + 1}\, dx\, dy$

18. $\int_0^1 \int_0^1 x^2ye^{-y^2}\, dx\, dy$

Calculate the double integral.

19. $\iint_R (6x^2y^3 - 5y^4)\, dA, \quad R = \{(x, y) \mid 0 \le x \le 3, 0 \le y \le 1\}$

20. $\iint_R \cos(x + 2y)\, dA, \quad R = \{(x, y) \mid 0 \le x \le \pi, 0 \le y \le \pi/2\}$

21. $\iint_R \dfrac{xy^2}{x^2 + 1}\, dA, \quad R = \{(x, y) \mid 0 \le x \le 1, -3 \le y \le 3\}$

22. $\iint_R \dfrac{1 + x^2}{1 + y^2}\, dA, \quad R = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$

23. $\iint_R x \sin(x + y)\, dA, \quad R = \left[0, \dfrac{\pi}{6}\right] \times \left[0, \dfrac{\pi}{3}\right]$

24. $\iint_R \dfrac{x}{1 + xy}\, dA, \quad R = [0, 1] \times [0, 1]$

25. $\iint_R xye^{x^2y}\, dA, \quad R = [0, 1] \times [0, 2]$

26. $\iint_R \dfrac{x}{x^2 + y^2}\, dA, \quad R = [1, 2] \times [0, 1]$

27. $\iint_R \dfrac{\ln x}{xy}\, dA, \quad R = [1, 3] \times [1, 3]$

28. $\iint_R \dfrac{\sin y}{1 + x^2}\, dA, \quad R = [0, 1] \times \left[0, \dfrac{\pi}{3}\right]$

Sketch the solid whose volume is given by the iterated integral.

29. $\int_0^1 \int_0^1 (4 - x - 2y)\, dx\, dy$

30. $\int_0^1 \int_0^1 (2 - x^2 - y^2)\, dy\, dx$

31. $\int_{-1}^1 \int_{-1}^1 (y^2 - x^2 + 2)\, dx\, dy$

32. Find the volume of the solid that lies under the plane $3x + 2y + z = 12$ and above the rectangle $R = \{(x, y) \mid 0 \le x \le 1, 2 \le y \le 3\}$.

33. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 4 + x^2 - y^2$ and above the square $R = [-1, 1] \times [0, 2]$.

34. Find the volume of the solid that lies under the elliptic paraboloid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z = 1$ and above the rectangle $R = [-1, 1] \times [-2, 2]$.

35. Find the volume of the solid enclosed by the surface $z = 1 + e^x \sin y$ and the planes $x = \pm 1$, $y = 0$, $y = \pi$, and $z = 0$.

36. Find the volume of the solid enclosed by the surface $z = x \sec^2 y$ and the planes $z = 0$, $x = 0$, $x = 2$, $y = 0$, and $y = \pi/4$.

37. Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

38. Find the volume of the solid enclosed by the paraboloid $z = 2 + x^2 + (y - 2)^2$ and the planes $z = 1$, $x = 1$, $x = -1$, $y = 0$, and $y = 4$.

39. Use technology to graph the solid that lies between the surface $z = \dfrac{2xy}{x^2 + 1}$ and the plane $z = x + 2y$ and is bounded by the planes $x = 0$, $x = 2$, $y = 0$, and $y = 4$. Then find its volume.

40. Use technology to find the exact value of the integral $\displaystyle\iint_R x^5 y^3 e^{xy}\, dA$, where $R = [0, 1] \times [0, 1]$. Use technology to sketch the solid whose volume is given by the integral.

41. Use technology to sketch the solid that lies between the surfaces $z = e^{-x^2} \cos(x^2 + y^2)$ and $z = 2 - x^2 - y^2$ for $|x| \le 1$ and to approximate the volume of this solid.

Find the average value of f over the given rectangle.

42. $f(x, y) = x^2 y$, R has vertices $(-1, 0)$, $(-1, 5)$, $(1, 5)$, $(1, 0)$

43. $f(x, y) = e^y \sqrt{x + e^y}$, $R = [0, 4] \times [0, 1]$

Use symmetry to evaluate the double integral

44. $\displaystyle\iint_R \frac{xy}{1 + x^4}\, dA$, $R = \{(x, y) \mid -1 \le x \le 1, 0 \le y \le 1\}$

45. $\displaystyle\iint_R (1 + x^2 \sin y + y^2 \sin x)\, dA$, $R = [-\pi, \pi] \times [-\pi, \pi]$

46. Use technology to compute the iterated integrals

$$\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dy\, dx \qquad \text{and} \qquad \int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dx\, dy$$

Explain why these answers do not contradict Fubini's Theorem.

47. (a) Explain how the theorems of Fubini and Clairaut are similar.

(b) If $f(x, y)$ is continuous on $[a, b] \times [c, d]$ and

$$g(x, y) = \int_a^x \int_c^y f(s, t)\, dt\, ds$$

for $a < x < b$, $c < y < d$, show that $g_{xy} = g_{yx} = f(x, y)$.

12.3 Double Integrals over General Regions

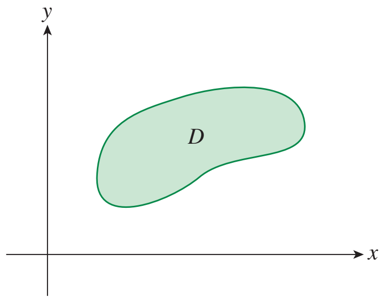


Figure 12.19
A general region D.

For single integrals, we usually integrate over an interval. However, for double integrals, we would like to be able to integrate a function f over a general region, not just over a rectangle.

General Regions

Consider a function f of two variables and a general region D like the one illustrated in Figure 12.19. We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 12.20. Then we define a new function F with domain R by

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases} \tag{1}$$

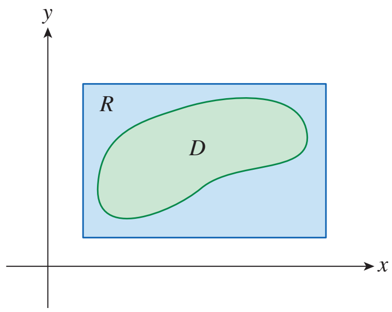


Figure 12.20
The region D is bounded; it can be enclosed in a rectangular region R.

If F is integrable over R, then we define the **double integral of f over D** by

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \quad \text{where } F \text{ is given in Equation 1} \tag{2}$$

The definition in Equation 2 seems reasonable because R is a rectangle and therefore $\iint_R F(x, y)\, dA$ is defined as in Section 12.1. The values of $F(x, y)$ are 0 when (x, y) lies outside of D and so they contribute nothing to the integral. This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geq 0$, we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x, y)$ (the graph of f). This interpretation seems reasonable; compare the graphs of f and F in Figures 12.21 and 12.22 and remember that $\iint_R F(x, y)\, dA$ is the volume under the graph of F.

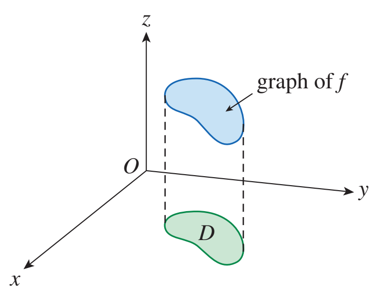


Figure 12.21
Graph of f over the region D.

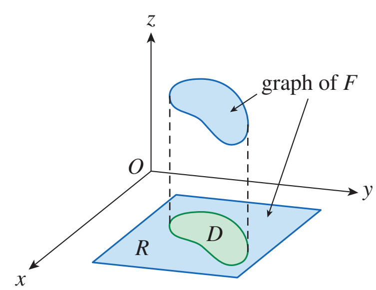


Figure 12.22
Graph of F over the rectangular region R.

Figure 12.22 also suggests that F is likely to have discontinuities at the boundary points of D. However, if f is continuous on D and the boundary curve of D is *well behaved* (in a sense discussed in upper-level mathematics courses), then it can be shown that $\iint_R F(x, y)\, dA$ exists and therefore $\iint_D f(x, y)\, dA$ exists. In particular, this is true for the following two types of regions.

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 12.23.

In order to evaluate $\iint_D f(x, y)\, dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as shown in Figure 12.24, and we let F be the function given in Equation 1; that is, F agrees with f on D and F is 0 outside of D.

Then, by Fubini's Theorem,

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

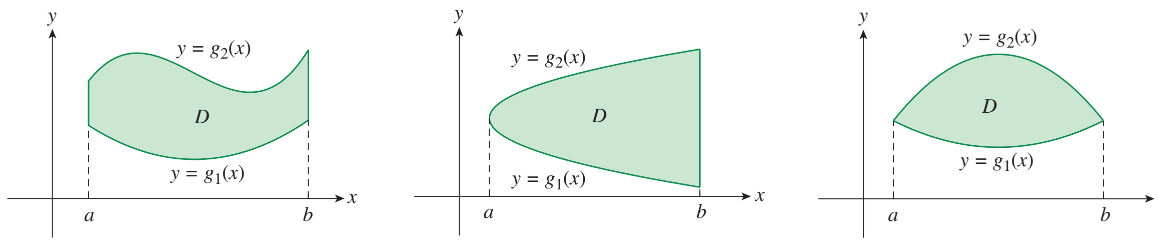


Figure 12.23
Some type I regions.

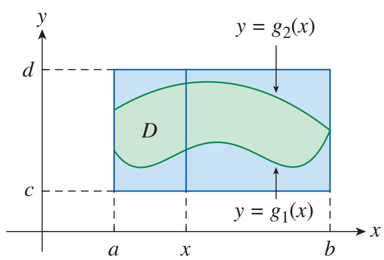


Figure 12.24
The rectangle R contains the region D.

Using the definition of F, if $y < g_1(x)$ or $y > g_2(x)$, then $F(x, y) = 0$ because (a, b) lies outside of the region D. Therefore,

$$\int_c^d F(x, y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \le y \le g_2(x)$. This leads to the following formula that enables us to evaluate the double integral as an iterated integral.

Double Integral, Type I Region

If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \le x \le b,\ g_1\,(x) \le y \le g_2\,(x)\}$$

then

$$\iint_D f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx \tag{3}$$

The integral on the right side of Equation 3 is an iterated integral that is similar to the ones we considered in the preceding section, except that in the inner integral, we treat x as a constant not only in $f(x, y)$ but also in the limits of integration $g_1(x)$ and $g_2(x)$.

We can also consider plane regions of **type II**, which can be defined by

$$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\} \tag{4}$$

where h_1 and h_2 are continuous. Three type II regions are illustrated in Figure 12.25.

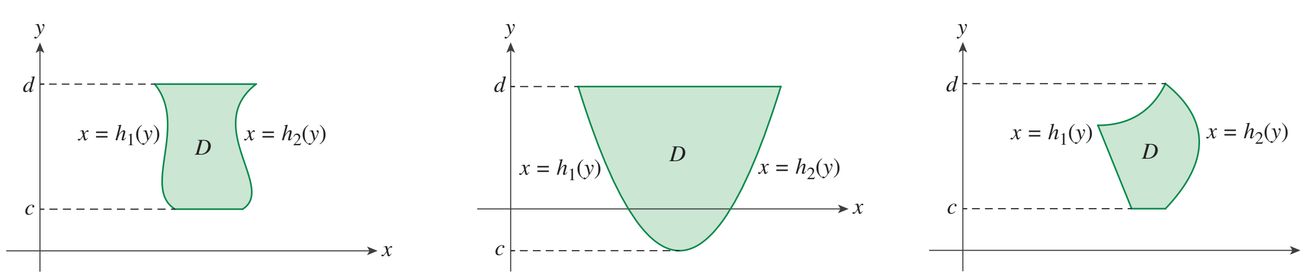


Figure 12.25
Some type II regions.

Using the same arguments that were used in deriving Equation 3, we can also evaluate double integral over a type II region.

Double Integral, Type II Region

If f is continuous on a type II region D such that

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

then $$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \tag{5}$$

Example 1 Evaluate a Double Integral over a Type I Region

Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

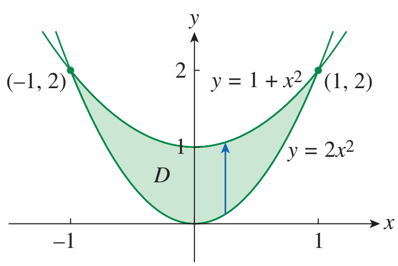


Figure 12.26
Graph of the region D, a type I region.

Solution

Find the point(s) where the parabolas intersect.

$2x^2 = 1 + x^2 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$

The parabolas intersect at the points $(1, 2)$ and $(-1, 2)$.

A graph of the region D is shown in Figure 12.26. Note that D is a type I region but not a type II region. We can write D as

$D = \{(x, y) \mid -1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}$.

The lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$.

Use Equation 3 to evaluate the double integral.

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx \qquad \text{Equation 3.}$$

$$= \int_{-1}^{1} \Big[xy + y^2 \Big]_{y=2x^2}^{y=1+x^2} dx \qquad \text{Integrate with respect to } y.$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx \qquad \text{FTC.}$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx \qquad \text{Simplify integrand.}$$

$$= \left[-3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \right]_{-1}^{1} = \frac{32}{15} \qquad \text{Antiderivative; FTC; simplify.}$$

■

Note: It is often very helpful to draw a diagram to help set up a double integral as in Example 1. It is also useful to draw a vertical arrow as in Figure 12.26. The limits of integration for the *inner* integral can then be read from the diagram as follows: The arrow starts at the lower boundary $y = g_1(x)$, which gives the lower limit in the integral, and the arrow ends at the upper boundary $y = g_2(x)$, which gives the upper limit of integration. For a type II region, the arrow is drawn horizontally from the left boundary to the right boundary.

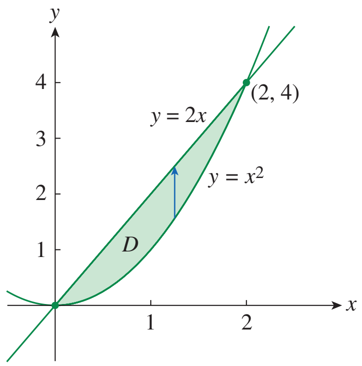


Figure 12.27
Graph of D as a type I region.

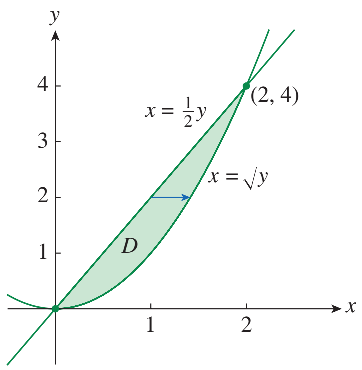


Figure 12.28
Graph of D as a type I region.

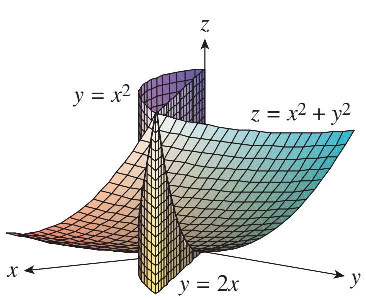


Figure 12.29
Graph of the solid described in Example 2.

Example 2 A Region That Is Both Type I and Type II

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution 1

Using Figure 12.27, we can interpret D as a type I region and write D as

$$D = \{(x, y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}.$$

The volume under $z = x^2 + y^2$ and above D is

$$
\begin{aligned}
V &= \iint_D (x^2 + y^2)\, dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx && \text{Equation 3.}\\
&= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx && \text{Antiderivative.}\\
&= \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2 x^2 - \frac{(x^2)^3}{3} \right] dx && \text{FTC.}\\
&= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx && \text{Simplify integrand.}\\
&= \left[-\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \right]_0^2 = \frac{216}{35} && \text{Antiderivative; FTC; simplify.}
\end{aligned}
$$

Solution 2

Using Figure 12.28, we can also interpret D as a type II region and write D as

$$D = \left\{(x, y) \mid 0 \le y \le 4,\ \tfrac{1}{2} y \le x \le \sqrt{y}\right\}.$$

Therefore, another expression for V is

$$
\begin{aligned}
V &= \iint_D (x^2 + y^2)\, dA = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy && \text{Equation 5.}\\
&= \int_0^4 \left[\frac{x^3}{3} + y^2 x \right]_{x=y/2}^{x=\sqrt{y}} dy && \text{Antiderivative.}\\
&= \int_0^4 \left[\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2} \right] dy && \text{FTC.}\\
&= \left[\frac{2}{15} y^{5/2} + \frac{2}{7} y^{7/2} - \frac{13}{96} y^4 \right]_0^4 = \frac{216}{35} && \text{Antiderivative; FTC; simplify.}
\end{aligned}
$$

A graph of this solid is shown in Figure 12.29. It lies above the xy-plane, below the paraboloid $z = x^2 + y^2$, and between the plane $y = 2x$ and the parabolic cylinder $y = x^2$. ■

Example 3 Choose the Better Description of a Region

Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution

The region D is shown in Figure 12.30. Again, we can interpret D as both a type I and type II region. However, the description of D as a type I region is more complicated because the lower boundary consists of two parts.

Therefore, let's express D as a type II region:

$$D = \left\{(x, y) \mid -2 \le y \le 4, \tfrac{1}{2}y^2 - 3 \le x \le y + 1\right\}$$

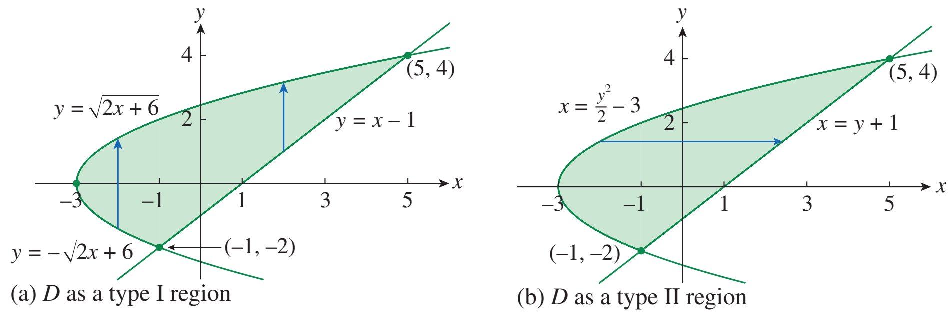


Figure 12.30
The region D can be interpreted as both a type I and type II region.

Use Equation 5 to evaluate the double integral.

$$\begin{aligned}
\iint_D xy\,dA &= \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy && \text{Equation 5.}\\
&= \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy && \text{Antiderivative.}\\
&= \frac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - \left(\frac{1}{2}y^2-3\right)^2\right] dy && \text{FTC.}\\
&= \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy && \text{Simplify integrand.}\\
&= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36 && \text{Antiderivative; FTC; simplify.}
\end{aligned}$$

Note that if we had expressed D as a type I region using Figure 12.30(a), then the double integral would be evaluated as

$$\iint_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

This would involve more work than the other method. ■

Example 4 Tetrahedron Volume

Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

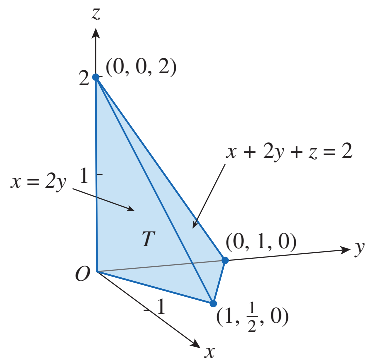


Figure 12.31
The tetrahedron T.

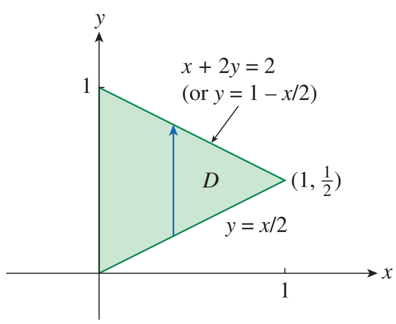


Figure 12.32
The region D.

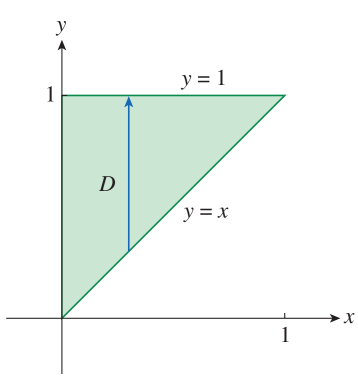


Figure 12.33
The region D interpreted as a type I region.

Solution

Start this solution by drawing two diagrams: one of the three-dimensional solid and another of the plane region D over which it lies.

Figure 12.31 shows the tetrahedron T bounded by the coordinate planes $x = 0$, $z = 0$, the vertical plane $x = 2y$, and the plane $x + 2y + z = 2$.

Since the plane $x + 2y + z = 2$ intersects the xy-plane (whose equation is $z = 0$) in the line $x + 2y = 2$, then T lies above the triangular region D in the xy-plane bounded by the lines $x = 2y$, $x + 2y = 2$, and $x = 0$, as shown in Figure 12.32.

The plane $x + 2y + z = 2$ can also be characterized by $z = 2 - x - 2y$.

Therefore, the volume of T lies under the graph of the function $z = 2 - x - 2y$ and above $D = \{(x, y) \mid 0 \le x \le 1, x/2 \le y \le 1 - x/2\}$.

Use Equation 3 to evaluate the double integral that represents the volume of T.

$$V = \iint_D (2 - x - 2y)\, dA \qquad \text{Expression for volume.}$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx \qquad \text{Equation 3.}$$

$$= \int_0^1 \Big[2y - xy - y^2 \Big]_{y=x/2}^{y=1-x/2} dx \qquad \text{Antiderivative.}$$

$$= \int_0^1 \left[2\left(1 - \frac{x}{2}\right) - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4} \right] dx \qquad \text{FTC.}$$

$$= \int_0^1 (x^2 - 2x + 1)\, dx = \left[\frac{x^3}{3} - x^2 + x \right]_0^1 = \frac{1}{3} \qquad \text{Antiderivative; FTC; simplify.}$$

Example 5 Reverse the Order of Integration

Evaluate the iterated integral $\displaystyle\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

Solution

If we try to evaluate the integral as presented, then we need to start by finding $\displaystyle\int \sin(y^2)\, dy$. However, it is impossible to find a closed-form, or finite term, antiderivative since $\displaystyle\int \sin(y^2)\, dy$ is not an elementary function. Therefore, we need to change the order of integration.

Write the given iterated integral as a double integral (using Equation 3 backward).

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \iint_D \sin(y_2)\, dA \quad \text{where } D = \{(x, y) \mid 0 \le x \le 1, x \le y \le 1\}$$

Figure 12.33 shows a sketch of the region D. Using the alternate graphical description in Figure 12.34, we can write D as

$D = \{(x, y) \mid 0 \le y \le 1, 0 \le x \le y\}$.

This characterization of the region D allows us to use Equation 5 to express the double integral as an iterated integral in the reverse order.

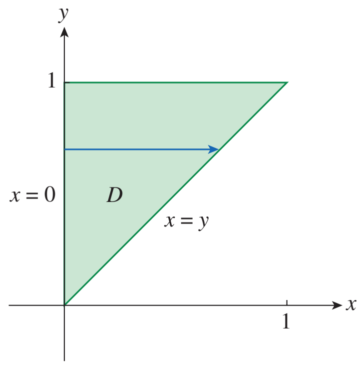


Figure 12.34
The region D interpreted as a type II region.

$$\begin{aligned}
\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx &= \iint_D \sin(y^2)\,dA && \text{Equation 3 backward.}\\
&= \int_0^1 \int_0^y \sin(y^2)\,dx\,dy && \text{Equation 5; type II region.}\\
&= \int_0^1 \Big[x\,\sin(y^2)\Big]_{x=0}^{x=y} dy && \text{Antiderivative.}\\
&= \int_0^1 y\,\sin(y^2)\,dy = -\frac{1}{2}\cos(y^2)\Big]_0^1 && \text{FTC; antiderivative.}\\
&= -\frac{1}{2}(\cos 1 - \cos 0) = \frac{1}{2}(1 - \cos 1) && \text{FTC; simplify.}
\end{aligned}$$

Properties of Double Integrals

In the integral properties that follow, we assume that all of the integrals exist. The first three properties of double integrals over a region D follow immediately from the definition of a double integral of f over a region D and from Properties 7, 8, and 9 in Section 12.1.

$$\iint_D [f(x, y) + g(x, y)]\,dA = \iint_D f(x, y)\,dA + \iint_D g(x, y)\,dA \tag{6}$$

$$\iint_D cf(x, y)\,dA = c\iint_D f(x, y)\,dA \quad \text{where } c \text{ is a constant} \tag{7}$$

If $f(x, y) \geq g(x, y)$ for all (x, y) in D, then

$$\iint_D f(x, y)\,dA \geq \iint_D g(x, y)\,dA \tag{8}$$

The next property of double integrals is similar to the property of single integrals given by the equation $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

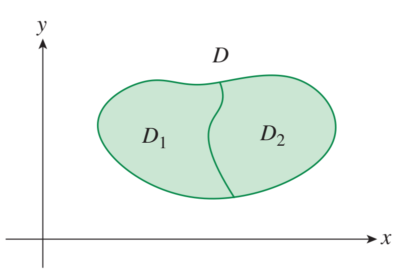
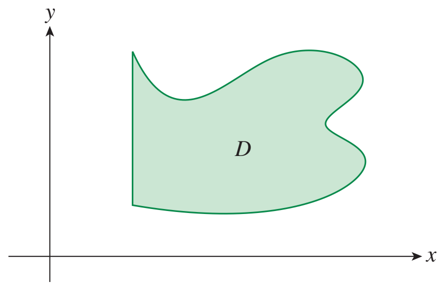


Figure 12.35
$D = D_1 \cup D_2$.

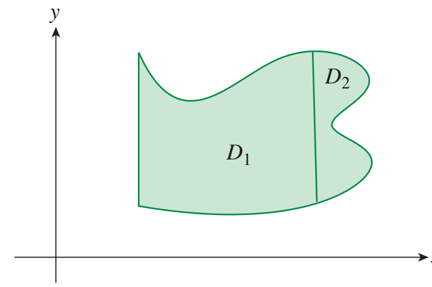
Suppose $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except possibly on their boundaries, as shown in Figure 12.35. Then

$$\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA \tag{9}$$

Property 9 can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 12.36 illustrates this procedure. (See Exercises 61 and 62.)



(a) D is neither a type I nor a type II region.



(b) $D = D_1 \cup D_2$; D_1 is a type I region and D_2 is a type II region.

Figure 12.36
The region D can be expressed as the union of a type I and a type II region.

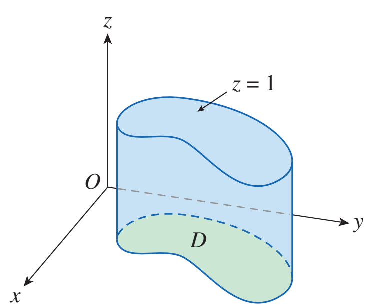


Figure 12.37
Cylinder with base D and height 1.

The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, then we get the area of D:

$$\iint_D 1\, dA = A(D) \tag{10}$$

Figure 12.37 illustrates this result: a solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write the volume of the solid as $\iint_D 1\, dA$.

Finally, we can combine Properties 7, 8, and 10 to prove the following property. (See Exercise 67.)

If $m \le f(x, y) \le M$ for all (x, y) in D, then

$$mA(D) \le \iint_D f(x, y)\, dA \le MA(D) \tag{11}$$

Example 6 Estimate the Value of a Double Integral

Use Property 11 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center at the origin and radius 2.

Solution

Try to bound the integrand.

Since $-1 \le \sin x \le 1$ and $-1 \le \cos y \le 1$, then $-1 \le \sin x \cos y \le 1$.

Therefore, $e^{-1} \le e^{\sin x \cos y} \le e^1 = e$.

Use $m = e^{-1} = \dfrac{1}{e}$, $M = e$, and $A(D) = \pi(2)^2$ in Property 11.

$$\frac{4\pi}{e} \le \iint_D e^{\sin x \cos y}\, dA \le 4\pi e$$

12.3 Exercises

Evaluate the iterated integral.

1. $\displaystyle\int_0^4 \int_0^{\sqrt{y}} xy^2\, dx\, dy$

2. $\displaystyle\int_0^1 \int_{2x}^2 (x - y)\, dy\, dx$

3. $\displaystyle\int_0^1 \int_0^{e^{x^2}} x\, dy\, dx$

4. $\displaystyle\int_1^4 \int_0^{\ln x} \frac{1}{x}\, dy\, dx$

5. $\displaystyle\int_0^1 \int_{x^2}^x (1 + 2y)\, dy\, dx$

6. $\displaystyle\int_0^2 \int_y^{2y} xy\, dx\, dy$

7. $\displaystyle\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\, dr\, d\theta$

8. $\displaystyle\int_0^1 \int_0^v \sqrt{1 - v^2}\, du\, dv$

Evaluate the double integral.

9. $\displaystyle\iint_D y^2\, dA,\ D = \{(x, y) \mid -1 \le y \le 1, -y - 2 \le x \le y\}$

10. $\displaystyle\iint_D \frac{y}{x^5 + 1}\, dA,\ D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le x^2\}$

11. $\displaystyle\iint_D x\, dA,\quad D = \{(x, y) \mid 0 \le x \le \pi, 0 \le y \le \sin x\}$

12. $\displaystyle\iint_D x^3\, dA,\quad D = \{(x, y) \mid 1 \le x \le e, 0 \le y \le \ln x\}$

13. $\displaystyle\iint_D e^{-y^2}\, dA,\quad D = \{(x, y) \mid 0 \le y \le 3, 0 \le x \le y\}$

14. $\displaystyle\iint_D y\sqrt{x^2 - y^2}\, dA,\ D = \{(x, y) \mid 0 \le x \le 2, 0 \le y \le x\}$

15. Draw an example of a region that is
(a) type I but not type II
(b) type II but not type I

16. Draw an example of a region that is
(a) both type I and type II
(b) neither type I nor type II

Express D as a region of type I and also as a region of type II. Then evaluate the double integral in two ways.

17. $\displaystyle\iint_D x\, dA,\quad D$ is enclosed by the lines $y = x$, $y = 0$, $x = 1$

18. $\displaystyle\iint_D xy\, dA,\quad D$ is enclosed by the curves $y = x^2$, $y = 3x$

Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

19. $\displaystyle\iint_D y\, dA,\quad D$ is bounded by $y = x - 2$, $x = y^2$

20. $\displaystyle\iint_D y^2 e^{xy}\, dA,\quad D$ is bounded by $y = x$, $y = 4$, $x = 0$

Evaluate the double integral.

21. $\displaystyle\iint_D x \cos y\, dA,\quad D$ is bounded by $y = 0$, $y = x^2$, $x = 1$

22. $\displaystyle\iint_D x\sqrt{y^2 - x^2}\, dA,\quad D$ is bounded by $x = 0$, $y = 1$, $y = x$

23. $\displaystyle\iint_D y^3\, dA,\quad D$ is the triangular region with vertices $(0, 2)$, $(1, 1)$, and $(3, 2)$

24. $\displaystyle\iint_D xy^2\, dA,\quad D$ is enclosed by $x = 0$ and $x = \sqrt{1 - y^2}$

25. $\displaystyle\iint_D (2x - y)\, dA$, D is bounded by the circle with center at the origin and radius 2

26. $\displaystyle\iint_D 2xy\, dA$, D is the triangular region with vertices $(0, 0)$, $(1, 2)$, and $(0, 3)$

Find the volume of the given solid.

27. Under the plane $x + 2y - z = 0$ and above the region bounded by $y = x$ and $y = x^4$

28. Under the surface $z = 2x + y^2$ and above the region bounded by $x = y^2$ and $x = y^3$

29. Under the surface $z = xy$ and above the triangle with vertices $(1, 1)$, $(4, 1)$ and $(1, 2)$

30. Enclosed by the paraboloid $z = x^2 + 3y^2$ and the planes $x = 0$, $y = 1$, $y = x$, and $z = 0$

31. Bounded by the coordinate planes and the plane $3x + 2y + z = 6$

32. Bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$

33. Enclosed by the cylinders $z = x^2$, $y = x^2$, and the planes $z = 0$, $y = 4$

34. Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant

35. Bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, $z = 0$ in the first octant

36. Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$

37. Use technology to find the x-coordinates of the points of intersection of the curves $y = x^4$ and $y = 3x - x^2$. Let D be the region bounded by these curves and find $\iint_D x\,dA$.

38. Use technology to find the volume of the solid in the first octant that is bounded by the planes $y = x$, $z = 0$, and $z = x$ and the cylinder $y = \cos x$.

Find the volume of the solid by subtracting two volumes.

39. The solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$, and the planes $x + y + z = 2$, $2x + 2y - z + 10 = 0$

40. The solid enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$

41. The solid under the plane $z = 3$, above the plane $z = y$, and between the parabolic cylinders $y = x^2$ and $y = 1 - x^2$

42. The solid in the first octant under the plane $z = x + y$, above the surface $z = xy$, and enclosed by the surfaces $x = 0$, $y = 0$, and $x^2 + y^2 = 4$

Sketch the solid whose volume is given by the iterated integral.

43. $\displaystyle\int_0^1 \int_0^{1-x} (1 - x - y)\,dy\,dx$

44. $\displaystyle\int_0^1 \int_0^{1-x^2} (1 - x)\,dy\,dx$

Use technology to find the exact volume of the solid.

45. Under the surface $z = x^3y^4 + xy^2$ and above the region bounded by the curves $y = x^3 - x$ and $y = x^2 + x$ for $x \geq 0$

46. Between the paraboloids $z = 2x^2 + y^2$ and $z = 8 - x^2 - 2y^2$ and inside the cylinder $x^2 + y^2 = 1$

47. Enclosed by $z = 1 - x^2 - y^2$ and $z = 0$

48. Enclosed by $z = x^2 + y^2$ and $z = 2y$

Sketch the region of integration and change the order of integration.

49. $\displaystyle\int_0^4 \int_0^{\sqrt{x}} f(x, y)\,dy\,dx$ **50.** $\displaystyle\int_0^1 \int_{4x}^{4} f(x, y)\,dy\,dx$

51. $\displaystyle\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\,dy\,dx$ **52.** $\displaystyle\int_0^3 \int_0^{\sqrt{9-y}} f(x, y)\,dx\,dy$

53. $\displaystyle\int_1^2 \int_0^{\ln x} f(x, y)\,dy\,dx$ **54.** $\displaystyle\int_0^1 \int_{\arctan x}^{\pi/4} f(x,y)\,dy\,dx$

Evaluate the integral by reversing the order of integration.

55. $\displaystyle\int_0^1 \int_{3y}^{3} e^{x^2}\,dx\,dy$ **56.** $\displaystyle\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\,dx\,dy$

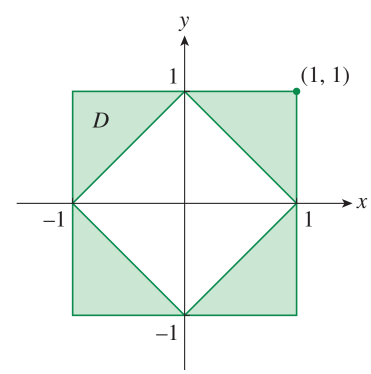
57. $\displaystyle\int_0^4 \int_{\sqrt{x}}^{2} \frac{1}{y^3 + 1}\,dy\,dx$ **58.** $\displaystyle\int_0^1 \int_x^1 e^{x/y}\,dy\,dx$

59. $\displaystyle\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\sqrt{1 + \cos^2 x}\,dx\,dy$

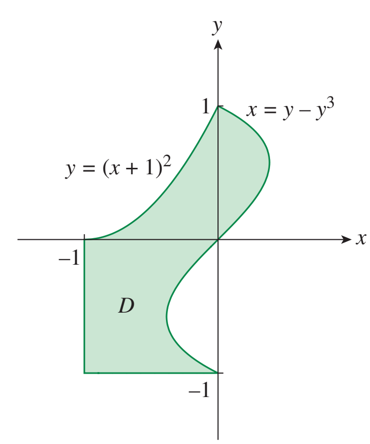
60. $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^{2} e^{x^4}\,dx\,dy$

Express D as a union of regions of type I or type II and evaluate the integral.

61. $\displaystyle\iint_D x^2\,dA$



62. $\displaystyle\iint_D y\,dA$



Use Property 11 to estimate the value of the integral.

63. $\displaystyle\iint_Q e^{-(x^2+y^2)^2}\,dA$, Q is the quarter-circle with center at the origin and radius 1/2 in the first quadrant

64. $\displaystyle\iint_T \sin^4(x+y)\,dA$, T is the triangle enclosed by the lines $y=0$, $y=2x$, and $x=1$

Find the average value of f over the region D.

65. $f(x, y) = xy$, D is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$

66. $f(x, y) = x \sin y$, D is enclosed by the curves $y = 0$, $y = x^2$, and $x = 1$

67. Prove Property 11.

68. Suppose that in evaluating a double integral over a region D, the following sum of integrals was obtained:

$$\iint_D f(x, y)\,dA = \int_0^1\int_0^{2y} f(x, y)\,dx\,dy + \int_1^3\int_0^{3-y} f(x, y)\,dx\,dy$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.

Use geometry or symmetry or both to evaluate the double integral.

69. $\displaystyle\iint_D (x+2)\,dA$, $D = \left\{(x, y) \mid 0 \le y \le \sqrt{9-x^2}\right\}$

70. $\displaystyle\iint_D \sqrt{R^2-x^2-y^2}\,dA$, D is the disk with center at the origin and radius R.

71. $\displaystyle\iint_D (2x+3y)\,dA$, D is the rectangle $0 \le x \le a$, $0 \le y \le b$

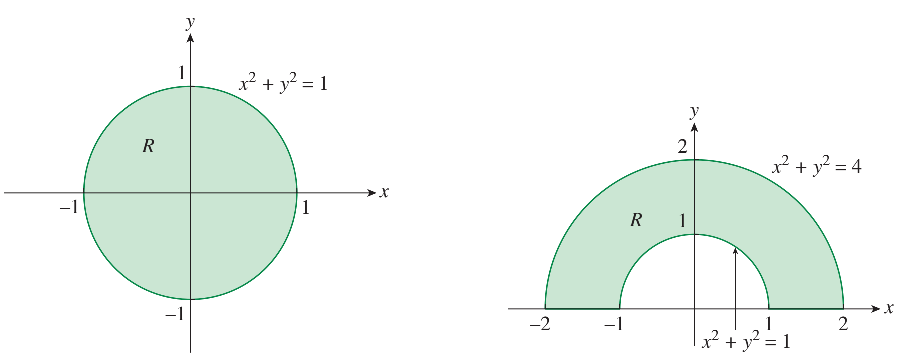
72. $\displaystyle\iint_D (2 + x^2y^3 - y^2\sin x)\,dA$, $D = \{(x, y) \mid |x| + |y| \le 1\}$

73. $\displaystyle\iint_D (ax^3 + by^3 + \sqrt{a^2-x^2})\,dA$, $D = [-a, a] \times [-b, b]$

74. Use technology to graph the solid bounded by the plane $x + y + z = 1$ and the paraboloid $z = 4 - x^2 - y^2$ and to find its exact volume.

12.4 Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\displaystyle\iint_R f(x, y)\,dA$, where R is one of the regions shown in Figure 12.38. In either case, the description of R in terms of rectangular coordinates is possible, but involves some difficult algebra. However, in each case, R is easily described using polar coordinates.



(a) $R = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

(b) $R = \{(r, \theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$

Figure 12.38
Regions described using polar coordinates.

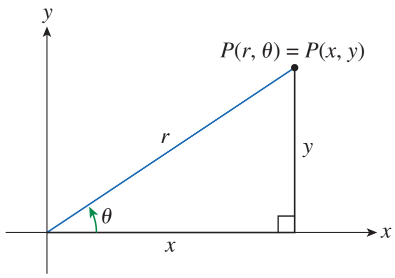


Figure 12.39
The connection between polar and Cartesian coordinates.

Recall from Figure 12.39 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

The regions in Figure 12.38 are special cases of a **polar rectangle**

$$R = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$$

which is illustrated in Figure 12.40. In order to compute the double integral $\iint_R f(x, y)\, dA$, where R is a polar rectangle, we divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = \dfrac{b-a}{m}$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = \dfrac{\beta - \alpha}{n}$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles R_{ij} shown in Figure 12.41.

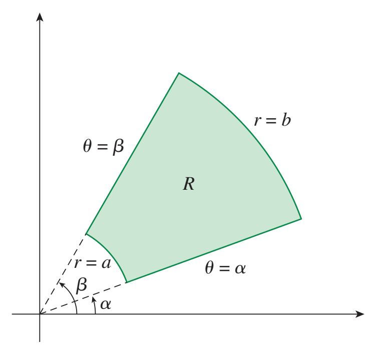


Figure 12.40
Polar rectangle.

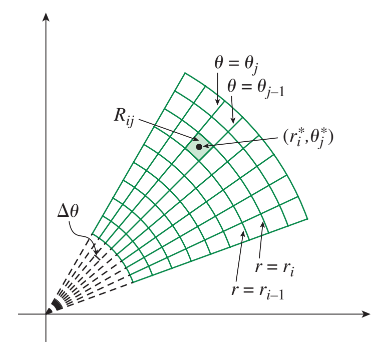


Figure 12.41
Dividing R into polar subrectangles.

The *center* of the polar subrectangle

$$R_{ij} = \{(r, \theta) \mid r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$$

has polar coordinates

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$$

We can find the area of R_{ij} by using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$. If we subtract the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$ then the area of R_{ij} is

$$\begin{aligned}\Delta A_i &= \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\,\Delta\theta \\ &= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\,\Delta\theta \; = r_i^*\,\Delta r\,\Delta\theta\end{aligned}$$

Although we have defined the double integral $\iint_R f(x, y)\, dA$ in terms of ordinary rectangles, it can be shown that, for continuous functions f, we always obtain the same answer using polar rectangles. The rectangular coordinates of the center of R_{ij} are $(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$, so a typical Riemann sum is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, \Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_i^*\, \Delta r\, \Delta\theta \qquad (1)$$

If we write $g(r, \theta) = rf(r \cos\theta, r \sin\theta)$, then the Riemann sum in Equation 1 can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\, \Delta r\, \Delta\theta$$

which is a Riemann sum for the double integral

$$\int_\alpha^\beta \int_a^b g(r, \theta)\, dr\, d\theta$$

Therefore, we have

$$\begin{aligned}\iint_R f(x, y)\, dA &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, \Delta Ai \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\, \Delta r\, \Delta\theta = \int_\alpha^\beta \int_a^b g(r, \theta)\, dr\, d\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta\end{aligned}$$

Change to Polar Coordinates in a Double Integral

If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r \cos\theta, r \sin\theta)\, r\, dr\, d\theta \qquad (2)$$

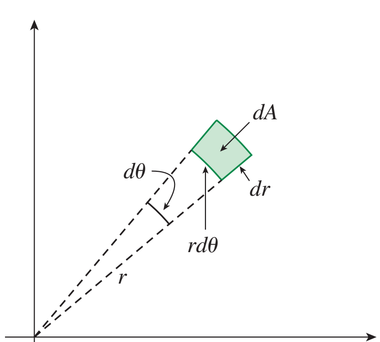


Figure 12.42
A small polar rectangle interpreted as an ordinary rectangle.

Equation 2 says that we convert from rectangular to polar coordinates in a double integral by writing $x = r \cos\theta$ and $y = r \sin\theta$, using the appropriate limits of integration for r and θ, and replacing dA by $r\, dr\, d\theta$. Don't forget that extra factor of r on the right side of Equation 2. A classical method to remember this is shown in Figure 12.42, where the *infinitesimal* polar rectangle can be thought of as an ordinary rectangle with dimensions $r\, d\theta$ and dr and therefore has *area* $dA = r\, dr\, d\theta$.

Example 1 Integrate over a Region Best Described in Polar Coordinates

Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution

The region R can be described as $R = \{(x, y) \mid y \geq 0,\ 1 \leq x^2 + y^2 \leq 4\}$.

The region R is the half-ring shown in Figure 12.38(b), and in polar coordinates it is given by $1 \leq r \leq 2$, $0 \leq \theta \leq \pi$.

Use Equation 2 to evaluate the double integral.

$$\iint_R (3x + 4y^2)\, dA = \int_0^{\pi} \int_1^2 (3r \cos\theta + 4r^2 \sin^2\theta)\, r\, dr\, d\theta \qquad \text{Equation 2.}$$

$$= \int_0^{\pi} \int_1^2 (3r^2 \cos\theta + 4r^3 \sin^2\theta)\, dr\, d\theta \qquad \text{Distribute } r.$$

$$= \int_0^{\pi} \Big[r^3 \cos\theta + r^4 \sin^2\theta \Big]_{r=1}^{r=2} d\theta \qquad \text{Antiderivative.}$$

$$= \int_0^{\pi} (7 \cos\theta + 15 \sin^2\theta)\, d\theta \qquad \text{FTC; simplify.}$$

$$= \int_0^{\pi} \left[7 \cos\theta + \frac{15}{2}(1 - \cos 2\theta) \right] d\theta \qquad \text{Trigonometric identity.}$$

$$= \left[7 \sin\theta + \frac{15\theta}{2} - \frac{15}{4} \sin 2\theta \right]_0^{\pi} = \frac{15\pi}{2} \qquad \text{Antiderivative; FTC; simplify.}$$

Example 2 Find a Volume Using Polar Coordinates

Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

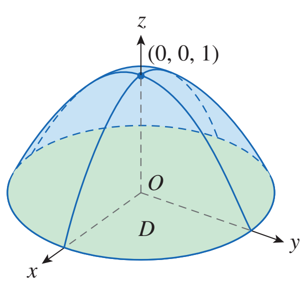


Figure 12.43
The solid lies under the paraboloid and above the circular disk D.

Solution

If we let $z = 0$ in the equation of the paraboloid, then $x^2 + y^2 = 1$.

Therefore, the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$.

So, the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \leq 1$, as illustrated in Figure 12.43.

In polar coordinates, D is described by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

Convert the integrand to polar form: $1 - x^2 - y^2 = 1 - r^2$.

Use Equation 2 to evaluate the double integral.

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta \qquad \text{Equation 2.}$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr \qquad \text{Integrand is a product of a function in } \theta \text{ times a function in } r.$$

$$= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2} \qquad \text{Antiderivatives; FTC; simplify.}$$

If we had used rectangular coordinates instead of polar coordinates, then the expression for volume would be

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx.$$

This is just not as easy to evaluate because it involves finding $\int (1 - x^2)^{3/2}\, dx$. ■

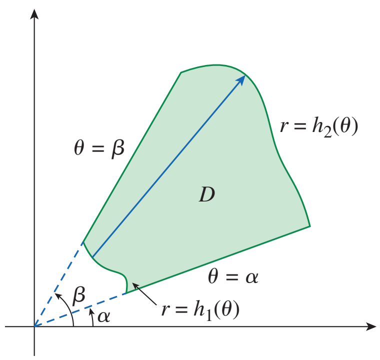


Figure 12.44
A more complicated polar region D described by $D = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$.

Evaluating double integrals in polar coordinates can be extended to more complicated regions, like the one shown in Figure 12.44. The region D is similar to the type II rectangular regions considered in Section 12.3. In fact, by combining Equation 2 in this section with Equation 12.3.5, we obtain the following formula.

If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos\theta, r \sin\theta)\, r\, dr\, d\theta \tag{3}$$

Consider the special case in Equation 3 in which $f(x, y) = 1$, $h_1(\theta) = 0$, and $h_2(\theta) = h(\theta)$. The area of the region D bounded by $\theta = \alpha$, $\theta = \beta$, and $r = h(\theta)$ is

$$A(D) = \iint_D 1\, dA = \int_\alpha^\beta \int_0^{h(\theta)} r\, dr\, \theta \qquad \text{Equation 2.}$$

$$= \int_\alpha^\beta \left[\frac{r^2}{2}\right]_0^{h(\theta)} d\theta = \int_\alpha^\beta \frac{1}{2}[h(\theta)]^2\, d\theta \qquad \text{Antiderivative; FTC.}$$

This result agrees with Formula 3 in Appendix H.2.

Example 3 Volume of a Solid

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Solution

The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$.

Complete the square: $x^2 + y^2 = 2x \Rightarrow (x - 1)^2 + y^2 = 1$.
The disk D is illustrated in Figure 12.45.

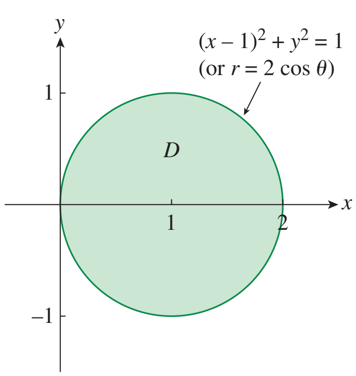


Figure 12.45
The disk D.

Convert this equation to polar form using $x^2 + y^2 = r^2$ and $x = r\cos\theta$.

The disk D is described in polar form as $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$.

Therefore, the disk D is described by $D = \left\{(r, \theta) \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, 0 \le r \le 2\cos\theta\right\}$.

Write a double integral to represent the volume and then use Equation 3 to convert to polar form.

$$V = \iint_D (x^2 + y^2)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta \qquad \text{Equation 3.}$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta \qquad \text{Antiderivative; FTC.}$$

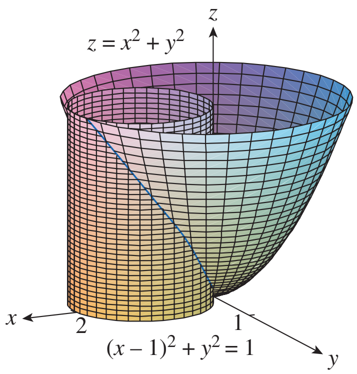


Figure 12.46
The solid lies above the disk D and below the paraboloid $z = x^2 + y^2$.

$$= 8\int_0^{\pi/2} \cos^4\theta \, d\theta \qquad \text{Even function.}$$

$$= 8\left(\frac{1}{4}\cos^3\theta \sin\theta\Big]_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2}\cos^2\theta \, d\theta\right) \qquad \text{Formula 74, } n = 4\text{, Table of Integrals.}$$

$$= 6\int_0^{\pi/2} \cos^2\theta \, d\theta \qquad \text{FTC; simplify.}$$

$$= 6\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} \qquad \text{Formula 64, Table of Integrals.}$$

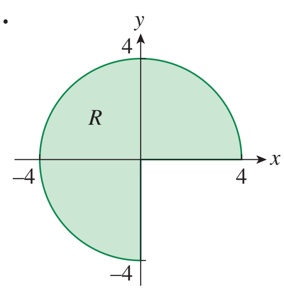
$$= 6 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{2} \qquad \text{FTC; simplify.}$$

The solid is illustrated in Figure 12.46. ■

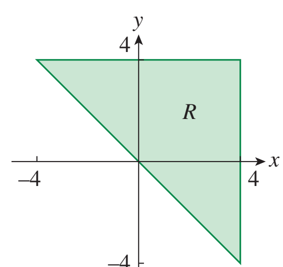
12.4 Exercises

A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x, y)\, dA$ as an iterated integral, where f is an arbitrary function on R.

1.



2.



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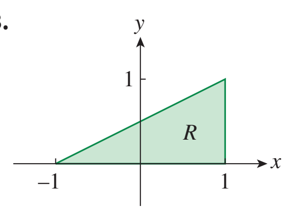
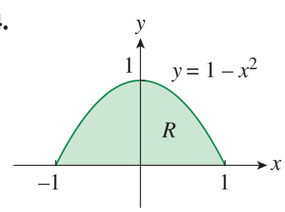
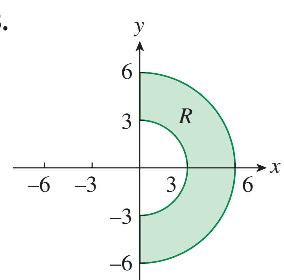


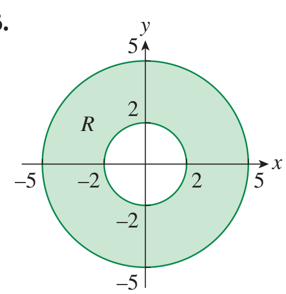
4.



5.



6.



Sketch the region whose area is given by the integral and evaluate the integral.

7. $\displaystyle\int_\pi^{2\pi}\int_4^7 r\, dr\, d\theta$ **8.** $\displaystyle\int_0^{\pi/2}\int_0^{4\cos\theta} r\, dr\, d\theta$

Evaluate the integral by changing to polar coordinates.

9. $\displaystyle\iint_D xy\, dA$, where D is the disk with center at the origin and radius 3

10. $\displaystyle\iint_R (x + y)\, dA$, where R is the region that lies to the left of the y-axis between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

11. $\displaystyle\iint_R \cos(x^2 + y^2)\, dA$, where R is the region that lies above the x-axis within the circle $x^2 + y^2 = 9$

12. $\displaystyle\iint_R \sqrt{4 - x^2 - y^2}\, dA$, where $R = \{(x, y) \mid x^2 + y^2 \le 4, x \ge 0\}$

13. $\displaystyle\iint_D e^{-x^2-y^2}\, dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis

14. $\displaystyle\iint_R ye^x\, dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 25$

15. $\displaystyle\iint_R \arctan\frac{y}{x}\, dA$, where $R = \{(x, y) \mid 1 \le x^2 + y^2 \le 4, 0 \le y \le x\}$

16. $\iint_D x\,dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$

17. $\iint_R \frac{y^2}{x^2 + y^2}\,dA$, where R is the region that lies between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ with $0 < a < b$

Use a double integral to find the area of the region.

18. One loop of the rose $r = \sin 3\theta$

19. The region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$

20. The region enclosed by both cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

21. The region inside the circle $(x - 1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$

Use polar coordinates to find the volume of the given solid.

22. Under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \le 4$

23. Below the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy-plane

24. Enclosed by the hyperboloid $-x^2 - y^2 + z^2 = 1$ and the plane $z = 2$

25. Inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$

26. A sphere of radius a

27. Bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant

28. Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$

29. Bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$

30. Inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$

31. (a) A cylindrical drill with radius r_1 is used to bore a hole through the center of a sphere of radius r_2. Find the volume of the ring-shaped solid that remains.
(b) Express the volume in part (a) in terms of the height of the ring. Notice that the volume depends only on h, not on r_1 or r_2.

Evaluate the iterated integral by converting to polar coordinates.

32. $\displaystyle\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin(x^2 + y^2)\,dy\,dx$

33. $\displaystyle\int_0^a \int_{-\sqrt{a^2-y^2}}^{0} x^2 y\,dx\,dy$

34. $\displaystyle\int_0^1 \int_y^{\sqrt{2-y^2}} (x + y)\,dx\,dy$

35. $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2}\,dy\,dx$

36. A swimming pool is circular with a 40-ft diameter. The depth is constant along east-west lines and increases linearly from 2 ft at the south end to 7 ft at the north end. Find the volume of water in the pool.

37. An agricultural sprinkler distributes water in a circular pattern of radius 100 ft. It supplies water to a depth of e^{-r} feet per hour at a distance of r feet from the sprinkler.
(a) If $0 < R \le 100$, what is the total amount of water supplied per hour to the region inside the circle of radius R centered at the sprinkler?
(b) Determine an expression for the average amount of water per hour per square foot supplied to the region inside the circle of radius R.

38. Find the average value of the function $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$ on the annular region $a^2 \le x^2 + y^2 \le b^2$, where $0 < a < b$.

39. Let D be the disk with center at the origin and radius a. What is the average distance from points in D to the origin?

40. Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_1^{\sqrt{2}}\int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\,dy\,dx$$

into one double integral. Then evaluate the double integral.

41. (a) Here is a definition of an improper integral over the entire plane $\mathbb{R}^2$.

$$\begin{aligned} I &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx \\ &= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\,dA \end{aligned}$$

where D_a is the disk with radius a and center at the origin. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx = \pi$$

(b) An equivalent definition of the improper integral in part (a) is

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\,dA$$

where S_a is the square with vertices $(\pm a, \ \pm a)$. Use this definition to show that

$$\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \pi$$

(c) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

(d) Use the change of variable $t = \sqrt{2}x$, to show that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

This is a fundamental result in probability and statistics related to the normal distribution.

42. Use the result of Exercise 41 part (c) to evaluate the following integrals.

(a) $\displaystyle\int_0^{\infty} x^2 e^{-x^2}\, dx$ (b) $\displaystyle\int_0^{\infty} \sqrt{x}e^{-x}\, dx$

12.5 Applications of Double Integrals

We have already seen an application of double integrals: computing the volume of certain solids. Another geometric application is finding the areas of surfaces, and we will consider this problem in the next section. In this section, we explore physical applications such as computing mass, electric charge, center of mass, and moment of inertia. We will see that these physical concepts are also important when applied to probability density functions of two random variables.

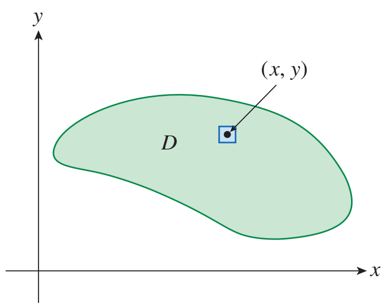


Figure 12.47
A small rectangle in the region D that contains (x, y).

Density and Mass

In Chapter 6, we were able to use single integrals to compute moments and the center of mass of a thin plate or lamina with constant density. Now, using a double integral, we can consider a lamina with variable density. Suppose the lamina occupies a region D of the xy-plane and its **density** (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D. This means that

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$

where Δm and ΔA are the mass and area of a small rectangle that contains (x, y) and the limit is taken as the dimensions of the rectangle approach 0. (See Figure 12.47.)

To find the total mass m of the lamina, divide a rectangle R containing D into subrectangles R_{ij} of the same size (as shown in Figure 12.48) and consider $\rho(x, y)$ to be 0 outside of D. If we choose a point (x_{ij}^*, y_{ij}^*) in R_{ij}, then the mass of the part of the lamina that occupies R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\, \Delta A$, where ΔA is the area of R_{ij}.

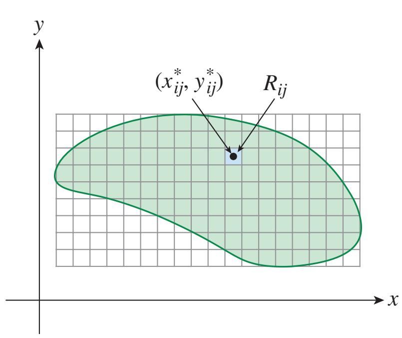


Figure 12.48
A point (x_{ij}^*, y_{ij}^*) in the subrectangle R_{ij}.

If we add all such masses, we get an approximation to the total mass:

$$m \approx \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A$$

If we increase the number of subrectangles, then the sum is a better approximation, and the total mass m of the lamina is the limiting value of these approximations:

$$m = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D \rho(x, y)\, dA \qquad (1)$$

Physicists also consider other types of density that can be treated in the same manner. For example, if an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ at a point (x, y) in D, then the total charge Q is given by

$$Q = \iint_D \sigma(x, y)\, dA \tag{2}$$

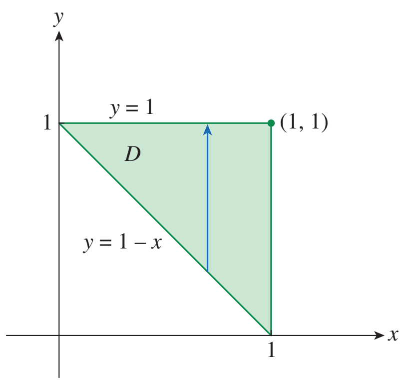


Figure 12.49
Charge is distributed over the region D.

Example 1 Find Total Charge by Integrating Charge Density

Charge is distributed over the triangular region D in Figure 12.49 so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter $(\mathrm{C/m^2})$. Find the total charge.

Solution

Use Equation 2 and Figure 12.49 to find the total charge.

$$Q = \iint_D \sigma(x, y)\, dA = \int_0^1 \int_{1-x}^1 xy\, dy\, dx \qquad \text{Equation 2; type I region.}$$

$$= \int_0^1 \left[x\frac{y^2}{2} \right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2}\left[1^2 - (1-x)^2\right] dx \qquad \text{Antiderivative; FTC.}$$

$$= \frac{1}{2}\int_0^1 (2x^2 - x^3)\, dx = \frac{1}{2}\left[\frac{2x^2}{3} - \frac{x^4}{4}\right]_0^1 = \frac{5}{24} \qquad \text{Antiderivative; FTC.}$$

The total charge is $\frac{5}{24}$ C. ■

Moments and Centers of Mass

In Section 6.6, we found the center of mass of a lamina with constant density; here we consider a lamina with variable density. Suppose the lamina occupies a region D and has density function $\rho(x, y)$. Recall from Chapter 6 that we defined the moment of a particle about an axis as the product of its mass and its directed distance from the axis.

Divide D into small rectangles as in Figure 12.48. Then the mass of R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\,\Delta A$, so we can approximate the moment of R_{ij} with respect to the x-axis by

$$[\rho(x_{ij}^*, y_{ij}^*)\Delta A]\, y_{ij}^*$$

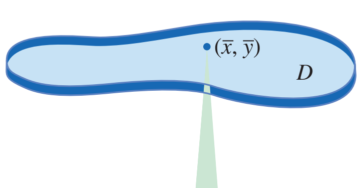


Figure 12.50
The center of mass of a lamina that occupies the region D.

If we add these quantities and take the limit as the number of subrectangles becomes large, we obtain the **moment** of the entire lamina **about the *x*-axis**:

$$M_x = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*))\, \Delta A = \iint_D y\rho(x, y)\, dA \tag{3}$$

Similarly, the **moment about the *y*-axis** is

$$M_y = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x\rho(x, y)\, dA \tag{4}$$

As before, we define the center of mass $(\bar{x}, \bar{y})$ so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass. Therefore, the lamina balances horizontally when supported at its center of mass (see Figure 12.50).

Center of Mass

The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)\, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)\, dA \tag{5}$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\, dA$$

Example 2 Center of Mass of a Nonuniform Triangle

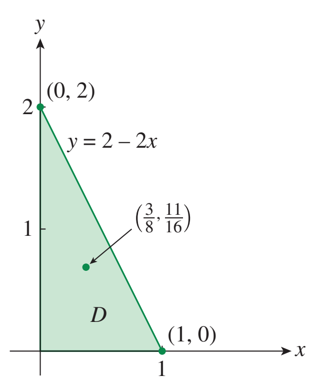


Figure 12.51
The lamina occupies the triangular region D.

Find the mass and center of mass of a triangular lamina with vertices (0, 0), (1, 0), and (0, 2) if the density function is $\rho(x, y) = 1 + 3x + y$.

Solution

The triangle is shown in Figure 12.51. Note that the equation of the upper boundary is $y = 2 - 2x$.

Use the definition to find the mass of the lamina.

$$m = \iint_D \rho(x, y)\, dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y)\, dy\, dx \qquad \text{Type I region.}$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx \qquad \text{Antiderivative.}$$

$$= 4\int_0^1 (1 - x^2)\, dx = 4\left[x - \frac{x^3}{3} \right]_0^1 = \frac{8}{3} \qquad \text{FTC; antiderivative; simplify.}$$

Use Equation 5 to find the center of mass.

$$\bar{x} = \frac{1}{m}\iint_D \rho(x, y)\, dA = \frac{3}{8}\int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\, dy\, dx \qquad \text{Equation 5; type I region.}$$

$$= \frac{3}{8}\int_0^1 \left[xy + 3x^2y + x\frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx \qquad \text{Antiderivative.}$$

$$= \frac{3}{2}\int_0^1 (x - x^3)\, dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{8} \qquad \text{FTC; antiderivative; simplify.}$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(\text{x, y})\text{dA} = \frac{3}{8}\int_0^1 \int_0^{2-2x} (y + 3xy + y^2)\, dy\, dx \qquad \text{Equation 5; type I region.}$$

$$= \frac{3}{8}\int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3}\right]_{y=0}^{y=2-2x}$$ Antiderivative.

$$= \frac{1}{4}\int_0^1 (7 - 9x - 3x^2 + 5x^3)\,dx$$ FTC; simplify.

$$= \frac{1}{4}\left[7x - 9\frac{x^2}{2} - x^3 + 5\frac{x^4}{4}\right]_0^1 = \frac{11}{16}$$ Antiderivative; FTC; simplify.

The center of mass is at the point $\left(\frac{3}{8}, \frac{11}{16}\right)$, illustrated in Figure 12.51. ■

Example 3 Center of Mass of a Semicircular Lamina

The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

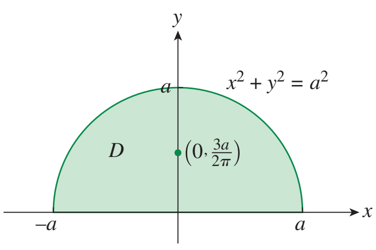


Figure 12.52
The lamina as the upper half of the circle.

Solution

For convenience, place the lamina as the upper half of the circle $x^2 + y^2 = a^2$, as shown in Figure 12.52.

The distance from a point (x, y) to the center of the circle (the origin) is $\sqrt{x^2 + y^2}$.

Therefore, the density function is $\rho(x, y) = K\sqrt{x^2 + y^2}$.
Find the mass of the lamina.

$$m = \iint_D \rho(x, y)\,dA = \iint_D K\sqrt{x^2 + y^2}\,dA$$ Definition of mass.

$$= \int_0^{\pi}\int_0^a (Kr)r\,dr\,d\theta$$ D is a polar region.

$$= K\int_0^{\pi} d\theta \int_0^a r^2\,dr$$ Double integral can be written as the product of two single integrals.

$$= \left[K\pi\frac{r^3}{3}\right]_0^a = \frac{K\pi a^3}{3}$$ Antiderivatives; FTC; simplify.

Both the lamina and the density function are symmetric with respect to the y-axis, so the center of mass must lie on the y-axis, that is, $\bar{x} = 0$.

The y-coordinate is given by

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\,dA = \frac{3}{K\pi a^3}\int_0^{\pi}\int_0^a r\,\sin\,\theta(Kr)\,r\,dr\,d\theta$$ Equation 5; D is a polar region.

$$= \frac{3}{\pi a^3}\int_0^{\pi}\sin\,\theta\,d\theta\int_0^a r^3 dr$$ Double integral can be written as the product of two single integrals.

$$= \frac{3}{\pi a^3}\Big[-\cos\,\theta\Big]_0^{\pi}\left[\frac{r^4}{4}\right]_0^a$$ Antiderivatives.

$$= \frac{3}{\pi a^3}\cdot 2\cdot\frac{a^4}{4} = \frac{3a}{2\pi}$$ FTC; simplify.

Compare the location of the center of mass in Example 3 with Example 7 in Section 6.6, where we found the center of mass of a lamina with the same shape but uniform density is located at the point $\left(0, \frac{4a}{3\pi}\right)$.

Therefore, the center of mass is located at the point $\left(0, \frac{3a}{2\pi}\right)$. ■

Moment of Inertia

The **moment of inertia** (also called the **second moment**) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. We can extend this concept to a lamina with density function $\rho(x, y)$ occupying a region D by proceeding as we did for ordinary moments. Divide D into small rectangles, approximate the moment of inertia of each subrectangle about the x-axis, and take the limit of the sum as the number of rectangles becomes large. The result is the **moment of inertia** of the lamina **about the *x*-axis**:

$$I_x = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \, \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint\limits_D y^2 \rho(x, y) \, dA \tag{6}$$

Similarly, the **moment of inertia about the *y*-axis** is

$$I_y = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \, \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint\limits_D x^2 \rho(x, y) \, dA \tag{7}$$

It is also of interest to consider the **moment of inertia about the origin**, also called the **polar moment of inertia**:

$$I_0 = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} [(x_{ij}^*)^2 + (y_{ij}^*)^2] \, \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint\limits_D (x^2 + y^2) \rho(x, y) \, dA \tag{8}$$

Example 4 Moments of Inertia of a Uniform Disk

Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center at the origin, and radius a.

Solution

The boundary of D is the circle $x^2 + y^2 = a^2$.

In polar coordinates, D is described by $0 \le \theta \le 2\pi, 0 \le r \le a$.

Compute I_0 first.

$$I_0 = \iint\limits_D (x^2 + y^2) \rho \, dA = \rho \int_0^{2\pi} \int_0^a r^2 \, r \, dr \, d\theta$$ Definition of I_0; D is a polar region.

$$= \rho \int_0^{2\pi} d\theta \int_0^a r^3 \, dr$$ Double integral can be written as the product of two single integrals.

$$= \rho \cdot 2\pi \left[\frac{r^4}{4} \right]_0^a = \frac{\pi \rho a^4}{2}$$ Antiderivatives; FTC; simplify.

Instead of computing I_x and I_y directly, use the facts that $I_x + I_y = I_0$ and $I_x = I_y$ (from the symmetry in the problem).

Therefore, $I_x = I_y = \dfrac{I_0}{2} = \dfrac{\pi \rho a^4}{4}$. ■

In Example 4, notice that the mass of the disk is

$$m = \text{density} \times \text{area} = \rho(\pi a^2)$$

so the moment of inertia of the disk about the origin (like a wheel about its axle) can be written as

$$I_0 = \frac{\pi \rho a^4}{2} = \frac{1}{2} (\rho \pi a^2) a^2 = \frac{1}{2} m a^2$$

Thus, if we increase the mass or the radius of the disk, we thereby increase the moment of inertia. In general, the moment of inertia plays much the same role in rotational motion that mass plays in linear motion. The moment of inertia of a wheel is what makes it difficult to start or stop the rotation of the wheel, just as the mass of a car is what makes it difficult to start or stop the motion of the car.

Probability

A *probability density function* f of a continuous random variable X completely describes the random variable. A valid probability density function f is defined such that $f(x) \geq 0$ for all x, $\int_{-\infty}^{\infty} f(x)\,dx = 1$, and the probability that X takes on a value between a and b is the definite integral from a to b of f, that is,

$$\mathrm{P}(a \leq X \leq b) = \int_a^b f(x)\,dx$$

We can now consider a pair of continuous random variables X and Y that are jointly distributed, for example, the lifetimes of two components of a machine, the height and weight of an adult female chosen at random, or the wind speed and barometric pressure of a hurricane. The **joint density function** of X and Y is a function f of two variables such that the probability that (X, Y) lies in a region D is

$$\mathrm{P}((X, Y) \in D) = \iint_D f(x, y)\,dA$$

In particular, if the region is a rectangle, the probability that X lies between a and b, and Y lies between c and d is

$$\mathrm{P}(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x, y)\,dA$$

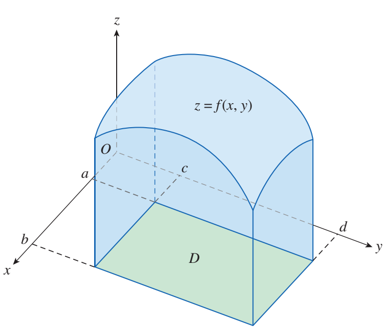


Figure 12.53
The expression for the probability that X lies between a and b and Y lies between c and d can be interpreted as the volume of a solid.

This double integral expression can be interpreted as the volume of a solid that lies above the rectangle $D = [a, b] \times [c, d]$ and below the graph of the joint density function, as illustrated in Figure 12.53.

The probability of any event must be positive and must also be a number between 0 and 1 (inclusive). Therefore, a joint density function f has the following properties:

$$f(x, y) \geq 0 \qquad \iint_{\mathbb{R}^2} f(x, y)\,dA = 1$$

As in Section 12.4, the double integral over $\mathbb{R}^2$ is an improper integral defined as the limit of double integrals over expanding circles or squares, and we can write

$$\iint_{\mathbb{R}^2} f(x, y)\,dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\,dx\,dy = 1$$

Example 5 Valid Joint Density Function

If the joint density function of X and Y is given by

$$f(x, y) = \begin{cases} C(x + 2y) & \text{if } 0 \leq x \leq 10, 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

find the value of the constant C. Then find $\mathrm{P}(X \leq 7, Y \geq 2)$.

Solution

The value of C must ensure that the double integral of f is equal to 1.

Because $f(x, y) = 0$ outside the rectangle $D = [0, 10] \times [0, 10]$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dy\,dx = \int_0^{10}\int_0^{10} C(x + 2y)\,dy\,dx \qquad f(x, y) = 0 \text{ outside the rectangle } D.$$

$$= C\int_0^{10}\Big[xy + y^2\Big]_{y=0}^{y=10} dx \qquad \text{Antiderivative.}$$

$$= C\int_0^{10}(10x + 100)\,dx = C\Big[5x^2 + 100x\Big]_0^{10} \qquad \text{FTC; antiderivative.}$$

$$= C(5 \cdot 100 + 100 \cdot 10) = 1500C \qquad \text{FTC.}$$

Therefore, $1500C = 1 \;\Rightarrow\; C = \dfrac{1}{1500}$.

Compute the probability that X is at most 7 and Y is at least 2.

$$\mathrm{P}(X \le 7, Y \ge 2) = \int_{-\infty}^{7}\int_2^{\infty} f(x, y)\,dy\,dx \qquad \text{Probability expression by definition.}$$

$$= \int_0^{7}\int_2^{10} \frac{1}{1500}(x + 2y)\,dy\,dx \qquad f(x, y) = 0 \text{ outside } D; \text{ expression for } f(x, y).$$

$$= \frac{1}{1500}\int_0^{7}\Big[xy + y^2\Big]_{y=2}^{y=10} dx \qquad \text{Antiderivative.}$$

$$= \frac{1}{1500}\int_0^{7}(8x + 96)\,dx \qquad \text{FTC; simplify.}$$

$$= \frac{1}{1500}\big[4x^2 + 96x\big]_0^7 = \frac{868}{1500} \approx 0.5787 \qquad \text{Antiderivative; FTC.}$$

■

Suppose X is a random variable with probability density function $f_1(x)$ and Y is a random variable with density function $f_2(y)$. Then X and Y are called **independent random variables** if their joint density function is the product of their individual density functions:

$$f(x, y) = f_1(x)\,f_2(y)$$

Waiting times are often modeled using the exponential density function:

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mu^{-1}e^{-t/\mu} & \text{if } t \ge 0 \end{cases}$$

where μ is the mean waiting time. In the next example, we consider a situation with two independent waiting times.

Example 6 Waiting Times at the Theater

The manager of a movie theater determines that the mean time moviegoers wait in line to buy a ticket for this week's film is 10 minutes and the mean time they wait to buy popcorn is 5 minutes. Assuming that the waiting times are independent, find the probability that a moviegoer waits a total of less than 20 minutes before taking their seat.

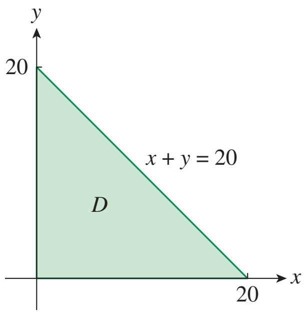


Figure 12.54
We need to find $P((X, Y) \in D)$.

Solution

Assume that both the waiting time X for the ticket purchase and the waiting time Y in the refreshment line are modeled by exponential probability density functions.

Use the given means to write each density function.

$$f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{1}{10} e^{-x/10} & \text{if } x \geq 0 \end{cases} \qquad f_2(y) = \begin{cases} 0 & \text{if } y < 0 \\ \dfrac{1}{5} e^{-y/5} & \text{if } y \geq 0 \end{cases}$$

Since X and Y are independent, the joint density function is the product:

$$f(x, y) = f_1(x) f_2(y) = \begin{cases} \dfrac{1}{50} e^{-x/10} e^{-y/5} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We need to find the probability that $X + Y < 20$:

$P(X + Y < 20) = P((X, Y) \in D)$ where D is the triangular region shown in Figure 12.54.

$$P(X + Y < 20) = \iint_D f(x, y)\, dA = \int_0^{20} \int_0^{20-x} \frac{1}{50} e^{-x/10} e^{-y/5}\, dy\, dx$$

Probability computation; type I region.

$$= \frac{1}{50} \int_0^{20} \left[e^{-x/10} (-5) e^{-y/5} \right]_{y=0}^{y=20-x} dx \qquad \text{Antiderivative.}$$

$$= \frac{1}{10} \int_0^{20} e^{-x/10} (1 - e^{(x-20)/5})\, dx \qquad \text{FTC.}$$

$$= \frac{1}{10} \int_0^{20} (e^{-x/10} - e^{-4} e^{x/10})\, dx \qquad \text{Distribute.}$$

$$= \frac{1}{10} \left[-10 e^{-x/10} - e^{-4} (10) e^{x/10} \right]_0^{20} \qquad \text{Antiderivative.}$$

$$= 1 + e^{-4} - 2e^{-2} \approx 0.7476 \qquad \text{FTC; simplify.}$$

Therefore, approximately 75% of the moviegoers wait less than 20 minutes before taking their seats.

Expected Values

If X is a continuous random variable with probability density function f, then the **mean**, or **expected value**, of X is

$$\mu = \int_{-\infty}^{\infty} x f(x), dx$$

If X and Y are random variables with joint density function f, we define the ***X*-mean** and ***Y*-mean**, also called the **expected values** of X and Y, to be

$$\mu_1 = \iint_{\mathbb{R}^2} x f(x, y)\, dA \qquad \mu_2 = \iint_{\mathbb{R}^2} y f(x, y)\, dA \tag{9}$$

Notice how closely the expressions for μ_1 and μ_2 in Equation 9 look like the moments M_x and M_y of a lamina with density functions ρ in Equations 3 and 4. In fact, we can think of probability as being like continuously distributed mass. We calculate probability the way we calculate mass—by integrating a density function. And because the total *probability mass* is 1, the expressions for $\bar{x}$ and $\bar{y}$ in Equation 5 show that we can think of the expected values of X and Y, μ_1 and μ_2, as the coordinates of the *center of mass* of the probability distribution.

We will consider normal distributions in the next example. A single random variable X is *normally distributed* if its probability density function is for the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}$$

where μ is the mean and σ is the standard deviation.

Example 7 Jointly Distributed Normal Random Variables

A factory produces (cylindrically shaped) roller bearings that are sold as having diameter 4.0 cm and length 6.0 cm. In fact, the diameters X are normally distributed with mean 4.0 cm and standard deviation 0.01 cm while the lengths Y are normally distributed with mean 6.0 cm and standard deviation 0.01 cm. Assuming that X and Y are independent, write the joint density function and graph it. Find the probability that a bearing randomly chosen from the production line has either length or diameter that differs from the mean by more than 0.02 cm.

Solution

We are given that X and Y are normally distributed with $\mu_1 = 4.0$, $\mu_2 = 6.0$, and $\sigma_1 = \sigma_2 = 0.01$.

The individual density functions for X and Y are

$$f_1(x) = \frac{1}{0.01\sqrt{2\pi}}\, e^{-(x-4)^2/0.0002} \qquad f_2(y) = \frac{1}{0.01\sqrt{2\pi}}\, e^{-(y-6)^2/0.0002}$$

Since X and Y are independent, the joint density function is the product:

$$\begin{aligned} f(x, y) = f_1(x)\, f_2(y) &= \frac{1}{0.0002\pi}\, e^{-(x-4)^2/0.0002}\, e^{-(y-6)^2/0.0002} \\ &= \frac{5000}{\pi}\, e^{-5000[(x-4)^2+(y-6)^2]} \end{aligned}$$

A graph of this function is shown in Figure 12.55.

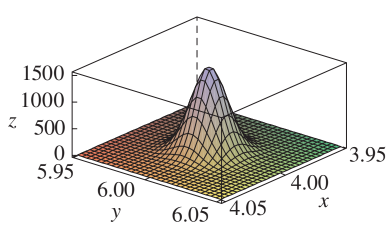


Figure 12.55
Graph of the bivariate normal joint probability density function in Example 7.

We'll use technology to help find the probability that both X and Y differ from their means by less than 0.02 cm. First find the probability that both X and Y are within 0.02 cm of their means, and then use the complement rule.

$$\begin{aligned} P(3.98 < X < 4.02,\, 5.98 < Y < 6.02) &= \int_{3.98}^{4.02}\int_{5.98}^{6.02} f(x, y)\, dy\, dx \\ &= \frac{5000}{\pi}\int_{3.98}^{4.02}\int_{5.98}^{6.02} e^{-5000[(x-4)^2+(y-6)^2]}\, dy\, dx \\ &\approx 0.9111 \end{aligned}$$

Therefore, the probability that either X or Y differs from its mean by more than 0.02 cm is $1 - 0.9111 \approx 0.0889$. ■

12.5 Exercises

1. Electric charge is distributed over the rectangle $1 \le x \le 3$, $0 \le y \le 2$ so that the charge density at (x, y) is $\sigma(x, y) = 2xy + y^2$ (measured in coulombs per square meter). Find the total charge on the rectangle.

2. Electric charge is distributed over the disk $x^2 + y^2 \le 4$ so that the charge density at (x, y) is $\sigma(x, y) = x + y + x^2 + y^2$ (measured in coulombs per square meter). Find the total charge on the disk.

Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ.

3. $D = \{(x, y) \mid 0 \le x \le 2, -1 \le y \le 1\}$; $\rho(x, y) = xy^2$

4. $D = \{(x, y) \mid 0 \le x \le a, 0 \le y \le b\}$; $\rho(x, y) = cxy$

5. D is the triangular region with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

6. D is the triangular region enclosed by the lines $x = 0$, $y = x$, and $2x + y = 6$; $\rho(x, y) = x^2$

7. D is bounded by $y = e^x$, $x = 0$, and $x = 1$; $\rho(x, y) = y$

8. D is bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$; $\rho(x, y) = x$

9. $D = \left\{(x, y) \mid 0 \le y \le \sin \dfrac{\pi x}{L}, 0 \le x \le L\right\}$; $\rho(x, y) = y$

10. D is bounded by the parabolas $y = x^2$ and $x = y^2$; $\rho(x, y) = \sqrt{x}$

11. A lamina occupies the part of the disk $x^2 + y^2 \le 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis.

12. Find the center of mass of the lamina in Exercise 11 if the density at any point is proportional to the square of its distance from the origin.

13. The boundary of a lamina consists of the semicircles $y = \sqrt{1 - x^2}$ and $y = \sqrt{4 - x^2}$ together with the portions of the x-axis that join them. Find the center of mass of the lamina if the density at any point is proportional to its distance from the origin.

14. Find the center of mass of the lamina in Exercise 13 if the density at any point is inversely proportional to its distance from the origin.

15. Find the center of mass of a lamina in the shape of an isosceles right triangle with equal sides of length a if the density at any point is proportional to the square of the distance from the vertex opposite the hypotenuse.

16. A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ but outside the circle $x^2 + y^2 = 1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

17. Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 7.

18. Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 12.

19. Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 15.

20. Consider a square fan blade with sides of length 2 and the lower left corner placed at the origin. If the density of the blade is $\rho(x, y) = 1 + 0.1x$, it is more difficult to rotate the blade about the x-axis or the y-axis? Justify your answer.

Use technology to find the mass, center of mass, and moments of inertia of the lamina that occupies the region D and has the given density function.

21. $D = \{(x, y) \mid 0 \le y \le \sin x, 0 \le x \le \pi\}$; $\rho(x, y) = xy$

22. D is enclosed by the cardioid $r = 1 + \cos\theta$; $\rho(x, y) = \sqrt{x^2 + y^2}$

23. The joint density function for a pair of random variables X and Y is

$$f(x, y) = \begin{cases} Cx(1 + y) & \text{if } 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant C.
(b) Find $P(X \le 1, Y \le 1)$.
(c) Find $P(X + Y \le 1)$.

24. (a) Verify that

$$f(x, y) = \begin{cases} 4xy & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

is a valid joint density function.

(b) If X and Y are random variables whose joint density function is the function f in part (a), find

(i) $P\left(X \ge \frac{1}{2}\right)$ (ii) $P\left(X \ge \frac{1}{2}, Y \le \frac{1}{2}\right)$

(c) Find the expected values of X and Y.

25. Suppose X and Y are random variables with joint density function

$$f(x, y) = \begin{cases} 0.1e^{-(0.5x + 0.2y)} & \text{if } x \ge 0, y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that f is indeed a valid joint density function.
(b) Find the following probabilities.
(i) $P(Y \geq 1)$ (ii) $P(X \leq 2, Y \leq 4)$
(c) Find the expected values of X and Y.

26. (a) A lamp has two bulbs of a type with an average lifetime of 1000 hours. Assuming that we can model the probability of failure of these bulbs by an exponential density function with mean $\mu = 1000$, find the probability that both of the lamp's bulbs fail within 1000 hours.
(b) Another lamp has just one bulb of the same type as in part (a). If one bulb burns out and is replaced by a bulb of the same type, find the probability that the two bulbs fail within a total of 1000 hours.

27. Suppose X and Y are independent random variables, where X is normally distributed with mean 45 and standard deviation 0.5 and Y is normally distributed with mean 20 and standard deviation 0.1.
(a) Find $P(40 \leq X \leq 50, 20 \leq Y \leq 25)$.
(b) Find $P(4(X - 45)^2 + 100(Y - 20)^2 \leq 2)$.

28. Two students both have classes that end at noon and they agree to meet every day after class. They arrive at the coffee shop independently. One student's arrival time is X and the other's arrival time is Y, where X and Y are measured in minutes after noon. The individual density functions are

$$f_1(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f_2(y) = \begin{cases} \frac{1}{50}y & \text{if } 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

(The first person arrives sometime after noon and is more likely to arrive promptly than late. The second person always arrives by 12:10 PM and is more likely to arrive late than promptly.) After the second person arrives, they will wait for up to half an hour for the first person, but the first person won't wait for the second. Find the probability that they meet.

29. When studying the spread of an epidemic, we assume that the probability that an infected individual will spread the disease to an uninfected individual is a function of the distance between them. Consider a circular city of radius 10 miles in which the population is uniformly distributed. For an uninfected individual at a fixed point $A(x_0, y_0)$, assume that the probability function is given by

$$f(P) = \frac{1}{20}[20 - d(P, A)]$$

where $d(P, A)$ denotes the distance between P and A.
(a) Suppose the exposure of a person to the disease is the sum of the probabilities of catching the disease from all members of the population. Assume that the infected people are uniformly distributed throughout the city, with k infected individuals per square mile. Find a double integral that represents the exposure of a person residing at A.
(b) Evaluate the integral for the case in which A is the center of the city and for the case in which A is located on the edge of the city. Where would you prefer to live?

12.6 Surface Area

In this section, we apply double integrals to the problem of computing the area of a surface. We'll start by finding a formula for the area of a parametric surface and then, as a special case, we will find a formula for the surface area of the graph of a function of two variables.

Area of a Parametric Surface

Recall from Section 10.5 that a parametric surface S is defined by a vector-valued function of two parameters

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \tag{1}$$

or, equivalently, by parametric equations

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

where (u, v) varies throughout a region D in the uv-plane.

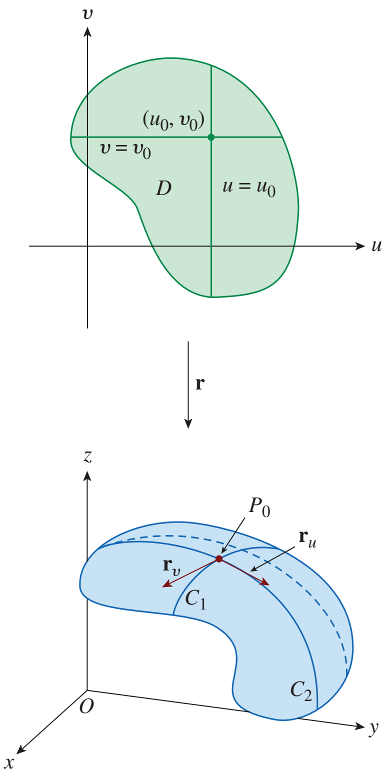


Figure 12.56
The grid curves C_1 and C_2, and the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$.

We will find the area of S by dividing S into patches and approximating the area of each patch by the area of a piece of a tangent plane. So, first let's recall from Section 11.4 how to find tangent planes to parametric surfaces.

Let P_0 be a point on S with position vector $\mathbf{r}(u_0, v_0)$. If we keep u constant by letting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1 lying on S. (See Figure 12.56.) The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v:

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\,\mathbf{k} \tag{2}$$

Similarly, if we keep v constant by letting $v = v_0$, we obtain a grid curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S, and its tangent vector at P_0 is

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\mathbf{k} \tag{3}$$

If the **normal vector** $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface S is called **smooth**. (It has no *corners*.) In this case, the tangent plane to S at P_0 exists and can be found using the normal vector.

Now let's consider the surface area of a general parametric surface given by Equation 1. For simplicity, start by considering a surface whose parameter domain D is a rectangle, and divide D into subrectangles R_{ij}. Choose (u_i^*, v_j^*) to be the lower left corner of R_{ij}, as shown in Figure 12.57. The part S_{ij} of the surface S that corresponds to R_{ij} is called a *patch* and has the point P_{ij} with position vector $\mathbf{r}(u_i^*, v_j^*)$ as one of its corners. Let

$$\mathbf{r}_u^* = \mathbf{r}_u(u_i^*, v_j^*) \quad \text{and} \quad \mathbf{r}_v^* = \mathbf{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij} as given by Equations 3 and 2.

Figure 12.58(a) shows how the two edges of the patch that meet at P_{ij} can be approximated by vectors. These vectors, in turn, can be approximated by the vectors $\Delta u\ \mathbf{r}_u^*$ and $\Delta v\ \mathbf{r}_v^*$ because partial derivatives can be approximated by difference quotients. So, we approximate S_{ij} by the parallelogram determined by the vectors $\Delta u\ \mathbf{r}_u^*$ and $\Delta v\ \mathbf{r}_v^*$. As shown in Figure 12.58(b), this parallelogram lies in the tangent plane to S at P_{ij}.

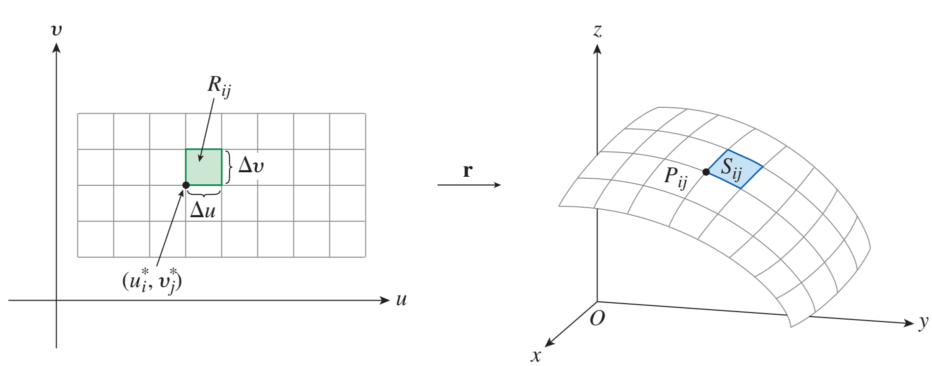
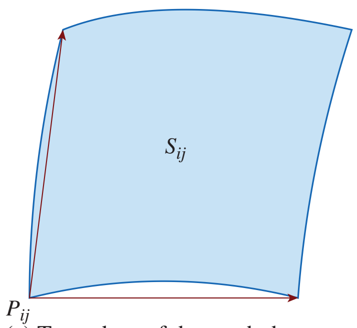
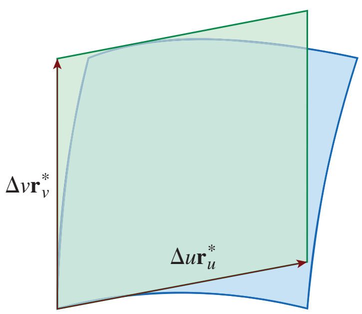


Figure 12.57
The image of the subrectangle R_{ij} is the patch S_{ij}.



(a) Two edges of the patch that meet at P_{ij} approximated by vectors.



(b) Approximate the patch by the parallelogram.

Figure 12.58
Approximating a patch by a parallelogram.

The area of this parallelogram is

$$|(\Delta u\ \mathbf{r}_u^*) \times (\Delta v\ \mathbf{r}_v^*)| = |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u\ \Delta v$$

Therefore, an approximation to the area of S is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*|\ \Delta u\ \Delta v$$

It seems reasonable that this approximation gets better as we increase the number of subrectangles, and we recognize the double sum as a Riemann sum for the double integral $\iint_D |\mathbf{r}_u \times \mathbf{r}_v| du\ dv$. This suggests the following definition.

Definition • Surface Area

If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\ dA$$

where $\mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$

Example 1 Area of a Parametric Surface

Find the surface area of a sphere of radius a.

Solution

In Example 4 in Section 10.5, we found the parametric representation

$$x = a\ \sin\phi\ \cos\theta \qquad y = a\ \sin\phi\ \sin\theta \qquad z = a\ \cos\phi$$

where the parameter domain is $D = \{(\phi, \theta) | 0 \le \phi \le \pi, 0 \le \theta \le 2\pi\}$

Compute the cross product of the tangent vectors:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi\ \cos\theta\ \mathbf{i} + a^2\sin^2\phi\sin\theta\ \mathbf{j} + a^2\sin\phi\cos\phi\ \mathbf{k}$$

Find the magnitude of this product.

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi} = a^2\sqrt{\sin^2\phi} = a^2\sin\phi$$

since $\sin \phi \geq 0$ for $0 \leq \phi \leq \pi$.

Therefore, using the definition of surface area, the area of the sphere is

$$A = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, dA = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi \, d\phi \, d\theta$$

$$= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi = a^2(2\pi)2 = 4\pi a^2 \qquad \blacksquare$$

Surface Area of a Graph

For the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, we consider x and y as parameters. The parametric equations are

$$x = x \quad y = y \quad z = f(x, y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \quad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{df}{dy}\mathbf{j} + \mathbf{k} \tag{4}$$

Notice the similarity between the surface area formula in Equation 5 and the arc length formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

from Section 6.4.

Therefore, the surface area formula becomes

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \tag{5}$$

Example 2 Surface Area of a Graph

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution

The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$.
Therefore, the given surface lies above the disk D with center at the origin and radius 3. See Figure 12.59.

Use Equation 6 to find the surface area.

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + 2(x)^2 + 2(y)^2}\, dA$$

$$= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA$$

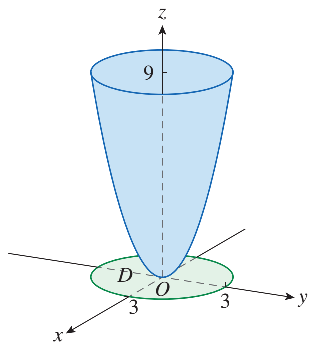


Figure 12.59
The surface lies above the disk D.

Convert to polar coordinates to evaluate this double integral.

$$A = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\; r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1 + 4r^2}\, dr$$

$$= 2\pi\left(\frac{1}{8}\right)\frac{2}{3}(1 + 4r^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}(37\sqrt{37} - 1)$$ ■

A common type of surface is a **surface of revolution** S obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis, where $f(x) \ge 0$ and f' is continuous. In Exercise 23 you are asked to use a parametric representation of S and the definition of surface area to prove the following formula for the area of a surface of revolution:

$$A = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx \tag{6}$$

12.6 Exercises

Find the area of the surface.

1. The part of the plane $x + 2y + 3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$

2. The part of the plane $2x - 5y + z = 10$ that lies above the triangle with vertices $(0, 0)$, $(0, 6)$, and $(4, 0)$

3. The part of the plane $3x + 2y + z = 6$ that lies in the first octant

4. The part of the plane with vector equation $\mathbf{r}(u, v) = \langle u + v, 2 - 3u, 1 + u - v\rangle$ that is given by $0 \le u \le 2$, $-1 \le v \le 1$

5. The part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$

6. The part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0)$, $(0, 1)$, and $(2, 1)$

7. The surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \le u \le 1$, $0 \le v \le 2$

8. The helicoid (or spiral ramp) with vector equation $\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + v\, \mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi$

9. The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$

10. The part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$

11. The part of the sphere $x^2 + y^2 + z^2 = b^2$ that lies inside the cylinder $x^2 + y^2 = a^2$, where $0 < a < b$

12. The surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \le x \le 1$, $0 \le y \le 1$

Express the area in terms of a single integral and use technology to find the indicated surface area.

13. The part of the surface $z = e^{-x^2-y^2}$ that lies above the disk $x^2 + y^2 \le 4$

14. The part of the surface $z = \cos(x^2 + y^2)$ that lies inside the cylinder $x^2 + y^2 = 1$

15. (a) Use the Midpoint Rule for double integrals (see Section 12.1) with six squares to estimate the area of the surface $z = \dfrac{1}{1 + x^2 + y^2}$, $0 \le x \le 6$, $0 \le y \le 4$.
(b) Use technology to approximate the surface area in part (a). Compare your answer to part (a).

16. (a) Use the Midpoint Rule for double integrals with $m = n = 2$ to estimate the area of the surface $z = xy + x^2 + y^2$, $0 \le x \le 2$, $0 \le y \le 2$.
(b) Use technology to approximate the surface area in part (a). Compare your answer to part (a).

17. Use technology to find the area of the surface with vector equation $\mathbf{r}(u, v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v\rangle$, $0 \le u \le \pi$, $0 \le v \le 2\pi$.

18. Use technology to find the area of the part of the surface $z = \dfrac{1 + x^2}{1 + y^2}$ that lies above the square $|x| + |y| \le 1$. Illustrate your answer by graphing this part of the surface.

19. Use technology to find the exact area of the surface $z = 1 + 2x + 3y + 4y^2$, $1 \le x \le 4$, $0 \le y \le 1$.

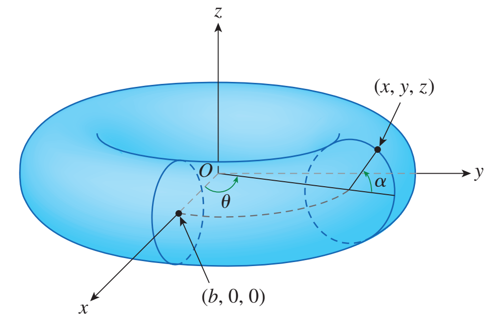
20. (a) Set up, but do not evaluate, a double integral for the area of the surface with parametric equations $x = au\cos v$, $y = bu\sin v$, $z = u^2$, $0 \le u \le 2$, $0 \le v \le 2\pi$.
(b) Eliminate the parameters to show that the surface is an elliptic paraboloid and set up another double integral for the surface area.
(c) Use the parametric equations in part (a) with $a = 2$ and $b = 3$ to graph the surface.
(d) For the case $a = 2$, $b = 3$, use technology to find the surface area.

21. (a) Show that the parametric equations $x = a\sin u\cos v$, $y = b\sin u\sin v$, $z = c\cos u$, $0 \le u \le \pi$, $0 \le v \le 2\pi$, represent an ellipsoid.
(b) Use the parametric equations in part (a) to graph the ellipsoid for the case $a = 1$, $b = 2$, $c = 3$.
(c) Set up, but do not evaluate, a double integral for the surface area of the ellipsoid in part (b).

22. The figure shows the torus obtained by rotating about the z-axis the circle in the xz-plane with center $(b, 0, 0)$ and radius $a < b$. Parametric equations for the torus are

$$x = b\cos\theta + a\cos\alpha\cos\theta$$
$$y = b\sin\theta + a\cos\alpha\sin\theta$$
$$z = a\sin\alpha$$

where θ and α are the angles shown in the figure. Find the surface area of the torus.



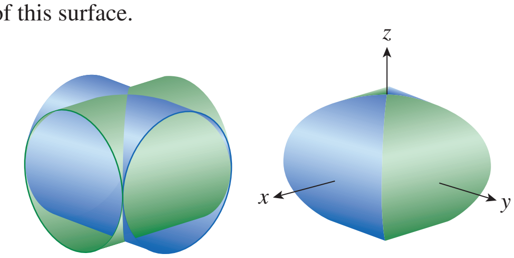
23. Use the definition of surface area and the parametric equations for a surface of revolution (see Section 10.5) to derive Equation 6.

Use Equation 7 to find the area of the surface obtained by rotating the given curve about the x-axis.

24. $y = x^3$, $0 \le x \le 2$

25. $y = \sqrt{1 + 4x}$, $1 \le x \le 5$

26. The figure shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.



27. Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

12.7 Triple Integrals

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, we can also define triple integrals for functions of three variables.

Triple Integrals over Rectangular Boxes

Let's start with the simplest case, where f is defined on a rectangular box:

$$B = \{(x, y, z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\} \tag{1}$$

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz. The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

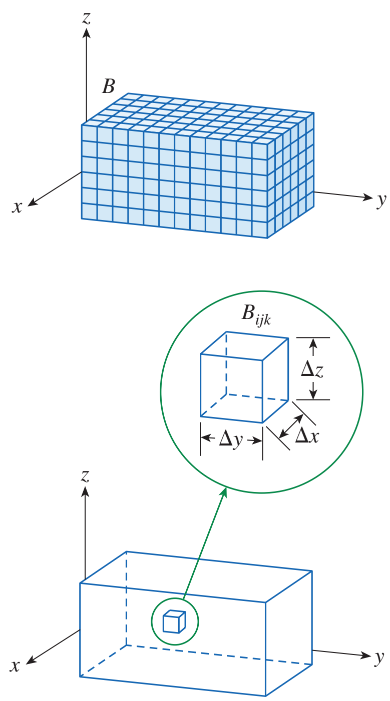


Figure 12.60
The box B divided into lmn sub-boxes.

which are shown in Figure 12.60. Each sub-box has volume $\Delta V = \Delta x\ \Delta y\ \Delta z$.

Then we form the **triple Riemann sum**

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*})\ \Delta V \tag{2}$$

where the sample point $(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*})$ is in B_{ijk}. By analogy with the definition of a double integral (12.1.5), we define the triple integral as the limit of the triple Riemann sums in Equation 2.

Definition • Triple Integral

The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\ dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*})\ \Delta V$$

if this limit exists.

Again, the triple integral always exists if f is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i, y_j, z_k), we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x, y, z)\ dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\,\Delta V$$

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

Fubini's Theorem for Triple Integrals

If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\ dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\ dx\ dy\ dz \tag{3}$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z. There are five other possible orders in which we can integrate, all of which give the same value. For instance, if we integrate with respect to y, then z, and then x, we have

$$\iiint_B f(x, y, z)\ dV = \int_a^b \int_r^s \int_c^d f(x, y, z)\ dy\ dz\ dx$$

Example 1 Triple Integral over a Box

Evaluate the triple integral $\iiint_B xyz^2\ dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$$

Solution

We could use any of the six possible orders of integration.
Let's integrate with respect to x, then y, and then z.

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz \qquad \text{Fubini's Theorem.}$$

$$= \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\, dz \qquad \text{Antiderivative.}$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz = \int_0^3 \left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz \qquad \text{FTC; antiderivative.}$$

$$= \int_0^3 \frac{3z^2}{4}\, dz = \frac{z^3}{4}\bigg]_0^3 = \frac{27}{4} \qquad \text{FTC; antiderivative; simplify.}$$

Triple Integrals over General Regions

We can now define the **triple integral over a general bounded region E** in three dimensional space (a solid) by much the same procedure that we used for double integrals (Section 12.3). First, enclose E in a box B of the type given by Equation 1. Then define a function F so that it agrees with f on E but is 0 for points in B that are outside E. By definition,

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

This integral exists if f is continuous and the boundary of E is *reasonably smooth*. The triple integral has essentially the same properties as the double integral (Properties 6–9 in Section 12.3).

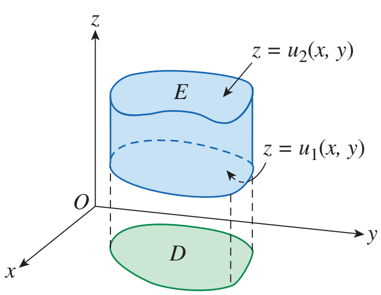


Figure 12.61
A type I solid region.

We restrict our attention to continuous functions f and to certain simple types of regions. A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\} \tag{4}$$

where D is the projection of E onto the xy-plane as shown in Figure 12.61. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

By the same sort of argument that led to Equation 12.3.3, it can be shown that if E is a type 1 region given by Equation 4, then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\right] dA \tag{5}$$

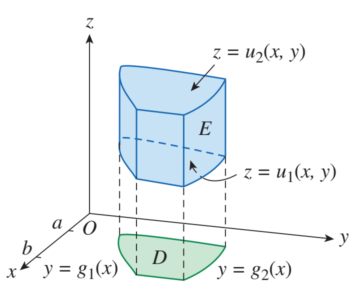


Figure 12.62
A type 1 solid region where the projection D is a type I plane region.

The meaning of the inner integral on the right side of Equation 5 is that x and y are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are treated as constants, while $f(x, y, z)$ is integrated with respect to z.

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 12.62), then

$$E = \{(x, y, z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x), u_1(x, y) \le z \le u_2(x, y)\}$$

and Equation 5 becomes

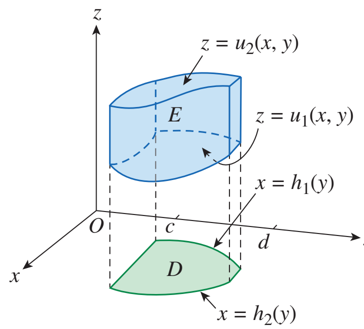


Figure 12.63
A type 1 solid region with a type II projection.

$$\iiint_E f(x, y, z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\,dz\,dy\,dx \tag{6}$$

If, on the other hand, D is a type II plane region (as in Figure 12.63), then

$$E = \{(x, y, z) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y),\ u_1(x, y) \le z \le u_2(x, y)\}$$

and Equation 6 becomes

$$\iiint_E f(x, y, z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\,dz\,dx\,dy \tag{7}$$

Example 2 Triple Integral over a Tetrahedron

Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

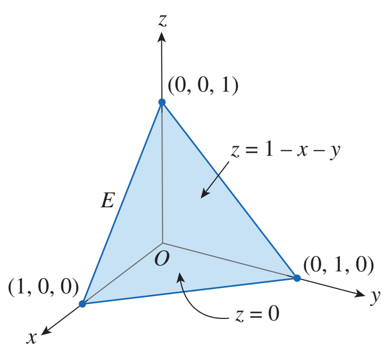


Figure 12.64
The solid region E.

Solution

When setting up a triple integral, it is often a good idea to draw two diagrams: one of the solid region E (see Figure 12.64) and one of its projection onto the xy-plane (see Figure 12.65).

The lower boundary of the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$ (or $z = 1 - x - y$), so we use $u_1(x, y) = 0$ and $u_2(x, y) = 1 - x - y$ in Equation 7.

Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ (or $y = 1 - x$) in the xy-plane.

Therefore, the projection of E is the triangular region shown in Figure 12.65, and we have

$$E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}. \tag{8}$$

Using this description of E as a type 1 region, we can evaluate the integral as follows:

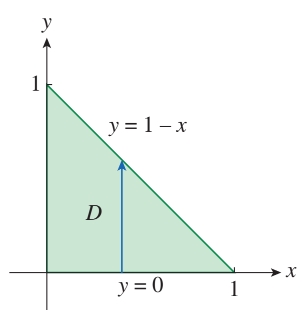


Figure 12.65
The region D is the projection of E onto the xy-plane.

$$\iiint_E z\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx \qquad E \text{ is a type 1 region.}$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\,dx \qquad \text{Antiderivative.}$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (1 - x - y)^2\,dy\,dx \qquad \text{FTC.}$$

$$= \frac{1}{2}\int_0^1 \left[-\frac{(1 - x - y)^3}{3}\right]_{y=0}^{y=1-x} dx \qquad \text{Antiderivative.}$$

$$= \frac{1}{6}\int_0^1 (1 - x)^3\,dx \qquad \text{FTC.}$$

$$= \frac{1}{6}\left[-\frac{(1 - x)^4}{4}\right]_0^1 = \frac{1}{24} \qquad \text{Antiderivative; FTC; simplify.}$$

■

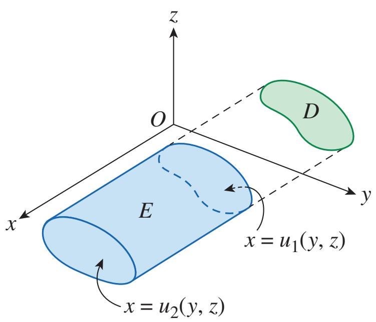


Figure 12.66
A type 2 region.

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \le x \le u_2(y, z)\}$$

where D is the projection of E onto the yz-plane (see Figure 12.66). The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx \right] dA \tag{9}$$

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \le y \le u_2(x, z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 12.67). For this type of region, we have

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy \right] dA \tag{10}$$

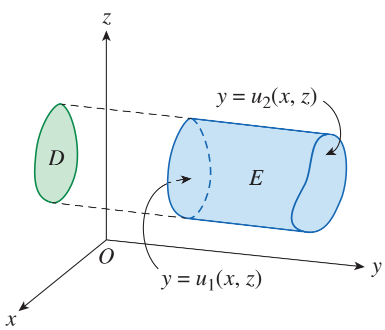


Figure 12.67
A type 3 region.

In each of Equations 9 and 10, there may be two possible expressions for the integral depending on whether D is a type I or type II plane region (and corresponding to Equations 7 and 8).

Example 3 Choosing the Best Order of Integration

Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Solution

The solid E is shown in Figure 12.68.

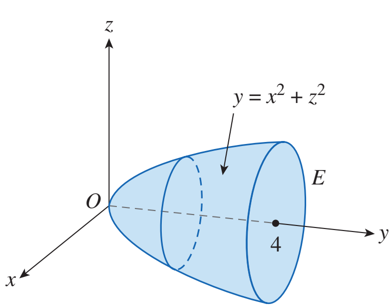


Figure 12.68
Region of integration.

If we interpret E as a type 1 region, then we need to consider its projection D_1 onto the xy-plane, which is the parabolic region in Figure 12.69. The trace of $y = x^2 + z^2$ in the plane $z = 0$ is the parabola $y = x^2$.

From $y = x^2 + z^2 \quad \Rightarrow \quad z = \pm\sqrt{y - x^2}$.
So, the lower boundary surface of E is $z = -\sqrt{y - x^2}$ and the upper surface is $z = \sqrt{y - x^2}$.

Therefore, the description of E as a type 1 region is

$E = \left\{(x, y, z) \mid -2 \le x \le 2, x^2 \le y \le 4, -\sqrt{y - x^2} \le z \le \sqrt{y - x^2}\right\}$.

The triple integral can be written as

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y - x^2}}^{\sqrt{y - x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx.$$

Although this expression is correct, it is difficult to evaluate. So, let's instead consider E as a type 3 region. Then its projection D_3 onto the xz-plane is the disk $x^2 + z^2 \le 4$ as shown in Figure 12.70.

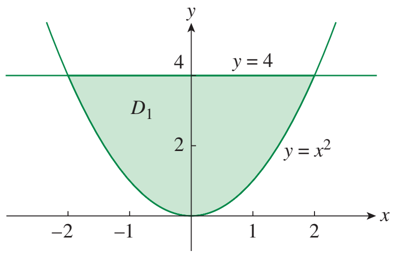


Figure 12.69
Projection onto the *xy*-plane.

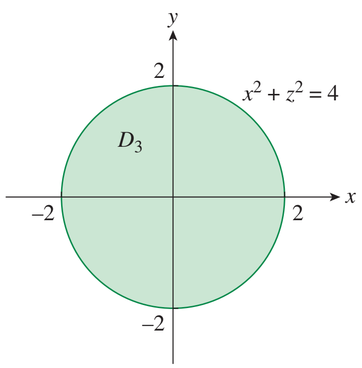


Figure 12.70
Projection onto the *xz*-plane.

Then the left boundary of E is the paraboloid $y = x^2 + z^2$ and the right boundary is the plane $y = 4$.

So, let $u_1(x, z) = x^2 + z^2$ and $u_2(x, z) = 4$ in Equation 10,

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_{D_3}\left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\right]dA = \iint_{D_3}(4-x^2-z^2)\sqrt{x^2+z^2}\,dA.$$

This double integral can be written as

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-x^2-z^2)\sqrt{x^2+z^2}\,dz\,dx.$$

but it is easier to evaluate if we convert to polar coordinates in the *xz*-plane: $x = r\cos\theta,\ z = r\sin\theta$

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_{D_3}(4-x^2-z^2)\sqrt{x^2+z^2}\,dA$$ *E as a type 3 region.*

$$= \int_0^{2\pi}\int_0^2 (4-r^2)r\,r\,dr\,d\theta$$ *Convert to polar coordinates.*

$$= \int_0^{2\pi} d\theta \int_0^2 (4r^2-r^4)\,dr$$ *Double integral can be written as the product of two single integrals.*

$$= 2\pi\left[\frac{4r^3}{3}-\frac{r^5}{5}\right]_0^2 = \frac{128\pi}{15}$$ *Antiderivatives; FTC; simplify.*

Applications of Triple Integrals

Recall that if $f(x) \geq 0$, then the single integral $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b, and if $f(x, y) \geq 0$, then the double integral $\iint_D f(x, y)\,dA$ represents the volume under the surface $z = f(x, y)$ and above D.

The corresponding interpretation of a triple integral $\iiint_E f(x, y, z)\,dV$, where $f(x, y, z) \geq 0$, is not very useful because it would be interpreted as the *hypervolume* of a four-dimensional object, and of course, that is very difficult to visualize. (Remember that E is just the *domain* of the function f; the graph of f lies in four-dimensional space.) Nonetheless, the triple integral $\iiint_E f(x, y, z)\,dV$ can be interpreted in meaningful ways in various physical situations, depending on the different interpretations of x, y, z, and $f(x, y, z)$.

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does indeed represent the volume of E:

$$V(E) = \iiint_E dV \tag{11}$$

For example, in the case of a type 1 region, let $f(x, y, z) = 1$ in Equation 5:

$$\iiint_E 1\ dV = \iint_D \left[\int_{u_1(x,\,y)}^{u_2(x,\,y)} dz \right] dA = \iint_D [u_2(x, y) - u_1(x, y)]\ dA$$

and from Section 12.3, we know this represents the volume that lies between the surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

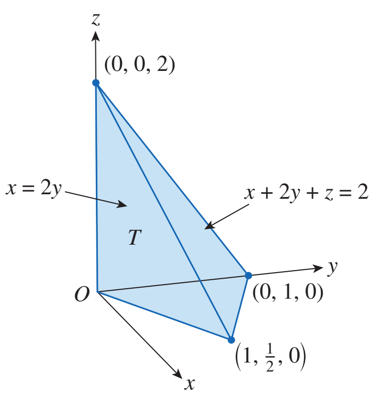


Figure 12.71
The tetrahedron T.

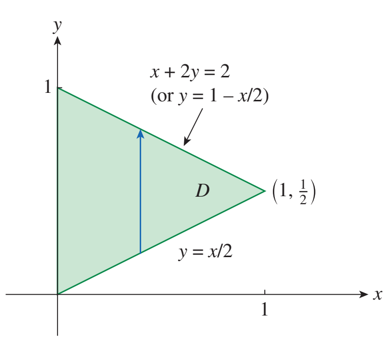


Figure 12.72
The projection onto the xy-plane.

Example 4 Tetrahedron Volume

Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Solution

The tetrahedron T and its projection D onto the xy-plane are shown in Figures 12.71 and 12.72.

The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is $z = 2 - x - 2y$.

Evaluate the triple integral.

$$V(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\ dy\ dx \qquad \text{Type 1 region.}$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\ dy\ dx \qquad \text{Antiderivative; FTC.}$$

$$= \frac{1}{3} \qquad \text{Calculation from Example 4 Section 12.3.}$$

Notice that it is not necessary to use triple integrals to compute volumes. They simply provide an alternate method for setting up the calculation. ■

All the applications of double integrals in Section 12.5 can be immediately extended to triple integrals. For example, if the density function of a solid object that occupies the region E is $\rho(x, y, z)$, in units of mass per unit volume, at any given point (x, y, z), then its **mass** is

$$m = \iiint_E \rho(x, y, z)\ dV \tag{12}$$

and its **moments** about the three coordinate planes are

$$M_{yz} = \iiint_E x\rho(x, y, z)\ dV \quad M_{xz} = \iiint_E y\rho(x, y, z)\ dV \quad M_{xy} = \iiint_E z\rho(x, y, z)\ dV \tag{13}$$

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m} \tag{14}$$

If the density is constant, the center of mass of the solid is called the **centroid** of E. The **moments of inertia** about the three coordinate axes are

$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\, dV \qquad I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\, dV \tag{15}$$

$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\, dV$$

As in Section 12.5, the total **electric charge** on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\, dV$$

If we have three continuous random variables X, Y, and Z, their **joint density function** is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$\mathrm{P}[(X, Y, Z) \in E] = \iiint_E f(x, y, z)\, dV$$

In particular,

$$\mathrm{P}(a \le X \le b,\ c \le Y \le d,\ r \le Z \le s) = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx$$

The joint density function must satisfy the two properties

$$f(x, y, z) \ge 0 \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = 1$$

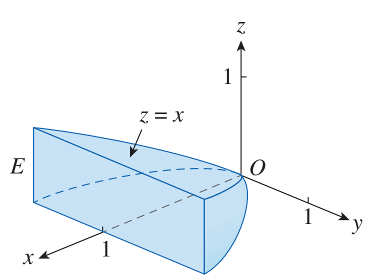


Figure 12.73
The solid region E.

Example 5 Center of Mass

Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

Solution

The solid E and its projection onto the xy-plane are shown in Figures 12.73 and 12.74.

The lower and upper surfaces of E are the planes $z = 0$ and $z = x$, so we describe E as a type 1 region:

$$E = \{(x, y, z) \mid -1 \le y \le 1,\ y^2 \le x \le 1, 0 \le z \le x\}$$

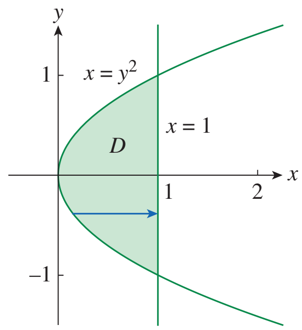


Figure 12.74
The region D.

If the density is $\rho(x, y, z) = \rho$, then the mass is

$$m = \iiint_E \rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x \rho\, dz\, dx\, dy \qquad E \text{ type 1; } D \text{ type II.}$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x\, dx\, dy = \rho \int_{-1}^{1} \left[\frac{x^2}{2}\right]_{x=y^2}^{x=1} dy \qquad \text{Antiderivatives; FTC.}$$

$$= \frac{\rho}{2} \int_{-1}^{1} (1 - y^4)\, dy = \rho \int_0^1 (1 - y^4)\, dy \qquad \text{FTC; even function.}$$

$$= \rho\left[y - \frac{y^5}{5}\right]_0^1 = \frac{4\rho}{5} \qquad \text{Antiderivative; FTC; simplify.}$$

Because of the symmetry of E and ρ about the xz-plane, we can conclude that $M_{xz} = 0$ and therefore $\overline{y} = 0$.

Find the other moments.

$$M_{yz} = \iiint_E x\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x x\rho\, dz\, dx\, dy \qquad E \text{ type 1; } D \text{ type II.}$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x^2\, dx\, dy = \rho \int_{-1}^{1} \left[\frac{x^3}{3}\right]_{x=y^2}^{x=1} dy \qquad \text{Antiderivatives; FTC.}$$

$$= \frac{2\rho}{3} \int_0^1 (1 - y^6)\, dy = \frac{2\rho}{3}\left[y - \frac{y^7}{7}\right]_0^1 = \frac{4\rho}{7} \qquad \text{Antiderivative; FTC; simplify.}$$

$$M_{xy} = \iiint_E z\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x z\rho\, dz\, dx\, dy \qquad E \text{ type 1; } D \text{ type II.}$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} \left[\frac{z^2}{2}\right]_{z=0}^{z=x} dx\, dy = \frac{\rho}{2} \int_{-1}^{1} \int_{y^2}^{1} x^2\, dx\, dy \qquad \text{Antiderivatives; FTC.}$$

$$= \frac{\rho}{3} \int_0^1 (1 - y^6)\, dy = \frac{2\rho}{7} \qquad \text{Antiderivative; FTC; simplify.}$$

Therefore, the center of mass is

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(\frac{5}{7}, 0, \frac{5}{14}\right).$$ ■

12.7 Exercises

1. Evaluate the integral in Example 1, integrating first with respect to y, then z, and then x.

2. Evaluate the integral $\iiint_E (xz - y^3)\, dV$, where

$$E = \{(x, y, z) \mid -1 \le x \le 1, 0 \le y \le 2, 0 \le z \le 1\}$$

using three different orders of integration.

Evaluate the iterated integral.

3. $\displaystyle\int_0^1 \int_0^z \int_0^{x+z} 6xz\, dy\, dx\, dz$

4. $\displaystyle\int_0^1 \int_x^{2x} \int_0^y 2xyz\, dz\, dy\, dx$

5. $\displaystyle\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y\, dx\, dz\, dy$

6. $\displaystyle\int_0^1 \int_0^z \int_0^y ze^{-y^2}\, dx\, dy\, dz$

7. $\displaystyle\int_0^{\pi/2} \int_0^y \int_0^x \cos(x + y + z)\, dz\, dx\, dy$

8. $\displaystyle\int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \sin y\, dy\, dz\, dx$

9. $\displaystyle\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\, dx\, dz\, dy$

Evaluate the triple integral.

10. $\displaystyle\iiint_E 2x\, dV$, where
$E = \{(x, y, z) \mid 0 \le y \le 2, 0 \le x \le \sqrt{4 - y^2}, 0 \le z \le y\}$

11. $\displaystyle\iiint_E yz \cos(x^5)\, dV$, where
$E = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le x, x \le z \le 2x\}$

12. $\displaystyle\iiint_E 6xy\, dV$, where E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$

13. $\displaystyle\iiint_E y\, dV$, where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 4$

14. $\displaystyle\iiint_E x^2 e^y\, dV$, where E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0$, $x = 1$, and $x = -1$

15. $\displaystyle\iiint_E xy\, dV$, where E is bounded by the parabolic cylinders $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = x + y$

16. $\displaystyle\iiint_T x^2\, dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

17. $\displaystyle\iiint_T xyz\, dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 0, 1)$

18. $\displaystyle\iiint_E x\, dV$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$

19. $\displaystyle\iiint_E z\, dV$, where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant

Use a triple integral to find the volume of the given solid.

20. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

21. The solid bounded by the cylinder $y = x^2$ and the planes $z = 0$, $z = 4$, and $y = 9$

22. The solid enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 5$ and $z = 1$

23. The solid enclosed by the paraboloid $x = y^2 + z^2$ and the plane $x = 16$

24. (a) Express the volume of the wedge in the first octant that is cut from the cylinder $y^2 + z^2 = 1$ by the planes $y = x$ and $x = 1$ as a triple integral.
(b) Use technology to find the value of the triple integral in part (a).

25. (a) In the **Midpoint Rule for triple integrals**, we use a triple Riemann sum to approximate a triple integral over a box B, where $f(x, y, z)$ is evaluated at the center $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ of the box B_{ijk}. Use the Midpoint Rule to estimate

$$\iiint_B \sqrt{x^2 + y^2 + z^2}\, dV$$

where B is the cube defined by $0 \le x \le 4$, $0 \le y \le 4$, $0 \le z \le 4$. Divide B into eight cubes of equal size.
(b) Use technology to find the integral in part (a). Compare your answer with the approximation in part (a).

Use the Midpoint Rule for triple integrals (Exercise 25) to estimate the value of the integral. Divide B into eight sub-boxes of equal size.

26. $\displaystyle\iiint_B \frac{1}{\ln(1 + x + y + z)}\, dV$, where
$B = \{(x, y, z) \mid 0 \le x \le 4, 0 \le y \le 8, 0 \le z \le 4\}$

27. $\displaystyle\iiint_B \sin(xy^2z^3)\, dV$, where
$B = \{(x, y, z) \mid 0 \le x \le 4, 0 \le y \le 2, 0 \le z \le 1\}$

28. $\displaystyle\iiint_B \sqrt{x}e^{xyz}\,dV$, where

$B = \{(x, y, z) \mid 0 \le x \le 4,\ 0 \le y \le 1,\ 0 \le z \le 2\}$

Sketch the solid whose volume is given by the iterated integral.

29. $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\,dz\,dx$

30. $\displaystyle\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\,dz\,dy$

Express the integral $\displaystyle\iiint_E f(x, y, z)\,dV$ as an iterated integral in six different ways, where E is the solid bounded by the given surface.

31. $y = 4 - x^2 - 4z^2, \quad y = 0$

32. $y^2 + z^2 = 9, \quad x = -2, \quad x = 2$

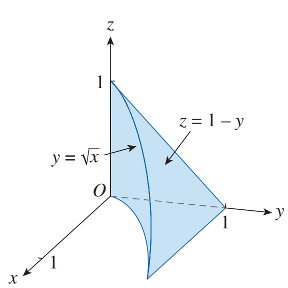
33. $y = x^2, \quad z = 0, \quad y + 2z = 4$

34. $x = 2, \quad y = 2, \quad z = 0, \quad x + y - 2z = 2$

35. The figure shows the region of integration for the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\,dz\,dy\,dx$$

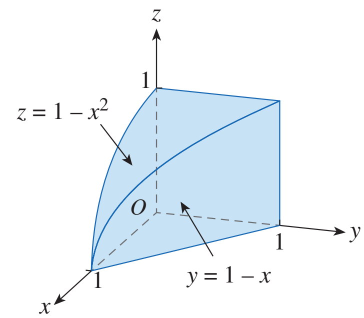
Rewrite this integral as an equivalent iterated integral in the five other orders.



36. The figure shows the region of integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\,dy\,dz\,dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.



Write five other iterated integrals that are equal to the given iterated integral.

37. $\displaystyle\int_0^1 \int_y^1 \int_0^y f(x, y, z)\,dz\,dx\,dy$

38. $\displaystyle\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\,dz\,dy\,dx$

Find the mass and center of mass of the solid E with the given density function ρ.

39. E is the solid of Exercise 12; $\quad \rho(x, y, z) = 2$

40. E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $x + z = 1$, $x = 0$, and $z = 0$; $\quad \rho(x, y, z) = 4$

41. E is the cube given by $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$; $\rho(x, y, z) = x^2 + y^2 + z^2$

42. E is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$; $\quad \rho(x, y, z) = y$

Assume the solid has constant density K.

43. Find the moments of inertia for a cube with side length L if one vertex is located at the origin and three edges lie along the coordinate axes.

44. Find the moments of inertia for a rectangular brick with dimensions a, b, and c and mass M if the center of the brick is situated at the origin and the edges are parallel to the coordinate axes.

45. Find the moment of inertia about the z-axis of the solid cylinder $x^2 + y^2 \le a^2$, $0 \le z \le h$.

46. Find the moment of inertia about the z-axis of the solid cone $\sqrt{x^2 + y^2} \le z \le h$.

Set up, but do not evaluate, integral expressions for (a) the mass, (b) the center of mass, and (c) the moment of inertia about the z-axis.

47. The solid of Exercise 22; $\rho(x, y, z) = \sqrt{x^2 + y^2}$

48. The hemisphere $x^2 + y^2 + z^2 \leq 1, z \geq 0$;
$\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

49. Let E be the solid in the first octant bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, and $z = 0$ with the density function $\rho(x, y, z) = 1 + x + y + z$. Use technology to find the following quantities for E.
(a) The mass
(b) The center of mass
(c) The moment of inertia about the z-axis

50. If E is the solid of Exercise 19 with density function $\rho(x, y, z) = x^2 + y^2$, use technology to find the following quantities.
(a) The mass
(b) The center of mass
(c) The moment of inertia about the z-axis

51. The joint density function for the random variables X, Y, and Z is $f(x, y, z) = Cxyz$ if $0 \leq x \leq 2$, $0 \leq y \leq 2$, $0 \leq z \leq 2$, and $f(x, y, z) = 0$ otherwise.
(a) Find the value of the constant C.
(b) Find $P(X \leq 1, Y \leq 1, Z \leq 1)$.
(c) Find $P(X + Y + Z \leq 1)$.

52. Suppose X, Y, and Z are random variables with joint density function given by

$$f(x, y, z) = \begin{cases} Ce^{-(0.5x + 0.2y + 0.1z)} & \text{if } x \geq 0,\ y \geq 0,\ z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant C.
(b) Find $P(X \leq 1, Y \leq 1)$.
(c) Find $P(X \leq 1, Y \leq 1, Z \leq 1)$.

The **average value** of a function $f(x, y, z)$ over a solid region E is defined to be

$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV$$

where $V(E)$ is the volume of E. For instance, if ρ is a density function, then ρ_{ave} is the average density of E.

53. Find the average value of the function $f(x, y, z) = xyz$ over the cube with side length L that lies in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

54. Find the average value of the function $f(x, y, z) = x^2z + y^2z$ over the region enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

55. (a) Find the region E for which the triple integral

$$\iiint_E (1 - x^2 - 2y^2 - 3z^2)\, dV$$

is a maximum.

(b) Use technology to find the maximum value of the triple integral in part (a).

Discovery Project | Volumes of Hyperspheres

The purpose of this project is to find formulas for the volume enclosed by a hypersphere in n-dimensional space. The hypersphere in $\mathbb{R}^n$ of radius r centered at the origin has equation

$$x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = r^2$$

Let $V_n(r)$ denote the volume enclosed by this hypersphere. A hypersphere in $\mathbb{R}^2$ is a circle and in $\mathbb{R}^3$, a sphere.

1. Use a double integral, the trigonometric substitution $y = r \sin\theta$, and technology to find the area of a circle with radius r.

2. Use a triple integral and trigonometric substitution to find the volume $V_3(r)$ enclosed by a sphere with radius r in $\mathbb{R}^3$.

3. Use a quadruple integral to find the hypervolume enclosed by the hypersphere $x^2 + y^2 + z^2 + w^2 = r^2$ in $\mathbb{R}^4$. (Use only trigonometric substitution and the reduction formulas for $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$.)

4. Use an n-tuple integral to find the volume $V_n(r)$ enclosed by a hypersphere of radius r in n-dimensional space $\mathbb{R}^n$. Hint: The formulas are different for n even and n odd.

12.8 Triple Integrals in Cylindrical and Spherical Coordinates

We saw in Section 12.4 that some double integrals are easier to evaluate using polar coordinates. In this section, we'll discover that some triple integrals are easier to evaluate using cylindrical or spherical coordinates.

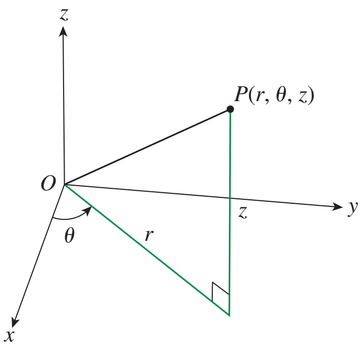


Figure 12.75
Geometric illustration of the cylindrical coordinates of a point P.

Cylindrical Coordinates

Recall from Section 9.7 that the cylindrical coordinates of a point P in three-dimensional space are (r, θ, z) where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P. See Figure 12.75.

Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates (see Figure 12.76). In particular, suppose that f is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

Since E is a type 1 region, we can use Equation 12.7.6 to write

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA \qquad (1)$$

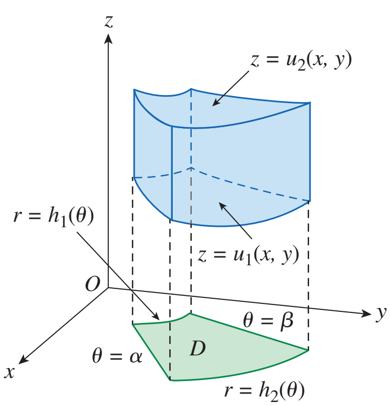


Figure 12.76
E is a type 1 region whose projection onto the xy-plane is D.

But we also know how to evaluate double integrals in polar coordinates. Therefore, combining Equation 1 and Equation 12.4.3, we obtain

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\ r\sin\theta)}^{u_2(r\cos\theta,\ r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta \qquad (2)$$

Equation 2 is the **formula for triple integration in cylindrical coordinates**. It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ, and replacing dV by $r\,dz\,dr\,d\theta$. Figure 12.77 shows a geometric approach to remember this. It is worthwhile to use this formula when E is a solid region easily described in cylindrical coordinates, and especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

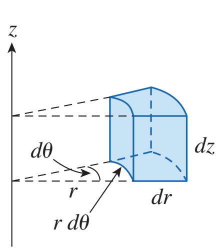


Figure 12.77
Volume element in cylindrical coordinates: $dV = r\,dz\,dr\,d\theta$.

Example 1 Find Mass Using Cylindrical Coordinates

A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 12.78.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Solution

In cylindrical coordinates, the cylinder is described by $r = 1$ and the paraboloid by $z = 1 - r^2$.

Therefore, we can describe the solid E as

$$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 1 - r^2 \le z \le 4\}.$$

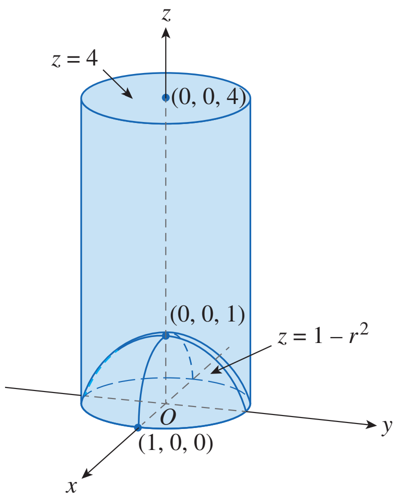


Figure 12.78
The solid E.

The density at (x, y, z) is proportional to the distance from the z-axis. So, the density function is $f(x, y, z) = K\sqrt{x^2+y^2} = Kr$ where K is the proportionality constant.

Use Equation 12.7.13 to find the mass of E.

$$m = \iiint_E K\sqrt{x^2+y^2}\, dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (Kr)\, r\, dz\, dr\, d\theta$$ Convert to cylindrical coordinates.

$$= \int_0^{2\pi}\int_0^1 Kr^2[4-(1-r^2)]\, dr\, d\theta$$ Antiderivatives; FTC.

$$= K\int_0^{2\pi} d\theta \int_0^1 (3r^2+r^4)\, dr$$ Integrand is a product of a function in θ times a function in r.

$$= 2\pi K\left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5}$$ Antiderivative; FTC; simplify.

■

Example 2 Integrate over a Solid Best Described in Cylindrical Coordinates

Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx.$

Solution

This iterated integral is a triple integral over the solid region

$E = \{(x, y, z)\,|\,-2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \sqrt{x^2+y^2} \le z \le 2\}$

and the projection of E onto the xy-plane is the disk $x^2 + y^2 \le 4$.

The lower surface of E is the cone $z = \sqrt{x^2+y^2}$ and its upper surface is the plane $z = 2$. See Figure 12.79.

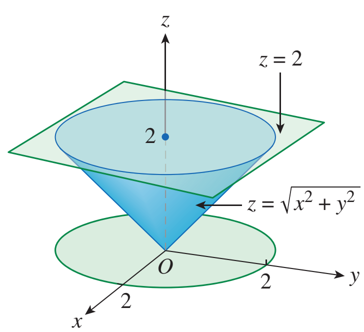


Figure 12.79
The solid region E and the projection onto the xy-plane.

The solid E has a simpler description in cylindrical coordinates:

$E = \{(r, \theta, z)\,|\,0 \le \theta \le 2\pi, 0 \le r \le 2, r \le z \le 2\}$

Convert the iterated integral to cylindrical coordinates and evaluate.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx$$

$$= \iiint_E (x^2+y^2)\, dV$$ Triple integral over the region E.

$$= \int_0^{2\pi}\int_0^2\int_r^2 r^2\, r\, dz\, dr\, d\theta$$ Convert to cylindrical coordinates.

$$= \int_0^{2\pi} d\theta \int_0^2 r^3(2-r)\, dr$$ Integrate with respect to z; FTC; remaining integrand is a product of a function in θ times a function in r.

$$= 2\pi\left[\frac{1}{2}r^4 - \frac{1}{5}r^5\right]_0^2 = \frac{16}{5}\pi$$ Antiderivative; FTC; simplify.

■

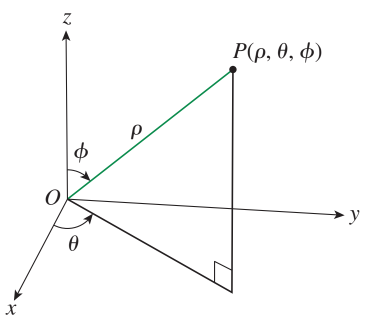


Figure 12.80
Geometric illustration of the spherical coordinates of a point P.

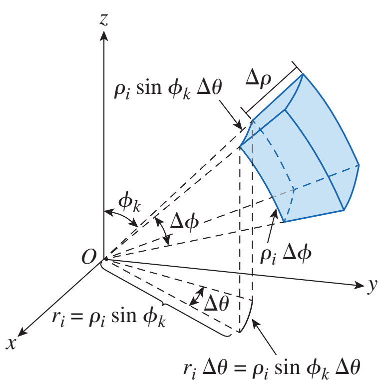


Figure 12.81
A small spherical wedge E_{ijk}.

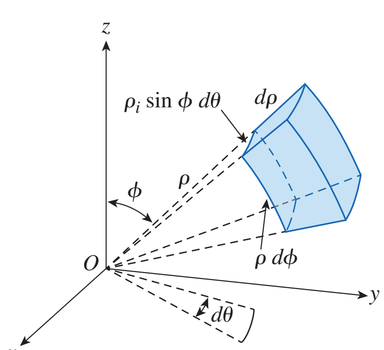


Figure 12.82
Volume element in spherical coordinates: $dV = \rho^2 \sin \phi \; d\rho \; d\theta \; d\phi$.

Spherical Coordinates

In Section 9.7, we defined the spherical coordinates (ρ, θ, ϕ) of a point P in three-dimensional space as illustrated in Figure 12.80. We also demonstrated the following relationships between rectangular coordinates and spherical coordinates:

$$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi \tag{3}$$

It this coordinate system, **spherical wedge** is analogous to a rectangular box, and is defined by

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

where $a \geq 0$ and $\beta - \alpha \leq 2\pi$, and $d - c \leq \pi$. Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result.

So, we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$. Figure 12.81 shows that E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i \, \Delta\phi$ (arc of a circle with radius ρ_i, angle $\Delta\phi$), and $\rho_i \sin \phi_k \Delta\theta$ (arc of a circle with radius $\rho_i \sin \phi_k$, angle $\Delta\theta$). So, an approximation to the volume of E_{ijk} is given by

$$(\Delta\rho) \times (\rho_i \, \Delta\phi) \times (\rho_i \sin \phi_k \, \Delta\theta) = \rho_i^2 \sin \phi_k \, \Delta\rho \, \Delta\theta \, \Delta\phi$$

Therefore, an approximation to a typical triple Riemann sum is

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho_i \sin \phi_k \cos \theta_j, \rho_i \sin \phi_k \sin \theta_j, \rho_i \cos \phi_k) \, \rho_i^2 \sin \phi_k \, \Delta\rho \, \Delta\theta \, \Delta\phi$$

But this is a Riemann sum for the function

$$F(\rho, \theta, \phi) = f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \cdot \rho^2 \sin\phi$$

Consequently, the following **formula for triple integration in spherical coordinates** is reasonable.

$$\iiint_E f(x, y, z) \, dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos \phi) \, \rho^2 \sin\phi \; d\rho \; d\theta \; d\phi \tag{4}$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

Equation 4 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

using the appropriate limits of integration, and replacing dV by $\rho^2 \sin\phi \; d\rho \; d\theta \; d\phi$. This conversion is illustrated geometrically in Figure 12.82.

This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \phi) \mid \alpha \le \theta \le \beta, c \le \phi \le d, g_1(\theta, \phi) \le \rho \le g_2(\theta, \phi)\}$$

In this case, the formula is the same as in Equation 4 except that the limits of integration for ρ are $g_1(\theta, \phi)$ and $g_2(\theta, \phi)$.

Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

Example 3 Triple Integral in Spherical Coordinates

Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$$

Solution

Since the boundary of B is a sphere, it seems reasonable to use spherical coordinates:

$$B = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$$

In addition, spherical coordinates are appropriate because $x^2 + y^2 + z^2 = \rho^2$.

Use Equation 4 to evaluate the triple integral.

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{(\rho^2)^{3/2}}\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi \qquad \text{Equation 4.}$$

$$= \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3}\, d\rho$$

$$= \Big[-\cos\phi\Big]_0^{\pi}(2\pi)\left[\frac{1}{3}e^{\rho^3}\right]_0^1 = \frac{4}{3}\pi(e-1) \qquad \text{Antiderivative; FTC; simplify.}$$

■

Note: It would have been extremely awkward to evaluate the integral in Example 3 without spherical coordinates. In rectangular coordinates, the iterated integral would have been

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx$$

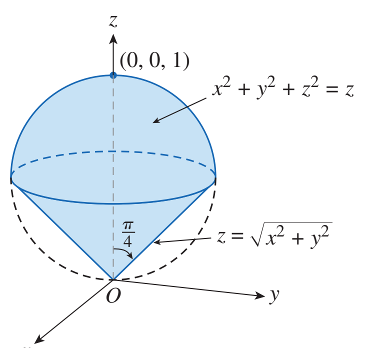


Figure 12.83
The solid lies above the cone and below the sphere.

Example 4 A Volume Calculation in Spherical Coordinates

Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See Figures 12.83 and 12.84.)

Solution

The sphere passes through the origin and has center $(0, 0, \frac{1}{2})$.
Write the equation of the sphere in spherical coordinates:

$$\rho^2 = \rho\cos\phi \quad \text{or} \quad \rho = \cos\phi$$

The equation of the cone can be written as

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} = \rho\sin\phi.$$

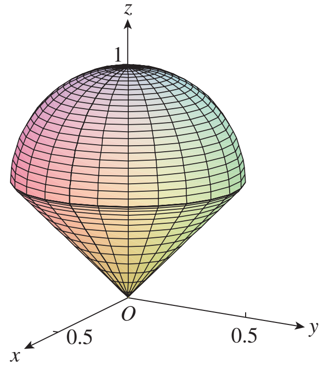


Figure 12.84
Another view of the solid in Example 4.

This implies that $\sin\phi = \cos\phi$, or $\phi = \dfrac{\pi}{4}$.

Therefore, the description of the solid E in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \mid 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4,\ 0 \le \rho \le \cos\phi\}.$$

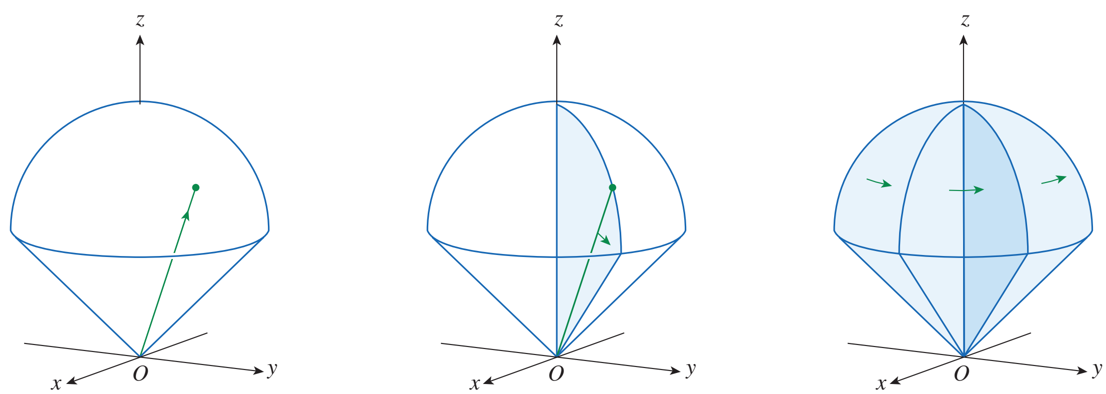
Figure 12.85 shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ. The volume of E is

$$V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2 \sin\phi \; d\rho\; d\phi\; d\theta \qquad \text{Equation 4.}$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\cos\phi} d\phi \qquad \text{Antiderivative.}$$

$$= \frac{2\pi}{3}\int_0^{\pi/4} \sin\phi\, \cos^3\phi \; d\phi \qquad \text{FTC.}$$

$$= \frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\pi/4} = \frac{\pi}{8} \qquad \text{Antiderivative; FTC; simplify.}$$



ρ varies from 0 to cos ϕ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π.

Figure 12.85
An illustration of how the solid E is swept out if the order of integration is first ρ, then ϕ, and then θ. ■

12.8 Exercises

Sketch the solid whose volume is given by the integral and evaluate the integral.

1. $\displaystyle\int_0^4\int_0^{2\pi}\int_r^4 r\, dz\, d\theta\, dr$

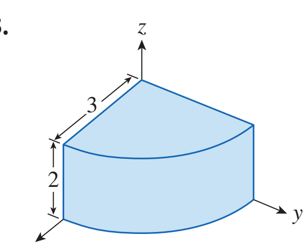
2. $\displaystyle\int_0^{\pi/2}\int_0^2\int_0^{9-r^2} r\, dz\, dr\, d\theta$

3. $\displaystyle\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

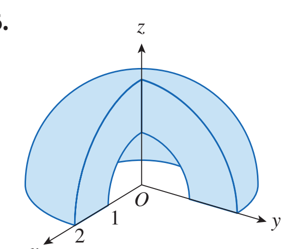
4. $\displaystyle\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

Set up the triple integral of an arbitrary continuous function $f(x, y, z)$ in cylindrical or spherical coordinates over the solid shown.

5.



6.



Use cylindrical coordinates.

7. Evaluate $\iiint_E \sqrt{x^2 + y^2}\, dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

8. Evaluate $\iiint_E (x^3 + xy^2)\, dV$, where E is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$.

9. Evaluate $\iiint_E e^z\, dV$, where E is enclosed by the paraboloid $z = 1 + x^2 + y^2$, the cylinder $x^2 + y^2 = 5$ and the xy-plane.

10. Evaluate $\iiint_E x\, dV$, where E is enclosed by the planes $z = 0$ and $z = x + y + 5$ and by the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

11. Evaluate $\iiint_E x^2\, dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

12. Find the volume of the solid that lies within the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

13. (a) Find the volume of the region E bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.
(b) Find the centroid of E (the center of mass in the case where the density is constant).

14. (a) Find the volume of the solid that the cylinder $r = a \cos\theta$ cuts out of the sphere of radius a centered at the origin.
(b) Illustrate the solid of part (a) by graphing the sphere and the cylinder on the same coordinate axes.

15. Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a\,(a > 0)$ if S has constant density K.

16. Find the mass of a ball B given by $x^2 + y^2 + z^2 \le a^2$ if the density at any point is proportional to its distance from the z-axis.

Use spherical coordinates.

17. Evaluate $\iiint_B (x^2 + y^2 + z^2)^2\, dV$, where B is the ball with center at the origin and radius 5.

18. Evaluate $\iiint_H (9 - x^2 - y^2)\, dV$, where H is the solid hemisphere $x^2 + y^2 + z^2 \le 9$, $z \ge 0$.

19. Evaluate $\iiint_E z\, dV$, where E lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.

20. Evaluate $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\, dV$, where E is enclosed by the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

21. Evaluate $\iiint_E x^2\, dV$, where E is bounded by the xz-plane and the hemispheres $y = \sqrt{9 - x^2 - z^2}$ and $y = \sqrt{16 - x^2 - z^2}$.

22. Evaluate $\iiint_E xyz\, dV$, where E lies between the spheres $\rho = 2$ and $\rho = 4$ and above the cone $\phi = \pi/3$.

23. Find the volume of the part of the ball $\rho \le a$ that lies between the cones $\phi = \dfrac{\pi}{6}$ and $\phi = \dfrac{\pi}{3}$.

24. Find the average distance from a point in a ball of radius a to its center.

25. (a) Find the volume of the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 4 \cos \phi$.
(b) Find the centroid of the solid in part (a).

26. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

27. Find the centroid of the solid in Exercise 21.

28. Let H be a solid hemisphere of radius a whose density at any point is proportional to its distance from the center of the base.
(a) Find the mass of H.
(b) Find the center of mass of H.
(c) Find the moment of inertia of H about its axis.

29. (a) Find the centroid of a solid homogeneous hemisphere of radius a.
(b) Find the moment of inertia of the solid in part (a) about a diameter of its base.

30. Find the mass and center of mass of a solid hemisphere of radius a if the density at any point is proportional to its distance from the base.

Use cylindrical or spherical coordinates, whichever seems more appropriate.

31. Find the volume and centroid of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

32. Find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\pi/6$.

33. Use technology to evaluate $\iiint_E z\, dV$, where E lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 2y$.

34. (a) Find the volume enclosed by the torus $\rho = \sin\phi$.
(b) Use technology to draw the torus.

Evaluate the integral by changing to cylindrical coordinates.

35. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy$

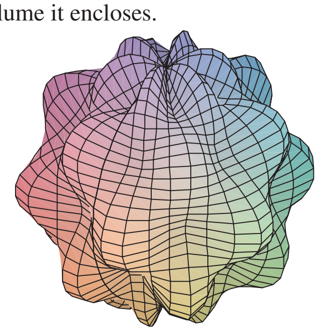
36. $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\,dz\,dy\,dx$

Evaluate the integral by changing to spherical coordinates.

37. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$

38. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2z+y^2z+z^3)\,dz\,dx\,dy$

39. In the Laboratory Project in Section 9.7, we investigated the family of surfaces $\rho = 1 + \frac{1}{5}\sin m\theta\,\sin n\phi$ that have been used as models for tumors. The *bumpy sphere* with $m = 6$ and $n = 5$ is shown in the figure. Use technology to find the volume it encloses.



40. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = 2\pi.$$

(The improper triple integral is defined as the limit of a triple integral over a solid sphere as the radius of the sphere increases indefinitely.)

41. When studying the formation of mountain ranges, geologists estimate the amount of work required to lift a mountain from sea level. Consider a mountain that is essentially in the shape of a right circular cone. Suppose that the weight density of the material in the vicinity of a point P is $g(P)$ and the height is $h(P)$.
(a) Find a definite integral that represents the total work done in forming the mountain.
(b) Assume that Mount Fuji in Japan is in the shape of a right circular cone with radius 62,000 ft, height 12,400 ft, and density a constant 200 lb/ft^3. How much work was done in forming Mount Fuji if the land was initially at sea level?

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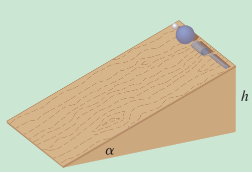
Applied Project | Roller Derby



Suppose that a solid ball (a marble), a hollow ball (a squash ball), a solid cylinder (a steel bar), and a hollow cylinder (a lead pipe) roll down a slope. Which of these objects reaches the bottom first? (Make a guess before proceeding.)

To answer this question, we consider a ball or cylinder with mass m, radius r, and moment of inertia I (about the axis of rotation). If the vertical drop is h, then the potential energy at the top is mgh. Suppose the object reaches the bottom with velocity v and angular velocity ω, so $v = \omega r$. The kinetic energy at the bottom consists of two parts: $\frac{1}{2}mv^2$ from translation (moving down the slope) and $\frac{1}{2}I\omega^2$ from rotation. If we assume that energy loss from rolling friction is negligible, then conservation of energy gives

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

1. Show that

$$v^2 = \frac{2gh}{1+I^*} \quad \text{where } I^* = \frac{I}{mr^2}$$

2. If $y(t)$ is the vertical distance traveled at time t, then the same reasoning as used in Problem 1 shows that $v^2 = \dfrac{2gy}{1 + I^*}$ at any time t. Use this result to show that y satisfies the differential equation

$$\frac{dy}{dt} = \sqrt{\frac{2g}{1 + I^*}}\,(\sin\,\alpha)\sqrt{y}$$

where α is the angle of inclination of the plane.

3. Solve the differential equation in Problem 2 and show that the total travel time is

$$T = \sqrt{\frac{2h(1 + I^*)}{g\,\sin^2\,\alpha}}$$

This shows that the object with the smallest value of I^* wins the race.

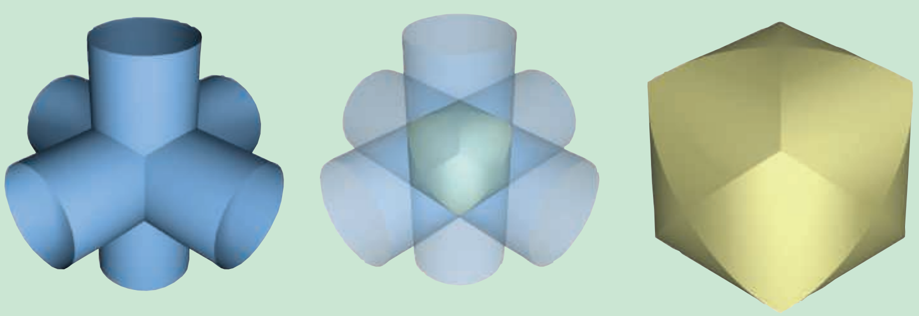
4. Show that $I^* = \frac{1}{2}$ for a solid cylinder and $I^* = 1$ for a hollow cylinder.

5. Calculate I^* for a partly hollow ball with inner radius a and outer radius r. Express your answer in terms of $b = \dfrac{a}{r}$. What happens as $a \to 0$ and as $a \to r$?

6. Show that $I^* = \frac{2}{5}$ for a solid ball and $I^* = \frac{2}{3}$ for a hollow ball. Therefore, the objects finish in the following order: solid ball, solid cylinder, hollow ball, hollow cylinder.

Discovery Project | The Intersection of Three Cylinders

The figure shows the solid enclosed by three circular cylinders with the same diameter that intersect at right angles. In this project, we compute its volume and determine how its shape changes if the cylinders have different diameters.



1. Carefully sketch the solid enclosed by the three cylinders $x^2 + y^2 = 1$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$. Indicate the positions of the coordinate axes and label the faces with the equations of the corresponding cylinders.

2. Find the volume of the solid in Problem 1.

3. Use technology to draw the edges of the solid.

4. What happens to the solid in Problem 1 if the radius of the first cylinder is different from 1? Sketch a graph to illustrate this change or construct a graph using technology.

5. If the first cylinder is $x^2 + y^2 = a^2$, where $a < 1$, set up, but do not evaluate, a double integral for the volume of the solid. What if $a > 1$?

12.9 Change of Variables in Multiple Integrals

In one-dimensional calculus, we often use a change of variable (a u-substitution) to simplify an integral. By reversing the roles of x and u, we can write the Substitution Rule (Section 5.5) as

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du \tag{1}$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$. Here's another way of writing Equation 1:

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du \tag{2}$$

Change of Variables in Double Integrals

A change of variables can also be useful in double integrals and we have already seen one example of this: conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

and the change of variables formula (Equation 12.4.2) can be written as

$$\iint_R f(x, y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

More generally, we consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

$$x = g(u, v) \qquad y = h(u, v) \tag{3}$$

or, as we sometimes write,

$$x = x(u, v) \qquad y = y(u, v)$$

We usually assume that T is a $\boldsymbol{C^1}$ **transformation**, which means that g and h have continuous first-order partial derivatives.

A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$. If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1). If no two points have the same image, T is called **one-to-one**. Figure 12.86 shows the effect of a transformation T on a region S in the uv-plane. T transforms S into a region R in the xy-plane called the **image of *S***, consisting of the images of all points in S.

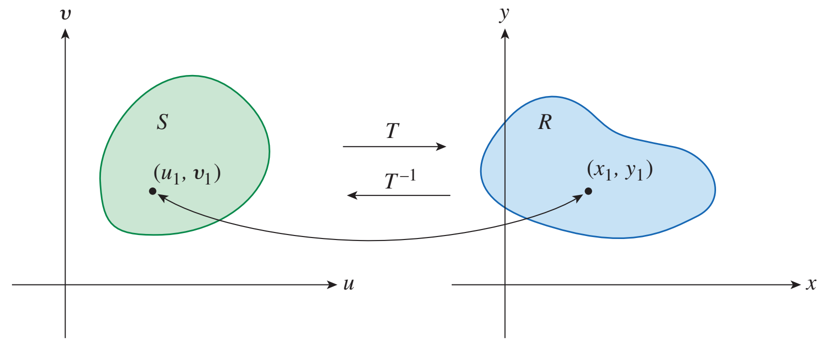


Figure 12.86
A geometric illustration of the effect of a transformation T on a region S in the uv-plane.

If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy-plane to the uv-plane and it may be possible to solve Equation 3 for u and v in terms of x and y:

$$u = G(x, y) \qquad v = H(x, y)$$

Example 1 Determine the Image of a Region Under a Transformation

A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$.

Solution

The transformation maps the boundary of S onto the boundary of the image. Therefore, begin by finding the images of the sides of S.

The first side, S_1 is given by $v = 0$ $(0 \le u \le 1)$. (See Figure 12.87.)
Using the given equations: $x = u^2, y = 0 \quad \Rightarrow \quad 0 \le x \le 1$.
Therefore, S_1 is mapped into the line segment from (0, 0) to (1, 0) in the xy-plane.

The second side, S_2 is $u = 1$ $(0 \le v \le 1)$.
Let $u = 1$ in the given equations: $x = 1 - v^2$, $y = 2v$.

Eliminate the variable v: $x = 1 - \dfrac{y^2}{4}$, $0 \le x \le 1$ which is part of a parabola.

Similarly, S_3 is given by $v = 1(0 \le u \le 1)$.

The image is the parabolic arc $x = \dfrac{y^2}{4} - 1$, $-1 \le x \le 0$.

Finally, S_4 is given by $u = 0$ $(0 \le v \le 1)$ whose image is $x = -v^2$, $y = 0$, that is, $-1 \le x \le 0$.

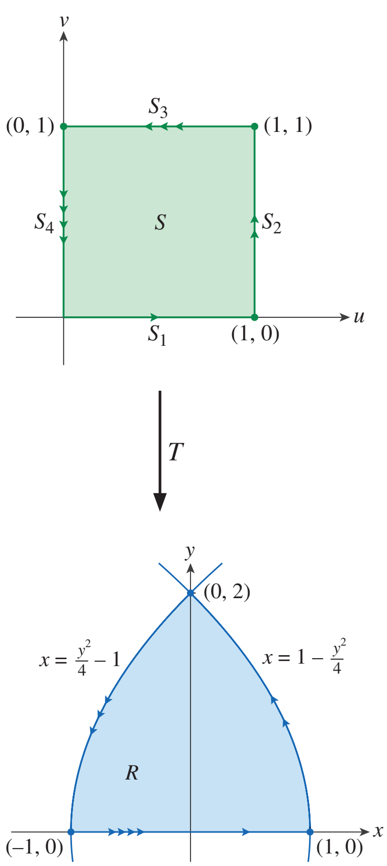


Figure 12.87
The image of the square S.

Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.

The image of S is the region R (shown in Figure 12.87) bounded by the x-axis and the parabolas $x = 1 - \dfrac{y^2}{4}$ and $x = \dfrac{y^2}{4} - 1$. ■

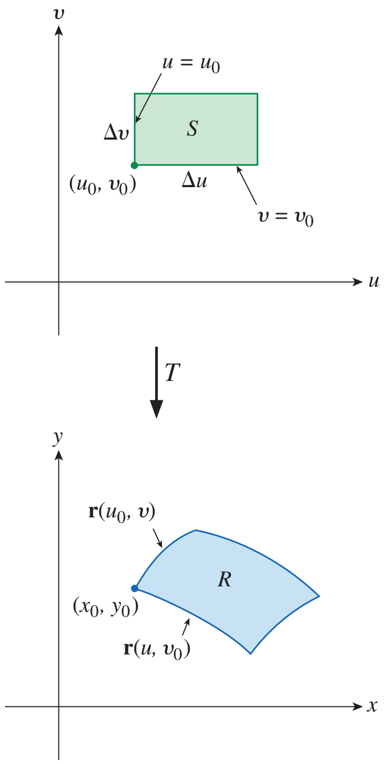


Figure 12.88
The image of S is the region R.

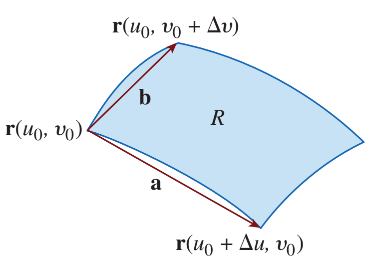


Figure 12.89
We can approximate R by a parallelogram.

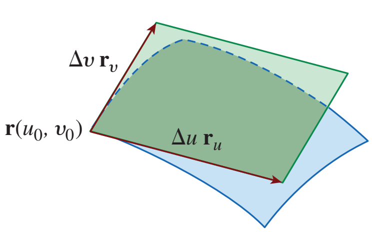


Figure 12.90
Use the vectors $\Delta u\ \mathbf{r}_u$ and $\Delta v\ \mathbf{r}_v$ to determine a parallelogram.

Now let's consider how a change of variables affects a double integral. Start with a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv. (See Figure 12.88.)

The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$. The vector

$$\mathbf{r}(u, v) = g(u, v)\ \mathbf{i} + h(u, v)\ \mathbf{j}$$

is the position vector of the image of the point (u, v). The equation of the lower boundary of S is $v = v_0$, whose image is given by the vector function $\mathbf{r}(u, v_0)$. The tangent vector at (x_0, y_0) to this image curve is

$$\mathbf{r}_u = g_u(u_0, v_0)\ \mathbf{i} + h_u(u_0, v_0)\ \mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}$$

Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely, $u = u_0$) is

$$\mathbf{r}_v = g_v(u_0, v_0)\ \mathbf{i} + h_v(u_0, v_0)\ \mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \qquad \mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$$

as shown in Figure 12.89. Write $\mathbf{r}_u$ using the limit definition.

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

then

$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u\ \mathbf{r}_u$$

Similarly,

$$\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v\ \mathbf{r}_v$$

This suggests that we can approximate R by a parallelogram determined by the vectors $\Delta u\ \mathbf{r}_u$ and $\Delta v\ \mathbf{r}_v$. (See Figure 12.90.) Therefore, we can approximate the area of R by the area of this parallelogram, which from Section 9.4, is

$$\left|(\Delta u\ \mathbf{r}_u) \times (\Delta v\ \mathbf{r}_v)\right| = \left|\mathbf{r}_u \times \mathbf{r}_v\right| \Delta u\ \Delta v \tag{4}$$

Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

The determinant that arises in this calculation is called the *Jacobian* of the transformation and is given a special notation.

The Jacobian is named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851). Although the French mathematician Cauchy first used these special determinants involving partial derivatives, Jacobi developed them into a method for evaluating multiple integrals.

Definition • Jacobian

The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

With this notation, we can use Equation 4 to find an approximation to the area ΔA of R:

$$\Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u \ \Delta v \tag{5}$$

where the Jacobian is evaluated at (u_0, v_0).

Next we divide a region S in the uv-plane into rectangles S_{ij} and denote their images in the xy-plane as R_{ij}. (See Figure 12.91.)

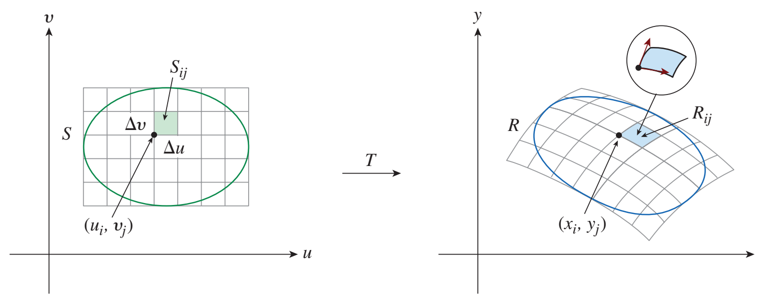


Figure 12.91
A typical rectangle S_j and its image R_{ij}.

Apply the approximation in Equation 5 to each image R_{ij}, and approximate the double integral of f over R as follows:

$$\iint_R f(x, y) dA \approx \sum_{i=j}^{m} \sum_{j=1}^{n} f(x_i, y_i) \ \Delta A$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\,(g(u_i, v_j), h\,(u_i, v_j)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \ \Delta u \ \Delta v$$

where the Jacobian is evaluated at (u_i, v_j). Notice that this double sum is a Riemann sum for the integral

$$\iint_S f(g(u, v), \ h(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du \, dv$$

This argument suggests that the following theorem is true. A full proof is given in books on advanced calculus.

Change of Variables in a Double Integral

Suppose T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y) \ dA = \iint_S f(x(u, v), \ y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du \, dv \tag{6}$$

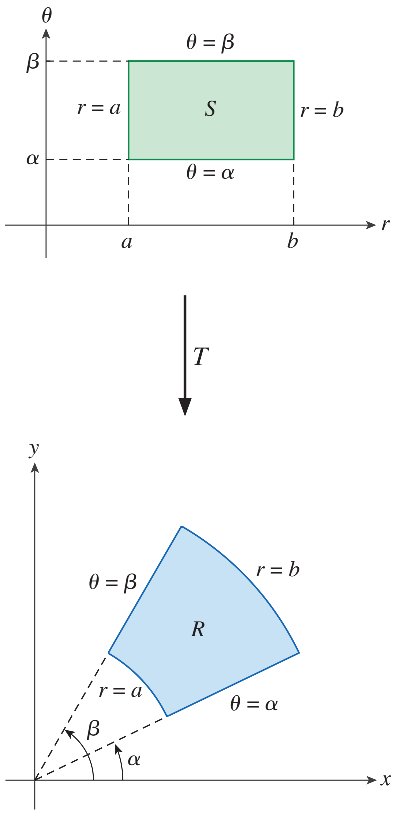


Figure 12.92
The geometry of the polar coordinate transformation T.

A Closer Look

1. This result says that we change from an integral in x and y to an integral in u and v by expressing x and y in terms of u and v and writing

$$dA = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

2. Notice the similarity between this result and the one-dimensional case in Equation 2. Instead of the derivative $\dfrac{dx}{du}$, we have the absolute value of the Jacobian, that is, $\left|\dfrac{\partial(x, y)}{\partial(u, v)}\right|$. ■

Let's show that the Formula for integration in polar coordinates is a special case of Equation 6. In this case, the transformation T from the $r\theta$-plane to the xy-plane is given by

$$x = g(r, \theta) = r\cos\theta \qquad y = h(r, \theta) = r\sin\theta$$

and the geometry of the transformation is shown in Figure 12.92: T maps an ordinary rectangle in the $r\theta$-plane to a polar rectangle in the xy-plane. The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2mm] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$$

Therefore, Equation 6 gives

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| dr\, d\theta$$
$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

which is the same as Equation 12.4.2.

Example 2 Evaluate a Double Integral with a Change of Variables

Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\displaystyle\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

Solution

The region R is shown in Figure 12.87.
In Example 1, we discovered that $T(S) = R$, where S is the square $[0, 1] \times [0, 1]$.

The reason for making this change of variables to evaluate the integral is that S is a much simpler region than R.

Compute the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

Therefore, by Equation 6,

$$\iint_R y\, dA = \iint_S 2uv \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA \qquad \text{Equation 6.}$$

$$= \int_0^1 \int_0^1 (2uv)4(u^2 + v^2)\, du\, dv \qquad \text{Iterated integral; Jacobian.}$$

$$= 8 \int_0^1 \int_0^1 (u^3 v + uv^3)\, du\, dv \qquad \text{Simplify integrand.}$$

$$= 8 \int_0^1 \left[\frac{1}{4}u^4 v + \frac{1}{2}u^2 v^3 \right]_{u=0}^{u=1} dv \qquad \text{Antiderivative.}$$

$$= \int_0^1 (2v + 4v^3)\, dv \qquad \text{FTC.}$$

$$= \Big[v^2 + v^4 \Big]_0^1 = 2 \qquad \text{Antiderivative; FTC; simplify.}$$

Note: Example 2 was not difficult to solve because we were given a suitable change of variables. If we are not given a transformation, then the first step is to determine an appropriate change of variables. If $f(x, y)$ is difficult to integrate, then the form of $f(x, y)$ may suggest a transformation. If the region of integration R is awkward, then the transformation should be chosen so that the corresponding region S in the uv-plane has a convenient description.

Example 3 Integrate over a Trapezoidal Region

Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\, dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.

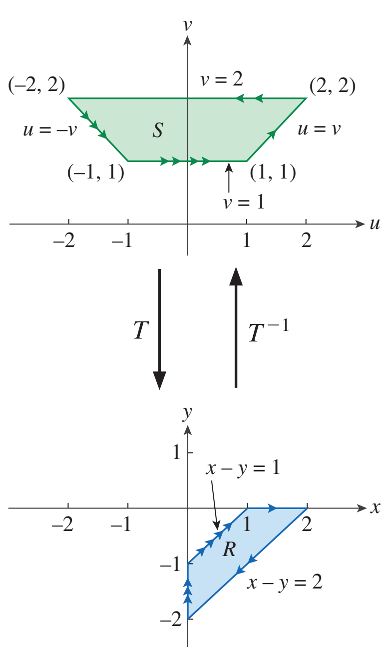


Figure 12.93
The geometry of the transformation T.

Solution

It isn't easy to integrate $e^{(x+y)/(x-y)}$.

Consider a change of variables suggested by the form of this function:

$$u = x + y \qquad v = x - y \tag{7}$$

These equations define a transformation T^{-1} from the xy-plane to the uv-plane. Equation 6 involves a transformation T from the uv-plane to the xy-plane.
Find this transformation by solving the equations in (7) for x and y:

$$x = \frac{1}{2}(u + v) \qquad y = \frac{1}{2}(u - v) \tag{8}$$

The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

To find the region S in the uv-plane corresponding to R, note that the sides of R lie on the lines

$$y = 0 \qquad x - y = 2 \qquad x = 0 \qquad x - y = 1$$

Therefore, the region S is the trapezoidal region with vertices $(1, 1)$, $(2, 2)$, $(-2, 2)$, and $(-1, 1)$ shown in Figure 12.93.

The region S is described by $S = \{(u, v) \mid 1 \le v \le 2, -v \le u \le v\}$.

Use Equation 6 to evaluate the integral.

$$\iint_R e^{(x+y)/(x-y)}\, dA = \iint_S e^{u/v} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv \qquad \text{Equation 6.}$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v} \left(\frac{1}{2}\right) du\, dv \qquad \text{Iterated integral; Jacobian.}$$

$$= \frac{1}{2} \int_1^2 \left[v e^{u/v} \right]_{u=-v}^{u=v} dv \qquad \text{Antiderivative.}$$

$$= \frac{1}{2} \int_1^2 (e - e^{-1}) v\, dv = \frac{3}{4}(e - e^{-1}) \qquad \text{FTC; antiderivative; simplify.}$$

■

Triple Integrals

There is a similar change of variables formula for triple integrals. Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$$

The **Jacobian** of T is the following 3×3 determinant:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} \tag{9}$$

Under hypotheses similar to those for a change of variables in a double integral, we have the following formula for triple integrals:

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw \tag{10}$$

Example 4 Spherical Coordinate Transformation

Use Equation 10 to derive the formula for triple integration in spherical coordinates.

Solution

In this case, the change of variables is given by

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

Compute the Jacobian.

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi(-\rho^2 \sin\phi \cos\phi \sin^2\theta - \rho^2 \sin\phi \cos\phi \cos^2\theta)$$
$$- \rho \sin\phi(\rho \sin^2\phi \cos^2\theta + \rho \sin^2\phi \sin^2\theta)$$
$$= -\rho^2 \sin\phi \cos^2\phi - \rho^2 \sin\phi \sin^2\phi = -\rho^2 \sin\phi$$

Since $0 \leq \phi \leq \pi$, then $\sin\phi \geq 0$.

Therefore, $\left|\dfrac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)}\right| = |-\rho^2 \sin\phi| = \rho^2 \sin\phi$.

Use Equation 10.

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

which is equivalent to Equation 12.8.4. ■

12.9 Exercises

Find the Jacobian of the transformation.

1. $x = 5u - v, \quad y = u + 3v$

2. $x = uv, \quad y = \dfrac{u}{v}$

3. $x = ue^v, \quad y = ve^{-u}$

4. $x = e^{-r} \sin\theta, \quad y = e^r \cos\theta$

5. $x = e^{s+t}, \quad y = e^{s-t}$

6. $x = \dfrac{u}{v}, \quad y = \dfrac{v}{w}, \quad z = \dfrac{w}{u}$

7. $x = v + w^2, \quad y = w + u^2, \quad z = u + v^2$

8. $x = u \tan^{-1} v, \quad y = w \sin v, \quad z = w \cos v$

Find the image of the set S under the given transformation.

9. $S = \{(u, v) \mid 0 \leq u \leq 3, 0 \leq v \leq 2\}$;
$x = 2u + 3v, \ y = u - v$

10. S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $\quad x = v, y = u(1 + v^2)$

11. S is the triangular region with vertices $(0, 0)$, $(1, 1)$, $(0, 1)$; $x = u^2, y = v$

12. S is the disk given by $u^2 + v^2 \leq 1$; $\quad x = au, y = bv$

A region R in the xy-plane is given. Find equations for a transformation T that maps a rectangular region S in the uv-plane onto R, where the sides of S are parallel to the u- and v-axes.

13. R is bounded by $y = 2x - 1$, $y = 2x + 1$, $y = 1 - x$, $y = 3 - x$

14. R is the parallelogram with vertices $(0, 0)$, $(4, 3)$, $(2, 4)$, $(-2, 1)$

15. R lies between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ in the first quadrant

16. R is bounded by the hyperbolas $y = \dfrac{1}{x}$, $y = \dfrac{4}{x}$ and the lines $y = x$, $y = 4x$ in the first quadrant

Use the given transformation to evaluate the integral.

17. $\displaystyle\iint_R (x - 3y)\, dA$, where R is the triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$; $\quad x = 2u + v, y = u + 2v$

18. $\displaystyle\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$;
$x = \dfrac{1}{4}(u + v), y = \dfrac{1}{4}(v - 3u)$

19. $\displaystyle\iint_R x^2\, dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$; $\quad x = 2u, \quad y = 3v$

20. $\displaystyle\iint_R (x^2 - xy + y^2)\, dA$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$;
$x = \sqrt{2}u - \sqrt{\dfrac{2}{3}}v, y = \sqrt{2}u + \sqrt{\dfrac{2}{3}}v$

21. $\displaystyle\iint_R xy\, dA$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$; $\quad x = \dfrac{u}{v}, \quad y = v$

22. $\displaystyle\iint_R y^2\, dA$, where R is the region bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$, $xy^2 = 2$;
$u = xy$, $v = xy^2$. Use technology to sketch the region R.

23. (a) Evaluate $\iiint_E dV$, where E is the solid enclosed by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Use the transformation $z = au$, $y = bv$, $z = cw$.

(b) Earth is not a perfect sphere; rotation has resulted in flattening at the poles. So the shape can be approximated by an ellipsoid with $a = b = 6378$ km and $c = 6356$ km. Use part (a) to estimate the volume of Earth.

24. If the solid in Exercise 23(a) has constant density k, find its moment of inertia about the z-axis.

Evaluate the integral by making an appropriate change of variables.

25. $\displaystyle\iint_R \frac{x - 2y}{3x - y}\, dA$, where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$

26. $\displaystyle\iint_R (x + y)\, e^{x^2 - y^2}\, dA$, where R is the rectangle enclosed by the lines $x - y = 0$, $x - y = 2$, $x + y = 0$, and $x + y = 3$

27. $\displaystyle\iint_R \cos\left(\frac{y - x}{y + x}\right) dA$, where R is the trapezoidal region with vertices (1 ,0), (2, 0), (0, 2), and (0, 1)

28. $\displaystyle\iint_R \sin(9x^2 + 4y^2)\, dA$, where R is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$

29. $\displaystyle\iint_R e^{x+y}\, dA$, where R is given by the inequality $|x| + |y| \leq 1$

30. Let f be continuous on [0, 1] and let R be the triangular region with vertices (0, 0), (1, 0), and (0, 1). Show that

$$\iint_R f(x + y)\, dA = \int_0^1 uf(u)\, du$$

12 Review

Concepts and Vocabulary

1. Suppose f is a continuous function defined on a rectangle $R = [a, b] \times [c, d]$.

(a) Write an expression for a double Riemann sum of f. If $f(x, y) \geq 0$, what does the sum represent?

(b) Write the definition of $\iint_R f(x, y)\, dA$ as a limit.

(c) What is the geometric interpretation of $\iint_R f(x, y)\, dA$ if $f(x, y) \geq 0$? What if f takes on both positive and negative values?

(d) Explain how to evaluate $\iint_R f(x, y)\, dA$.

(e) Explain the Midpoint Rule for double integrals.

(f) Write an expression for the average value of f.

2. (a) How is $\iint_D f(x, y)\, dA$ defined if D is a bounded region that is not a rectangle?

(b) What is a type I region? Explain how to evaluate $\iint_D f(x, y)\, dA$ if D is a type I region.

(c) What is a type II region? Explain how to evaluate $\iint_D f(x, y)\, dA$ if D is a type II region.

(d) Write the properties of double integrals.

3. Explain the procedure to change from rectangular coordinates to polar coordinates in a double integral. Give a reason for making this change.

4. If a lamina occupies a plane region D and has density function $\rho(x, y)$, write expressions for each of the following in terms of double integrals.

(a) The mass

(b) The moments about the axes

(c) The center of mass

(d) The moments of inertia about the axes and the origin

5. Let f be a valid joint density function of a pair of continuous random variables X and Y.

(a) Write a double integral for the probability that X lies between a and b and Y lies between c and d.

(b) What two properties must f satisfy?

(c) Write expressions involving double integrals for the expected value of X and the expected value of Y.

6. Write an expression for the area of a surface S for each of the following cases.

(a) S is a parametric surface given by a vector function $\mathbf{r}(u, v)$, $(u, v) \in D$.

(b) S has the equation $z = f(x, y)$, $(x, y) \in D$.

(c) S is the surface of revolution obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis.

7. (a) Write the definition of the triple integral of f over a rectangular box B.

(b) Explain how to evaluate $\iiint_B f(x, y, z)\, dV$.

(c) Explain how $\iiint_E f(x, y, z)\, dV$ is defined if E is a bounded solid region that is not a box.

(d) What is a type 1 solid region? Explain how to evaluate $\iiint_E f(x, y, z)\, dV$ if E is a type 1 region.

(e) What is a type 2 solid region? Explain how to evaluate $\iiint_E f(x, y, z)\, dV$ if E is a type 2 region.

(f) What is a type 3 solid region? Explain how to evaluate $\iiint_E f(x, y, z)\, dV$ if E is a type 3 region.

8. Suppose a solid object occupies the region E and has density function $\rho(x, y, z)$. Write expressions for each of the following.

(a) The mass

(b) The moments about the coordinate planes

(c) The coordinates of the center of mass

(d) The moments of inertia about the axes

9. (a) Explain the procedure for changing from rectangular coordinates to cylindrical coordinates in a triple integral.

(b) Explain the procedure for changing from rectangular coordinates to spherical coordinates in a triple integral.

(c) In what situations would it be helpful to change to cylindrical or spherical coordinates?

10. (a) If a transformation T is given by $x = g(u, v)$, $y = h(u, v)$, write the Jacobian of T.

(b) Explain how to change variables in a double integral.

(c) Explain how to change variables in a triple integral.

True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. $\int_{-1}^{2}\int_{0}^{6} x^2 \sin(x-y)\, dx\, dy = \int_{0}^{6}\int_{-1}^{2} x^2 \sin(x-y)\, dy\, dx$

2. $\int_{0}^{1}\int_{0}^{x} \sqrt{x+y^2}\, dy\, dx = \int_{0}^{x}\int_{0}^{1} \sqrt{x+y^2}\, dx\, dy$

3. $\int_{1}^{2}\int_{3}^{4} x^2 e^y\, dy\, dx = \int_{1}^{2} x^2\, dx \int_{3}^{4} e^y\, dy$

4. $\int_{-1}^{1}\int_{0}^{1} e^{x^2+y^2} \sin y\, dx\, dy = 0$

5. If f is continuous on $[0, 1]$, then

$$\int_{0}^{1}\int_{0}^{1} f(x)\, f(y)\, dy\, dx = \left[\int_{0}^{1} f(x)\, dx\right]^2$$

6. $\int_{1}^{4}\int_{0}^{1} (x^2 + \sqrt{y}) \sin(x^2y^2)\, dx\, dy \leq 9$

7. If D is the disk given by $x^2 + y^2 \leq 4$, then

$$\iint_D \sqrt{4 - x^2 - y^2}\, dA = \frac{16}{3}\pi$$

8. The integral $\iiint_E kr^3\, dz\, dr\, d\theta$ represents the moment of inertia about the z-axis of a solid E with constant density k.

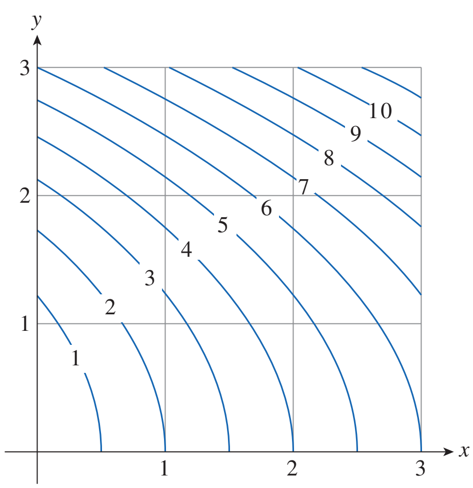
9. The integral

$$\int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2} dz\, dr\, d\theta$$

represents the volume enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

Exercises

1. A contour map is shown for a function f on the square $R = [0, 3] \times [0, 3]$. Use a Riemann sum with nine terms to estimate the value of $\iint_R f(x, y)\, dA$. Take sample points to be the upper right corners of the squares.



2. Use the Midpoint Rule to estimate the integral in Exercise 1.

Calculate the iterated integral.

3. $\int_{1}^{2}\int_{0}^{2} (y + 2xe^y)\, dx\, dy$

4. $\int_{0}^{1}\int_{0}^{1} ye^{xy}\, dx\, dy$

5. $\int_{0}^{1}\int_{0}^{x} \cos(x^2)\, dy\, dx$

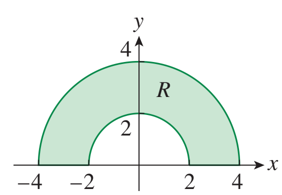
6. $\int_{0}^{1}\int_{x}^{e^x} 3xy^2\, dy\, dx$

7. $\int_{0}^{\pi}\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} y \sin x\, dz\, dy\, dx$

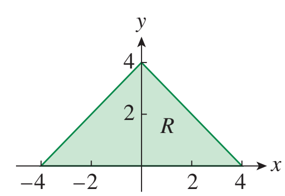
8. $\int_{0}^{1}\int_{0}^{y}\int_{x}^{1} 6xyz\, dz\, dy\, dx$

Write $\iint_R f(x, y)\, dA$ as an iterated integral, where R is the region shown and f is an arbitrary continuous function on R.

9.



10.



11. Describe the region whose area is given by the integral

$$\int_{0}^{\pi/2}\int_{0}^{\sin 2\theta} r\, dr\, d\theta$$

12. Describe the solid whose volume is given by the integral

$$\int_{0}^{\pi/2}\int_{0}^{\pi/2}\int_{1}^{2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

and evaluate the integral.

Calculate the iterated integral by first reversing the order of integration.

13. $\int_{0}^{1}\int_{x}^{1} \cos(y^2)\, dy\, dx$

14. $\int_{0}^{1}\int_{\sqrt{y}}^{1} \frac{ye^{x^2}}{x^3}\, dx\, dy$

Calculate the value of the multiple integral.

15. $\displaystyle\iint_R ye^{xy}\, dA$, where $R = \{(x, y) \mid 0 \le x \le 2, 0 \le y \le 3\}$

16. $\displaystyle\iint_D xy\, dA$, where $D = \{(x, y) \mid 0 \le y \le 1, y^2 \le x \le y + 2\}$

17. $\displaystyle\iint_D \frac{y}{1 + x^2}\, dA$, where D is bounded by $y = \sqrt{x}$, $y = 0$, $x = 1$

18. $\displaystyle\iint_D \frac{1}{1 + x^2}\, dA$, where D is the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$

19. $\displaystyle\iint_D y\, dA$, where D is the region in the first quadrant bounded by the parabolas $x = y^2$ and $x = 8 - y^2$

20. $\displaystyle\iint_D y\, dA$, where D is the region in the first quadrant that lies above the hyperbola $xy = 1$ and the line $y = x$ and below the line $y = 2$

21. $\displaystyle\iint_D (x^2 + y^2)^{3/2}\, dA$, where D is the region in the first quadrant bounded by the lines $y = 0$ and $y = \sqrt{3}\, x$ and the circle $x^2 + y^2 = 9$

22. $\displaystyle\iint_D x\, dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$

23. $\displaystyle\iiint_E xy\, dV$, where

$E = \{(x, y, z) \mid 0 \le x \le 3, 0 \le y \le x, 0 \le z \le x + y\}$

24. $\displaystyle\iiint_T xy\, dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $\left(\frac{1}{3}, 0, 0\right)$, $(0, 1, 0)$, and $(0, 0, 1)$

25. $\displaystyle\iiint_E y^2z^2\, dV$, where E is bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$

26. $\displaystyle\iiint_E z\, dV$, where E is bounded by the planes $y = 0$, $z = 0$, $x + y = 2$ and the cylinder $y^2 + z^2 = 1$ in the first octant

27. $\displaystyle\iiint_E yz\, dV$, where E lies above the plane $z = 0$, below the plane $z = y$, and inside the cylinder $x^2 + y^2 = 4$

28. $\displaystyle\iiint_H z^3\sqrt{x^2 + y^2 + z^2}\, dV$, where H is the solid hemisphere that lies above the xy-plane and has center at the origin and radius 1

Find the volume of the given solid.

29. Under the paraboloid $z = x^2 + 4y^2$ and above the rectangle $R = [0, 2] \times [1, 4]$

30. Under the surface $z = x^2y$ and above the triangle in the xy-plane with vertices $(1, 0)$, $(2, 1)$, and $(4, 0)$

31. The solid tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(0, 2, 0)$, and $(2, 2, 0)$

32. Bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$

33. One of the wedges cut from the cylinder $x^2 + 9y^2 = a^2$ by the planes $z = 0$ and $z = mx$

34. Above the paraboloid $z = x^2 + y^2$ and below the half-cone $z = \sqrt{x^2 + y^2}$

35. Consider a lamina that occupies the region D bounded by the parabola $x = 1 - y^2$ and the coordinate axes in the first quadrant with density function $\rho(x, y) = y$.

(a) Find the mass of the lamina.

(b) Find the center of mass.

(c) Find the moments of inertia about the origin and about the x- and y-axes.

36. A lamina occupies the part of the disk $x^2 + y^2 \le a^2$ that lies in the first quadrant.

(a) Find the centroid of the lamina.

(b) Find the center of mass of the lamina if the density function is $\rho(x, y) = xy^2$.

37. (a) Find the centroid of a right circular cone with height h and base radius a. (Place the cone so that its base is in the xy-plane with center at the origin and its axis along the positive z-axis.)

(b) Find the moment of inertia of the cone about its axis (the z-axis).

38. (a) Set up, but do not evaluate, an integral for the surface area of the parametric surface given by the vector function $\mathbf{r}(u, v) = v^2\mathbf{i} - uv\,\mathbf{j} + u^2\mathbf{k}$, $0 \le u \le 3$, $-3 \le v \le 3$.

(b) Use technology to find the surface area.

39. Find the area of the part of the surface $z = x^2 + y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$.

40. Use technology to graph the surface $z = x \sin y$, $-3 \le x \le 3$, $-\pi \le y \le \pi$, and to find its surface area.

41. Use polar coordinates to evaluate

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\, dy\, dx$$

42. Use spherical coordinates to evaluate

$$\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2 + y^2 + z^2}\, dz\, dx\, dy$$

43. If D is the region bounded by the curves $y = 1 - x^2$ and $y = e^x$, use technology to find the value of the integral $\iint_D y^2\, dA$.

44. Use technology to find the center of mass of the solid tetrahedron with vertices (0, 0, 0), (1, 0, 0), (0, 2, 0), (0, 0, 3) and density function $\rho(x, y, z) = x^2 + y^2 + z^2$.

45. The joint density function for the random variables X and Y is

$$f(x, y) = \begin{cases} C(x + y) & \text{if } 0 \le x \le 3,\ 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant C.

(b) Find $P(X \le 2,\ Y \ge 1)$.

(c) Find $P(X + Y \le 1)$.

46. A lamp has three bulbs, each of a type with average lifetime 800 hours. If we model the probability of failure of the bulbs by an exponential density function with mean 800, find the probability that all three bulbs fail within a total of 1000 hours.

47. Rewrite the integral

$$\int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$$

as an iterated integral in the order $dx\, dy\, dz$.

48. Give five other iterated integrals that are equal to

$$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x, y, z)\, dz\, dx\, dy$$

49. Use the transformation $u = x - y$, $v = x + y$ to evaluate

$$\iint_R \frac{x - y}{x + y}\, dA$$

where R is the square with vertices (0, 2), (1, 1) (2, 2), and (1, 3).

50. Use the transformation $x = u^2$, $y = v^2$, $z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

51. Use the change of variables formula and an appropriate transformation to evaluate $\iint_R xy\, dA$, where R is the square with vertices (0, 0), (1, 1), (2, 0) and (1, −1).

52. (a) Evaluate $\displaystyle\iint_D \frac{1}{(x^2 + y^2)^{n/2}}\, dA$ where n is an integer and D is the region bounded by the circles with center at the origin and radii r and R, $0 < r < R$.

(b) For what values of n does the integral in part (a) have a limit as $r \to 0^+$?

(c) Find $\displaystyle\iiint_E \frac{1}{(x^2 + y^2 + z^2)^{n/2}}\, dV$ where E is the region bounded by the spheres with center at the origin and radii r and R, $0 < r < R$.

(d) For what values of n does the integral in part (c) have a limit as $r \to 0^+$?

Focus on Problem Solving

1. If $[\![x]\!]$ denotes the greatest integer in x, evaluate the integral

$$\iint_R [\![x+y]\!]\,dA$$

where $R = \{(x, y) \mid 1 \le x \le 3,\ 2 \le y \le 5\}$

2. Evaluate the integral

$$\int_0^1 \int_0^1 e^{\max\{x^2,\, y^2\}}\,dy\,dx$$

where max $\{x^2, y^2\}$ means the larger of the numbers x^2 and y^2.

3. Find the average value of the function $f(x) = \displaystyle\int_x^1 \cos(t^2)\,dt$ on the interval $[0, 1]$.

4. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are constant vectors, $\mathbf{r}$ is the position vector $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and E is given by the inequalities $0 \le \mathbf{a}\cdot\mathbf{r} \le \alpha$, $0 \le \mathbf{b}\cdot\mathbf{r} \le \beta$, $0 \le \mathbf{c}\cdot\mathbf{r} \le \gamma$, show that

$$\iiint_E (\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})(\mathbf{c}\cdot\mathbf{r})\,dV = \frac{(\alpha\beta\gamma)^2}{8\,|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}$$

5. The double integral $\displaystyle\int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy$ is an improper integral and could be defined as the limit of double integrals over the rectangle $[0, t] \times [0, t]$ as $t \to 1^-$. But if we expand the integrand as a geometric series, we can express the integral as the sum of an infinite series. Show that

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

6. Leonhard Euler was able to find the exact sum of the series in Problem 5. In 1736, he proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

In this problem, you are asked to prove this fact by evaluating the double integral in Problem 5. Start by making the change of variables

$$x = \frac{u-v}{\sqrt{2}} \qquad y = \frac{u+v}{\sqrt{2}}$$

This provides a rotation about the origin through the angle $\pi/4$. You will need to sketch the corresponding region in the uv-plane.

Hint: If, in evaluating the integral, you encounter either of the expressions $\dfrac{1-\sin\theta}{\cos\theta}$ or $\dfrac{\cos\theta}{1+\sin\theta}$, you might consider using the identity $\cos\theta = \sin\left(\dfrac{\pi}{2} - \theta\right)$ and the corresponding identity for $\sin\theta$.

7. (a) Show that

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1 - xyz}\, dx\, dy\, dz = \sum_{i=1}^{\infty} \frac{1}{n^3}$$

(No one has ever been able to find the exact value of the sum of this series.)

(b) Show that

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1 + xyz}\, dx\, dy\, dz = \sum_{i=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$$

Use this equation to evaluate the triple integral.

8. Show that

$$\int_0^{\infty} \frac{\arctan \pi x - \arctan x}{x}\, dx = \frac{\pi}{2} \ln \pi$$

by first expressing the integral as an iterated integral.

9. If f is continuous, show that

$$\int_0^x \int_0^y \int_0^z f(t)\, dt\, dz\, dy = \frac{1}{2} \int_0^x (x - t)^2 f(t)\, dt$$

10. (a) A lamina has constant density ρ and takes the shape of a disk with center at the origin and radius R. Use Newton's Law of Gravitation (Section 6.6) to show that the magnitude of the force of attraction that the lamina exerts on a body with mass m located at the point $(0, 0, d)$ on the positive z-axis is

$$F = 2\pi G m \rho d \left(\frac{1}{d} - \frac{1}{\sqrt{R^2 + d^2}} \right)$$

Hint: Divide the disk as in Figure 12.41 in Section 12.4 and first compute the vertical component of the force exerted by the polar subrectangle R_{ij}.

(b) Show that the magnitude of the force of attraction of a lamina with density ρ that occupies an entire plane on an object with mass m located at a distance d from the plane is

$$F = 2\pi G m \rho$$

Notice that this expression does not depend on d.

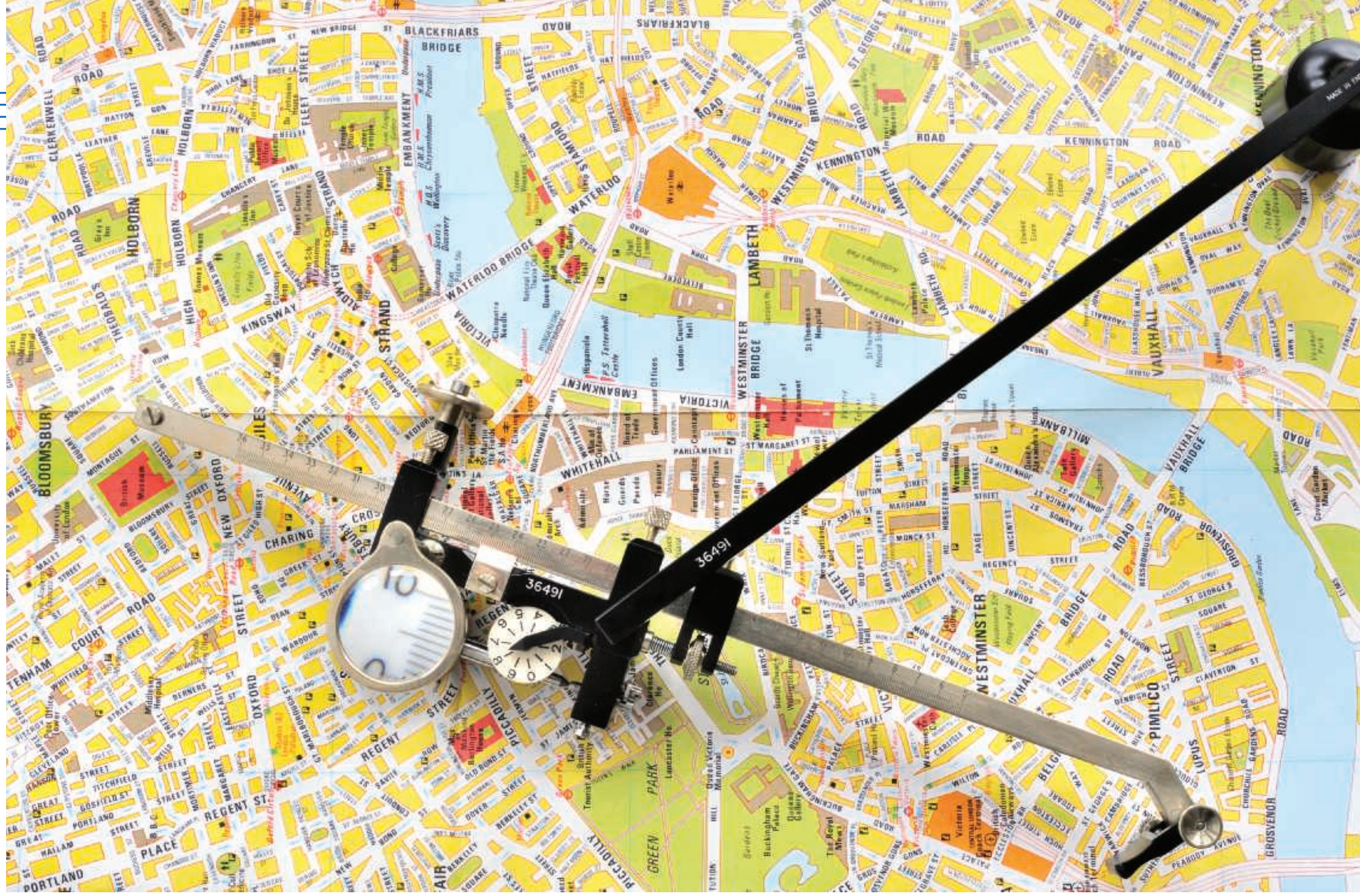
11. The plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad a > 0, \quad b > 0, \quad c > 0$$

cuts the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

into two pieces. Find the volume of the smaller piece.



godrick/Shutterstock.com

The concept of a planimeter was developed by Johann Martin Hermann in 1814 and was first constructed in 1854 by the Swiss mathematician Jakob Amsler-Laffon. This device is used to trace around an arbitrary two-dimensional shape and to measure the area of the enclosed region. The area measurement is based on Green's Theorem.

13 Vector Calculus

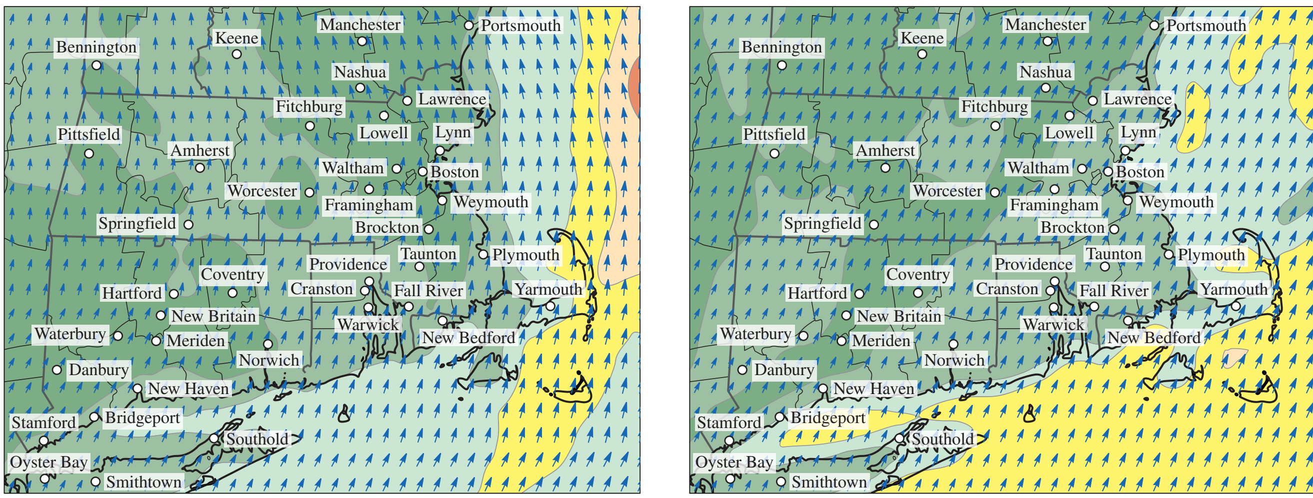
Contents

In this chapter, we study the calculus of vector fields. These are functions that assign vectors to points in space. In particular, we define line integrals (which can be used to find the work done by a force field in moving an object along a curve). Then we define surface integrals (which can be used to find the rate of fluid flow across a surface). The connections between these new types of integrals and the single, double, and triple integrals that we have already studied are given by the higher-dimensional versions of the Fundamental Theorem of Calculus: Green's Theorem, Stokes' Theorem, and the Divergence Theorem.

13.1 Vector Fields

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$

The vectors in Figure 13.1 are air velocity vectors that indicate the mean wind speed and direction at points in the New England area. Notice that the wind patterns on consecutive days are quite different. Every point on the map is associated with a wind velocity vector. This is an example of a *velocity vector field.*

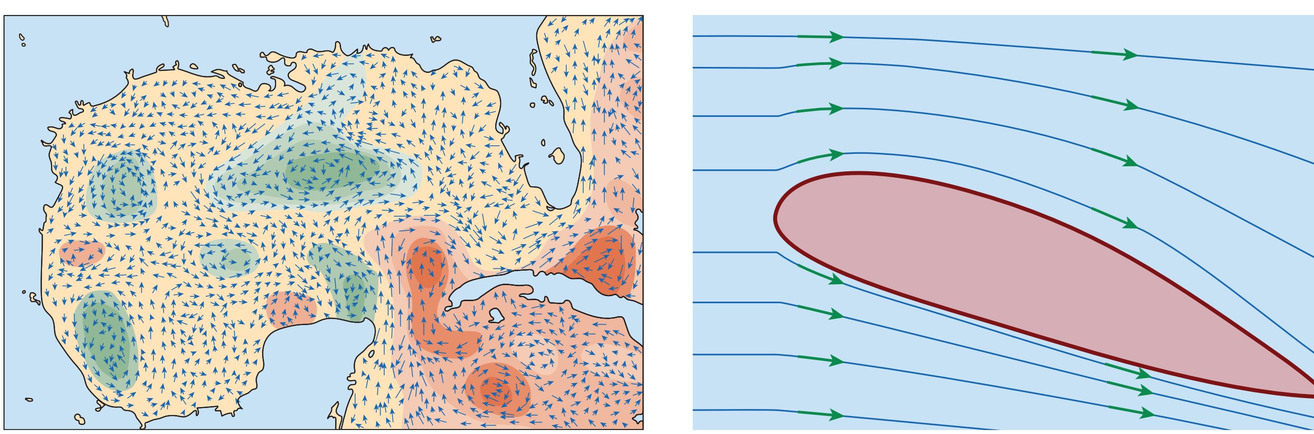


(a) 9:00 AM CST December 1, 2020

(b) 9:00 AM CST December 2, 2020

Figure 13.1
Velocity vector fields showing New England wind patterns. (weather.us)

Other examples of velocity vector fields are illustrated in Figure 13.2: ocean currents and air flow past an airfoil.



(a) Ocean currents in the Gulf of Mexico (NOAA)

(b) Airflow past an inclined air foil (Wikimedia Commons https://en.wikipedia.org/wiki/File:Airfoil-with-flow.png)

Figure 13.2
Velocity vector fields.

Another type of vector field, called a *force field*, associates a force vector with each point in a region. An example is the gravitational force field that we will consider in Example 4.

In general, a vector field is a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in V_2 (or V_3).

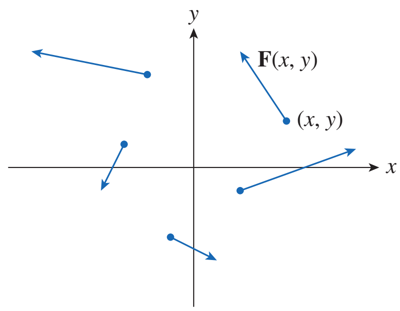


Figure 13.3
A vector field on $\mathbb{R}^2$.

Definition • Vector Field on $\mathbb{R}^2$

Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

The best way to picture a vector field is to draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y). Of course, it's impossible to draw arrows for all points (x, y), but we can construct a reasonable visualization of $\mathbf{F}$ by sketching a few representative points in D, as illustrated in Figure 13.3. Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

or, for short,
$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

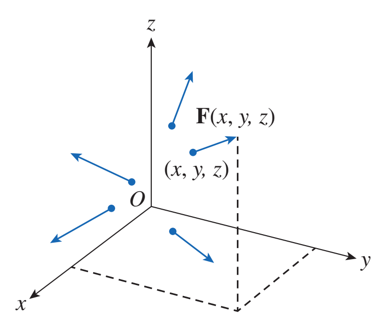


Figure 13.4
A vector field on $\mathbb{R}^3$.

Definition • Vector Field on $\mathbb{R}^3$

Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Figure 13.4 shows a vector field $\mathbf{F}$ on $\mathbb{R}^3$. We can also characterize $\mathbf{F}$ in terms of its component functions P, Q, and R as

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

As with the vector functions in Section 10.1, we can define continuity of vector fields and show that $\mathbf{F}$ is continuous if and only if its component functions P, Q, and R are continuous.

We sometimes identify a point (x, y, z) with its position vector $\mathbf{x} = \langle x, y, z\rangle$ and write $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x, y, z)$. Then $\mathbf{F}$ becomes a function that assigns a vector $\mathbf{F}(\mathbf{x})$ to a vector $\mathbf{x}$.

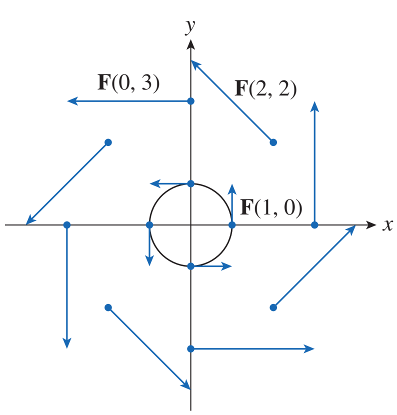


Figure 13.5
Several vectors in the vector field $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$.

Example 1 Sketch a Two-Dimensional Vector Field

A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Describe $\mathbf{F}$ by sketching several vectors $\mathbf{F}(x, y)$ as in Figure 13.5.

Solution

Consider $\mathbf{F}(1, 0) = -(0)\,\mathbf{i} + 1\,\mathbf{j} = \mathbf{j}$.
Draw the vector $\mathbf{j} = \langle 0, 1\rangle$ starting at the point (1, 0), as shown in Figure 13.5.

Continue in this manner; the table shows several other representative values of $\mathbf{F}(x, y)$ and the corresponding vectors are shown in Figure 13.5.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1\rangle$	$(-1, 0)$	$\langle 0, -1\rangle$
$(2, 2)$	$\langle -2, 2\rangle$	$(-2, 2)$	$\langle 2, -2\rangle$
$(3, 0)$	$\langle 0, 3\rangle$	$(-3, 0)$	$\langle 0, -3\rangle$
$(0, 1)$	$\langle -1, 0\rangle$	$(0, -1)$	$\langle 1, 0\rangle$
$(-2, 2)$	$\langle -2, -2\rangle$	$(2, -2)$	$\langle 2, 2\rangle$
$(0, 3)$	$\langle -3, 0\rangle$	$(0, -3)$	$\langle 3, 0\rangle$

Figure 13.5 suggests that each arrow is tangent to a circle with center at the origin.

To confirm this, consider the dot product of the position vector $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$ with the vector $\mathbf{F}(\mathbf{x}) = \mathbf{F}(x, y)$.

$$\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) = (x\,\mathbf{i} + y\,\mathbf{j}) \cdot (-y\,\mathbf{i} + x\,\mathbf{j}) = -xy + yx = 0$$

This shows that $\mathbf{F}(x, y)$ is perpendicular to the position vector $\langle x, y\rangle$ and is therefore tangent to a circle with center at the origin and radius $|\mathbf{x}| = \sqrt{x^2 + y^2}$.

Notice also that $|\mathbf{F}(x, y)| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = |\mathbf{x}|$.

Therefore, the magnitude of the vector $\mathbf{F}(x, y)$ is equal to the radius of the circle. ■

Some graphing calculators and computer algebra systems are capable of plotting vector fields in two or three dimensions. These graphs provide a better visualization of the vector field because a large number of representative vectors are presented. Figure 13.6 shows a computer plot of the vector field in Example 1; Figures 13.7 and 13.8 show two other vector fields. In these figures, each vector is scaled proportional to its magnitude.

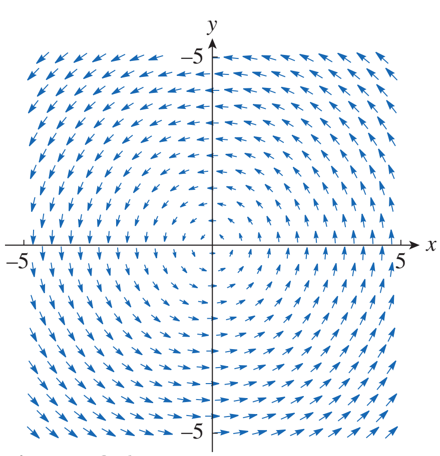


Figure 13.6
The vector field for $\mathbf{F}(x, y) = \langle -y, x\rangle$.

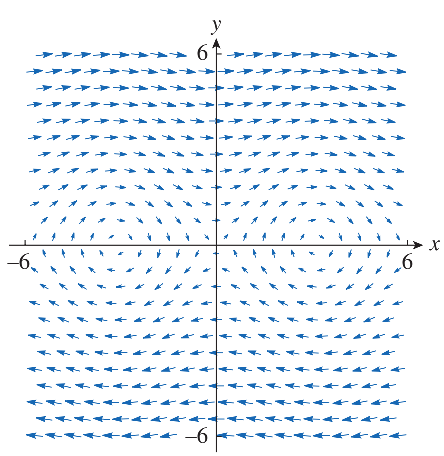


Figure 13.7
The vector field for $\mathbf{F}(x, y) = \langle y, \sin x\rangle$.

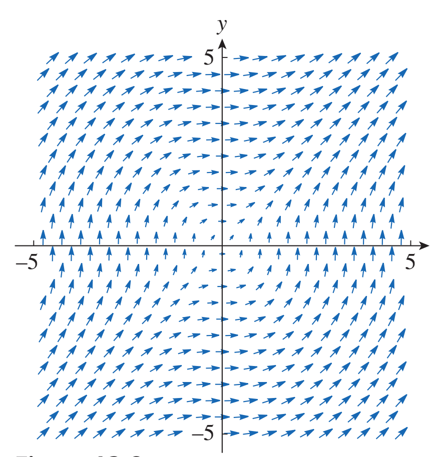


Figure 13.8
The vector field for $\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2)\rangle$.

Example 2 Sketch a Three-Dimensional Vector Field

Sketch the vector field on $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = z\,\mathbf{k}$.

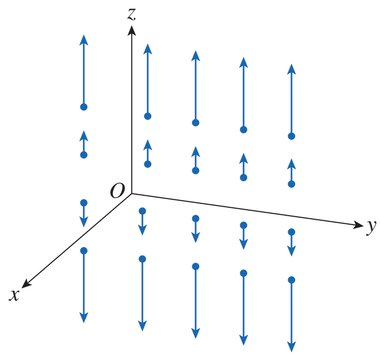


Figure 13.9
Several vectors in the vector field
$\mathbf{F}(x, y, z) = z\mathbf{k}$.

Solution

The sketch is shown in Figure 13.9.

All vectors are vertical and point upward above the xy-plane and downward below it.

The magnitude of each vector increases with the distance from the xy-plane. ■

We were able to draw the vector field in Example 2 because the formula was straightforward. However, most three-dimensional vector fields are almost impossible to sketch without technology. Some examples are shown in Figures 13.10, 13.11, and 13.12. Notice that the vector fields in Figures 13.10 and 13.11 have similar formulas, but all the vectors in Figure 13.11 point in the general direction of the negative y-axis because their y-components are all -2. If the vector field in Figure 13.12 represents a velocity field, then a particle would be swept upward and would spiral around the z-axis in the clockwise direction as viewed from above.

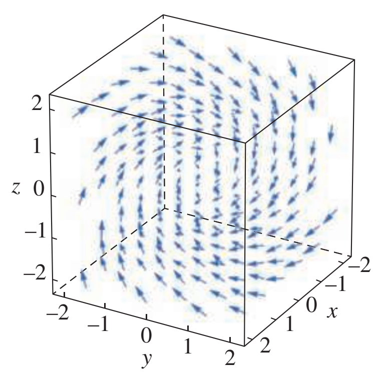


Figure 13.10
The vector field for
$\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.

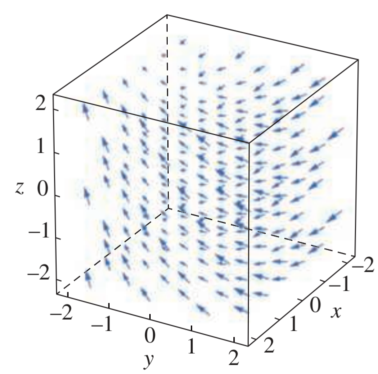


Figure 13.11
The vector field for
$\mathbf{F}(x, y, z) = y\mathbf{i} - 2\mathbf{j} + x\mathbf{k}$.

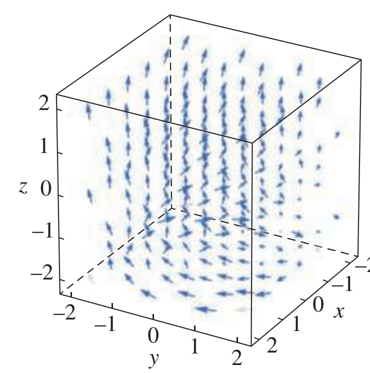


Figure 13.12
The vector field for
$\mathbf{F}(x, y, z) = \frac{y}{z}\mathbf{i} - \frac{x}{z}\mathbf{j} + \frac{z}{4}\mathbf{k}$.

Example 3 Velocity Fields

Suppose a fluid is flowing steadily along a pipe and let $\mathbf{V}(x, y, z)$ be the velocity vector at a point (x, y, z). Then $\mathbf{V}$ assigns a vector to each point (x, y, z) in a certain domain E (the interior of the pipe); $\mathbf{V}$ is a vector field on $\mathbb{R}^3$ called a **velocity field**. A possible velocity field is illustrated in Figure 13.13. The speed at any given point is indicated by the length of the arrow.

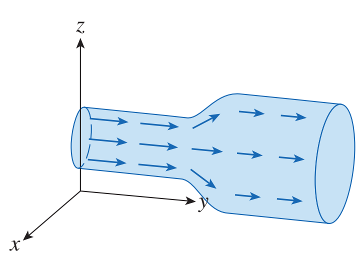


Figure 13.13
A velocity field in fluid flow.

Velocity fields also occur in other areas of physics. For instance, the vector field in Example 1 could be used as the velocity field describing the counterclockwise rotation of a wheel. We have seen other examples of velocity fields in Figures 13.1 and 13.2. ■

Example 4 The Gravitational Field

Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{mM\,G}{r^2}$$

This is an example of an inverse square law.

where R is the distance between the objects and G is the gravitational constant.
Let's assume that the object with mass M is located at the origin in $\mathbb{R}^3$. (For instance, M could be the mass of Earth and the origin would be at its center.)

Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z \rangle$.

Then $r = |\mathbf{x}| \quad \Rightarrow \quad r^2 = |\mathbf{x}|^2$.

The gravitational force exerted on this second object acts toward the origin.

The unit vector in this direction is $-\dfrac{\mathbf{x}}{|\mathbf{x}|}$.

Physicists often use the notation $\mathbf{r}$ instead of $\mathbf{x}$ for the position vector, so Equation 3 may be written in the form $\mathbf{F} = -\dfrac{mMG}{r^3}\mathbf{r}$.

Therefore, the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z \rangle$ is

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x} \tag{1}$$

The function given by Equation 1 is an example of a vector field, called the **gravitational field**, because it associates a vector [the force $\mathbf{F}(\mathbf{x})$] with every point in space.

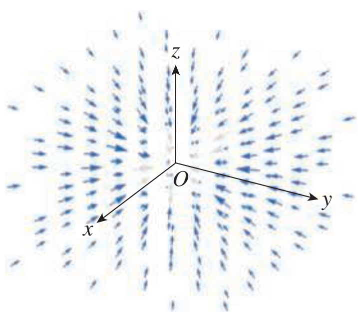


Figure 13.14
Gravitational force field.

Equation 1 is a compact way of writing the gravitational field, but we can also write this equation in terms of its component functions by using the facts that $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$.

$$\mathbf{F}(x, y, z) = \frac{-mM\,Gx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mM\,Gy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mM\,Gz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

The gravitational field $\mathbf{F}$ is illustrated in Figure 13.14. ■

Example 5 Electric Force Fields

Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = \langle x, y, z \rangle$ is

$$\mathbf{F}(\mathbf{x}) = \frac{\epsilon qQ}{|\mathbf{x}|^3}\mathbf{x} \tag{2}$$

where ϵ is a constant (that depends on the units used).

For like charges, we have $qQ > 0$ and the force is repulsive.

For unlike charges, we have $qQ < 0$ and the force is attractive.

Notice the similarity between Equations 1 and 2. Both vector fields are examples of **force fields**.

Instead of considering the electric force $\mathbf{F}$, physicists often consider the force per unit charge:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q}\,\mathbf{F}(\mathbf{x}) = \frac{\epsilon Q}{|\mathbf{x}|^3}\,\mathbf{x}.$$

Then $\mathbf{E}$ is a vector field on $\mathbb{R}^3$ called the **electric field** of Q. ■

Gradient Fields

If f is a scalar function of two variables, recall from Section 11.6 that its gradient ∇f (grad f) is defined by

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Therefore, ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

Example 6 Sketch a Gradient Vector Field

Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are the two graphs related?

Solution

The gradient vector field is given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = 2xy\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}.$$

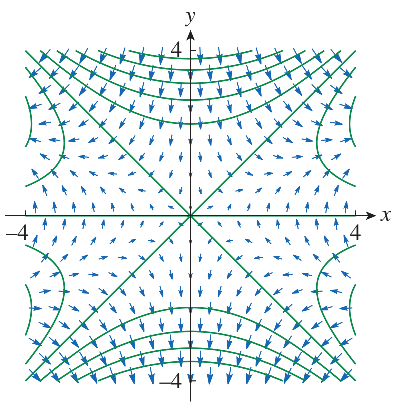


Figure 13.15
A contour map of f with the gradient vector field.

Figure 13.15 shows a contour map of f with the gradient vector field. Notice that the gradient vectors are perpendicular to the level curves, as we would expect from Section 11.6.

Notice also that the gradient vectors are long where the level curves are close to each other and short where the curves are farther apart. That's because the length of the gradient vector is the value of the directional derivative of f and closely spaced level curves indicate a steep graph. ■

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this case, f is called a **potential function** of $\mathbf{F}$.

Not all vector fields are conservative, but such fields do arise frequently in physics. For example, the gravitational field $\mathbf{F}$ in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\\ &= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}\\ &= \mathbf{F}(x, y, z)\end{aligned}$$

In Sections 13.3 and 13.5, we will learn how to determine whether or not a given vector field is conservative.

13.1 Exercises

Sketch the vector field **F** by drawing a diagram like Figure 13.5 or Figure 13.9.

1. $\mathbf{F}(x, y) = 0.3\mathbf{i} - 0.4\mathbf{j}$

2. $\mathbf{F}(x, y) = \frac{1}{2}x\,\mathbf{i} + y\,\mathbf{j}$

3. $\mathbf{F}(x, y) = y\,\mathbf{i} + \frac{1}{2}\mathbf{j}$

4. $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + x\,\mathbf{j}$

5. $\mathbf{F}(x, y) = \mathbf{i} - 2\,\mathbf{j}$

6. $\mathbf{F}(x, y) = x\,\mathbf{i} + (x + y)\,\mathbf{j}$

7. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

8. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

9. $\mathbf{F}(x, y, z) = \mathbf{k}$

10. $\mathbf{F}(x, y, z) = -y\mathbf{k}$

11. $\mathbf{F}(x, y, z) = x\,\mathbf{k}$

12. $\mathbf{F}(x, y, z) = \mathbf{j} - \mathbf{i}$

Match the vector fields **F** with the plots labeled I–VI. Give reasons for your choices.

13. $\mathbf{F}(x, y) = \langle y, x\rangle$

14. $\mathbf{F}(x, y) = \langle 1, \sin y\rangle$

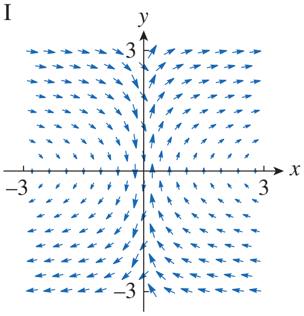
15. $\mathbf{F}(x, y) = \langle x - 2, x + 1\rangle$

16. $\mathbf{F}(x, y) = \langle y, 1/x\rangle$

17. $\mathbf{F}(x, y) = \langle \sin x, \cos y\rangle$

18. $\mathbf{F}(x, y) = \langle 1, e^{-y}\rangle$

I



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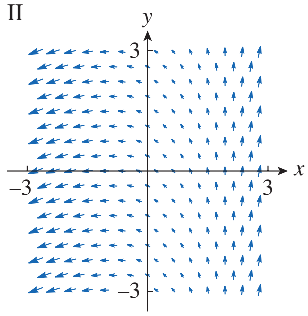

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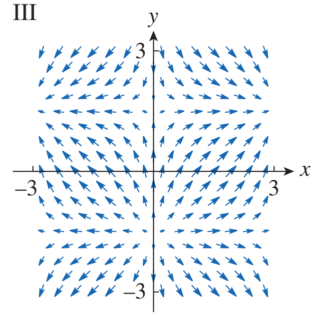

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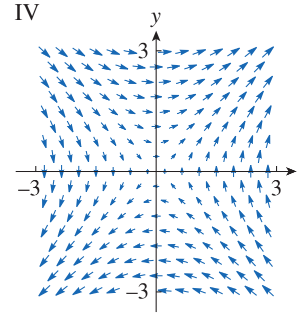

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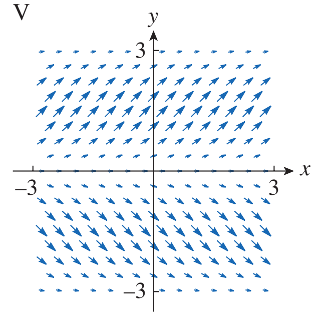
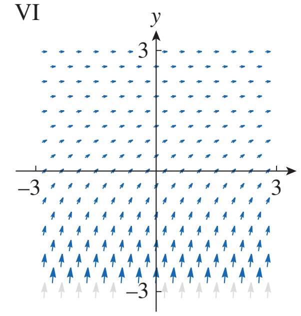


VI



Match the vector fields **F** on $\mathbb{R}^3$ with the plots labeled I–IV. Give reasons for your choices.

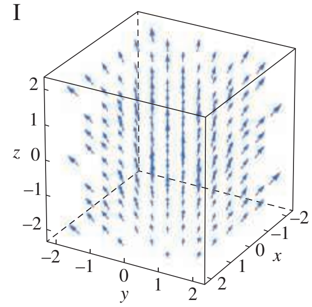
19. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$

20. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$

21. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$

22. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

I



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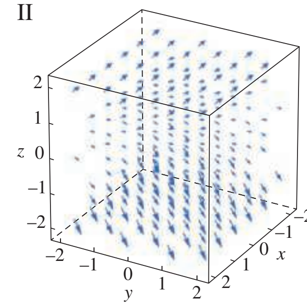

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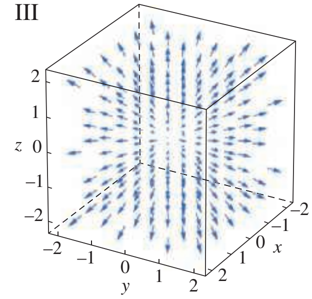

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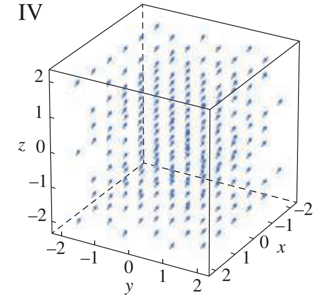


23. Use technology to plot the vector field.

$$\mathbf{F}(x, y) = (y^2 - 2xy)\,\mathbf{i} + (3xy - 6x^2)\,\mathbf{j}$$

Explain the appearance by finding the set of points (x, y) such that $\mathbf{F}(x, y) = \mathbf{0}$.

24. Let $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$, where $\mathbf{x} = \langle x, y\rangle$ and $r = |\mathbf{x}|$. Use technology to plot this vector field in various domains until you can recognize a pattern. Describe the appearance of the plot and explain it by finding points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

Find the gradient vector field of f.

25. $f(x, y) = xe^{xy}$

26. $f(x, y) = \tan(3x - 4y)$

27. $f(x, y) = y\sin(xy)$

28. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

29. $f(x, y, z) = x\ln(y - 2z)$

30. $f(x, y, z) = x^2ye^{y/z}$

Find the gradient vector field ∇f of f and sketch it.

31. $f(x, y) = x^2 - y$

32. $f(x, y) = \sqrt{x^2 + y^2}$

Use technology to plot the gradient vector field of f together with a contour map of f. Explain how they are related to each other.

33. $f(x, y) = \sin x + \sin y$

34. $f(x, y) = \sin(x + y)$

35. $f(x, y) = \ln(1 + x^2 + 2y^2)$

36. $f(x, y) = \cos x - 2\sin y$

Match the function f with the plots of their gradient vector fields labeled I–IV. Give reasons for your choices.

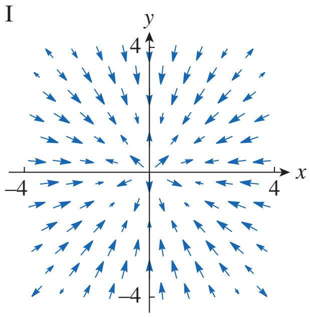
37. $f(x, y) = x^2 + y^2$

38. $f(x, y) = x(x + y)$

39. $f(x, y) = (x + y)^2$

40. $f(x, y) = \sin\sqrt{x^2 + y^2}$

I



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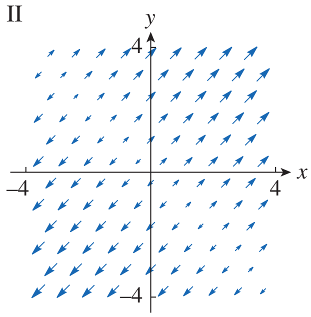

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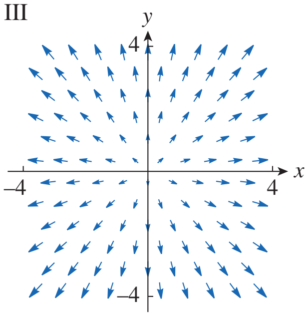
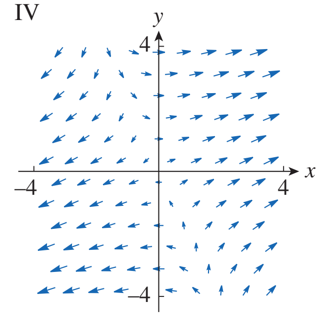


IV



41. A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2\rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

42. At time $t = 1$, a particle is located at position $(1, 3)$. If it moves in a velocity field

$$\mathbf{F}(x, y) = \langle xy - 2, y^2 - 10\rangle$$

find its approximate location at time $t = 1.05$.

43. The **flow lines** (or **streamlines**) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Therefore, the vectors in a vector field are tangent to the flow lines.

(a) Use a sketch of the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ to draw some flow lines. Use your sketches to guess the equations of the flow lines.

(b) If parametric equations of a flow line are $x = x(t)$, $y = y(t)$, explain why these functions satisfy the differential equations $\dfrac{dx}{dt} = x$ and $\dfrac{dy}{dt} = -y$. Then solve the differential equations to find an equation of the flow line that passes through the point $(1, 1)$.

44. (a) Sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\,\mathbf{j}$ and then sketch some flow lines. What shape do these flow lines appear to have?

(b) If parametric equations of the flow lines are $x = x(t)$, $y = y(t)$, what differential equations do these functions satisfy? Deduce that $\dfrac{dy}{dx} = x$.

(c) If a particle starts at the origin in the velocity field given by $\mathbf{F}$, find an equation of the path it follows.

13.2 Line Integrals

In this section, we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C. These integrals are called *line integrals*, even though *curve integrals* might be better terminology. They originated in the early 19th century and were devised to solve problems involving fluid flow, forces, electricity, and magnetism.

Line Integrals in the Plane

Consider a plane curve C given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b \tag{1}$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, and assume that C is a smooth curve. [This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$. See Section 10.3.] If we divided the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. (See Figure 13.16.)

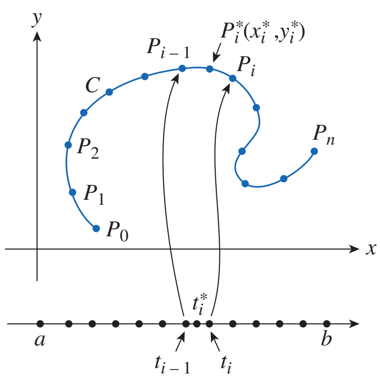


Figure 13.16
The curve C divided into n subarcs.

Choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc. (This point corresponds to a value t_i^* in $[t_{i-1}, t_i]$.) If f is any function of two variables whose domain includes the curve C, then we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\ \Delta s_i$$

which is similar to a Riemann sum. Then take the limit of these sums and make the following definition by analogy with a single integral.

Definition • Line Integral

If f is defined on a smooth curve C given by $x = x(t)$, $y = y(t)$, $a \le t \le b$, then the **line integral of f along C** is

$$\int_C f(x, y)\ ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\ \Delta s_i$$

if this limit exists.

In Section 6.4, we found that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

A similar type of argument can be used to show that if f is a continuous function, then the limit in the definition of a line integral always exists and the following formula can be used to evaluate the line integral:

$$\int_C f(x, y)\ ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \tag{2}$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

The arc length function is discussed in Section 10.3.

If $s(t)$ is the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

So, a way to remember Equation 2 is to express everything in terms of the parameter t. Use the parametric equations to express x and y in terms of t and write ds as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In the special case where C is the line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, we can write the parametric equations of C as follows: $x = x$, $y = 0$, $a \le x \le b$. Equation 2 then becomes

$$\int_C f(x, y)\, ds = \int_a^b f(x, 0)\, dx$$

and so the line integral reduces to an ordinary single integral in this case.

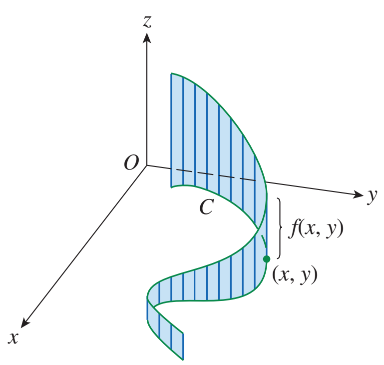


Figure 13.17
A geometric interpretation of a line integral of a positive function.

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area. In fact, if $f(x, y) \ge 0$, $\int_C f(x, y)\, ds$ represents the area of one side of the *fence* or *curtain* in Figure 13.17, whose base is C and whose height above the point (x, y) is $f(x, y)$.

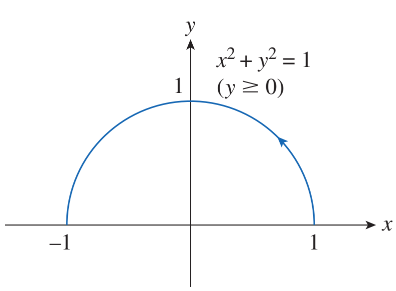


Figure 13.18
The upper half of the unit circle centered at the origin.

Example 1 Integrating Along a Semicircle

Evaluate $\int_C (2 + x^2y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

Solution

In order to use Equation 2, we need parametric equations to represent C.

Recall that the unit circle can be parameterized using the equations $x = \cos t$, $y = \sin t$.

The upper half of the circle is described by the parameter interval $0 \le t \le \pi$. (See Figure 13.18.)

Use Equation 2:

$$\int_C (2 + x^2y)\, ds = \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{Equation 2.}$$

$$= \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}\, dt \qquad \text{Use } x'(t) = -\sin t \text{ and } y'(t) = \cos t.$$

$$= \int_0^\pi (2 + \cos^2 t \sin t)\, dt = \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi \qquad \text{Simplify; antiderivative.}$$

$$= 2\pi + \frac{2}{3} \qquad \text{FTC; simplify.}$$

■

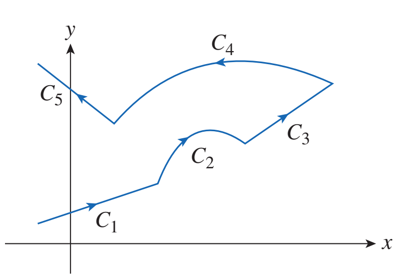


Figure 13.19
A piecewise-smooth curve.

Suppose now that C is a **piecewise-smooth curve**: that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where, as illustrated in Figure 13.19, the initial point of C_{i+1} is the terminal point of C_i. Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds$$

Example 2 Integrating Along a Piecewise-Smooth Curve

Evaluate $\int_C 2x\ ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1), followed by the vertical line segment C_2 from (1, 1) to (1, 2).

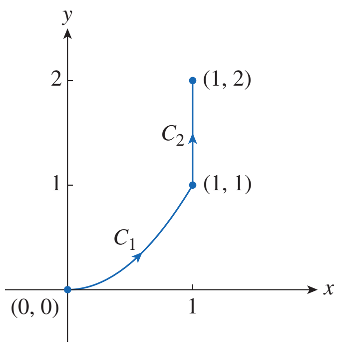


Figure 13.20
The curve $C = C_1 \cup C_2$.

Solution

The curve C is shown in Figure 13.20.

C_1 is a function of x; choose x as the parameter.

The equations for C_1 become: $x = x$, $y = x^2$, $0 \le x \le 1$.

Find the line integral along C_1.

$$\int_{C_1} 2x\ ds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\ dx \qquad \text{Equation 2.}$$

$$= \int_0^1 2x\sqrt{1 + 4x^2}\ dx \qquad \text{Use derivatives.}$$

$$= \frac{1}{4}\cdot\frac{2}{3}(1 + 4x^2)^{3/2}\Big]_0^1 = \frac{5\sqrt{5} - 1}{6} \qquad \text{Antiderivative; FTC; simplify.}$$

Choose y as the parameter on the curve C_2.

The equations for C_2 are $x = 1$, $y = y$, $1 \le y \le 2$.

The line integral along C_2 is:

$$\int_{C_2} 2x\ ds = \int_1^2 2(1)\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\ dy = \int_1^2 2\ dy = 2.$$

The line integral along C is:

$$\int_C 2x\ ds = \int_{C_1} 2x\ ds + \int_{C_2} 2x\ ds = \frac{5\sqrt{5} - 1}{6} + 2.$$ ■

Any physical interpretation of a line integral $\int_C f(x, y)\ ds$ depends on the physical interpretation of the function f. Suppose a thin wire is shaped like a curve C and the linear density at a point (x, y) is $\rho(x, y)$. Then the mass of the part of the wire from P_{i-1} to P_i in Figure 13.16 is approximately $\rho(x_i^*, y_i^*)\Delta s_i$ and so the total mass of the wire is approximately $\sum_{i=1}^n \rho(x_i^*, y_i^*)\Delta s_i$.

By taking more and more points on the curve, we obtain the **mass** m of the wire as the limiting value of these approximations:

$$m = \lim_{n\to\infty} \sum_{i=1}^n \rho(x_i^*, y_i^*)\,\Delta s_i = \int_C \rho(x, y)\ ds$$

For example, if $f(x, y) = 2 + x^2y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire. The **center of mass** of the wire with density function ρ is located at the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{m}\int_C x\rho(x, y)\ ds \qquad \bar{y} = \frac{1}{m}\int_C y\rho(x, y)\ ds \tag{3}$$

Other physical interpretations of line integrals will be discussed later in this chapter.

Example 3 Center of Mass of a Wire

A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \ge 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

Solution

As in Example 1, use the parametrization

$x = \cos t,\ y = \sin t,\ 0 \le t \le \pi \quad \Rightarrow \quad ds = dt.$

The linear density is $\rho(x, y) = k(1 - y)$ where k is a constant.

Find the mass of the wire.

$$m = \int_C k(1 - y)\ ds = \int_0^\pi k(1 - \sin t)\ dt \qquad \text{Equation 2.}$$

$$= k\Big[t + \cos t\Big]_0^\pi = k(\pi - 2) \qquad \text{Antiderivative; FTC.}$$

Find the center of mass.

$$\bar{y} = \frac{1}{m}\int_C y\rho(x, y)\ ds = \frac{1}{k(\pi + 2)}\int_C yk(1 - y)\ ds \qquad \text{Equation 3; use expressions for } m \text{ and } \rho.$$

$$= \frac{1}{\pi - 2}\int_0^\pi (\sin t - \sin^2 t)\ dt \qquad \text{Equation 4.}$$

$$= \frac{1}{\pi - 2}\left[-\cos t - \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^\pi \qquad \text{Antiderivative.}$$

$$= \frac{4 - \pi}{2(\pi - 2)} \qquad \text{FTC; simplify.}$$

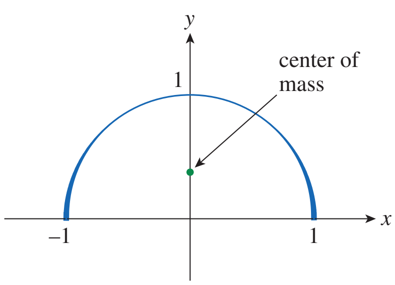


Figure 13.21
The wire in the shape of a semicircle and its center of mass.

By symmetry, $\bar{x} = 0$, so the center of mass is $\left(0,\ \dfrac{4 - \pi}{2(\pi - 2)}\right) \approx (0,\ 0.376)$.

Figure 13.21 shows a graph of the wire and the center of mass. ■

Line Integrals with Respect to *x* or *y*

Two additional line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in the definition of a line integral. They are called the **line integrals of *f* along *C* with respect to *x* and *y***:

$$\int_C f(x, y)\ dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\ \Delta x_i \tag{4}$$

$$\int_C f(x, y)\ dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\ \Delta y_i \tag{5}$$

To distinguish the original line integral $\int_C f(x, y)\ ds$ from those in Equations 4 and 5, it is called the **line integral with respect to arc length**.

The following formulas indicate that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t: $x = x(t)$, $y = y(t)$, $dx = x'(t)\ dt$, $dy = y'(t)\ dt$

$$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt \qquad (6)$$
$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

Line integrals with respect to x and y frequently occur together in real-world applications. When this happens, it is customary to abbreviate by writing

$$\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

When setting up a line integral, often the most difficult step is to write a parametric representation for a curve whose geometric description is given. In particular, we often need to parameterize a line segment. So, it is useful to remember that a vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1 \quad 0 \le t \le 1 \qquad (7)$$

(See Equation 9.5.4.)

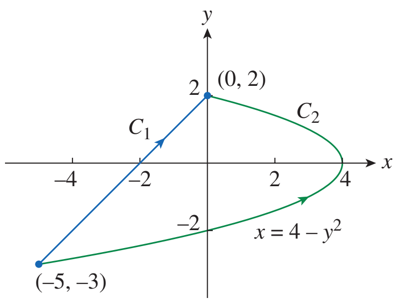


Figure 13.22
The curves C_1 and C_2.

Example 4 Integrate Along Two Curves with the Same Endpoints

Evaluate $\int_C y^2\, dx + x\, dy$, where (a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. (See Figure 13.22.)

Solution

(a) Use Equation 7 with $\mathbf{r}_0 = \langle -5, -3\rangle$ and $\mathbf{r}_1 = \langle 0, 2\rangle$ to write a parametric representation for the line segment:

$x = 5t - 5, \quad y = 5t - 3, \quad 0 \le t \le 1.$

Then $dx = 5\ dt$, $dy = 5dt$. Use the Equations in 6:

$$\int_{C_1} y^2\, dx + x\, dy = \int_0^1 (5t - 3)^2(5\ dt) + (5t - 5)(5\ dt) \qquad \text{Equations in 6.}$$
$$= 5\int_0^1 (25t^2 - 25t + 4)\ dt \qquad \text{Simplify integrand.}$$
$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = -\frac{5}{6} \qquad \text{Antiderivative; FTC.}$$

(b) Since the parabola is given as a function of y, let y be the parameter and write C_2 as

$x = 4 - y^2, \quad y = y, \quad -3 \le y \le 2.$

Then $dx = -2y\ dy$ and we can use the Equations in 7.

$$\int_{C_2} y^2\, dx + x\, dy = \int_{-3}^2 y^2(-2y)\ dy + (4 - y^2)\ dy \qquad \text{Equations in 6.}$$
$$= \int_{-3}^2 (-2y^3 - y^2 + 4)\ dy \qquad \text{Simplify integrand.}$$
$$= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right]_{-3}^2 = \frac{245}{6} \qquad \text{Antiderivative; FTC.}$$

■

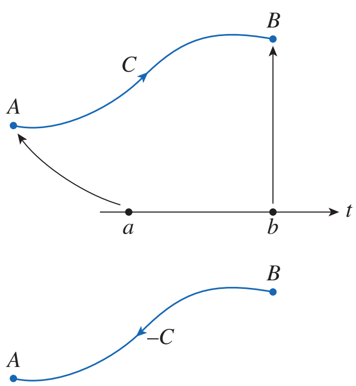


Figure 13.23
The curves C and $-C$.

Notice that we got different answers in parts (a) and (b) of Example 4 even though the two curves had the same endpoints. Therefore, in general, the value of a line integral depends not just on the endpoints of the curve but also on the path. (But see Section 13.3 for conditions under which the integral is independent of the path.)

Notice also that the answers in Example 4 depend on the direction, or orientation, of the curve. If $-C_1$ denotes the line segment from $(0, 2)$ to $(-5, -3)$, you can verify, using the parametrization

$$x = -5t \qquad y = 2 - 5t \qquad 0 \le t \le 1$$

that

$$\int_{-C_1} y^2\, dx + x\, dy = \frac{5}{6}$$

In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t. For example, in Figure 13.23, the initial point A corresponds to the parameter value a and the terminal point B corresponds to $t = b$.

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 13.23), then we have

$$\int_{-C} f(x, y)\, dx = -\int_C f(x, y)\, dx \qquad \int_{-C} f(x, y)\, dy = -\int_C f(x, y)\, dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y)\, ds = \int_C f(x, y)\, ds$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C.

Line Integrals in Space

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

or by a vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. If f is a function of three variables that is continuous on some region containing C, then we define the **line integral of f along C** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta s_i$$

We evaluate this line integral using an equation similar to Equation 3:

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \qquad (8)$$

Observe that the integrals in both Equations 2 and 8 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt$$

For the special case $f(x, y, z) = 1$, we get

$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\,dt = L$$

where L is the length of the curve C (see Formula 10.3.3).

Line integrals along C with respect to x, y, and z can also be defined. For example,

$$\int_C f(x, y, z)\,dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta z_i$$

$$= \int_a^b f(x(t), y(t), z(t))\,z'(t)\,dt$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\int_C P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz \qquad (9)$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

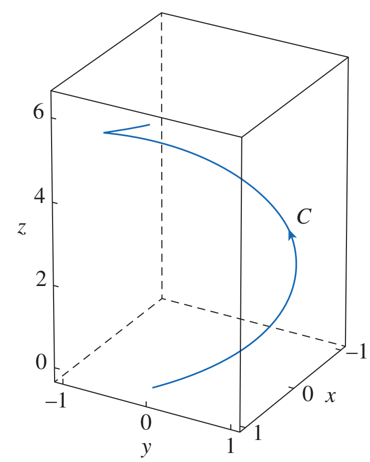


Figure 13.24
The curve C is a circular helix.

Example 5 A Line Integral in Space

Evaluate $\int_C y \sin z\,ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$ (as shown in Figure 13.24.)

Solution

Use Equation 8 to evaluate the integral.

$$\int_C y \sin z\,ds = \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \qquad \text{Equation 8.}$$

$$= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1}\,dt \qquad \text{Use derivatives.}$$

$$= \sqrt{2} \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t)\,dt \qquad \text{Simplify integrand.}$$

$$= \frac{\sqrt{2}}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{2\pi} = \sqrt{2}\pi \qquad \text{Antiderivative; FTC.}$$

■

Example 6 A Line Integral Along Two Curves

Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from (2, 0, 0) to (3, 4, 5), followed by the vertical line segment C_2 from (3, 4, 5) to (3, 4, 0).

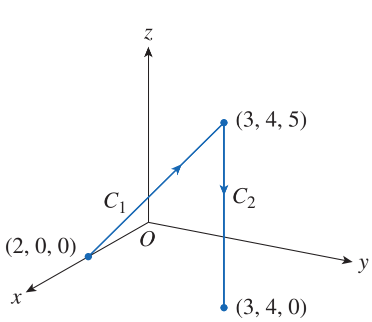


Figure 13.25
The curve $C = C_1 \cup C_2$.

Solution

The curve C is shown in Figure 13.25.

Use Equation 7 to write C_1 as

$$\mathbf{r}(t) = (1 - t)\langle 2, 0, 0\rangle + t\langle 3, 4, 5\rangle = \langle 2 + t, 4t, 5t\rangle$$

or, in parametric form, as

$$x = 2 + t \quad y = 4t \quad z = 5t \quad 0 \le t \le 1$$

Evaluate the integral along C_1 in terms of t.

$$\begin{aligned}\int_{C_1} y\,dx + z\,dy + x\,dz &= \int_0^1 (4t)\,dt + (5t)4\,dt + (2+t)5\,dt \\ &= \int_0^1 (10+29t)\,dt = \left[10t + 29\frac{t^2}{2}\right]_0^1 = 24.5\end{aligned}$$

Similarly, the line segment C_2 can be written in the form

$$\mathbf{r}(t) = (1-t)\langle 3, 4, 5\rangle + t\langle 3, 4, 0\rangle = \langle 3, 4, 5-5t\rangle$$

or $\quad x = 3 \quad y = 4 \quad z = 5 - 5t \quad 0 \le t \le 1$

Then $dx = 0 = dy$, so the integral along C_2 is

$$\int_{C_2} y\,dx + z\,dy + x\,dz = \int_0^1 3(-5)\,dt = -15.$$

Add the values of these two integrals to obtain

$$\int_C y\,dx + z\,dy + x\,dz = 24.5 - 15 = 9.5.$$

Line Integrals of Vector Fields

Recall from Section 6.6 that the work done by a variable force $f(x)$ in moving a particle from a to b along the x-axis is $W = \int_a^b f(x)\,dx$. Then in Section 11.3, we found that the work done by a constant force $\mathbf{F}$ in moving an object from a point P to another point Q in space is $W = \mathbf{F} \cdot \mathbf{D}$, where $D = \overrightarrow{PQ}$ is the displacement vector.

Now suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$, such as the gravitational field of Example 4 in Section 13.1 or the electric force field of Example 5 in Section 13.1. (A force field on $\mathbb{R}^2$ could be considered a special case where $R = 0$ and P and Q depend only on x and y.) We would like to compute the work done by this force in moving a particle along a smooth curve C.

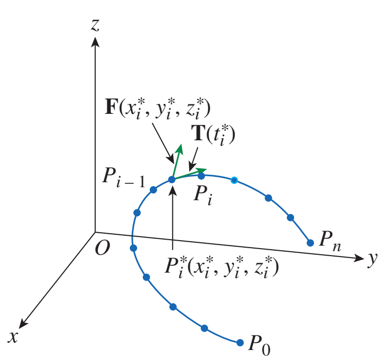


Figure 13.26
The subarcs of C and an arbitrary point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the ith subarc.

Start by dividing the parameter interval $[a, b]$ into subintervals of equal width. This leads to the division of C into subarcs $P_{i-1}P_i$ with lengths Δs_i. Figure 13.16 illustrates this process for the two-dimensional case and Figure 13.26 for the three-dimensional case. Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the ith subarc corresponding to the parameter value t_i^*. If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction of $\mathbf{T}(t_i^*)$, the unit tangent vector at P_i^*. Therefore, the work done by the force $\mathbf{F}$ in moving the particle from P_{i-1} to P_i is approximately

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)]\,\Delta s_i$$

and the total work done in moving the particle along C is approximately

$$\sum_{i=1}^{n} [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)]\,\Delta s_i \tag{10}$$

where $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C. It seems reasonable that these approximations should become better as n becomes larger. Therefore,

we define the **work** W done by the force field $\mathbf{F}$ as the limit of the Riemann sums in Equation 11, namely,

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\, ds = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \tag{11}$$

Equation 11 says that *work is the line integral with respect to arc length of the tangential component of the force.*

If the curve C is given by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$. Using Equation 8, we can rewrite Equation 11 in the form

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right] |\mathbf{r}'(t)|\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

This integral is often abbreviated as $\int_C \mathbf{F} \cdot d\mathbf{r}$ and occurs in other areas of physics as well. Therefore, the following definition for the line integral applies to *any* continuous vector field.

Definition • Line Integral of F along *C*

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along *C*** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

When using this definition, remember that $\mathbf{F}(\mathbf{r}(t))$ is just an abbreviation for $\mathbf{F}(x(t), y(t), z(t))$, so we evaluate $\mathbf{F}(\mathbf{r}(t))$ simply by using $x = x(t)$, $y = y(t)$, and $z = z(t)$ in the expression for $\mathbf{F}(x, y, z)$. Notice also that we can formally write $d\mathbf{r} = \mathbf{r}'(t)\, dt$.

Example 7 Work Done by a Force Field

Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le \pi/2$.

Solution

Since $x = \cos t$ and $y = \sin t$:

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\,\mathbf{i} - \cos t \sin t\,\mathbf{j} \quad \text{and} \quad \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}.$$

Therefore, the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^{\pi/2} (-2\cos^2 t \sin t)\, dt$$

$$= \left[2\frac{\cos^3 t}{3}\right]_0^{\pi/2} = -\frac{2}{3}$$

Figure 13.27 shows a graph of the force field and the curve C. ■

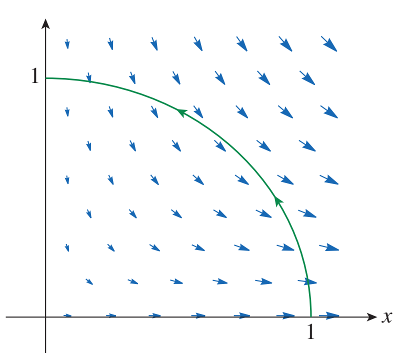


Figure 13.27
The force field $\mathbf{F}$ and the curve C. The work done is negative because the field impedes movement along the curve.

Note: Even though $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

because the unit tangent vector $\mathbf{T}$ is replaced by its negative when C is replaced by $-C$.

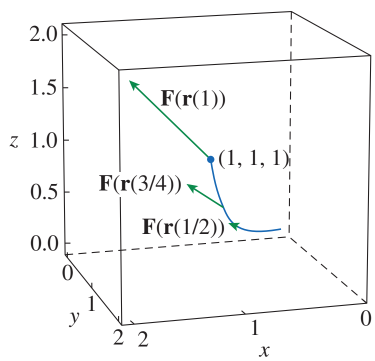


Figure 13.28
The twisted cubic C and some typical vectors acting at three points on C.

Example 8 Line integral of a Vector Field

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\, \mathbf{i} + yz\, \mathbf{j} + zx\, \mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \le t \le 1$$

Solution

Identify $\mathbf{r}(t)$, find $\mathbf{r}'(t)$ and $\mathbf{F}(\mathbf{r}(t))$.

$\mathbf{r}(t) = t\, \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\, \mathbf{j} + 3t^2 \mathbf{k}$

$\mathbf{F}(\mathbf{r}(t)) = t^3 \mathbf{i} + t^5 \mathbf{j} + t^4 \mathbf{k}$

Figure 13.28 shows some specific vectors $\mathbf{F}(\mathbf{r}(t))$.

Use these expressions to evaluate the integral.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^0 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \qquad \text{Definition of a line integral.}$$

$$= \int_0^1 (t^3 + 5t^6)\, dt \qquad \text{Compute dot product; simplify integrand.}$$

$$= \left[\frac{t^4}{4} + \frac{5t^7}{7}\right]_0^1 = \frac{27}{28} \qquad \text{Antiderivative; FTC.}$$

■

Finally, note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given in component form by the equation $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Use the definition to compute its line integral along C:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (x'(t)\, \mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k})\, dt$$

$$= \int_a^b [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)]\, dt$$

But this last integral is precisely the line integral in Equation 9. Therefore, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz \quad \text{where } \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

For example, the integral $\int_C y\, dx + z\, dy + x\, dz$ in Example 6 could be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

13.2 Exercises

Evaluate the line integral, where C is the given curve.

1. $\int_C y^3\, ds,\quad C\colon x = t^3,\ y = t,\ 0 \le t \le 2$

2. $\int_C xy\, ds,\quad C\colon x = t^2,\ y = 2t,\ 0 \le t \le 1$

3. $\int_C xy^4\, ds,$ C is the right half of the circle $x^2 + y^2 = 16$

4. $\int_C x \sin y\, ds,$ C is the line segment from $(0, 3)$ to $(4, 6)$

5. $\int_C (x^2y^3 - \sqrt{x},$ C is the arc of the curve $y = \sqrt{x}$ from $(1, 1)$ to $(4, 2)$

6. $\int_C xe^y\, dx,$ C is the arc of the curve $x = e^y$ from $(1, 0)$ to $(e, 1)$

7. $\int_C xy\, dx + (x - y)\, dy,$ C consists of line segments from $(0, 0)$ to $(2, 0)$ and from $(2, 0)$ to $(3, 2)$

8. $\int_C \sin x\, dx + \cos y\, dy,$ C consists of the top half of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$ and the line segment from $(-1, 0)$ to $(-2, 3)$

9. $\int_C xyz\, ds,\quad C\colon x = 2\sin t,\ y = t,\ z = -2\cos t,\ 0 \le t \le \pi$

10. $\int_C xyz^2\, ds,$ C is the line segment from $(-1, 5, 0)$ to $(1, 6, 4)$

11. $\int_C xe^{yz}\, ds,$ C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$

12. $\int_C (2x + 9z)\, ds,\quad C\colon x = t,\ y = t^2,\ z = t^3,\ 0 \le t \le 1$

13. $\int_C x^2 y \sqrt{z}\, dz \quad C\colon x = t^3,\ y = t,\ z = t^2,\ 0 \le t \le 1$

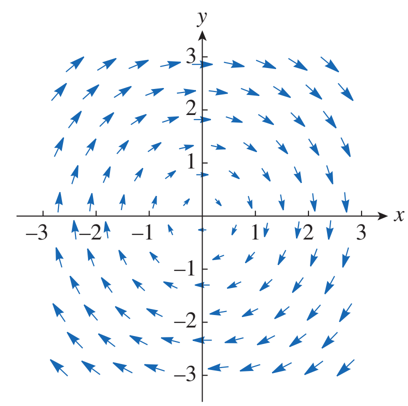
14. $\int_C z\, dx + x\, dy + y\, dz,\quad C\colon x = t^2,\ y = t^3,\ z = t^2,\ 0 \le t \le 1$

15. $\int_C (x + yz)\, dx + 2x\, dy + xyz\, dz,$ C consists of line segments from $(1, 0, 1)$ to $(2, 3, 1)$ and from $(2, 3, 1)$ to $(2, 5, 2)$

16. $\int_C x^2\, dx + y^2\, dy + z^2\, dz,$ C consists of line segments from $(0, 0, 0)$ to $(1, 2, -1)$ and from $(1, 2, -1)$ to $(3, 2, 0)$

17. $\int_C (y + z)\, dx + (x + z)\, dy + (x + y)\, dz,$ C consists of line segments from $(0, 0, 0)$ to $(1, 0, 1)$ and from $(1, 0, 1)$ to $(0, 1, 2)$

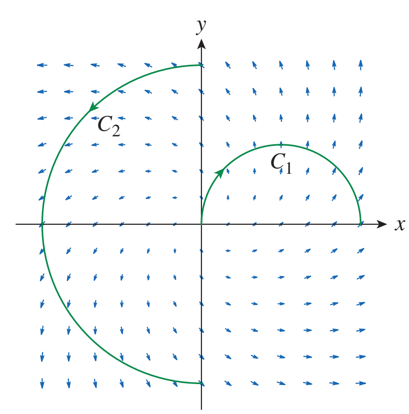
18. Let $\mathbf{F}$ be the vector field shown in the figure.



(a) If C_1 is the vertical line segment from $(-3, -3)$ to $(-3, 3)$, determine whether $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.

(b) If C_2 is the counterclockwise-oriented circle with radius 3 and center at the origin, determine whether $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.

19. The figure shows a vector field $\mathbf{F}$ and two curves C_1 and C_2. Are the line integrals of $\mathbf{F}$ over C_1 and C_2 positive, negative, or zero? Explain your answer.



Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

20. $\mathbf{F}(x, y) = xy\,\mathbf{i} + 3y^2\,\mathbf{j},\ \mathbf{r}(t) = 11t^4\,\mathbf{i} + t^3\,\mathbf{j},\quad 0 \le t \le 1$

21. $\mathbf{F}(x, y, z) = (x + y)\,\mathbf{i} + (y - z)\,\mathbf{j} + z^2\,\mathbf{k},$
$\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} + t^2\,\mathbf{k},\quad 0 \le t \le 1$

22. $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + xz\,\mathbf{k},$
$\mathbf{r}(t) = t^3\,\mathbf{i} - t^2\,\mathbf{j} + t\,\mathbf{k},\quad 0 \le t \le 1$

23. $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} - x\,\mathbf{k}$,
$\mathbf{r}(t) = t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$, $0 \le t \le \pi$

Use technology to evaluate the line integral.

24. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = xy\,\mathbf{i} + \sin y\,\mathbf{j}$ and
$\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t^2}\mathbf{j}$, $1 \le t \le 2$

25. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where
$\mathbf{F}(x, y, z) = y \sin z\,\mathbf{i} + z \sin x\,\mathbf{j} + x \sin y\,\mathbf{k}$ and
$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sin 5t\,\mathbf{k}$, $0 \le t \le \pi$

26. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = yze^x\,\mathbf{i} + zxe^y\,\mathbf{j} + xye^z\,\mathbf{k}$
and $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \tan t\,\mathbf{k}$, $0 \le t \le \pi/4$

27. $\int_C x \sin(y + z)\, ds$, where C has parametric equations $x = t^2$,
$y = t^3$, $z = t^4$, $0 \le t \le 5$

28. $\int_C ze^{-xy}\, ds$, where C has parametric equations $x = t$,
$y = t^2$, $z = e^{-t}$, $0 \le t \le 1$

29. $\int_C z \ln(x + y)\, ds$, where C has parametric equations
$x = 1 + 3t$, $y = 2 + t^2$, $z = t^4$, $-1 \le t \le 1$

Use a graph of the vector field **F** and the curve C to guess whether the line integral of **F** over C is positive, negative, or zero. Then evaluate the line integral.

30. $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + xy\,\mathbf{j}$, C is the arc of the circle
$x^2 + y^2 = 4$ traversed counter clockwise from $(2, 0)$
to $(0, \ -2)$

31. $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j}$, C is the parabola
$y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$

32. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where
$\mathbf{F}(x, y) = e^{x-1}\,\mathbf{i} + xy\,\mathbf{j}$ and C is given by
$\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$, $0 \le t \le 1$.
(b) Illustrate part (a) by using technology to graph C and the vectors from the vector field corresponding to $t = 0$, $1/\sqrt{2}$, and 1 (as in Figure 13.28).

33. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where
$\mathbf{F}(x, y, z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$ and C is given by
$\mathbf{r}(t) = 2t\,\mathbf{i} + 3t\,\mathbf{j} - t^2\,\mathbf{k}$, $-1 \le t \le 1$.
(b) Illustrate part (a) by using technology to graph C and the vectors from the vector field corresponding to $t = \pm 1$ and $t = \pm\frac{1}{2}$ (as in Figure 13.28).

34. Use technology to find the exact value of $\int_C x^3y^2z\, ds$, where C is the curve with parametric equations $x = e^{-t} \cos 4t$, $y = e^{-t} \sin 4t$, $z = e^{-t}$, $0 \le t \le 2\pi$.

35. (a) Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + xy\,\mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counter-clockwise direction.
(b) Use technology to graph the force field and circle on the same coordinate axes. Use the graph to explain your answer to part (a).

36. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $x \ge 0$. If the linear density is a constant k, find the mass and center of mass of the wire.

37. A thin wire has the shape of the first-quadrant part of the circle with center at the origin and radius a. If the density function is $\rho(x, y) = kxy$, find the mass and center of mass of the wire.

38. (a) Write the formulas similar to Equation 4 for the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of a thin wire in the shape of a space curve C if the wire has density function $\rho(x, y, z)$.
(b) Find the center of mass of a wire in the shape of the helix $x = 2 \sin t$, $y = 2 \cos t$, $z = 3t$, $0 \le t \le 2\pi$, if the density is a constant k.

39. Find the mass and center of mass of a wire in the shape of the helix $x = t$, $y = \cos t$, $z = \sin t$, $0 \le t \le 2\pi$, if the density at any point is equal to the square of the distance from the origin.

40. If a wire with linear density $\rho(x, y)$ lies along a plane curve C, its **moments of inertia** about the x- and y-axes are defined as

$$I_x = \int_C y^2\rho(x, y)\, ds \qquad I_y = \int_C x^2\rho(x, y)\, ds$$

Find the moments of inertia for the wire in Example 3.

41. If a wire with linear density $\rho(x, y, z)$ lies along a space curve C, its **moments of inertia** about the x-, y-, and z-axes are defined as

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds$$
$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds$$
$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds$$

Find the moments of inertia for the wire in Exercise 38.

42. Find the work done by the force field $\mathbf{F}(x, y) = x\,\mathbf{i} + (y + 2)\,\mathbf{j}$ in moving an object along an arch of the cycloid $\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}$, $0 \le t \le 2\pi$.

43. Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + ye^x\,\mathbf{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from $(1, 0)$ to $(2, 1)$.

44. Find the work done by the force field $\mathbf{F}(x, y, z) = \langle y + z, x + z, x + y \rangle$ on a particle that moves along the line segment from $(1, 0, 0)$ to $(3, 4, 2)$.

45. The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z \rangle$ is $\mathbf{F}(\mathbf{r}) = \dfrac{K\mathbf{r}}{|\mathbf{r}|^3}$ where K is a constant. (See Example 5 of Section 13.1.) Find the work done as the particle moves along a straight line from $(2, 0, 0)$ to $(2, 1, 5)$.

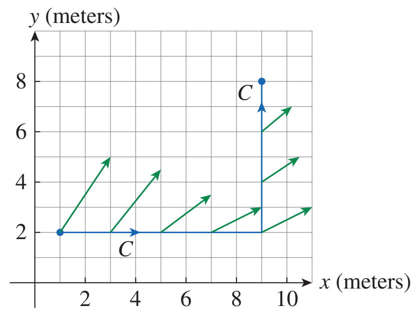
46. A 160-lb person carries a 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the person makes exactly three complete revolutions climbing to the top, how much work is done by the person against gravity?

47. Suppose there is a hole in the can of paint in Exercise 46 and 9 lb of paint leaks steadily out of the can during the person's ascent. How much work is done?

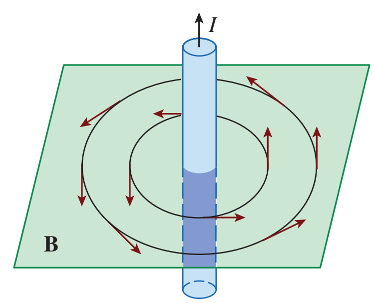
48. (a) Show that a constant force field does zero work on a particle that moves once uniformly around the circle $x^2 + y^2 = 1$.
(b) Is this result also true for a force field $\mathbf{F}(\mathbf{x}) = k\,\mathbf{x}$, where k is a constant and $\mathbf{x} = \langle x, y \rangle$?

49. The base of a circular fence with radius 10 m is given by $x = 10 \cos t$, $y = 10 \sin t$. The height of the fence at position (x, y) is given by the function $h(x, y) = 4 + 0.01(x^2 - y^2)$, so the height varies from 3 m to 5 m. Suppose that 1 L of paint covers 100 m^2. Sketch the fence and determine how much paint is required to paint both sides of the fence.

50. An object moves along the curve C shown in the figure from $(1, 2)$ to $(9, 8)$. The lengths of the vectors in the force field $\mathbf{F}$ are measured in newtons by the scales on the axes. Estimate the work done by $\mathbf{F}$ on the object.



51. Experiments show that a steady current I in a long wire produces a magnetic field $\mathbf{B}$ that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as illustrated in the figure).



Ampére's Law relates the electric current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where I is the net current that passes through any surface bounded by a closed curve C, and μ_0 is a constant called the permeability of free space. Let C be a circle of radius r. Show that the magnitude $B = |\mathbf{B}|$ of the magnetic field at a distance r from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$

13.3 The Fundamental Theorem for Line Integrals

Recall from Section 5.4 that Part 2 of the Fundamental Theorem of Calculus can be written as

$$\int_a^b F'(x)\, dx = F(b) - F(a) \tag{1}$$

where F' is continuous on $[a, b]$. We also referred to Equation 1 as the Net Change Theorem: the integral of a rate of change is the net change.

The Fundamental Theorem for Line Integrals

If we interpret the gradient vector ∇f of a function of two or three variables as a kind of derivative of f, then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

Theorem • Fundamental Theorem for Line Integrals

Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

A Closer Look

1. This theorem says that we can evaluate the line integral of a conservative vector field (the gradient vector field of the potential function f) simply by knowing the value of f at the endpoints of C. In fact, it says that the line integral of ∇f is the net change in f.

If f is a function of two variables and C is a plane curve with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, as in Figure 13.29(a), then Theorem 2 becomes

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$

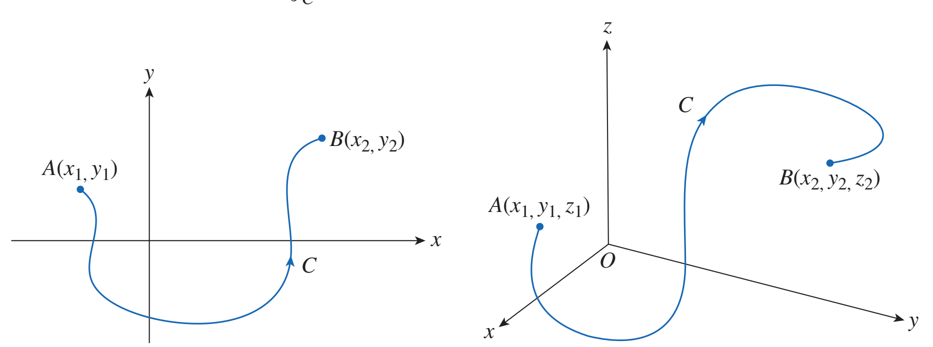


Figure 13.29
A plane curve and a space curve.

(a) A plane curve with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$

(b) A space curve with initial point $A(x_1, y_1, z_1)$ and terminal point $B(x_2, y_2, z_2)$.

If f is a function of three variables and C is a space curve joining the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$ as in Figure 13.29(b), then we have

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Let's prove this theorem for this case.

Proof

Use the definition of a line integral of $\mathbf{F}$ along C (Section 13.2).

$$\begin{aligned}
\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt && \text{Definition.} \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt && \text{Dot product.} \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\,dt && \text{Chain Rule.} \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) && \text{FTC.}
\end{aligned}$$

■

2. Under the hypotheses of this theorem, if C_1 and C_2 are smooth curves with the same initial points and the same terminal points, then we can conclude that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

3. Although we have proved this theorem for smooth curves, it is also true for piecewise-smooth curves. This can be seen by subdividing C into a finite number of smooth curves and adding the resulting integrals. ■

Example 1 Apply the Fundamental Theorem to the Calculation of Work

Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

in moving a particle with mass m from the point (3, 4, 12) to the point (2, 2, 0) along a piecewise-smooth curve C. (See Example 4 of Section 13.1.)

Solution

From Section 13.1: $\mathbf{F}$ is a conservative vector field and, in fact, $\mathbf{F} = \nabla f$, where $f(x, y, z) = \dfrac{mMG}{\sqrt{x^2 + y^2 + z^2}}$.

Therefore, by the fundamental theorem for line integrals, the work done is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} \\ &= f(2, 2, 0) - f(3, 4, 12) \\ &= \frac{mMG}{\sqrt{2^2 + 2^2}} - \frac{mMG}{\sqrt{3^2 + 4^2 + 12^2}} = mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right) \end{aligned}$$ ■

Independence of Path

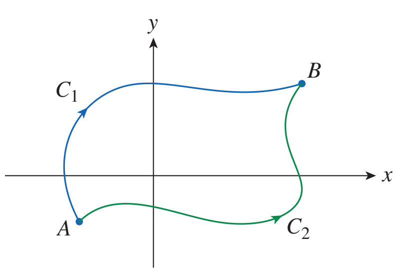


Figure 13.30
$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$.

Suppose C_1 and C_2 are two piecewise-smooth curves (called **paths**) that have the same initial point A and terminal point B. We know from Example 4 in Section 13.2 that, in general, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. But one implication of the fundamental theorem for line integrals is that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

whenever ∇f is continuous (see Figure 13.30). In other words, the line integral of a *conservative* vector field depends only on the initial point and terminal point of a curve.

In general, if $\mathbf{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial and terminal points. Using this terminology, we can say that *line integrals of conservative vector fields are independent of path.*

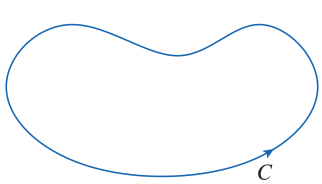


Figure 13.31
A closed curve.

A curve is called **closed** if its terminal point coincides with its initial point, that is $\mathbf{r}(b) = \mathbf{r}(a)$. (See Figure 13.31.) If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D and C is

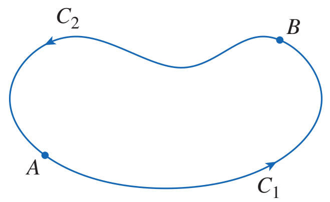


Figure 13.32
The path $C = C_1 \cup C_2$.

any closed path in D, we can choose any two points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A. (See Figure 13.32.) Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

since C_1 and $-C_2$ have the same initial and terminal points.

Conversely, if it is true that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever C is a closed path in D, then we demonstrate independence of path as follows. Take any two paths C_1 and C_2 from A to B in D and define C to be the curve consisting of C_1 followed by $-C_2$. Then

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

and so $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Therefore, we have proved the following theorem.

Theorem • Independence of Path and Closed Path

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Since we know that the line integral of any conservative vector field $\mathbf{F}$ is independent of path, it follows that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path. The physical interpretation is that the work done by a conservative force field (such as the gravitational or electric field in Section 13.1) as it moves an object around a closed path is 0.

The following theorem says that the *only* vector fields that are independent of path are conservative. It is stated and proved for plane curves, but there is a similar version for space curves. We assume that D is **open**, which means that for every point P in D there is a disk with center P that lies entirely in D. (So, D doesn't contain any of its boundary points.) In addition, we assume that D is **connected**: this means that any two points in D can be joined by a path that lies in D.

Theorem • Independence of Path and a Conservative Vector Field

Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Proof

Let $A(a, b)$ be a fixed point in D.

Construct the desired potential function f by defining

$$f(x, y) = \int_{(a,\,b)}^{(x,\,y)} \mathbf{F} \cdot d\mathbf{r} \quad \text{for any point } (x, y) \text{ in } D.$$

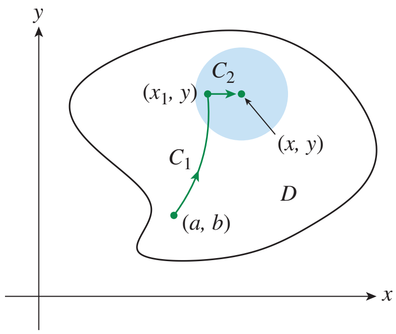


Figure 13.33
The paths C_1 from (a, b) to (x_1, y) and C_2 from (x_1, y) to (x, y).

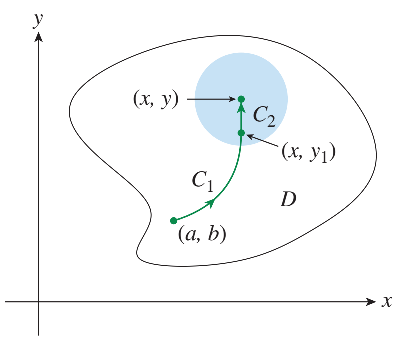


Figure 13.34
The paths C_1 from (a, b) to (x, y_1) and C_2 from (x, y_1) to (x, y).

Since $\int_C \mathbf{F} \cdot d\,\mathbf{r}$ is independent of path, it does not matter which path C from (a, b) to (x, y) is used to evaluate $f(x, y)$.

Since D is open, there exists a disk contained in D with center (x, y).

Choose any point (x_1, y) in the disk with $x_1 < x$ and let C consist of any path C_1 from (a, b) to (x_1, y) followed by the horizontal line segment C_2 from (x_1, y) to (x, y). (See Figure 13.33.) Then

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{(a,\,b)}^{(x_1,\,y)} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Notice that the first of these integrals does not depend on x, so

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\,\mathbf{r}.$$

If we write $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, then $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} P\,dx + Q\,dy$.

On C_2, y is constant, so $dy = 0$.

Using t as the parameter, where $x_1 \le t \le x$, by Part 1 of the Fundamental Theorem of Calculus we have

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial x} \int_{x_1}^{x} P(t, y)\,dt = P(x, y).$$

A similar argument, using a vertical line segment (see Figure 13.34), shows that

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial y} \int_{y_1}^{y} Q(x, t)\,dt = Q(x, y).$$

Therefore, $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} = \nabla f$, which says that $\mathbf{F}$ is conservative. ■

Conservative Vector Fields and Potential Functions

We still need a method to determine whether or not a vector field $\mathbf{F}$ is conservative. Suppose it is known that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative, where P and Q have continuous first-order partial derivatives. Then there is a function f such that $\mathbf{F} = \nabla f$, that is,

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial f}{\partial y}$$

Therefore, by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial Q}{\partial x}$$

Theorem • Conservative Vector Field and Partial Derivatives

If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

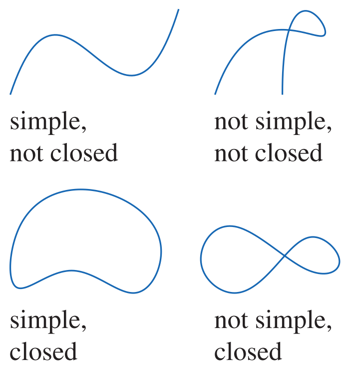


Figure 13.35
Types of curves.

The converse of the theorem involving a conservative vector field and partial derivatives is true only for a special type of region. To explain this, we first need the concept of a **simple curve**, which is a curve that does not intersect itself anywhere between its endpoints. In Figure 13.35, $\mathbf{r}(a) = \mathbf{r}(b)$ for a simple closed curve, but $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$.

In the theorem about independence of path and a conservative vector field, we needed an open connected region. For the next theorem, we need a stronger condition. A **simply connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D. Figures 13.36 and 13.37 suggest that a simply connected region contains no holes and cannot consist of two separate pieces.

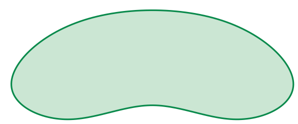


Figure 13.36
Simply connected region.

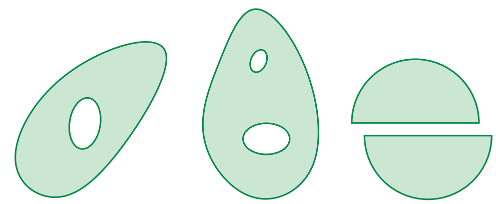


Figure 13.37
Regions that are not simply connected.

In terms of simply connected regions, we can now state a partial converse to the theorem involving a conservative vector field and partial derivatives that gives a convenient method for verifying that a vector field on $\mathbb{R}^2$ is conservative. The proof will be sketched in the next section as a consequence of Green's Theorem.

Theorem • Partial Derivative Conditions for a Conservative Vector Field

Let $\mathbf{F} = P\ \mathbf{i} + Q\ \mathbf{j}$ be a vector field on an open simply connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

Example 2 Conservative Vector Field Determination

Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\ \mathbf{i} + (x - 2)\ \mathbf{j}$$

is conservative.

Solution

Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$. Then

$$\frac{\partial P}{\partial y} = -1 \quad \frac{\partial Q}{\partial x} = 1.$$

Since $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$, $\mathbf{F}$ is not conservative by the theorem above. ■

Example 3 Conservative Vector Field Determination

Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\ \mathbf{i} + (x^2 - 3y^2)\ \mathbf{j}$$

is conservative.

Solution

Let $P(x, y) = 3 + 2xy$ and $Q(x, y) = x^2 - 3y^2$. Then

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}.$$

Also, the domain of $\mathbf{F}$ is the entire plane ($D = \mathbb{R}^2$), which is open and simply connected.

Therefore, we can apply the theorem above and conclude that $\mathbf{F}$ is conservative. ■

Figures 13.38 and 13.39 show the vector fields in Examples 2 and 3, respectively. The vectors in Figure 13.38 that start on the closed curve C all appear to point in roughly the same directions as C. So, it looks as if $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$ and therefore $\mathbf{F}$ is not conservative. The calculation in Example 2 confirms this observation. Some of the vectors near the curves C_1 and C_2 in Figure 13.39 point in approximately the same direction as the curves, whereas others point in the opposite direction. So it appears plausible that the line integrals around all closed paths are 0. Example 3 shows that $\mathbf{F}$ is indeed conservative.

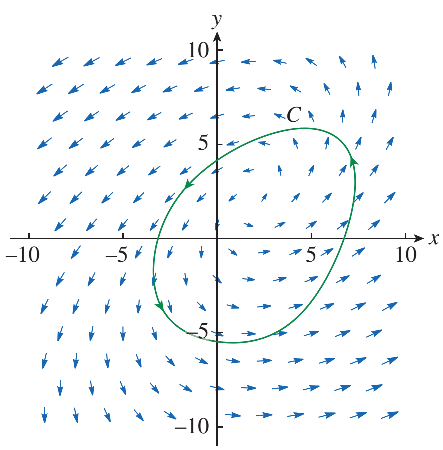


Figure 13.38
The vector field
$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$ is not conservative.

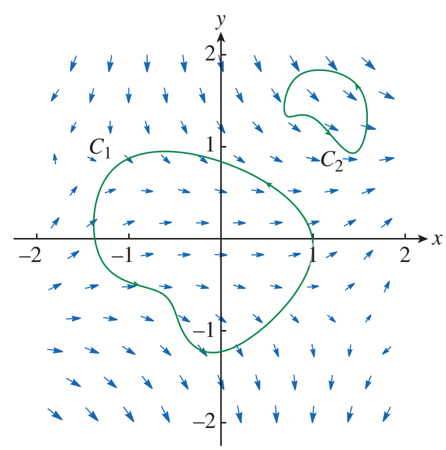


Figure 13.39
The vector field
$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$ is conservative.

In Example 3, we used the partial derivative conditions to justify that $\mathbf{F}$ is conservative. But the theorem involving these conditions does not provide a method for finding the (potential) function f such that $\mathbf{F} = \nabla f$. The proof of the theorem involving independence of path and a conservative vector field provides a clue for finding f. We use *partial integration* as in the following example.

Example 4 Find the Line Integral of a Conservative Vector Field

(a) If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j}, \quad 0 \le t \le \pi$$

Solution

(a) From Example 3, we know that **F** is conservative; there must exist a function f with $\nabla f = \mathbf{F}$, that is,

$$f_x(x, y) = 3 + 2xy \tag{2}$$

$$f_y(x, y) = x^2 - 3y^2 \tag{3}$$

Integrate Equation 2 with respect to x:

$$f(x, y) = 3x + x^2y + g(y) \tag{4}$$

Notice that the constant of integration is a constant with respect to x, that is, a function of y, which we have called $g(y)$.

Next, differentiate both sides of Equation 4 with respect to y.

$$f_y(x, y) = x^2 + g'(y) \tag{5}$$

Compare Equations 3 and 5; this implies $g'(y) = -3y^2$.

Integrate with respect to y:

$g(y) = -y^3 + K$ where K is a constant

Use this expression in Equation 4:

$f(x, y) = 3x + x^2y - y^3 + K$ is the desired potential function

(b) To use the fundamental theorem for line integrals, we simply need to know the initial and terminal points of C: $\mathbf{r}(0) = (0, 1)$, $\mathbf{r}(\pi) = (0, -e^{\pi})$.

In the expression for $f(x, y)$ in part (a), any value of the constant K will suffice, so let $K = 0$.

Evaluate the line integral.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, -e^{\pi}) - f(0, 1) = e^{3\pi} - (-1) = e^{3\pi} + 1$$

This method is much shorter than the straightforward method for evaluating line integrals that we learned in Section 13.2. ■

A criterion for determining whether or not a vector field **F** on $\mathbb{R}^3$ is conservative is given in Section 13.5. Meanwhile, the next example shows that the technique for finding the potential function is much the same as for vector fields on $\mathbb{R}^2$.

Example 5 Find the Potential Function for a Three-Dimensional Vector Field

If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

Solution

If there is such a function f, then

$$f_x(x, y, z) = y^2 \tag{6}$$

$$f_y(x, y, z) = 2xy + e^{3z} \tag{7}$$

$$f_z(x, y, z) = 3ye^{3z} \tag{8}$$

Integrate Equation 6 with respect to x:

$$f(x, y, z) = xy^2 + g(y, z) \tag{9}$$

where $g(y, z)$ is a constant with respect to x.

Differentiate Equation 9 with respect to y:

$f_y(x, y, z) = 2xy + g_y(y, z)$

Compare this expression with Equation 7: $g_y(y, z) = e^{3z}$.

Integrate this expression with respect to y: $g(y, z) = ye^{3z} + h(z)$.

Rewrite Equation 9: $f(x, y, z) = xy^2 + ye^{3z} + h(z)$.

Finally, differentiate with respect to z and compare with Equation 8:

$h'(z) = 0 \quad \Rightarrow \quad h(z) = K$ a constant

The desired function is $f(x, y, z) = xy^2 + ye^{3z} + K$.

Note that you can verify $\nabla f = \mathbf{F}$. ■

Conservation of Energy

Let's apply the ideas of this chapter to a continuous force field $\mathbf{F}$ that moves an object along a path C given by $\mathbf{r}(t)$, $a \le t \le b$, where $\mathbf{r}(a) = A$ is the initial point and $\mathbf{r}(b) = B$ is the terminal point of C. According to Newton's Second Law of Motion (see Section 10.4), the force $\mathbf{F}(\mathbf{r}(t))$ at a point on C is related to the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So, the work done by the force on the object is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b F(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\, dt$$

$$= \frac{m}{2}\int_a^b \frac{d}{dt}[\mathbf{r}'(t) \cdot \mathbf{r}'(t)]\, dt \qquad \text{Section 10.2; Differentiation Formula 4.}$$

$$= \frac{m}{2}\int_a^b \frac{d}{dt}|\mathbf{r}'(t)|^2\, dt = \frac{m}{2}\Big[\,|\mathbf{r}'(t)|^2\Big]_a^b \qquad \text{FTC.}$$

$$= \frac{m}{2}(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2)$$

Therefore,

$$W = \frac{1}{2}m|\mathbf{v}(b)|^2 - \frac{1}{2}m|\mathbf{v}(a)|^2 \qquad (10)$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity.

The quantity $\frac{1}{2}m|\mathbf{v}(t)|^2$, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore, we can rewrite Equation 10 as

$$W = K(B) - K(A) \qquad (11)$$

which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that **F** is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$. In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = -\nabla P$. Then using the fundamental theorem for line integrals, we have

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} = -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] = P(A) - P(B)$$

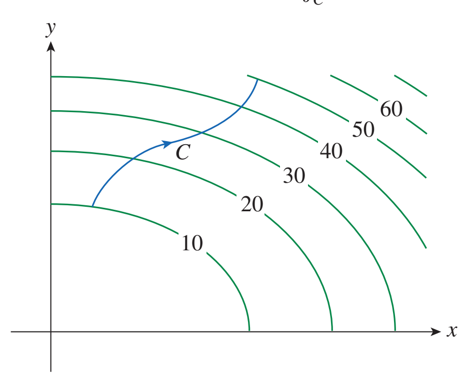
Comparing this equation with Equation 11, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.

13.3 Exercises

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



2. The table contains values of a function f with continuous gradient for selected values of x and y. Find $\int_C \nabla f \cdot d\mathbf{r}$, where C has parametric equations

$$x = t^2 + 1 \quad y = t^3 + t \quad 0 \le t \le 1$$

x \ y	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

Determine whether or not **F** is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

3. $\mathbf{F}(x, y) = (2x - 3y)\,\mathbf{i} + (-3x + 4y - 8)\,\mathbf{j}$

4. $\mathbf{F}(x, y) = (xy + y^2)\,\mathbf{i} + (x^2 + 2xy)\,\mathbf{j}$

5. $\mathbf{F}(x, y) = ye^x\,\mathbf{i} + (e^x + e^y)\,\mathbf{j}$

6. $\mathbf{F}(x, y) = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j}$

7. $\mathbf{F}(x, y) = e^x \cos y\,\mathbf{i} + e^x \sin y\,\mathbf{j}$

8. $\mathbf{F}(x, y) = (2xy + y^{-2})\,\mathbf{i} + (x^{-2}xy^{-3})\,\mathbf{j}, \quad y > 0$

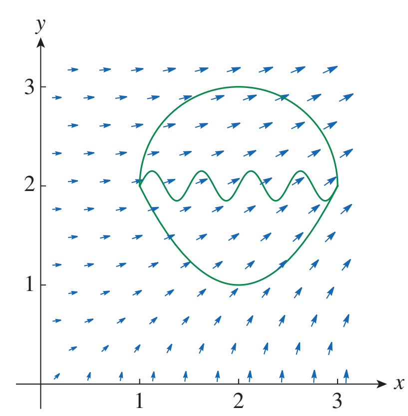
9. $\mathbf{F}(x, y) = (ye^x + \sin y)\,\mathbf{i} + (e^x + x\cos y)\,\mathbf{j}$

10. $\mathbf{F}(x, y) = (3x^2 - 2y^2)\,\mathbf{i} + (4xy + 3)\,\mathbf{j}$

11. $\mathbf{F}(x, y) = (\ln y + 2xy^3)\,\mathbf{i} + (3x^2y^2 + x/y)\,\mathbf{j}$

12. $\mathbf{F}(x, y) = (xy\cos xy + \sin xy)\,\mathbf{i} + (x^2 \cos xy)\,\mathbf{j}$

13. The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$ and three curves that start at $(1, 2)$ and end at $(3, 2)$.



(a) Explain why $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all three curves.

(b) What is this common value?

(a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

14. $\mathbf{F}(x, y) = x^2\,\mathbf{i} + y^2\,\mathbf{j}$, C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$

15. $\mathbf{F}(x, y) = xy^2\,\mathbf{i} + x^2y\,\mathbf{j}$, $\quad C: \mathbf{r}(t) = \left\langle t + \sin\frac{1}{2}\pi t,\ t + \cos\frac{1}{2}\pi t \right\rangle$, $0 \le t \le 1$

16. $\mathbf{F}(x, y) = \dfrac{y^2}{1 + x^2}\,\mathbf{i} + 2y\arctan x\,\mathbf{j}$, $C: \mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j}$, $0 \le t \le 1$

17. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + x^2e^{xy}\,\mathbf{j}$, $\quad C: \mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $0 \le t \le \pi/2$

18. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$, $\quad C$ is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$

19. $\mathbf{F}(x, y, z) = (2xz + y^2)\,\mathbf{i} + 2xy\,\mathbf{j} + (x^2 + 3z^2)\,\mathbf{k}$, $\quad C: x = t^2$, $y = t + 1$, $z = 2t - 1$, $\quad 0 \le t \le 1$

20. $\mathbf{F}(x, y, z) = y^2\cos z\,\mathbf{i} + 2xy\cos z\,\mathbf{j} - xy^2\sin z\,\mathbf{k}$, $C: \mathbf{r}(t) = t^2\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$, $\quad 0 \le t \le \pi$

21. $\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + xe^y\,\mathbf{j} + (z + 1)e^z\,\mathbf{k}$, $C: \mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $\quad 0 \le t \le 1$

Show that the line integral is independent of path and evaluate the integral.

22. $\int_C \tan y\,dx + x\sec^2 y\,dy$,
C is any path from $(0, 1)$ to $(2, \pi/4)$

23. $\int_C (1 - ye^{-x})\,dx + e^{-x}\,dy$, C is any path from $(0, 1)$ to $(1, 2)$

24. $\int_C \sin y\,dx + (x\cos y - \sin y)\,dy$,
C is any path from $(2, 0)$ to $(1, \pi)$

25. Suppose you're asked to determine the curve that requires the least work for a force field $\mathbf{F}$ to move a particle from one point to another point. You decide to check first whether $\mathbf{F}$ is conservative, and indeed it turns out that it is. How would you reply to the request?

26. Suppose an experiment determines that the amount of work required for a force field $\mathbf{F}$ to move a particle from the point $(1, 2)$ to the point $(5, -3)$ along a curve C_1 is 1.2 J and the work done by $\mathbf{F}$ in moving the particle along another curve C_2 between the same two points is 1.4 J. What can you say about $\mathbf{F}$? Why?

Find the work done by the force field $\mathbf{F}$ in moving an object from P to Q.

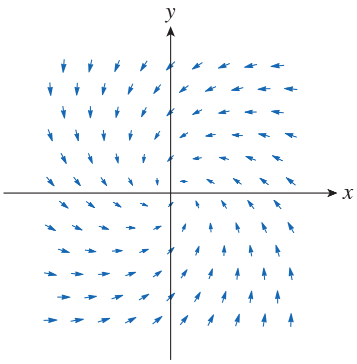
27. $\mathbf{F}(x, y) = x^3\,\mathbf{i} + y^3\,\mathbf{j}$; $\quad P(1, 0)$, $Q(2, 2)$

28. $\mathbf{F}(x, y) = 2y^{3/2}\,\mathbf{i} + 3x\sqrt{y}\,\mathbf{j}$; $\quad P(1, 1)$, $Q(2, 4)$

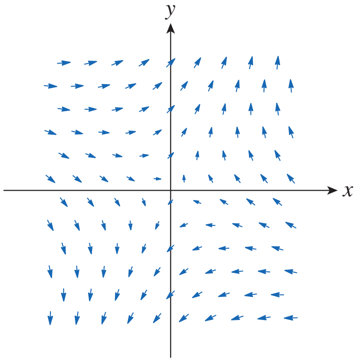
29. $\mathbf{F}(x, y) = e^{-y}\,\mathbf{i} - xe^{-y}\,\mathbf{j}$; $\quad P(0, 1)$, $Q(2, 0)$

Is the vector field shown in the figure conservative? Explain your answer.

30.



31.



32. If $\mathbf{F}(x, y) = \sin y\,\mathbf{i} + (1 + x\cos y)\,\mathbf{j}$, use a graph to guess whether $\mathbf{F}$ is conservative. Then determine whether your guess is correct.

33. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation.

(a) $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0 \qquad$ (b) $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$

34. Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

35. Use Exercise 34 to show that the line integral $\int_C y\,dx + x\,dy + xyz\,dz$ is not independent of path.

Determine whether or not the given set is (a) open, (b) connected, and (c) simply connected.

36. $\{(x, y) \mid 0 < y < 3\}$

37. $\{(x, y) \mid 1 < |x| < 2\}$

38. $\{(x, y) \mid 1 \le x^2 + y^2 \le 4,\ y \ge 0\}$

39. $\{(x, y) \mid (x, y) \ne (2, 3)\}$

40. Let $\mathbf{F}(x, y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$.

(a) Show that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) Show that $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path.

Hint: Compute $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$. Does this contradict Theorem 6?

41. (a) Suppose that $\mathbf{F}$ is an inverse square force field, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Find the work done by $\mathbf{F}$ in moving an object from a point P_1 along a path to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin.

(b) An example of an inverse square field is the gravitational field

$$\mathbf{F} = -\frac{mMG\mathbf{r}}{|\mathbf{r}|^3}$$

discussed in Example 4 in Section 13.1. Use part (a) to find the work done by the gravitational field when Earth moves from aphelion (at a maximum distance of 1.52×10^8 km from the sun) to perihelion (at a minimum distance of 1.47×10^8 km). Use the values $m = 5.97 \times 10^{24}$ kg, $M = 1.99 \times 10^{30}$ kg, and $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$.

(c) Another example of an inverse square field is the electric force field

$$\mathbf{F} = \frac{\epsilon qQ\mathbf{r}}{|\mathbf{r}|^3}$$

discussed in Example 5 in Section 13.1. Suppose that an electron with a charge of -1.6×10^{-19} C is located at the origin. A positive unit charge is positioned at a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric force field. Use the value $\epsilon = 8.985 \times 10^9$.

13.4 Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

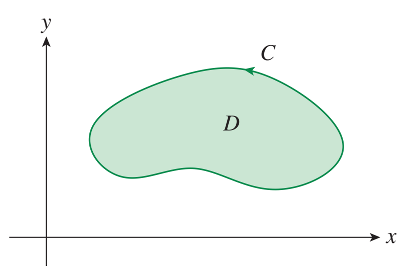
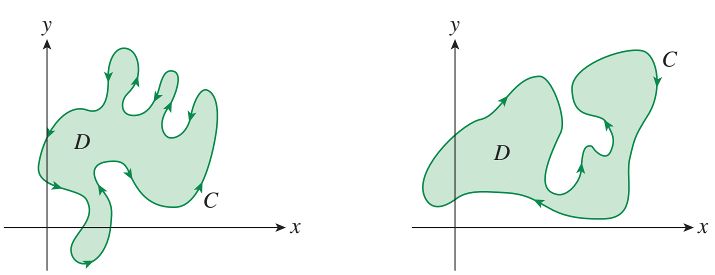


Figure 13.40
The region D is bounded by the simple closed curve C.

Green's Theorem

Let C be a simple closed curve and let D be the region bounded by C, as in Figure 13.40. (We assume that D consists of all points inside C as well as all points on C.) In stating Green's Theorem, we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus, if C is given by the vector function $\mathbf{r}(t)$, $a \le t \le b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C. Figure 13.41 shows examples of positive and negative orientation.



(a) The curve C has positive orientation.

(b) The curve C has negative orientation.

Figure 13.41
Examples of orientation.

Recall that the left side of this equation is another way of writing $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$.

Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

A Closer Look

1. The notation

$$\oint_C P\,dx + Q\,dy \quad \text{or} \quad \circlearrowleft\!\!\!\!\!\!\int_C P\,dx + Q\,dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C. Another notation for the positively oriented boundary curve of D is ∂D, so the equation in Green's Theorem can be written as

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q dy \tag{1}$$

2. Green's Theorem should be regarded as the counterpart of the Fundamental Theorem of Calculus for double integrals. Compare Equation 1 with the statement of the Fundamental Theorem of Calculus, Part 2, in the following equation:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

In both cases, there is an integral involving derivatives $\left(F', \dfrac{\partial Q}{\partial x}, \text{and } \dfrac{dP}{dy}\right)$ on the left side of the equation. And in both cases, the right side involves the values of the original functions (F, Q, and P) only on the *boundary* of the domain. (In the one-dimensional case, the domain is an interval $[a, b]$ whose boundary consists of just two points, a and b.) ■

Green's Theorem is named after the self-taught English scientist George Green (1793–1841). In 1828, he published privately *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, but only 100 copies were printed. This pamphlet contained a theorem that is equivalent to what we know as Green's Theorem, but it did not become widely known until 1846 when William Thomson (Lord Kelvin) located a copy of Green's essay, realized its significance, and had it reprinted. Green was the first person to try to formulate a mathematical theory of electricity and magnetism. His work was the basis for the subsequent electromagnetic theories of Thomson, Stokes, Rayleigh, and Maxwell.

Green's Theorem is not easy to prove in general, but we can give a proof for the special case where the region is both of type I and of type II (see Section 12.3). Let's call such regions **simple regions**.

Proof of Green's Theorem for the Case in Which *D* Is a Simple Region

Notice that Green's Theorem will be proved if we can show that

$$\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA \tag{2}$$

and

$$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA \tag{3}$$

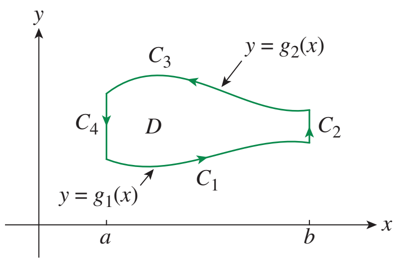


Figure 13.42
C is the union of the four curves C_1, C_2, C_3, and C_4.

We prove Equation 2 by expressing D as a type I region:

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous functions. This enables us to compute the double integral on the right side of Equation 2 as follows:

$$\iint_D \frac{\partial P}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y)\, dy\, dx = \int_a^b \left[P(x, g_2(x)) - P(x, g_1(x))\right] dx \qquad (4)$$

where the last step follows from the Fundamental Theorem of Calculus.

Now we compute the left side of Equation 2 by breaking up C into the union of four curves C_1, C_2, C_3, and C_4 as shown in Figure 13.42. On C_1, we take x as the parameter and write the parametric equations as $x = x$, $y = g_1(x)$, $a \le x \le b$. Thus,

$$\int_{C_1} P(x, y)\, dx = \int_a^b P(x, g_1(x))\, dx$$

Observe that C_3 goes from right to left but $-C_3$ goes from left to right, so we can write the parametric equations of $-C_3$ as $x = x$, $y = g_2(x)$, $a \le x \le b$. Therefore,

$$\int_{C_3} P(x, y)\, dx = -\int_{-C_3} P(x, y)\, dx = -\int_a^b P(x, g_2(x))\, dx$$

On C_2 or C_4 (either of which might reduce to just a single point), x is constant, so $dx = 0$ and

$$\int_{C_2} P(x, y)\, dx = 0 = \int_{C_4} P(x, y)\, dx$$

Hence,

$$\begin{aligned}\int_C P(x, y) &= \int_{C_1} P(x, y)\, dx + \int_{C_2} P(x, y)\, dx + \int_{C_3} P(x, y)\, dx + \int_{C_4} P(x, y)\, dx \\ &= \int_a^b P(x, g_1(x))\, dx - \int_a^b P(x, g_2(x))\, dx\end{aligned}$$

Comparing this expression with Equation 4 we see that

$$\int_C P(x, y)\, dx = -\iint_D \frac{\partial P}{\partial y}\, dA$$

Equation 3 can be proved in much the same way by expressing D as a type II region (see Exercise 32). Then, by adding Equations 2 and 3, we obtain Green's Theorem. ■

Example 1 Use Green's Theorem to Calculate a Line Integral

Evaluate $\int_C x^4\, dx + xy\, dy$, where C is the triangular curve consisting of the line segments from (0, 0) to (1, 0), from (1, 0) to (0, 1), and from (0, 1) to (0, 0).

Solution

The given line integral could be evaluated using the methods of Section 13.2; that would involve three separate integrals along the three sides of the triangle. So, let's use Green's Theorem instead.

Notice that the region D enclosed by C is simple and C has positive orientation (see Figure 13.43).

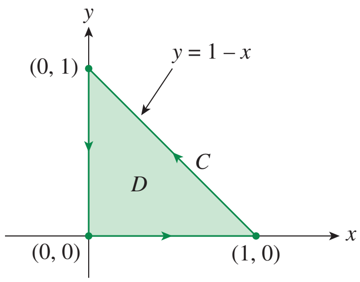


Figure 13.43
The triangular curve C.

Let $P(x, y) = x^4$ and $Q(x, y) = xy$. Then

$$\int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$ Green's Theorem.

$$= \int_0^1 \int_0^{1-x} (y - 0)\,dy\,dx$$ Use partial derivatives; D is a type I region.

$$= \int_0^1 \left[\frac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx$$ Antiderivative; FTC.

$$= -\frac{1}{6}\Big[(1-x)^3\Big]_0^1 = \frac{1}{6}$$ Antiderivative; FTC; simplify.

Example 2 Impossible Line Integral Without Green's Theorem

Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4 + 1})\,dy$, where C is the circle $x^2 + y^2 = 9$.

Solution

The region D bounded by C is the disk $x^2 + y^2 \le 9$.

So, let's change to polar coordinates after applying Green's Theorem.

$$\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}(7x + \sqrt{y^4+1}) - \frac{\partial}{\partial y}(3y - e^{\sin x})\right] dA$$ Green's Theorem.

$$= \int_0^{2\pi}\int_0^3 (7-3)\,r\,dr\,d\theta$$ Convert to polar coordinates.

$$= 4\int_0^{2\pi} d\theta \int_0^3 r\,dr = 36\pi$$ Integrand is a product of a function in θ times a function in $\boldsymbol{r}$.

Instead of using polar coordinates, we could simply use the fact that D is a disk of radius 3 and write

$$\iint_D 4\,dA = 4 \cdot \pi(3)^2 = 36\pi$$

In Examples 1 and 2, we found that the double integral was easier to evaluate than the line integral. (Try setting up the line integral in Example 2 and you'll soon be convinced!) But sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction. For instance, if it is known that $P(x, y) = Q(x, y) = 0$ on the curve C, then Green's Theorem gives

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy = 0$$

no matter what values P and Q assume in the region D.

Finding Areas with Green's Theorem

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint_D 1\,dA$, we would like to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$$P(x, y) = 0 \qquad P(x, y) = -y \qquad P(x, y) = -\frac{1}{2}y$$

$$Q(x, y) = x \qquad Q(x, y) = 0 \qquad Q(x, y) = \frac{1}{2}x$$

Then Green's Theorem gives the following formulas for the areas of D:

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx \tag{5}$$

Example 3 Area of an Ellipse

Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution

The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, where $0 \le t \le 2\pi$. Use the third formula in Equation 5:

$$A = \frac{1}{2}\int_C x\,dy - y\,dx \qquad \text{Equation 5.}$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt \qquad \text{Use parametric equations.}$$

$$= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab \qquad \text{Simplify; antiderivative; FTC.}$$

■

Equation 5 can be used to explain how planimeters work. A **planimeter** (one is shown at the beginning of this chapter) is a mechanical instrument used for measuring the area of a region by tracing its boundary curve. These devices are useful in all the sciences: in biology, for measuring the area of leaves or wings; in medicine, for measuring the size of cross-sections of organs or tumors; in forestry, for estimating the size of forested regions from photographs.

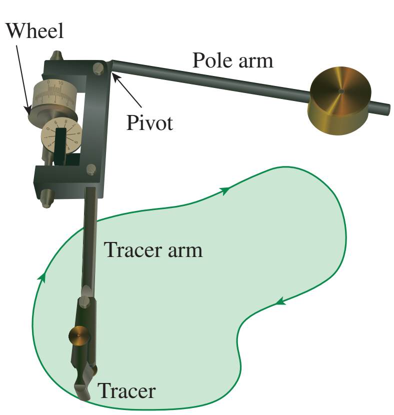


Figure 13.44
A Keuffel and Esser polar planimeter.

Figure 13.44 shows the operation of a polar planimeter: the pole is fixed and, as the tracer is moved along the boundary curve of the region, the wheel partly slides and partly rolls perpendicular to the tracer arm. The planimeter measures the distance that the wheel rolls and this is proportional to the area of the enclosed region. The explanation as a consequence of Equation 5 can be found in the following articles:

- R. W. Gatterdam "The planimeter as an example of Green's theorem," *Amer. Math. Monthly* 88 (1981): 701–4.
- Tanya Leise, "As the Planimeter Wheel Turns," *College Math. Journal* 38 (2007): 24–31.

Extended Versions of Green's Theorem

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a finite union of simple regions. For example, if D is the region shown in Figure 13.45, then we can write $D = D_1 \cup D_2$, where D_1 and D_2

are both simple. The boundary of D_1 is $C_1 \cup C_3$ and the boundary of D_2 is $C_2 \cup (-C_3)$ so, applying Green's Theorem to D_1 and D_2 separately, we get

$$\int_{C_1 \cup C_3} P\ dx + Q\ dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_{C_2 \cup (-C_3)} P\ dx + Q\ dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

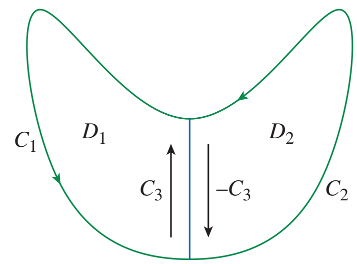


Figure 13.45
The region $D = D_1 \cup D_2$.

If we add these two equations, the line integrals along C_3 and $-C_3$ cancel, so we get

$$\int_{C_1 \cup C_2} P\ dx + Q\ dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

which is Green's Theorem for $D = D_1 \cup D_2$, since its boundary is $C = C_1 \cup C_2$.

The same sort of argument allows us to establish Green's Theorem for any finite union of nonoverlapping simple regions (see Figure 13.46).

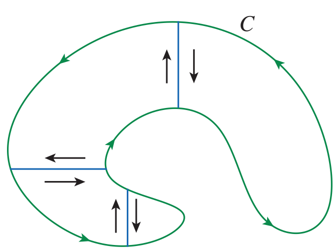


Figure 13.46
The region enclosed by C is a finite union of nonoverlapping simple regions.

Example 4 Union of Simple Regions

Evaluate $\oint_C y^2\ dx + 3xy\ dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution

The region D, shown in Figure 13.47, is not simple; but the y-axis divides it into two simple regions.

The region D described in polar coordinates is

$D = \{(r, \theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$.

Use Green's Theorem.

$$\oint_C y^2\ dx + 3xy\ dy = \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right] dA \qquad \text{Green's Theorem.}$$

$$= \iint_D y\ dA = \int_0^{\pi} \int_1^2 (r \sin\theta)\ r\ dr\ d\theta \qquad \text{Compute partial derivatives; convert to polar coordinates.}$$

$$= \int_0^{\pi} \sin\theta\ d\theta \int_1^2 r^2\ dr \qquad \text{Integrand is a product of a function in } \theta \text{ times a function in } r.$$

$$= \Big[-\cos\theta\Big]_0^{\pi} \left[\frac{1}{3} r^3\right]_1^2 = \frac{14}{3} \qquad \text{Antiderivatives; FTC; simplify.}$$

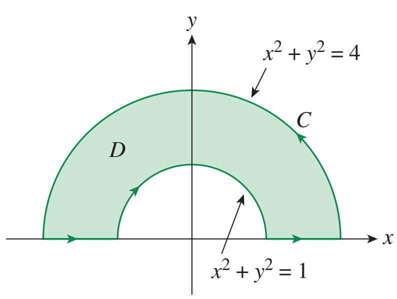


Figure 13.47
The region D can be divided into two simple regions.

Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply connected. Observe that the boundary C of the region D in Figure 13.48 consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. Thus, the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2. If we divide D into two regions D' and D'' by means of the lines shown in Figure 13.49 and then apply Green's Theorem to each of D' and D'', we get

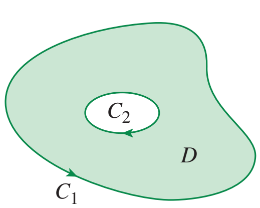


Figure 13.48
The boundary of D consists of two simple closed curves.

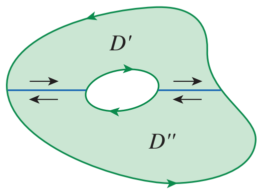


Figure 13.49
Divide the region D into two regions.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$
$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel, and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy,$$

which is Green's Theorem for the region D.

Example 5 Use the General Version of Green's Theorem

If $\mathbf{F}(x, y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

Solution

Since C is an *arbitrary* closed path that encloses the origin, it is difficult to compute the given integral directly.

So, let's consider a counterclockwise-oriented circle C' with center at the origin and radius a, where a is chosen to be small enough that C' lies inside C (see Figure 13.50).

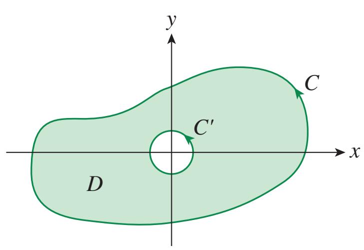


Figure 13.50
The region D is bounded by C and C'.

Let D be the region bounded by C and C'. Then its positively oriented boundary is $C \cup (-C')$ and the general version of Green's Theorem gives

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$
$$= \iint_D \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2}\right] dA = 0$$

Therefore, $\displaystyle\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$

that is, $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}.$

Evaluate the last integral using the parametrization:
$\mathbf{r}(t) = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j},\ 0 \le t \le 2\pi.$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$
$$= \int_0^{2\pi} \frac{(-a \sin t)(-a \sin t) + (a \cos t)(a \cos t)}{a^2\cos^2 t + a^2\sin^2 t} dt = \int_0^{2\pi} dt = 2\pi \quad ■$$

We end this section by using Green's Theorem to discuss a result that was stated in the preceding section.

Sketch of Proof of Theorem Involving Partial Derivative Conditions for a Conservative Vector Field

We are assuming that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is a vector field on an open simply connected region D, that P and Q have continuous first-order partial derivatives, and that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

If C is any simple closed path in D and R is the region that C encloses, then Green's Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R 0\,dA = 0$$

A curve that is not simple crosses itself at one or more points and can be broken up into a number of simple curves. We have shown that the line integrals of $\mathbf{F}$ around these simple curves are all 0 and, adding these integrals, we see that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C. Therefore, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D. It follows that $\mathbf{F}$ is a conservative vector field.

13.4 Exercises

Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1. $\oint_C (x - y)\,dx + (x + y)\,dy$,
C is the circle with center at the origin and radius 2

2. $\oint_C xy\,dx + x^2\,dy$,
C is the rectangle with vertices (0, 0), (3, 0), (3, 1), and (0, 1)

3. $\oint_C xy\,dx + x^2y^3\,dy$, C is the triangle with vertices (0, 0), (1, 0), and (1, 2)

4. $\oint_C x\,dx + y\,dy$, C consists of the line segments from (0, 1) to (0, 0) and from (0, 0) to (1, 0) and the parabola $y = 1 - x^2$ from (1, 0) to (0, 1)

Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

5. $\int_C xy^2\,dx + 2x^2y\,dy$,
C is the triangle with vertices (0, 0), (2, 2), and (2, 4)

6. $\int_C \cos y\,dx + x^2 \sin y\,dy$, C is the rectangle with vertices (0, 0), (5, 0), (5, 2), and (0, 2)

7. $\int_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy$, C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$

8. $\int_C xe^{-2x}dx + (x^4 + 2x^2y^2)\,dy$, C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

9. $\int_C y^4\,dx + 2xy^3\,dy$, C is the ellipse $x^2 + 2y^2 = 2$

10. $\int_C y^3\,dx - x^3\,dy$, C is the circle $x^2 + y^2 = 4$

11. $\int_C \sin y\,dx + x \cos y\,dy$, C is the ellipse $x^2 + xy + y^2 = 1$

Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

12. $\mathbf{F}(x, y) = \langle \sqrt{x} + y^3, x^2 + \sqrt{y} \rangle$, C consists of the arc of the curve $y = \sin x$ from (0, 0) to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to (0, 0)

13. $\mathbf{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$, C is the triangle from (0, 0) to (2, 6) to (2, 0) to (0, 0)

14. $\mathbf{F}(x, y) = \langle e^x + x^2y, e^y - xy^2 \rangle$, C is the circle $x^2 + y^2 = 25$ oriented clockwise

15. $\mathbf{F}(x, y) = \langle y - \ln(x^2 + y^2), 2 \tan^{-1}(y/x) \rangle$, C is the circle $(x - 2)^2 + (y - 3)^2 = 1$ oriented counterclockwise

16. $\mathbf{F}(x, y) = \langle \sqrt{x^2 + 1}, \tan^{-1} x \rangle$,
C is the triangle from $(0, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$

Verify Green's Theorem by using technology to evaluate both the line integral and the double integral.

17. $P(x, y) = y^2 e^x$, $Q(x, y) = x^2 e^y$,
C consists of the line segment from $(-1, 1)$ to $(1, 1)$ followed by the arc of the parabola $y = 2 - x^2$ from $(1, 1)$ to $(-1, 1)$

18. $P(x, y) = 2x - x^3 y^5$, $Q(x, y) = x^3 y^8$,
C is the ellipse $4x^2 + y^2 = 4$

19. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\,\mathbf{i} + xy^2\,\mathbf{j}$ in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis.

20. A particle starts at the point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and then along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle x, x^3 + 3xy^2 \rangle$.

21. Use one of the equations in (5) to find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

22. If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid*, with parametric equations $x = 5 \cos t - \cos 5t$, $y = 5 \sin t - \sin 5t$. Graph the epicycloid and use the equations in 5 to find the area it encloses.

23. (a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1 y_2 - x_2 y_1$$

(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, show that the area of the polygon is

$$A = \frac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)].$$

(c) Find the area of the pentagon with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, $(0, 2)$, and $(-1, 1)$.

24. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are

$$\bar{x} = \frac{1}{2A}\oint_C x^2\,dy \qquad \bar{y} = -\frac{1}{2A}\oint_C y^2\,dx$$

where A is the area of D.

25. Use Exercise 24 to find the centroid of a quarter-circular region of radius a.

26. Use Exercise 24 to find the centroid of the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, b), where $a > 0$ and $b > 0$.

27. A plane lamina with constant density $\rho(x, y) = \rho$ occupies a region in the xy-plane bounded by a simple closed path C. Show that its moments of inertia about the axes are

$$I_x = -\frac{\rho}{3}\oint_C y^3\,dx \qquad I_y = \frac{\rho}{3}\oint_C x^3\,dy$$

28. Use Exercise 27 to find the moment of inertia of a circular disk of radius a with constant density ρ about a diameter. (Compare with Example 4 in Section 12.5.)

29. Use the method of Example 5 to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y) = \frac{2xy\,\mathbf{i} + (y^2 - x^2)\,\mathbf{j}}{(x^2 + y^2)^2}$$

and C is any positively oriented simple closed curve that encloses the origin.

30. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x^2 + y, 3x - y^2 \rangle$ and C is the positively oriented boundary curve of a region D that has area 6.

31. If $\mathbf{F}$ is the vector field of Example 5, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.

32. Complete the proof of the special case of Green's Theorem by proving Equation 3.

33. Use Green's Theorem to prove the change of variables formula for a double integral (Equation 12.9.9) for the case where $f(x, y) = 1$:

$$\iint_R dx\,dy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv$$

Here R is the region in the xy-plane that corresponds to the region S in the uv-plane under the transformation given by $x = g(u, v)$, $y = h(u, v)$.

Hint: Note that the left side is $A(R)$ and apply the first part of Equation 5. Convert the line integral over ∂R to a line integral over ∂S and apply Green's Theorem in the uv-plane.

13.5 Curl and Divergence

In this section, we define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism. Each operation resembles differentiation, but one produces a vector field whereas the other produces a scalar field.

Curl

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \tag{1}$$

As a memory aid, let's rewrite Equation 1 using operator notation. The vector differential operator ∇ ("del") is defined as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

If we think of ∇ as a vector with components $\dfrac{\partial}{\partial x}$, $\dfrac{\partial}{\partial y}$, and $\dfrac{\partial}{\partial z}$, we can consider the formal cross product of ∇ with the vector field $\mathbf{F}$ as follows:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\ &= \text{curl } \mathbf{F}\end{aligned}$$

In the second term, note that $-\left(\dfrac{\partial R}{\partial x} - \dfrac{\partial P}{\partial z}\right) = \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)$.

Thus, the easiest way to remember Equation 1 is by means of the symbolic expression

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} \tag{2}$$

Example 1 Compute Curl F

If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

Solution

Use Equation 2.

$$\begin{aligned}\text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\mathbf{k} \\ &= (-2y - xy)\,\mathbf{i} - (0 - x)\,\mathbf{j} + (yz - 0)\,\mathbf{k} \\ &= -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}\end{aligned}$$

■

Recall that the gradient of a function f of three variables is a vector field on $\mathbb{R}^3$; therefore, we can compute its curl. The next theorem says that the curl of a gradient vector field is $\mathbf{0}$.

Theorem • Curl of ∇f

If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

Proof

Write the expression for curl(∇f) by definition and simplify.

$$\text{curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y\, \partial z} - \frac{\partial^2 f}{\partial z\, \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z\, \partial x} - \frac{\partial^2 f}{\partial x\, \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x\, \partial y} - \frac{\partial^2 f}{\partial y\, \partial x}\right)\mathbf{k}$$

$$= 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0} \qquad \text{Clairaut's Theorem.}$$

■

Notice the similarity to what we learned in Section 9.4: $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for every three-dimensional vector $\mathbf{a}$.

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, this theorem can be rephrased as follows:

$$\text{If } \mathbf{F} \text{ is conservative, then curl } \mathbf{F} = \mathbf{0}.$$

This gives us a way of verifying that a vector field is not conservative.

Example 2 Use Curl F to Show F Is Not Conservative

Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ is not conservative.

Solution

In Example 1, we showed that

curl $\mathbf{F} = -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$.

This shows that curl $\mathbf{F} \neq \mathbf{0}$ and so, $\mathbf{F}$ is not conservative. ■

The converse of the theorem involving curl(∇f) is not true in general, but the following theorem says the converse is true if $\mathbf{F}$ is defined everywhere. (More generally, it is true if the domain is simply connected, that is, "has no hole.") The next theorem is the three-dimensional version of the theorem involving partial derivative conditions for a conservative vector field. Its proof requires Stokes' Theorem and is sketched at the end of Section 13.7.

Theorem • Conditions for a Conservative Vector Field

If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Example 3 Find a Potential Function for a Conservative Vector Field

(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\ \mathbf{i} + 2xyz^3\ \mathbf{j} + 3xy^2z^2\ \mathbf{k}$$

is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

Solution

(a) Compute the curl of $\mathbf{F}$:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\ \mathbf{i} - (3y^2z^2 - 3y^2z^2)\ \mathbf{j} + (2yz^3 - 2yz^3)\ \mathbf{k}$$

$$= \mathbf{0}$$

Since curl $\mathbf{F} = \mathbf{0}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^3$, $\mathbf{F}$ is a conservative vector field by Theorem 4.

(b) The technique for finding f was presented in Section 13.3.

Start by writing expressions for the three partial derivatives.

$$f_x(x, y, z) = y^2z^3 \tag{3}$$

$$f_y(x, y, z) = 2xyz^3 \tag{4}$$

$$f_z(x, y, z) = 3xy^2z^2 \tag{5}$$

Integrate Equation 3 with respect to x.

$$f(x, y, z) = xy^2z^3 + g(y, z) \tag{6}$$

Differentiate Equation 6 with respect to y: $f_y(x, y, z) = 2xyz^3 + g_y(y, z)$.

Compare this expression with Equation 4, then $g_y(y, z) = 0$.

Thus, $g(y, z) = h(z)$ and $f_z(x, y, z) = 3xy^2z^2 + h'(z)$.

Use Equation 5, then $h'(z) = 0$.

Therefore, $f(x, y, z) = xy^2z^3 + K$. ■

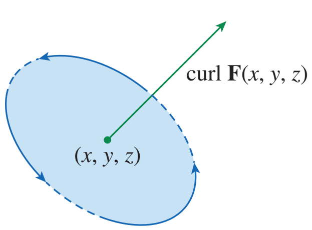


Figure 13.51
The length of the curl vector is a measure of how quickly the particles move around the axis.

The reason for the name *curl* is that the curl vector is associated with rotations. One connection is explained in Exercise 41. Another occurs when $\mathbf{F}$ represents the velocity field in fluid flow (see Example 3 in Section 13.1). Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl $\mathbf{F}(x, y, z)$ and the length of this curl vector is a measure of how quickly the particles move round the axis (see Figure 13.51). If curl $\mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P and $\mathbf{F}$ is called **irrotational** at P. In other words, there is no whirlpool or eddy at P. If curl $\mathbf{F} = \mathbf{0}$, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis. If curl $\mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis. There is a more detailed explanation in Section 13.7 as a consequence of Stokes' Theorem.

Divergence

If $\mathbf{F} = P\ \mathbf{i} + Q\ \mathbf{j} + R\ \mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\dfrac{\partial P}{\partial x}, \dfrac{\partial Q}{\partial y}$, and $\dfrac{\partial R}{\partial z}$ exist, then the **divergence of F** is the function of three variables defined by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{7}$$

Note that curl $\mathbf{F}$ is a vector field but div $\mathbf{F}$ is a scalar field. In terms of the gradient operator $\nabla = \dfrac{\partial}{\partial x}\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j} + \dfrac{\partial}{\partial z}\mathbf{k}$, the divergence of $\mathbf{F}$ can be written symbolically as the dot product of ∇ and $\mathbf{F}$:

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} \tag{8}$$

Example 4 Compute Divergence

If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find div $\mathbf{F}$.

Solution

Use the definition of divergence (Equation 7 or Equation 8).

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz$$

■

If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$, then curl $\mathbf{F}$ is also a vector field on $\mathbb{R}^3$. Therefore, we can compute its divergence. The next theorem shows that the result is 0.

Theorem • Divergence of the Curl of F

If $\mathbf{F} = P\ \mathbf{i} + Q\ \mathbf{j} + R\ \mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Proof

Use the definitions of divergence and curl.

Note the analogy with the scalar triple product: $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

$$\begin{aligned}
\text{div curl } \mathbf{F} &= \nabla \cdot (\nabla \times \mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \frac{\partial^2 R}{\partial x\,\partial y} - \frac{\partial^2 Q}{\partial x\,\partial z} + \frac{\partial^2 P}{\partial y\,\partial z} - \frac{\partial^2 R}{\partial y\,\partial x} + \frac{\partial^2 Q}{\partial z\,\partial x} - \frac{\partial^2 P}{\partial z\,\partial y} \\
&= 0 \qquad \text{Clairaut's Theorem: terms cancel in pairs.}
\end{aligned}$$

■

Example 5 No Possible Curl

Show that the vector field $\mathbf{F}(x, y, z) = xz\ \mathbf{i} + xyz\ \mathbf{j} - y^2\ \mathbf{k}$ cannot be written as the curl of another vector field, that is $\mathbf{F} \neq \text{curl}\ \mathbf{G}$.

Solution

In Example 4, we showed that $\text{div}\ \mathbf{F} = z + xz$; therefore, $\text{div}\ \mathbf{F} \neq 0$.

Suppose there exists a vector field $\mathbf{G}$ such that $\mathbf{F} = \text{curl}\ \mathbf{G}$.

Then, $\text{div}\ \mathbf{F} = \text{div}\ \text{curl}\ \mathbf{G} = 0$.

This contradicts $\text{div}\ \mathbf{F} \neq 0$. Therefore, $\mathbf{F}$ is not the curl of another vector field. ■

The reason for this interpretation of div **F** will be discussed at the end of Section 13.8 as a consequence of the Divergence Theorem.

Again, the reason for the name *divergence* can be understood in the context of fluid flow. If $\mathbf{F}(x, y, z)$ is the velocity of a fluid (or gas), then $\text{div}\ \mathbf{F}(x, y, z)$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume. In other words, $\text{div}\ \mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z). If $\text{div}\ \mathbf{F} = 0$, then $\mathbf{F}$ is said to be **incompressible**.

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f. If f is a function of three variables, we have,

$$\text{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This expression occurs so often that it is abbreviated as $\nabla^2 f$. The operator

$$\nabla^2 = \nabla \cdot \nabla$$

is called the **Laplace operator** because of its relation to **Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

We can also apply the Laplace operator ∇^2 to a vector field

$$\mathbf{F} = P\ \mathbf{i} + Q\ \mathbf{j} + R\ \mathbf{k}$$

in terms of its components:

$$\nabla^2 \mathbf{F} = \nabla^2 P\ \mathbf{i} + \nabla^2 Q\ \mathbf{j} + \nabla^2 R\ \mathbf{k}$$

Vector Forms of Green's Theorem

The curl and divergence operators allow us to rewrite Green's Theorem in versions that will be useful in later work. Suppose that the plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem. Then consider the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Its line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\ dx + Q\ dy$$

and, if we regard $\mathbf{F}$ as a vector field on $\mathbb{R}^3$ with third component 0, we have

$$\text{curl}\ \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Therefore,

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can now rewrite the equation in Green's Theorem in the vector form

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA \tag{9}$$

Equation 9 expresses the line integral of the tangential component of **F** along C as the double integral of the vertical component of curl **F** over the region D enclosed by C.

We now derive a similar formula involving the *normal* component of **F**.

If C is given by the vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad a \le t \le b$$

then the unit tangent vector (see Section 10.2) is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$$

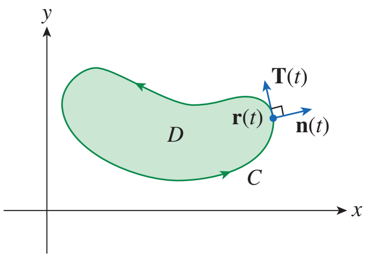


Figure 13.52
The unit tangent vector and the unit normal vector.

One can verify that the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$$

(See Figure 13.52.) Then, from Equation 13.2.2, we have

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds &= \int_a^b (\mathbf{F} \cdot \mathbf{n})(t)\,|\mathbf{r}'(t)|\,dt \\
&= \int_a^b \left[\frac{P(x(t),\ y(t))y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t),\ y(t))x'(t)}{|\mathbf{r}'(t)|}\right]|\mathbf{r}'(t)|\,dt \\
&= \int_a^b P(x(t),\ y(t))y'(t)\,dt - Q(x(t),\ y(t))x'(t)dt \\
&= \int_C P\,dy - Q dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)dA
\end{aligned}$$

by Green's Theorem. But the integrand in this double integral is just the divergence of **F**. So, we have a second vector form of Green's Theorem.

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \text{div } \mathbf{F}(x,\ y)\, dA \tag{10}$$

This version says that the line integral of the normal component of **F** along C is equal to the double integral of the divergence of **F** over the region D enclosed by C.

13.5 Exercises

Find (a) the curl and (b) the divergence of the vector field.

1. $\mathbf{F}(x, y, z) = xyz\ \mathbf{i} - x^2y\ \mathbf{k}$

2. $\mathbf{F}(x, y, z) = x^2yz\ \mathbf{i} + xy^2z\ \mathbf{j} + xyz^2\ \mathbf{k}$

3. $\mathbf{F}(x, y, z) = xye^z\ \mathbf{i} + yze^x\ \mathbf{k}$

4. $\mathbf{F}(x, y, z) = \sin yz\ \mathbf{i} + \sin zx\ \mathbf{j} + \sin xy\ \mathbf{k}$

5. $\mathbf{F}(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}(x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k})$

6. $\mathbf{F}(x, y, z) = e^{xy} \sin z\ \mathbf{j} + y \tan^{-1}(x/z)\ \mathbf{k}$

7. $\mathbf{F}(x, y, z) = \langle \ln x, \ln(xy), \ln(xyz) \rangle$

8. $\mathbf{F}(x, y, z) = \langle e^x, e^{xy}, e^{xyz} \rangle$

9. $\mathbf{F}(x, y, z) = \dfrac{\sqrt{x}}{1 + z}\mathbf{i} + \dfrac{\sqrt{y}}{1 + x}\mathbf{j} + \dfrac{\sqrt{z}}{1 + y}\mathbf{k}$

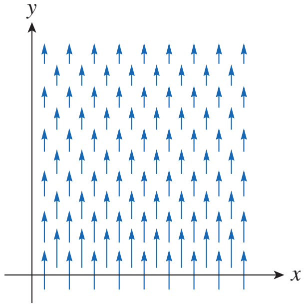
10. $\mathbf{F}(x, y, z) = \langle \tan^{-1}(xy), \tan^{-1}(yz), \tan^{-1}(xz) \rangle$

The vector field **F** is shown in the xy-plane and looks the same in all other horizontal planes. (In other words, **F** is independent of z and its z-component is 0.)

(a) Is div **F** positive, negative, or zero? Explain your reasoning.

(b) Determine whether curl $\mathbf{F} = \mathbf{0}$. If not, in which direction does curl **F** point?

11.



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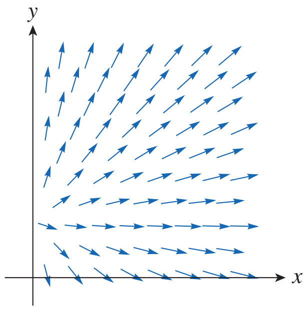

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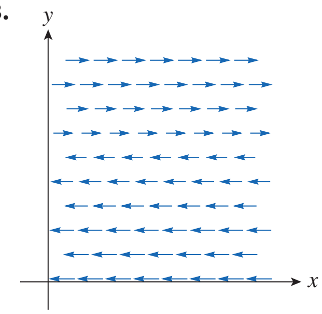

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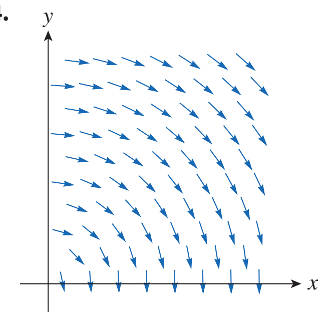


15. Let f be a scalar field and **F** a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether the result is a scalar field or a vector field.

(a) curl f (b) grad f
(c) div **F** (d) curl(grad f)
(e) grad **F** (f) grad(div **F**)
(g) div(grad f) (h) grad (div f)
(i) curl(curl **F**) (j) div(div **F**)
(k) (grad f) × (div **F**) (l) div(curl(grad f))

Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

16. $\mathbf{F}(x, y, z) = y^2z^3\ \mathbf{i} + 2xyz^3\ \mathbf{j} + 3xy^2z^2\ \mathbf{k}$

17. $\mathbf{F}(x, y, z) = xyz^2\ \mathbf{i} + x^2yz^2\ \mathbf{j} + x^2y^2z\ \mathbf{k}$

18. $\mathbf{F}(x, y, z) = 2xy\ \mathbf{i} + (x^2 + 2yz)\ \mathbf{j} + y^2\ \mathbf{k}$

19. $\mathbf{F}(x, y, z) = e^z\ \mathbf{i} + \mathbf{j} + xe^z\ \mathbf{k}$

20. $\mathbf{F}(x, y, z) = ye^{-x}\ \mathbf{i} + e^{-x}\ \mathbf{j} + 2z\ \mathbf{k}$

21. $\mathbf{F}(x, y, z) = y \cos xy\ \mathbf{i} + x \cos xy\ \mathbf{j} - \sin z\ \mathbf{k}$

22. $\mathbf{F}(x, y, z) = e^x \sin yz\ \mathbf{i} + ze^x \cos yz\ \mathbf{j} + ye^x \cos yz\ \mathbf{k}$

23. Is there a vector field **G** on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle x \sin y, \cos y, z - xy \rangle$? Explain your reasoning.

24. Is there a vector field **G** on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle xyz, -y^2z, yz^2 \rangle$? Explain your reasoning.

25. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$$

where f, g, h are differentiable functions, is irrotational.

26. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(y, z)\,\mathbf{i} + g(x, z)\,\mathbf{j} + h(x, y)\,\mathbf{k}$$

is incompressible.

Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and $\mathbf{F}$, $\mathbf{G}$ are vector fields, then $f\mathbf{F}$, $\mathbf{F}\cdot\mathbf{G}$, and $\mathbf{F}\times\mathbf{G}$ are defined by

$$(f\mathbf{F})(x, y, z) = f(x, y, z)\mathbf{F}(x, y, z)$$
$$(\mathbf{F}\cdot\mathbf{G})(x, y, z) = \mathbf{F}(x, y, z)\cdot\mathbf{G}(x, y, z)$$
$$(\mathbf{F}\times\mathbf{G})(x, y, z) = \mathbf{F}(x, y, z)\times\mathbf{G}(x, y, z)$$

27. $\operatorname{div}(\mathbf{F}+\mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$

28. $\operatorname{curl}(\mathbf{F}+\mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}$

29. $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$

30. $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}$

31. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$

32. $\operatorname{div}(\nabla f\times\nabla g) = 0$

33. $\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F}$

Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = |\mathbf{r}|$.

34. Verify each identity.

(a) $\nabla\cdot\mathbf{r} = 3$ (b) $\nabla\cdot(r\mathbf{r}) = 4r$

(c) $\nabla^2 r^3 = 12r$

35. Verify each identity.

(a) $\nabla r = \dfrac{\mathbf{r}}{r}$ (b) $\nabla\times\mathbf{r} = \mathbf{0}$

(c) $\nabla(1/r) = -\dfrac{\mathbf{r}}{r^3}$ (d) $\nabla\ln r = \dfrac{\mathbf{r}}{r^2}$

36. If $\mathbf{F} = \mathbf{r}/r^p$, find $\operatorname{div}\mathbf{F}$. Is there a value of p for which $\operatorname{div}\mathbf{F} = 0$?

37. Use Green's Theorem in the form of Equation 13 to prove **Green's first identity**:

$$\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. (The quantity $\nabla g\cdot\mathbf{n} = D_{\mathbf{n}}g$ occurs in the line integral. This is the directional derivative in the direction of the normal vector $\mathbf{n}$ and is called the **normal derivative** of g.)

38. Use Green's first identity (Exercise 37) to prove **Green's second identity**:

$$\iint_D (f\nabla^2 g - g\nabla^2 f)\,dA = \oint_C (f\nabla g - g\nabla f)\cdot\mathbf{n}\,ds$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous.

39. Recall from Section 11.3 that a function g is called *harmonic* on D if it satisfies Laplace's equation, that is, $\nabla^2 g = 0$ on D. Use Green's first identity (with the same hypotheses as in Exercise 37) to show that if g is harmonic on D, then $\oint_C D_{\mathbf{n}}g\,ds = 0$. Here $D_{\mathbf{n}}g$ is the normal derivative of g defined in Exercise 37.

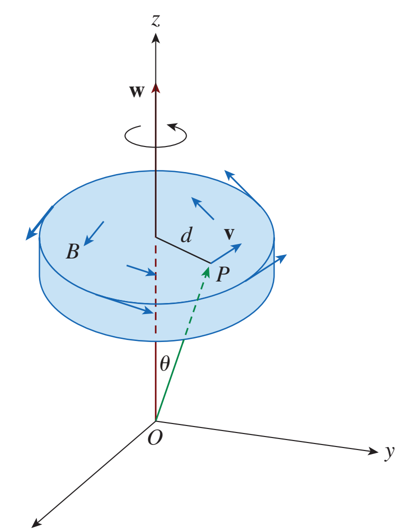
40. Use Green's first identity to show that if f is harmonic on D, and if $f(x, y) = 0$ on the boundary curve C, then $\iint_D |\nabla f|^2\,dA = 0$. (Assume the same hypotheses as in Exercise 37.)

41. This exercise demonstrates a connection between the curl vector and rotations. Let B be a rigid body rotating about the z-axis. The rotation can be described by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed of B, that is, the tangential speed of any point P in B divided by the distance d from the axis of rotation. Let $\mathbf{r} = \langle x, y, z\rangle$ be the position vector of P.

(a) By considering the angle θ in the figure, show that the velocity field of B is given by $\mathbf{v} = \mathbf{w}\times\mathbf{r}$.

(b) Show that $\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$.

(c) Show that $\operatorname{curl}\mathbf{v} = 2\mathbf{w}$.



42. Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\text{div } \mathbf{E} = 0 \qquad \text{div } \mathbf{H} = 0$$

$$\text{curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \text{curl } \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove the following:

(a) $\nabla \times (\nabla \times \mathbf{E}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}$

(b) $\nabla \times (\nabla \times \mathbf{H}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$

(c) $\nabla^2 \mathbf{E} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}$ Hint: Use Exercise 33.

(d) $\nabla^2 \mathbf{H} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$

43. We have seen that all vector fields of the form $\mathbf{F} = \nabla g$ satisfy the equation curl $\mathbf{F} = \mathbf{0}$ and that all vector fields of the form $\mathbf{F} = \text{curl } \mathbf{G}$ satisfy the equation div $\mathbf{F} = 0$ (assuming continuity of the appropriate partial derivatives). This suggests the question: are there any equations that all functions of the form $f = \text{div } \mathbf{G}$ must satisfy? Show that the answer to this question is "No" by proving that *every* continuous function f on $\mathbb{R}^3$ is the divergence of some vector field.

Hint: Let $\mathbf{G}(x, y, z) = \langle g(x, y, z), 0, 0\rangle$, where

$$g(x, y, z) = \int_0^x f(t, y, z)\, dt.$$

13.6 Surface Integrals

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length. Suppose f is a function of three variables whose domain includes a surface S. We will define the surface integral of f over S in such a way that, in the case where $f(x, y, z) = 1$, the value of the surface integral is equal to the surface area of S. We start with parametric surfaces and then deal with the special case where S is the graph of a function of two variables.

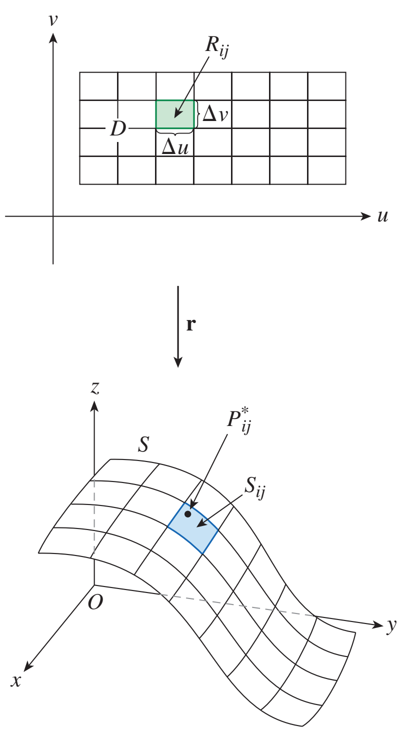


Figure 13.53
The vector function $\mathbf{r}$ maps the subrectangle R_{ij} into the patch S_{ij}.

Parametric Surfaces

Recall from Section 10.5 that a parametric surface S is defined by a vector function $\mathbf{r}(u, v)$ of two parameters u and v:

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

First assume that the parameter domain D is a rectangle. Divide D into subrectangles R_{ij} with dimensions Δu and Δv. Then the surface S is divided into corresponding patches S_{ij} as illustrated in Figure 13.53. Evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} of the patch, and form the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\, \Delta S_{ij}$$

Take the limit as the number of patches increases and define the **surface integral of f over the surface S** as

$$\iint_S f(x, y, z)\, dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\, \Delta S_{ij} \tag{1}$$

Notice the analogy with the definition of a line integral and also the analogy with the definition of a double integral.

To evaluate the surface integral in Equation 1, we approximate the patch area ΔS_{ij} by the area of an approximating parallelogram in the tangent plane. In the discussion of surface area in Section 12.6, we made the approximation

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v| \ \Delta u \ \Delta v$$

where $\mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$

are the tangent vectors at a corner of S_{ij}. If the components are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in the interior of D, it can be shown from Equation 1, even when D is not a rectangle, that

We assume that the surface is covered only once as (u, v) ranges throughout D. The value of the surface integral does not depend on the parametrization that is used.

$$\iint_S f(x, y, z) \ dS = \iint_D f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v| \ dA \tag{2}$$

Compare this result with the formula for a line integral:

$$\int_C f(x, y, z) \ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)| \ dt$$

In addition, observe that

$$\iint_S 1 \ dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \ dA = A(S)$$

Equation 2 allows us to compute a surface integral by converting it into a double integral over the parameter domain D. When using this formula, remember that $f(\mathbf{r}(u, v))$ is evaluated by writing $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ in the formula for $f(x, y, z)$.

Example 1 Integrate over a Sphere

Compute the surface integral $\iint_S x^2 \ dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution

As in Example 4, Section 10.5, use the parametric representation

$x = \sin\phi \ \cos\theta \qquad y = \sin\phi \ \sin\theta \qquad z = \cos\phi \qquad 0 \le \phi \le \pi \qquad 0 \le \theta \le 2\pi$

The vector function is

$\mathbf{r}(\phi, \theta) = \sin\phi \ \cos\theta \, \mathbf{i} + \sin\phi \ \sin\theta \, \mathbf{j} + \cos\phi \, \mathbf{k}.$

From Example 1, Section 12.6: $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$.

Use Equation 2 to evaluate the surface integral.

$$\iint_S x^2 \ dS = \iint_D (\sin\phi \ \cos\theta)^2|\mathbf{r}_\phi \times \mathbf{r}_\theta| \ dA$$

Equation 2.

$$= \int_0^{2\pi}\int_0^{\pi} \sin^2\phi \ \cos^2\theta \ \sin\phi \ d\phi \ d\theta$$

Write as a double integral; use expression for $|\mathbf{r}_\phi \times \mathbf{r}_\theta|$.

$$= \int_0^{2\pi} \cos^2\theta \ d\theta \int_0^{\pi} \sin^3\phi \ d\phi$$

Integrand is a product of a function in θ times a function in ϕ.

The trigonometric identities used:

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$\sin^2\phi = 1 - \cos^2\phi$

$$= \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta)\, d\theta \int_0^{\pi} (\sin\phi - \sin\phi \cos^2\phi)\, d\phi \qquad \text{Trigonometric identities.}$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} \left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi} = \frac{4\pi}{3} \qquad \text{Antiderivatives; FTC.}$$

Surface integrals have applications similar to those for the integrals we have previously considered. For example, if a thin sheet (say, of aluminum foil) has the shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$, then the total **mass** of the sheet is

$$m = \iint_S \rho(x, y, z)\, dS$$

and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_S x\rho(x, y, z)\, dS \quad \bar{y} = \frac{1}{m}\iint_S y\rho(x, y, z)\, dS \quad \bar{z} = \frac{1}{m}\iint_S z\rho(x, y, z)\, dS.$$

Moments of inertia can also be defined as before (see Exercise 40).

Graphs

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

and therefore

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$$

Thus,

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \tag{3}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Therefore, in this case, Equation 2 becomes

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \tag{4}$$

Similar formulas apply when it is more convenient to project S onto the yz-plane or xz-plane. For instance, if S is a surface with equation $y = h(x, z)$ and D is its projection onto the xz-plane, then

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, h(x, z), z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA$$

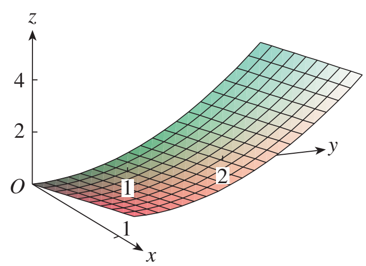


Figure 13.54
Graph of the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$.

Example 2 Integration over the Graph of a Function

Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$. (See Figure 13.54.)

Solution

Find the partial derivatives of z: $\dfrac{\partial z}{\partial x} = 1$ and $\dfrac{\partial z}{\partial y} = 2y$.

Use Equation 4.

$$\iint_S y\, dS = \iint_D y\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \qquad \text{Equation 4.}$$

$$= \int_0^1 \int_0^2 y\sqrt{1 + 1 + 4y^2}\, dy\, dx \qquad \text{Write as a double integral; use partial derivatives.}$$

$$= \int_0^1 \sqrt{2}\, dx \int_0^2 y\sqrt{1 + 2y^2}\, dy \qquad \text{Integrand is a product of a function in } x \text{ times a function in } y.$$

$$= \sqrt{2}\left(\frac{1}{4}\right)\left[\frac{2}{3}(1 + 2y^2)^{3/2}\right]_0^2 = \frac{13\sqrt{2}}{3} \qquad \text{Antiderivatives; FTC.}$$

■

If S is a piecewise-smooth surface, that is, a finite union of smooth surfaces $S_1, S_2, \ldots, S_n$ that intersect only along their boundaries, then the surface integral of f over S is defined by

$$\iint_S f(x, y, z)\, dS = \iint_{S_1} f(x, y, z)\, dS + \cdots + \iint_{S_n} f(x, y, z)\, dS$$

Example 3 Integration over a Piecewise-Smooth Surface

Evaluate $\iint_S z\, dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \le 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

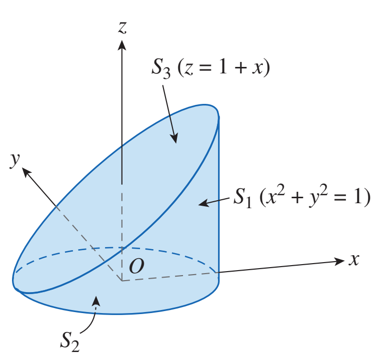


Figure 13.55
The surface S is piecewise smooth.

Solution

The surface S is shown in Figure 13.55. (The axes are in a different position so that we get a better look at S.)

For S_1, use θ and z as parameters (see Example 5, Section 12.5).

The parametric equations are $x = \cos\theta \quad y = \sin\theta \quad z = z$

where $0 \le \theta \le 2\pi$ and $0 \le z \le 1 + x = 1 + \cos\theta$.

Compute the appropriate cross product and magnitude.

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

$$|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Evaluate the surface integral over S_1.

$$\iint_{S_1} z\, dS = \iint_D z\left|\mathbf{r}_\theta \times \mathbf{r}_z\right|\, dA \qquad \text{Equation 2.}$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} z\, dz\, d\theta \qquad \text{Convert to a double integral.}$$

$$= \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2\, d\theta \qquad \text{Antiderivative with respect to } z\text{; FTC.}$$

$$= \frac{1}{2}\int_0^{2\pi}\left[1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right] d\theta \qquad \text{Expand; trigonometric identity.}$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi}{2} \qquad \text{Antiderivative; FTC.}$$

Since S_2 lies in the plane $z = 0$, then $\displaystyle\iint_{S_2} z\, dS = \iint_{S_2} 0\, dS = 0$.

The top surface, S_3, lies above the disk D and is part of the plane $z = 1 + x$.

Let $g(x, y) = 1 + x$ in Equation 4 and convert to polar coordinates.

$$\iint_{S_3} z\, dS = \iint_D (1+x)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \qquad \text{Equation 4.}$$

$$= \int_0^{2\pi}\int_0^1 (1 + r\cos\theta)\sqrt{1+1+0}\; r\, dr\, d\theta \qquad \text{Convert to a double integral in polar coordinates.}$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 (r + r^2\cos\theta) dr\, d\theta \qquad \text{Simplify the integrand.}$$

$$= \sqrt{2}\int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{3}\cos\theta\right) d\theta \qquad \text{Antiderivative with respect to } r\text{; FTC.}$$

$$= \sqrt{2}\left[\frac{\theta}{2} + \frac{\sin\theta}{3}\right]_0^{2\pi} = \sqrt{2}\pi \qquad \text{Antiderivative; FTC.}$$

Use these three results to evaluate the original surface integral.

$$\iint_S z\, dS = \iint_{S_1} z\, dS + \iint_{S_2} z\, dS + \iint_{S_3} z\, dS$$

$$= \frac{3\pi}{2} + 0 + \sqrt{2}\pi = \left(\frac{3}{2} + \sqrt{2}\right)\pi$$

Oriented Surfaces

To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip shown in Figure 13.56. [This surface is named after the German geometer August Möbius (1790–1868).] You can construct one for yourself by taking a long rectangular strip of paper, giving it a half-twist, and taping the short edges together as in Figure 13.57. If an ant were to crawl along the Möbius strip starting at a point P, it would end up on the "other side" of the strip (that is, with its upper side pointing in the opposite direction). Then, if the ant continued to crawl in the same direction, it

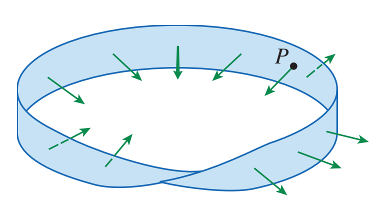


Figure 13.56
A Möbius strip.

would end up back at the same point P without ever having crossed an edge. (If you have constructed a Möbius strip, try drawing a pencil line down the middle.) Therefore, a Möbius strip really has only one side. You can graph the Möbius strip using the parametric equations in Exercise 34 in Section 10.5.

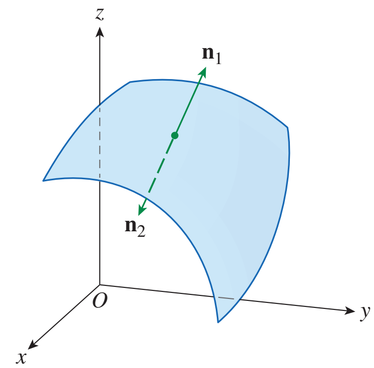


Figure 13.58
The two unit normal vectors at (x, y, z).

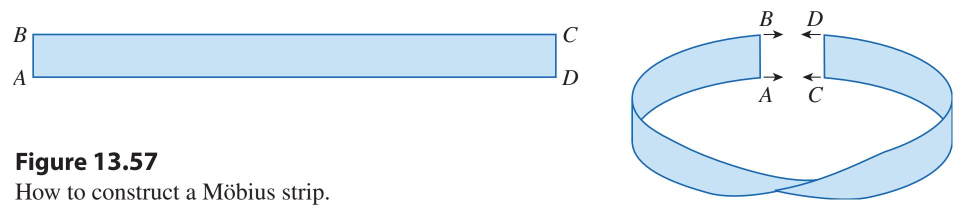


Figure 13.57
How to construct a Möbius strip.

From now on we consider only orientable (two-sided) surfaces. Start with a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point). There are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z). (See Figure 13.58.)

If it is possible to choose a unit normal vector $\mathbf{n}$ at every such point (x, y, z) so that $\mathbf{n}$ varies continuously over S, then S is called an **oriented surface** and the given choice of $\mathbf{n}$ provides S with an **orientation**. There are two possible orientations for any orientable surface, as illustrated in Figure 13.59.

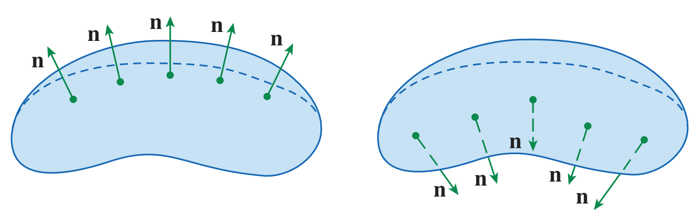


Figure 13.59
The two orientations of an orientable surface.

For a surface $z = g(x, y)$ given as the graph of g, we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

$$\mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}} \tag{5}$$

Since the $\mathbf{k}$-component is positive, this gives the *upward* orientation of the surface.

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \tag{6}$$

and the opposite orientation is given by $-\mathbf{n}$. For instance, in Example 4, Section 10.5, we found the parametric representation

$$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\,\mathbf{i} + a \sin\phi \sin\theta\,\mathbf{j} + a \cos\phi\,\mathbf{k}$$

for the sphere $x^2 + y^2 + z^2 = a^2$. Then in Example 1, Section 12.6, we found that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2\sin^2\phi \cos\theta\,\mathbf{i} + a^2\sin^2\phi \sin\theta\,\mathbf{j} + a^2\sin\phi \cos\phi\,\mathbf{k}$$

and $$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2\sin\phi$$

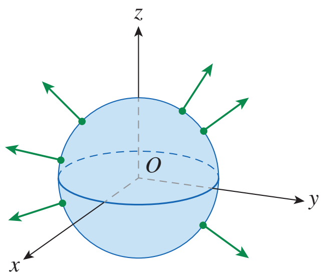


Figure 13.60
Positive orientation: the unit normal vector points outward from the sphere.

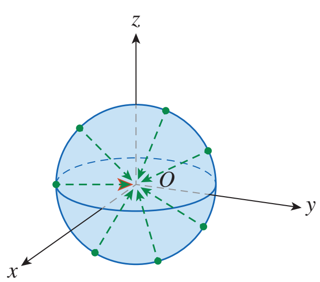


Figure 13.61
Negative orientation: the unit normal vector points inward from the sphere.

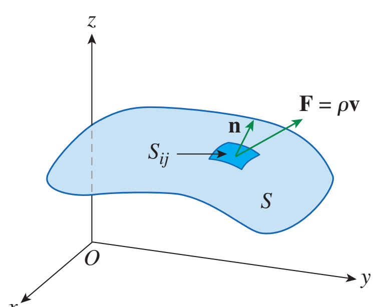


Figure 13.62
Divide S into small patches S_{ij}.

So, the orientation induced by $\mathbf{r}(\phi, \theta)$ is defined by the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} = \frac{1}{a}\mathbf{r}(\phi, \theta)$$

Observe that $\mathbf{n}$ points in the same direction as the position vector, that is, outward from the sphere (see Figure 13.60). The opposite (inward) orientation would have been obtained (see Figure 13.61) if we had reversed the order of the parameters because $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.

For a **closed surface**, that is, a surface that is the boundary of a solid region E, the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from E, and inward-pointing normals give the negative orientation (see Figures 13.60 and 13.61).

Surface Integrals of Vector Fields

Suppose that S is an oriented surface with unit normal vector $\mathbf{n}$, and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing through S. (Think of S as an imaginary surface that doesn't impede the fluid flow, like a fishing net across a stream.) Then the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$. If we divide S into small patches S_{ij}, as in Figure 13.62 (compare with Figure 13.53), then S_{ij} is nearly planar and so we can approximate the mass of fluid crossing S_{ij} in the direction of the normal $\mathbf{n}$ per unit time by the quantity

$$(\rho\mathbf{v} \cdot \mathbf{n})\, A(S_{ij})$$

where ρ, $\mathbf{v}$, and $\mathbf{n}$ are evaluated at some point on S_{ij}. (Recall that the component of the vector $\rho\mathbf{v}$ in the direction of the unit vector $\mathbf{n}$ is $\rho\mathbf{v} \cdot \mathbf{n}$.) If we sum these quantities and take the limit we get, according to Equation 1, the surface integral of the function $\rho\mathbf{v} \cdot \mathbf{n}$ over S:

$$\iint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS = \iint_S \rho(x, y, z)\mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z)\, dS \tag{7}$$

and this is interpreted physically as the rate of flow through S.

If we write $\mathbf{F} = \rho\mathbf{v}$, then $\mathbf{F}$ is also a vector field on $\mathbb{R}^3$ and the integral in Equation 7 becomes

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

A surface integral of this form occurs frequently in physics, even when $\mathbf{F}$ is not $\rho\mathbf{v}$, and is called the *surface integral* (or *flux integral*) of $\mathbf{F}$ over S.

Definition • Surface Integral of F Over *S*

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of F over *S*** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

In words, this definition says that the surface integral of a vector field over S is equal to the surface integral of its normal component over S (as previously defined).

If S is given by a vector function $\mathbf{r}(u, v)$, then $\mathbf{n}$ is given by Equation 6, and using the definition of a surface integral and Equation 2, we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} dS$$

$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v| \; dA$$

Compare Equation 9 to the similar expression for evaluating line integrals of vector fields in Definition 13.2.13:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \; dt$$

where D is the parameter domain. Thus, we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \; dA \qquad (8)$$

Example 4 Flux Across a Sphere

Find the flux of the vector field $\mathbf{F}(x, y, z) = z \, \mathbf{i} + y \, \mathbf{j} + x \, \mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution

Use the same parametric representation of the sphere as in Example 1.

$\mathbf{r}(\phi, \theta) = \sin \phi \cos \theta \, \mathbf{i} + \sin \phi \sin \theta \, \mathbf{j} + \cos \phi \, \mathbf{k}, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi$

Then $\mathbf{F}(\mathbf{r}(\phi, \theta)) = \cos \phi \, \mathbf{i} + \sin \phi \sin \theta \, \mathbf{j} + \sin \phi \cos \theta \, \mathbf{k}$
and, from Example 1, Section 12.6,

$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi \cos \theta \, \mathbf{i} + \sin^2\phi \sin \theta \, \mathbf{j} + \sin \phi \cos \phi \, \mathbf{k}.$

Therefore,

$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \cos \phi \sin^2\phi \cos \theta + \sin^3\phi \sin^2\theta + \sin^2\phi \cos \phi \cos \theta.$

Use Equation 9 to find the flux.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) \; dA \qquad \text{Equation 8.}$$

$$= \int_0^{2\pi} \int_0^{\pi} (2 \sin^2\phi \cos \phi \cos \theta + \sin^3\phi \sin^2\theta) \; d\phi \; d\theta$$

$$= 2 \int_0^{\pi} \sin^2\phi \cos \phi \; d\phi \int_0^{2\pi} \cos \theta \; d\theta + \int_0^{\pi} \sin^3\phi \; d\phi \int_0^{2\pi} \sin^2\theta \; d\theta$$

$$= 0 + \int_0^{\pi} \sin^3\phi \; d\phi \int_0^{2\pi} \sin^2\theta \; d\theta \qquad \int_0^{2\pi} \cos \theta \; d\theta = 0.$$

$$= \frac{4\pi}{3}$$

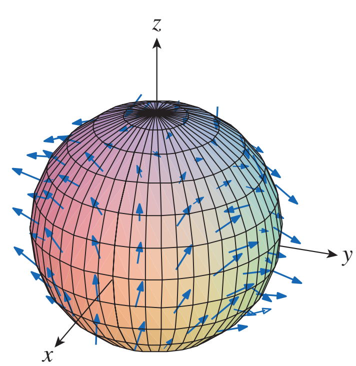


Figure 13.63
The vector field $\mathbf{F}$ at points on the unit sphere.

Figure 13.63 shows the vector field $\mathbf{F}$ at points on the unit sphere. ■

If, for instance, the vector field in Example 4 is a velocity field describing the flow of a fluid with density 1, then the answer, $4\pi/3$, represents the rate of flow through the unit sphere in units of mass per unit time.

In the case of a surface S given by a graph $z = g(x, y)$, we can think of x and y as parameters and use Equation 3 to write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right)$$

Thus, Equation 8 becomes

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \qquad (9)$$

This formula assumes the upward orientation of S; for a downward orientation, we multiply by -1. Similar formulas can be obtained if S is given by $y = h(x, z)$ or $x = k(y, z)$. (See Exercises 36 and 37.)

Example 5 Surface Integral of a Vector Field

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

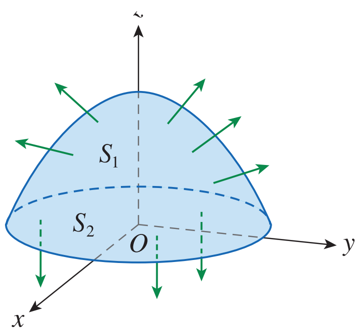


Figure 13.64
The surface S and unit normal vectors on S.

Solution

S consists of a parabolic top surface S_1 and a circular bottom surface S_2. Figure 13.64 shows S and the vector field $\mathbf{F}$ at some points on S.

Since S is a closed surface, we use the convention of positive (outward) orientation. This means that S_1 is oriented upward and we can use Equation 10 with D as the projection of S_1 on the xy-plane, namely, the disk $x^2 + y^2 \le 1$.

On S_1: $P(x, y, z) = y,\quad Q(x, y, z) = x,\quad R(x, y, z) = z = 1 - x^2 - y^2$

and $\dfrac{\partial g}{\partial x} = -2x,\quad \dfrac{\partial g}{\partial y} = -2y$

Use Equation 9 to evaluate the integral over S_1.

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA && \text{Equation 9.}\\
&= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2]\, dA && \text{Use known expressions.}\\
&= \iint_D (1 + 4xy - x^2 - y^2)\, dA && \text{Simplify integrand.}\\
&= \int_0^{2\pi}\int_0^1 (1 + 4r^2\cos\theta\sin\theta - r^2)\, r\, dr\, d\theta && \text{Convert to polar coordinates.}\\
&= \int_0^{2\pi}\int_0^1 (r - r^3 + 4r^3\cos\theta\sin\theta)\, dr\, d\theta && \text{Simplify integrand.}\\
&= \int_0^{2\pi}\left(\frac{1}{4} + \cos\theta\sin\theta\right) d\theta && \text{Antiderivative; FTC.}\\
&= \frac{1}{4}(2\pi) + 0 = \frac{\pi}{2} && \text{Antiderivative; FTC.}
\end{aligned}$$

The disk S_2 is oriented downward, so its unit normal vector is $\mathbf{n} = -\mathbf{k}$. Therefore,

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k})\, dS = \iint_D (-z)dA = \iint_D 0\, dA = 0 \qquad z = 0 \text{ on } S_2.$$

Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ by definition, as the sum of the surface integrals of $\mathbf{F}$ over the pieces S_1 and S_2.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2} \qquad \blacksquare$$

Although we used the example of fluid flow to motivate the surface integral of a vector field, this concept also arises in other physical situations. For instance, if $\mathbf{E}$ is an electric field (see Example 5, Section 13.1), then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of $\mathbf{E}$ through the surface S. One of the important laws of electrostatics is **Gauss' Law**, which says that the net charge enclosed by a closed surface S is

$$Q = \epsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S} \tag{10}$$

where ϵ_0 is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\epsilon_0 \approx 8.8542 \times 10^{-12}\ \mathrm{C}^2/\mathrm{N} \cdot \mathrm{m}^2$.) Therefore, if the vector field $\mathbf{F}$ in Example 4 represents an electric field, we can conclude that the charge enclosed by S is $Q = 4\pi\epsilon_0/3$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K\nabla u$$

where K is an experimentally determined constant called the **conductivity** of the substance. The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

Example 6 Heat Flow Across a Sphere

The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

Solution

Let the center of the ball be at the origin.

Then $u(x, y, z) = C(x^2 + y^2 + z^2)$ where C is the proportionality constant.

The heat flow is $\mathbf{F}(x, y, z) = -K\nabla u = -KC(2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k})$

where K is the conductivity of the metal.

Instead of using the usual parametrization of the sphere as in Example 4, notice that the outward unit normal to the sphere $x^2 + y^2 + z^2 = a^2$ at the point (x, y, z) is

$$\mathbf{n} = \frac{1}{a}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \quad \text{and} \quad \mathbf{F} \cdot \mathbf{n} = -\frac{2KC}{a}(x^2 + y^2 + z^2).$$

But on S we have $x^2 + y^2 + z^2 = a^2$, so $\mathbf{F} \cdot \mathbf{n} = -2aKC$.

Therefore, the rate of heat flow across S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -2aKC \iint_S dS$$
$$= -2aKCA(S) = -2aKC(4\pi a^2) = -8KC\pi a^3$$

13.6 Exercises

1. Let S be the boundary surface of the box enclosed by the planes $x = 0$, $x = 2$, $y = 0$, $y = 4$, $z = 0$, and $z = 6$. Approximate $\iint_S e^{-0.1(x+y+z)}\, dS$ by using a Riemann sum as in Equation 1, taking the patches S_{ij} to be the rectangles that are the faces of the box S and the points P_{ij}^* to be the centers of the rectangles.

2. A surface S consists of the cylinder $x^2 + y^2 = 1$, $-1 \leq z \leq 1$, together with its top and bottom disks. Suppose f is a continuous function with

$$f(\pm 1, 0, 0) = 2 \qquad f(0, \pm 1, 0) = 3 \qquad f(0, 0, \pm 1) = 4.$$

Estimate the value of $\iint_S f(x, y, z)\, dS$ by using a Riemann sum, taking the patches S_{ij} to be four quarter-cylinders and the top and bottom disks.

3. Let H be the hemisphere $x^2 + y^2 + z^2 = 50$, $z \geq 0$, and suppose f is a continuous function with $f(3, 4, 5) = 7$, $f(3, -4, 5) = 8$, $f(-3, 4, 5) = 9$, and $f(-3, -4, 5) = 12$. Divide H into four patches and estimate the value of $\iint_H f(x, y, z)\, dS$.

4. Suppose that $f(x, y, z) = g\left(\sqrt{x^2 + y^2 + z^2}\right)$, where g is a function of one variable such that $g(2) = -5$. Evaluate $\iint_S f(x, y, z)\, dS$, where S is the sphere $x^2 + y^2 + z^2 = 4$.

Evaluate the surface integral.

5. $\iint_S (x + y + z)\, dS$, S is the parallelogram with parametric equations $x = u + v$, $y = u - v$, $z = 1 + 2u + v$, $0 \leq u \leq 2$, $0 \leq v \leq 1$

6. $\iint_S xyz\, dS$, S is the cone with parametric equations $x = u \cos v$, $y = u \sin v$, $z = u$, $0 \leq u \leq 1$, $0 \leq v \leq \pi/2$

7. $\iint_S y\, dS$, S is the helicoid with vector equation $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle$, $0 \leq u \leq 1$, $0 \leq v \leq \pi$

8. $\iint_S (x^2 + y^2)\, dS$, S is the surface with vector equation $\mathbf{r}(u, v) = \langle 2uv, u^2 - v^2, u^2 + v^2 \rangle$, $u^2 + v^2 \leq 1$

9. $\iint_S x^2 yz\, dS$, S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$

10. $\iint_S xy\, dS$, S is the triangular region with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$

11. $\iint_S yz\, dS$, S is the part of the plane $x + y + z = 1$ that lies in the first octant

12. $\iint_S y\, dS$, S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leq x \leq 1$, $0 \leq y \leq 1$

13. $\iint_S x^2 z^2\, dS$, S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$

14. $\iint_S z\, dS$, S is the surface $x = y + 2z^2$, $0 \leq y \leq 1$, $0 \leq z \leq 1$

15. $\iint_S y\, dS$, S is the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$

16. $\displaystyle\iint_S y^2\, dS$, S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane

17. $\displaystyle\iint_S (x^2z + y^2z)\, dS$, S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$

18. $\displaystyle\iint_S xz\, dS$, S is the boundary of the region enclosed by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$ and $x + y = 5$

19. $\displaystyle\iint_S (z + x^2y)\, dS$, S is the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$ in the first octant

20. $\displaystyle\iint_S (x^2 + y^2 + z^2)\, dS$, S is the part of the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$, together with its top and bottom disks

Evaluate the surface integral $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces, use the positive (outward) orientation.

21. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, and has upward orientation

22. $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$, S is the helicoid described in Exercise 7 with upward orientation

23. $\mathbf{F}(x, y, z) = xze^y\,\mathbf{i} - xze^y\,\mathbf{j} + z\,\mathbf{k}$, S is the part of the plane $x + y + z = 1$ in the first octant and has downward orientation

24. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z^4\,\mathbf{k}$, S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation

25. $\mathbf{F}(x, y, z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$, S is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant, with orientation toward the origin

26. $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$, S is the hemisphere $x^2 + y^2 + z^2 = 25$, $y \geq 0$, orientated in the direction of the positive y-axis

27. $\mathbf{F}(x, y, z) = y\,\mathbf{j} - z\,\mathbf{k}$, S consists of the paraboloid $y = x^2 + z^2$, $0 \leq y \leq 1$, and the disk $x^2 + z^2 \leq 1$, $y = 1$

28. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$, S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$

29. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$, S is the cube with vertices $(\pm 1, \pm 1, \pm 1)$

30. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + (z - y)\,\mathbf{j} + x\,\mathbf{k}$, S is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

31. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$, S is the boundary of the solid half-cylinder $0 \leq z \leq \sqrt{1 - y^2}$, $0 \leq x \leq 2$

32. Let S be the surface $z = xy$, $0 \leq x \leq 1$, $0 \leq y \leq 1$.

(a) Use technology to approximate $\displaystyle\iint_S xyz\, dS$.

(b) Find the exact value of $\displaystyle\iint_S x^2yz\, dS$.

33. Use technology to find the exact value of $\displaystyle\iint_S xyz\, dS$ where S is the surface $z = x^2y^2$, $0 \leq x \leq 1$, $0 \leq y \leq 2$.

34. Use technology to find the value of $\displaystyle\iint_S x^2y^2z^2\, dS$ where S is the part of the paraboloid $z = 3 - 2x^2 - y^2$ that lies above the xy-plane.

35. Find the flux of

$$\mathbf{F}(x, y, z) = \sin(xyz)\,\mathbf{i} + x^2y\,\mathbf{j} + z^2e^{x/5}\,\mathbf{k}$$

across the part of the cylinder $4y^2 + z^2 = 4$ that lies above the xy-plane and between the planes $x = -2$ and $x = 2$ with upward orientation. Use technology to draw the cylinder and vector field on the same coordinate axes.

36. Find a formula for $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Equation 10 for the case where S is given by $y = h(x, z)$ and $\mathbf{n}$ is the unit normal that points toward the left.

37. Find a formula for $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Equation 10 for the case where S is given by $x = k(y, z)$ and $\mathbf{n}$ is the unit normal that points forward (that is, toward the viewer when the axes are drawn in the usual way).

38. Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, if it has constant density.

39. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leq z \leq 4$, if its density function is $\rho(x, y, z) = 10 - z$.

40. (a) Write an expression involving an integral for the moment of inertia I_z about the z-axis of a thin sheet in the shape of a surface S if the density function is ρ.

(b) Find the moment of inertia about the z-axis of the funnel described in Exercise 39.

41. Let S be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies above the plane $z = 4$. If S has constant density k, find (a) the center of mass and (b) the moment of inertia about the z-axis.

42. A fluid has density 870 kg/m^2 and flows with velocity $\mathbf{v} = z\,\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$, $0 \le z \le 1$.

43. Seawater has density 1025 kg/m^3 and flows in a velocity field $\mathbf{v} = y\,\mathbf{i} + x\,\mathbf{j}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9$, $z \ge 0$.

44. Use Gauss' Law to find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \le a^2$, $z \ge 0$, if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$$

45. Use Gauss' Law to find the charge enclosed by the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

46. The temperature at the point (x, y, z) in a substance with conductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 6$, $0 \le x \le 4$.

47. The temperature at a point in a ball with conductivity K is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

48. Let $\mathbf{F}$ be an inverse square field, that is $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Show that the flux of $\mathbf{F}$ across a sphere S with center at the origin is independent of the radius of S.

13.7 Stokes' Theorem

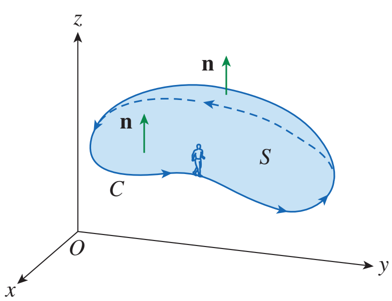


Figure 13.65
The orientation of S induces the positive orientation of the boundary curve C.

Stokes' Theorem can be thought of as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve). Figure 13.65 shows an oriented surface with unit normal vector $\mathbf{n}$. The orientation of S induces the **positive orientation of the boundary curve C** shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of $\mathbf{n}$ then the surface will always be on your left.

Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \quad \text{and} \quad \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS$$

Stokes' Theorem says that the line integral around the boundary of S of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$.

Stokes' Theorem is named after the Irish mathematical physicist Sir George Stokes (1819–1903). Stokes was a professor at Cambridge University (in fact, he held the same position as Newton, Lucasian Professor of Mathematics) and was especially noted for his studies of fluid flow and light. What we call Stokes' Theorem was actually discovered by the Scottish physicist Sir William Thomson (1824–1907, known as Lord Kelvin). Stokes learned of this theorem in a letter from Thomson in 1850 and asked students to prove it on an examination at Cambridge University in 1854. We don't know if any of those students were able to do so.

The positively oriented boundary curve of the oriented surface S is often written as ∂S, so Stokes' Theorem can be expressed as

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} \tag{1}$$

There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus. As before, there is an integral involving derivatives on the left side of Equation 1 (recall that curl $\mathbf{F}$ is sort of a derivative of $\mathbf{F}$) and the right side involves the values of $\mathbf{F}$ only on the *boundary* of S.

In fact, in the special case where the surface S is flat and lies in the xy-plane with upward orientation, the unit normal is $\mathbf{k}$, the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{k} dA$$

This is precisely the vector form of Green's Theorem given in Equation 13.5.9. Thus, we see that Green's Theorem is really a special case of Stokes' Theorem.

Stokes' Theorem is a challenge to prove in its full generality. However, we can give a proof when S is a graph and $\mathbf{F}$, S, and C are well behaved.

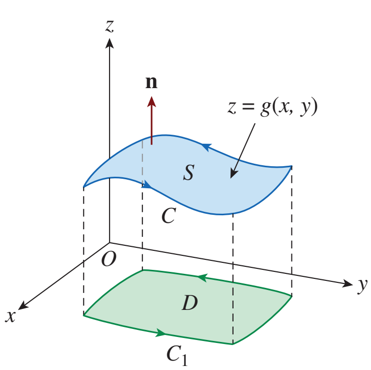


Figure 13.66
The positive orientation of C corresponds to the positive orientation of C_1.

Proof of a Special Case of Stokes' Theorem

We assume that the equation of S is $z = g(x, y)$, $(x, y) \in D$, where g has continuous second-order partial derivatives and D is a simple plane region whose boundary curve C_1 corresponds to C.

If the orientation of S is upward, then the positive orientation of C corresponds to the positive orientation of C_1. (See Figure 13.66.)

We are also given that $\mathbf{F} = P\ \mathbf{i} + Q\ \mathbf{j} + R\ \mathbf{k}$, where the partial derivatives of P, Q, and R are continuous.

Since S is a graph of a function, we can apply Equation 13.6.9 with $\mathbf{F}$ replaced by curl $\mathbf{F}$. The result is

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

$$= \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] dA \tag{2}$$

where the partial derivatives of P, Q, and R are evaluated at $(x, y, g(x, y))$.

If $x = x(t),\ y = y(t),\ a \le t \le b$ is a parametric representation of C_1, then a parametric representation of C is

$x = x(t),\ y = y(t),\ z = g(x(t), y(t)),\ a \le t \le b.$

Using the Chain Rule, we can now evaluate the line integral as follows:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt \\
&= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \right) \right] dt && \text{Chain Rule.} \\
&= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x} \right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y} \right)\frac{dy}{dt} \right] dt \\
&= \int_{C_1} \left(P + R\frac{\partial z}{\partial x} \right) dx + \left(Q + R\frac{\partial z}{\partial y} \right) dy \\
&= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x} \right) \right] dA && \text{Green's Theorem.}
\end{aligned}$$

Use the Chain Rule again and remember that P, Q, and R are functions of x, y, and z, and that z is itself a function of x and y.

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \Bigg[&\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x\,\partial y} \right) \\
&- \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 x}{\partial y \partial x} \right) \Bigg] dA
\end{aligned}$$

Four of the terms in this double integral cancel and the remaining six terms can be arranged to coincide with the right side of Equation 2.

Therefore, $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$. ■

Example 1 Use Stokes' Theorem to Calculate a Line Integral

Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)

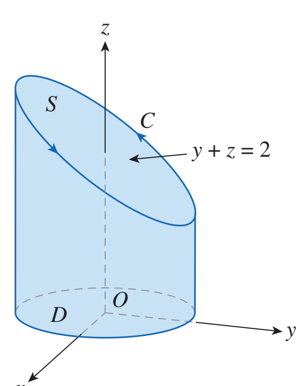


Figure 13.67
The curve C is an ellipse.

Solution

The curve C is an ellipse and is shown in Figure 13.67.

The integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ could be evaluated directly, but it's easier to use Stokes' Theorem. Find curl $\mathbf{F}$.

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y)\,\mathbf{k}$$

Although there are many surfaces with boundary C, the most convenient choice is the elliptical region S in the plane $y + z = 2$ that is bounded by C.

If we orient S upward, then C has the induced positive orientation.

The projection D of S onto the xy-plane is the disk $x^2 + y^2 \leq 1$.

Use Equation 13.6.10 with $z = g(x, y) = 2 - y$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y)\ dA \qquad \text{Stokes' Theorem; use curl } \mathbf{F}.$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r \sin \theta)\ r\ dr\ d\theta \qquad \text{Convert to double integral; use polar coordinates.}$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3} \sin \theta \right]_0^1 d\theta \qquad \text{Antiderivative with respect to } r.$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3} \sin \theta \right) d\theta \qquad \text{FTC.}$$

$$= \frac{1}{2}(2\pi) + 0 = \pi \qquad \text{Antiderivative with respect to } \theta\text{; FTC.}$$

Example 2 Use Stokes' Theorem to Calculate a Surface Integral

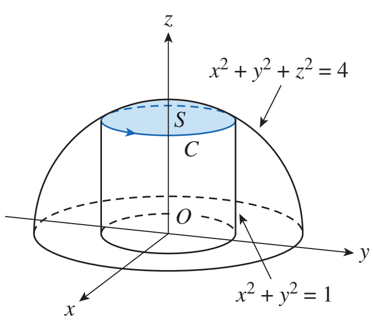


Figure 13.68
The surface S.

Use Stokes' Theorem to compute the integral $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\ \mathbf{i} + yz\ \mathbf{j} + xy\ \mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane (as illustrated in Figure 13.68).

Solution

Find the boundary curve C: solve the equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$.

Subtract: $z^2 = 3 \quad \Rightarrow \quad z = \sqrt{3}$ (since $z > 0$).

Therefore, C is the circle given by the equations $x^2 + y^2 = 1$, $z = \sqrt{3}$.

A vector equation of C is: $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + \sqrt{3}\ \mathbf{k}$, $0 \leq t \leq 2\pi$.

And $\mathbf{r}'(t) = -\sin t\ \mathbf{i} + \cos t\ \mathbf{j}$.

We also know $\mathbf{F}(\mathbf{r}(t)) = \sqrt{3} \cos t\ \mathbf{i} + \sqrt{3} \sin t\ \mathbf{j} + \cos t \sin t\ \mathbf{k}$.

Therefore, by Stokes' Theorem,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

$$= \int_0^{2\pi} (-\sqrt{3} \cos t \sin t + \sqrt{3} \sin t \cos t)\ dt$$

$$= \sqrt{3} \int_0^{2\pi} 0\ dt = 0$$

Note that in Example 2, we computed a surface integral simply by knowing the values of **F** on the boundary curve C. This means that if we have another oriented surface with the same boundary curve C, then we get exactly the same value for the surface integral!

In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

$$\iint_{S_1} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S} \tag{3}$$

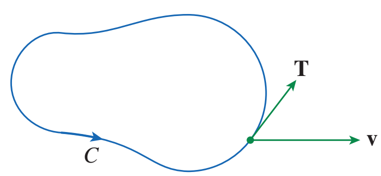
This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

Curl Vector Meaning

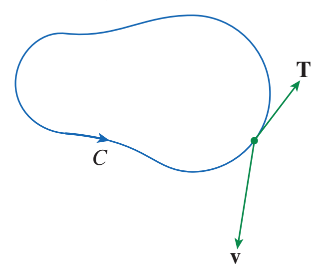
We can now use Stokes' Theorem to establish a physical interpretation of the curl vector. Suppose that C is an oriented closed curve and $\mathbf{v}$ represents the velocity field in fluid flow. Consider the line integral

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_C \mathbf{v} \cdot \mathbf{T}\, ds$$

and recall that $\mathbf{v} \cdot \mathbf{T}$ is the component of $\mathbf{v}$ in the direction of the unit tangent vector $\mathbf{T}$. This means that the closer the direction of $\mathbf{v}$ is to the direction of $\mathbf{T}$, the larger the value of $\mathbf{v} \cdot \mathbf{T}$. Thus, $\int_C \mathbf{v} \cdot d\mathbf{r}$ is a measure of the tendency of the fluid to move around C and is called the **circulation** of $\mathbf{v}$ around C. (See Figure 13.69.)



(a) $\int_C \mathbf{v} \cdot d\mathbf{r} > 0$, positive circulation



(b) $\int_C \mathbf{v} \cdot d\mathbf{r} < 0$, negative circulation

Figure 13.69
Examples of circulation.

Let $P_0(x_0, y_0, z_0)$ be a point in the fluid and let S_a be a small disk with radius a and center P_0. Then $(\text{curl } \mathbf{F})(P) \approx (\text{curl } \mathbf{F})(P_0)$ for all points P on S_a because curl **F** is continuous. Thus, by Stokes' Theorem, we get the following approximation to the circulation around the boundary circle C_a:

$$\begin{aligned}\int_{C_a} \mathbf{v} \cdot d\mathbf{r} &= \iint_{S_a} \text{curl } \mathbf{v} \cdot d\mathbf{S} = \iint_{S_a} \text{curl } \mathbf{v} \cdot \mathbf{n}\, dS \\ &\approx \iint_{S_a} \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0)\, dS = \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0)\pi a^2\end{aligned}$$

This approximation becomes better as $a \to 0$ and we have

$$\text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot d\mathbf{r} \tag{4}$$

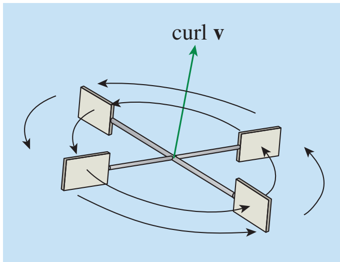


Figure 13.70
Imagine a tiny paddle wheel placed in the fluid at a point P; the paddle wheel rotates fastest when its axis is parallel to curl $\mathbf{v}$.

Equation 4 conveys the relationship between the curl and the circulation. It shows that curl $\mathbf{v} \cdot \mathbf{n}$ is a measure of the rotating effect of the fluid about the axis $\mathbf{n}$. The curling effect is greatest about the axis parallel to curl $\mathbf{v}$. (See Figure 13.70.)

One last comment on Stokes' Theorem: it can be used to prove the theorem involving conditions for a conservative vector field (which states that if curl $\mathbf{F} = 0$ on all of $\mathbb{R}^3$, then $\mathbf{F}$ is conservative). From previous work (Theorems 13.3.3 and 13.3.4), we know that $\mathbf{F}$ is conservative if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C. Given C, suppose we can find an orientable surface S whose boundary is C. (This can be done, but the proof requires some advanced techniques.) Then Stokes' Theorem gives

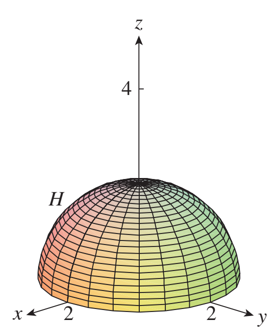
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$$

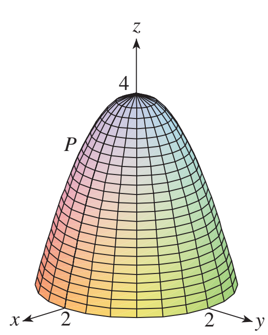
A curve that is not simple can be broken into a number of simple curves, and the integrals around these simple curves are all 0. Adding these integrals, we obtain $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C.

13.7 Exercises

1. A hemisphere H, and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$





Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

2. $\mathbf{F}(x, y, z) = 2y \cos z\, \mathbf{i} + e^x \sin z\, \mathbf{j} + xe^y\, \mathbf{k}$, S is the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$, oriented upward

3. $\mathbf{F}(x, y, z) = x^2z^2\, \mathbf{i} + y^2z^2\, \mathbf{j} + xyz\, \mathbf{k}$, S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$, oriented upward

4. $\mathbf{F}(x, y, z) = x^2y^3z\, \mathbf{i} + \sin(xyz)\, \mathbf{j} + xyz\, \mathbf{k}$, S is the part of the cone $y^2 = x^2 + z^2$ that lies between the planes $y = 0$ and $y = 3$, oriented in the direction of the positive y-axis

5. $\mathbf{F}(x, y, z) = \tan^{-1}(x^2yz^2)\, \mathbf{i} + x^2y\, \mathbf{j} + x^2z^2\, \mathbf{k}$, S is the cone $x = \sqrt{y^2 + z^2}$, $0 \leq x \leq 2$, oriented in the direction of the positive x-axis

6. $\mathbf{F}(x, y, z) = xyz\, \mathbf{i} + xy\, \mathbf{j} + x^2yz\, \mathbf{k}$, S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward
Hint: Use Equation 3.

7. $\mathbf{F}(x, y, z) = e^{xy} \cos z\, \mathbf{i} + x^2z\, \mathbf{j} + xy\, \mathbf{k}$, S is the hemisphere $x = \sqrt{1 - y^2 - z^2}$, oriented in the direction of the positive x-axis
Hint: Use Equation 3.

Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case, C is oriented counterclockwise as viewed from above.

8. $\mathbf{F}(x, y, z) = (x + y^2)\, \mathbf{i} + (y + z^2)\, \mathbf{j} + (z + x^2)\, \mathbf{k}$, C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

9. $\mathbf{F}(x, y, z) = e^{-x}\, \mathbf{i} + e^x\, \mathbf{j} + e^z\, \mathbf{k}$, C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant

10. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + 2xz\,\mathbf{j} + e^{xy}\,\mathbf{k}$, C is the circle $x^2 + y^2 = 16$, $z = 5$

11. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + 2z\,\mathbf{j} + 3y\,\mathbf{k}$, C is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$

12. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x^2 z\,\mathbf{i} + xy^2\,\mathbf{j} + z^2\,\mathbf{k}$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise as viewed from above.

(b) Use technology to graph both the plane and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use technology to graph C.

13. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x^2 y\,\mathbf{i} + \frac{1}{3}x^3\,\mathbf{j} + xy\,\mathbf{k}$$

and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$ oriented counterclockwise as viewed from above.

(b) Use technology to graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use technology to graph C.

Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and the surface S.

14. $\mathbf{F}(x, y, z) = -y\,\mathbf{i} + x\,\mathbf{j} - 2\,\mathbf{k}$, S is the cone $z^2 = x^2 + y^2$, $0 \le z \le 4$, oriented downward

15. $\mathbf{F}(x, y, z) = -2yz\,\mathbf{i} + y\,\mathbf{j} + 3x\,\mathbf{k}$, S is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward

16. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$, S is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \ge 0$, oriented in the direction of the positive y-axis

17. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\,dx - 2x\,dy + 3y\,dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

18. A particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2\,\mathbf{i} + 2xy\,\mathbf{j} + 4y^2\,\mathbf{k}$$

Find the work done.

19. Evaluate

$$\int_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \le t \le 2\pi$.
Hint: Observe that C lies on the surface $z = 2xy$.

20. If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

21. Suppose S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second-order partial derivatives. Use Exercises 28 and 30 of Section 13.5 to show the following:

(a) $\iint_C (f\nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$

(b) $\int_C (f\nabla f) \cdot d\mathbf{r} = 0$

(c) $\int_C (f\nabla g + g\nabla f) \cdot d\mathbf{r} = 0$

13.8 The Divergence Theorem

In Section 13.5, we rewrote Green's Theorem in a vector version as

$$\int_C \mathbf{F} \cdot \mathbf{n}\,ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\,dA$$

where C is the positively oriented boundary curve of the plane region D. If we wanted to extend this theorem to vector fields on $\mathbb{R}^3$, we might guess that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iiint_E \operatorname{div} \mathbf{F}(x, y, z)\,dV \qquad (1)$$

where S is the boundary surface of the solid region E. It turns out that Equation 1 is true, under appropriate hypotheses, and is called the Divergence Theorem. Notice its similarity to Green's Theorem and Stokes' Theorem in that it relates the integral of a derivative of a function (div $\mathbf{F}$ in this case) over a region to the integral of the original function $\mathbf{F}$ over the boundary of the region.

At this point, you may wish to review the various types of regions over which we were able to evaluate triple integrals in Section 12.7. The Divergence Theorem and its proof are given below for regions that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**. (For instance, regions bounded by ellipsoids or rectangular boxes are simple solid regions.) The boundary of E is a closed surface, and we use the convention, introduced in Section 13.6, that the positive orientation is outward; that is, the unit normal vector $\mathbf{n}$ is directed outward from E.

The Divergence Theorem is sometimes called Gauss' Theorem after the great German mathematician Carl Friedrich Gauss (1777–1855), who discovered this theorem during his investigation of electrostatics. In Eastern Europe, the Divergence Theorem is known as Ostrogradsky's Theorem after the Russian mathematician Mikhail Ostrogradsky (1801–1862), who published this result in 1826.

The Divergence Theorem

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

Thus, the Divergence Theorem states that, under the given conditions, the flux of $\mathbf{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\mathbf{F}$ over E.

Proof

Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. Then $\operatorname{div} \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$ so

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E \frac{\partial P}{\partial x}\, dV + \iiint_E \frac{\partial Q}{\partial y}\, dV + \iiint_E \frac{\partial R}{\partial z}\, dV.$$

If $\mathbf{n}$ is the unit outward normal of S, then the surface integral on the left side of the Divergence Theorem is

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_S (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \mathbf{n}\, dS \\ &= \iint_S P\,\mathbf{i} \cdot \mathbf{n}\, dS + \iint_S Q\,\mathbf{j} \cdot \mathbf{n}\, dS + \iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS\end{aligned}$$

Therefore, to prove the Divergence Theorem, it suffices to prove the following three equations:

$$\iint_S P\,\mathbf{i}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial P}{\partial x}dV \tag{2}$$

$$\iint_S Q\,\mathbf{j}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial Q}{\partial y}dV \tag{3}$$

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial R}{\partial z}dV \tag{4}$$

To prove Equation 4, use the fact that E is a type 1 region:

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$$

where D is the projection of $\boldsymbol{E}$ onto the xy-plane. By Equation 12.7.5,

$$\iiint_E \frac{\partial R}{\partial z}\,dV = \iint_D \left[\int_{u_1(x,\,y)}^{u_2(x,\,y)} \frac{\partial R}{\partial z}(x, y, z)\,dz\right] dA.$$

By the Fundamental Theorem of Calculus,

$$\iiint_E \frac{\partial R}{\partial z}\,dV = \iint_D [R(x, y, u_2(x, y)) - R(x, y, (u_1(x, y))]dA. \tag{5}$$

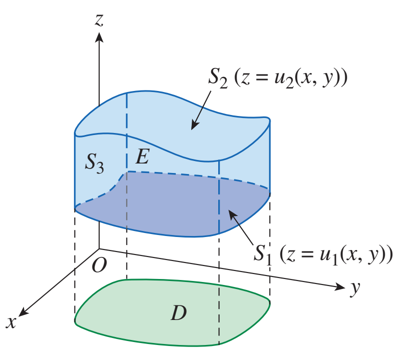


Figure 13.71
The three pieces that make up the boundary surface S.

The boundary surface S consists of three pieces: the bottom surface S_1, the top surface S_2, and possibly a vertical surface S_3, which lies above the boundary curve of D. (See Figure 13.71. It might happen that S_3 doesn't appear, as in the case of a sphere.)

Notice that on S_3: $\mathbf{k}\cdot\mathbf{n} = 0$, because $\mathbf{k}$ is vertical and $\mathbf{n}$ is horizontal. Therefore,

$$\iint_{S_3} R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_{S_3} 0\,dS = 0.$$

Thus, regardless of whether there is a vertical surface, we can write

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_{S_1} R\,\mathbf{k}\cdot\mathbf{n}\,dS + \iint_{S_2} R\,\mathbf{k}\cdot\mathbf{n}\,dS. \tag{6}$$

The equation of S_2 is $z = u_2(x, y)$, $(x, y) \in D$, and the outward normal $\mathbf{n}$ points upward. From Equation 13.6.9 (with $\mathbf{F}$ replaced by $R\,\mathbf{k}$), we have

$$\iint_{S_2} R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_D R(x, y, u_2(x, y))\,dA.$$

On S_1, we have $z = u_1(x, y)$, but here the outward normal $\mathbf{n}$ points downward, so we need to multiply by -1:

$$\iint_{S_1} R\ \mathbf{k}\cdot\mathbf{n}\ dS = -\iint_D R(x, y, u_1(x, y))\ dA.$$

Therefore, Equation 6 gives

$$\iint_S R\ \mathbf{k}\cdot\mathbf{n}\ dS = \iint_D \left[R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))\right]\ dA.$$

Comparing this expression with Equation 5 shows that

$$\iint_S R\ \mathbf{k}\cdot\mathbf{n}\ dS = \iiint_E \frac{\partial R}{\partial z} dV.$$

Notice that the method of proof of the Divergence Theorem is very similar to that of Green's Theorem.

Equations 2 and 3 are proved in a similar manner using the expressions for E as a type 2 or type 3 region, respectively. ■

Example 1 Use the Divergence Theorem to Calculate Flux

Find the flux of the vector field $\mathbf{F}(x, y, z) = z\ \mathbf{i} + y\ \mathbf{j} + x\ \mathbf{k}$ over the sphere $x^2 + y^2 + z^2 = 1$.

Solution

Compute the divergence of $\mathbf{F}$:

$$\text{div}\ \mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The unit sphere S is the boundary of the unit ball B given by $x^2 + y^2 + z^2 \le 1$.

Therefore, the Divergence Theorem gives the flux as

Compare the solution in Example 1 with the solution in Example 4, Section 13.6.

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_B \text{div}\ \mathbf{F}\ dV = \iiint_B 1\ dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4\pi}{3}.$$ ■

Example 2 A Surface Integral Made Easier with the Divergence Theorem

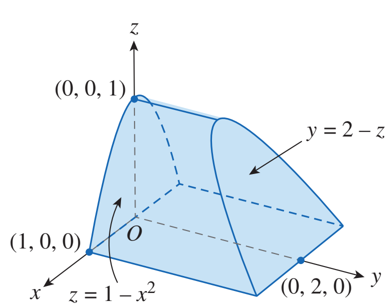


Figure 13.72
The region E.

Evaluate $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\ \mathbf{i} + (y^2 + e^{xz^2})\ \mathbf{j} + \sin(xy)\ \mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. (See Figure 13.72.)

Solution

It would be difficult to evaluate the given surface integral directly. We would have to evaluate four surface integrals corresponding to the four pieces of S.

The divergence of **F** is less complicated than **F** itself:

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 + e^{xz^2}) + \frac{\partial}{\partial z}(\sin xy) = y + 2y = 3y.$$

Use the Divergence Theorem to transform the given surface integral into a triple integral. The easiest way to evaluate the triple integral is to express E as a type 3 region:

$$E = \{(x, y, z) \mid -1 \le x \le 1,\ 0 \le z \le 1 - x^2,\ 0 \le y \le 2 - z\}.$$

Use this description of E to evaluate the triple integral.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F}\, dV = \iiint_E 3y\, dV \qquad \text{Divergence Theorem; expression for div } \mathbf{F}.$$

$$= 3\int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} y\, dy\, dz\, dx \qquad \text{Convert to a triple integral.}$$

$$= 3\int_{-1}^{1}\int_0^{1-x^2} \frac{(2-z)^2}{2}\, dz\, dx \qquad \text{Antiderivative with respect to } y\text{; FTC.}$$

$$= \frac{3}{2}\int_{-1}^{1}\left[-\frac{(2-z)^3}{3}\right]_0^{1-x^2} dx \qquad \text{Antiderivative with respect to } z.$$

$$= -\frac{1}{2}\int_{-1}^{1}\left[(x^2+1)^3 - 8\right] dx \qquad \text{FTC.}$$

$$= -\int_0^1 (x^6 + 3x^4 + 3x^2 - 7)\, dx = \frac{184}{35} \qquad \text{Antiderivative with respect to } x\text{; FTC.}$$

■

Divergence Theorem Extended

Although we have proved the Divergence Theorem only for simple solid regions, it can be proved for regions that are finite unions of simple solid regions. (The procedure is similar to the one we used in Section 13.4 to extend Green's Theorem.)

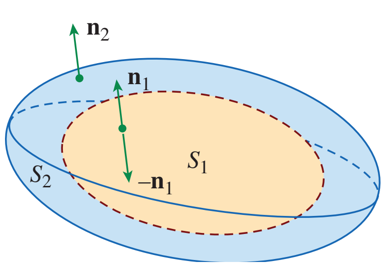


Figure 13.73
The region E lies between S_1 and S_2.

For example, let's consider the region E that lies between the closed surfaces S_1 and S_2, where S_1 lies inside S_2. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be outward normals of S_1 and S_2. Then the boundary surface of E is $S = S_1 \cup S_2$ and its normal $\mathbf{n}$ is given by $\mathbf{n} = -\mathbf{n}_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2. (See Figure 13.73.)

Apply the Divergence Theorem to S:

$$\iiint_E \text{div } \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \tag{7}$$

$$= \iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS$$

$$= -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$

Let's apply this result to the electric field (see Example 5, Section 13.1):

$$\mathbf{E}(\mathbf{x}) = \frac{\epsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

where S_1 is a small sphere with radius a and center at the origin. You can verify that div $\mathbf{E} = 0$. (See Exercise 24.) Therefore, Equation 7 gives

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} + \iiint_E \text{div } \mathbf{E}\; dV = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n}\; dS$$

The point of this calculation is that we can compute the surface integral over S_1 because S_1 is a sphere. The normal vector at $\mathbf{x}$ is $\mathbf{x}/|\mathbf{x}|$. Therefore,

$$\mathbf{E} \cdot \mathbf{n} = \frac{\epsilon Q}{|\mathbf{x}|^3}\mathbf{x} \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) = \frac{\epsilon Q}{|\mathbf{x}|^4}\mathbf{x} \cdot \mathbf{x} = \frac{\epsilon Q}{|\mathbf{x}|^2} = \frac{\epsilon Q}{a^2}$$

since the equation of S_1 is $|\mathbf{x}| = a$. Thus, we have

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n}\; dS = \frac{\epsilon Q}{a^2}\iint_{S_1} dS = \frac{\epsilon Q}{a^2}A(S_1) = \frac{\epsilon Q}{a^2}4\pi a^2 = 4\pi\epsilon Q$$

This shows that the electric flux of $\mathbf{E}$ is $4\pi\epsilon Q$ through *any* closed surface S_2 that contains the origin. [This is a special case of Gauss' Law (Equation 13.6.10) for a single charge. The relationship between ϵ and ϵ_0 is $\epsilon = 1/(4\pi\epsilon_0)$.]

Another application of the Divergence Theorem occurs in fluid flow. Let $\mathbf{v}(x, y, z)$ be the velocity field of a fluid with constant density ρ. Then $\mathbf{F} = \rho\mathbf{v}$ is the rate of flow per unit area. If $P_0(x_0, y_0, z_0)$ is a point in the fluid and B_a is a ball with center P_0 and very small radius a, then div $\mathbf{F}(P) \approx$ div $\mathbf{F}(P_0)$ for all points in B_a since div $\mathbf{F}$ is continuous. We approximate the flux over the boundary sphere S_a as follows:

$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_a} \text{div } \mathbf{F}\; dV = \iiint_{B_a} \text{div } \mathbf{F}(P_0)\; dV = \text{div } \mathbf{F}(P_0)V(B_a)$$

This approximation becomes better as $a \to 0$ and suggests that

$$\text{div } \mathbf{F}(P_0) = \lim_{a\to 0} \frac{1}{V(B_a)}\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} \tag{8}$$

Equation 8 says that div $\mathbf{F}(P_0)$ is the net rate of outward flux per unit volume at P_0. (This is the reason for the name *divergence*.) If div $\mathbf{F}(P) > 0$, the net flow is outward near P and P is called a **source**. If div $\mathbf{F}(P) < 0$, the net flow is inward near P and P is called a **sink**.

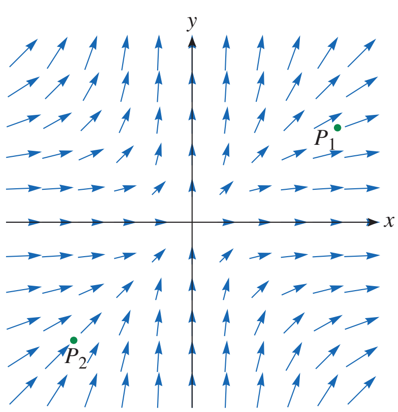


Figure 13.74
A graph of the vector field $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$.

Consider the vector field and the points P_1 and P_2 in Figure 13.74. It appears that the vectors that end near P_1 are shorter than the vectors that start near P_1. Thus, the net flow is outward near P_1, so div $\mathbf{F}(P_1) > 0$ and P_1 is a source. Near P_2, on the other hand, the incoming arrows are longer than the outgoing arrows. Here the net flow is inward, so div $\mathbf{F}(P_2) < 0$, and P_2 is a sink. We can use the formula for $\mathbf{F}$ to confirm this visual impression. Since $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$, then div $\mathbf{F} = 2x + 2y$, which is positive when $y > -x$. So, points above the line $y = -x$ are sources and those below are sinks.

13.8 Exercises

Verify that the Divergence Theorem is true for the vector field **F** on the region E.

1. $\mathbf{F}(x, y, z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2xz\,\mathbf{k}$, E is the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$

2. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$, E is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane

3. $\mathbf{F}(x, y, z) = \langle z, y, x \rangle$, E is the solid ball $x^2 + y^2 + z^2 \leq 16$

4. $\mathbf{F}(x, y, z) = \langle x^2, -y, z \rangle$, E is the solid cylinder $y^2 + z^2 \leq 9$, $0 \leq x \leq 2$

Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is, calculate the flux of **F** across S.

5. $\mathbf{F}(x, y, z) = xye^z\,\mathbf{i} + xy^2z^3\,\mathbf{j} - ye^z\,\mathbf{k}$, S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$

6. $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$, S is the surface of the box enclosed by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$ where a, b, and c are positive numbers

7. $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$, S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$

8. $\mathbf{F}(x, y, z) = (x^3 + y^3)\,\mathbf{i} + (y^3 + z^3)\,\mathbf{j} + (z^3 + x^3)\,\mathbf{k}$, S is the sphere with center at the origin and radius 2

9. $\mathbf{F}(x, y, z) = x^2 \sin y\,\mathbf{i} + x \cos y\,\mathbf{j} - xz \sin y\,\mathbf{k}$, S is the *fat sphere* $x^8 + y^8 + z^8 = 8$

10. $\mathbf{F}(x, y, z) = x^2y\,\mathbf{i} + xy^2\,\mathbf{j} + 2xyz\,\mathbf{k}$, S is the surface of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + z = 2$

11. $\mathbf{F}(x, y, z) = (\cos z + xy^2)\,\mathbf{i} + xe^{-z}\,\mathbf{j} + (\sin y + x^2z)\,\mathbf{k}$, S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$

12. $\mathbf{F}(x, y, z) = x^4\,\mathbf{i} - x^3z^2\,\mathbf{j} + 4xy^2z\,\mathbf{k}$, S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2$ and $z = 0$

13. $\mathbf{F}(x, y, z) = 4x^3z\,\mathbf{i} + 4y^3z\,\mathbf{j} + 3z^4\,\mathbf{k}$, S is the sphere with radius R and center at the origin

14. $\mathbf{F} = \mathbf{r}/|\mathbf{r}|$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, S consists of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the disk $x^2 + y^2 \leq 1$ in the xy-plane

15. $\mathbf{F} = |\mathbf{r}|^2\mathbf{r}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, S is the sphere with radius R and center at the origin

16. $\mathbf{F}(x, y, z) = e^y \tan z\,\mathbf{i} + y\sqrt{3 - x^2}\,\mathbf{j} + x \sin y\,\mathbf{k}$, S is the surface of the solid that lies above the xy-plane and below the surface $z = 2 - x^4 - y^4$, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$

17. Use technology to plot the vector field

$$\mathbf{F}(x, y, z) = \sin x \cos^2 y\,\mathbf{i} + \sin^3 y \cos^4 z\,\mathbf{j} + \sin^5 z \cos^6 x\,\mathbf{k}$$

in the cube cut from the first octant by the planes $x = \pi/2$, $y = \pi/2$, and $z = \pi/2$. Then compute the flux across the surface of the cube.

18. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

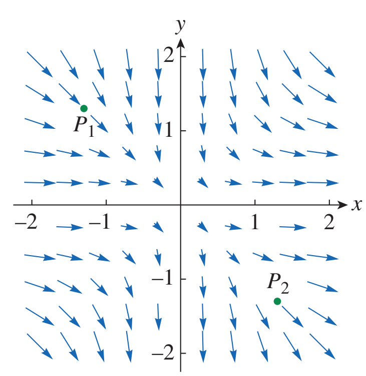
$$\mathbf{F}(x, y, z) = z^2x\,\mathbf{i} + \left(\frac{1}{3}y^3 + \tan z\right)\mathbf{j} + (x^2z + y^2)\,\mathbf{k}$$

and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$.

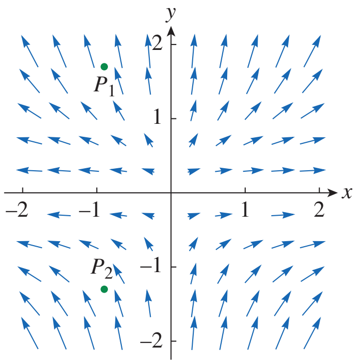
Hint: Note that S is not a closed surface. First compute integrals over S_1 and S_2, where S_1 is the disk $x^2 + y^2 \leq 1$, oriented downward, and $S_2 = S \cup S_1$.

19. Let $\mathbf{F}(x, y, z) = z \tan^{-1}(y^2)\,\mathbf{i} + z^3 \ln(x^2 + 1)\,\mathbf{j} + z\,\mathbf{k}$. Find the flux of **F** across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

20. A vector field **F** is shown in the figure. Use the interpretation of divergence derived in this section to determine whether div **F** is positive or negative at P_1 and at P_2.



21. (a) Are the points P_1 and P_2 sources or sinks for the vector field **F** shown in the figure? Explain your answer based solely on the figure.



(b) Given that $\mathbf{F}(x, y) = \langle x, y^2 \rangle$, use the definition of divergence to verify your answer to part (a).

Plot the vector field and guess where $\operatorname{div} \mathbf{F} > 0$ and where $\operatorname{div} \mathbf{F} < 0$. Then calculate $\operatorname{div} \mathbf{F}$ to check your guess.

22. $\mathbf{F}(x, y) = \langle xy, x + y^2 \rangle$

23. $\mathbf{F}(x, y) = \langle x^2, y^2 \rangle$

24. Verify that $\operatorname{div} \mathbf{E} = 0$ for the electric field $\mathbf{E}(\mathbf{x}) = \dfrac{\epsilon Q}{|\mathbf{x}|^3}\mathbf{x}$.

25. Use the Divergence Theorem to evaluate

$$\iint_S (2x + 2y + z^2)\, dS$$

where S is the sphere $x^2 + y^2 + z^2 = 1$.

Prove each identity, assuming that S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

26. $\displaystyle\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = 0$, where **a** is a constant vector

27. $\displaystyle V(E) = \frac{1}{3}\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

28. $\displaystyle\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$

29. $\displaystyle\iint_S D_{\mathbf{n}} f\, dS = \iiint_E \nabla^2 f\, dV$

30. $\displaystyle\iint_S (f\nabla g) \cdot \mathbf{n}\, dS = \iiint_E (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV$

31. $\displaystyle\iint_S (f\nabla g - g\nabla f) \cdot \mathbf{n}\, dS = \iiint_E (f\nabla^2 g - g\nabla^2 f)\, dV$

32. Suppose S and E satisfy the conditions of the Divergence Theorem and f is a scalar function with continuous partial derivatives. Prove that

$$\iint_S f\mathbf{n}\, dS = \iiint_E \nabla f\, dV$$

These surface and triple integrals of vector functions are vectors defined by integrating each component function.

Hint: Start by applying the Divergence Theorem to $\mathbf{F} = f\mathbf{c}$, where **c** is an arbitrary constant vector.

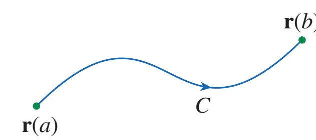
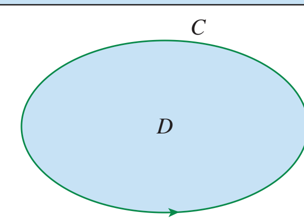
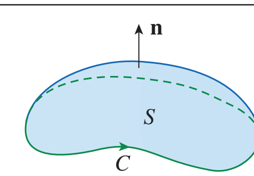
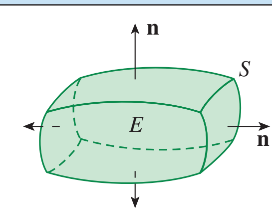
33. A solid occupies a region E with surface S and is immersed in a liquid with constant density ρ. We set up a coordinate system so that the xy-plane coincides with the surface of the liquid and positive values of z are measured downward into the liquid. Then the pressure at depth z is $p = \rho g z$, where g is the acceleration due to gravity (see Section 6.6). The total buoyant force on the solid due to the pressure distribution is given by the surface integral

$$\mathbf{F} = -\iint_S p\mathbf{n}\, dS$$

where **n** is the outer unit normal. Use the result of Exercise 32 to show that $\mathbf{F} = -W\mathbf{k}$, where W is the weight of the liquid displaced by the solid. (Note that **F** is directed upward because z is directed downward.) The result is *Archimedes' principle*: the buoyant force on an object equals the weight of the displaced liquid.

13.9 Summary

The main results of this chapter are all higher-dimensional versions of the Fundamental Theorem of Calculus. To help understand and remember these results, they are all presented here (without hypotheses) so that you can see more easily their essential similarity. Notice that in each case there is an integral of a *derivative* over a region on the left side, and the right side involves the values of the original function only on the *boundary* of the region.

Curves and their boundaries (endpoints)		
Fundamental Theorem of Calculus	$\int_a^b F'(x)\,dx = F(b) - F(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$	
Surfaces and their boundaries		
Green's Theorem	$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$	
Stokes' Theorem	$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$	
Solids and their boundaries		
Divergence Theorem	$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$	

13 Review

Concepts and Vocabulary

1. What is a vector field? Give three examples that have physical meaning.

2. (a) What is a conservative vector field?
   (b) What is a potential function?

3. (a) Write the definition of the line integral of a scalar function f along a smooth curve C with respect to arc length.
   (b) Explain how to evaluate such a line integral.
   (c) Write expressions for the mass and center of mass of a thin wire shaped like a curve C if the wire has linear density function $\rho(x, y)$.
   (d) Write the definitions of the line integrals along C of a scalar function f with respect to x, y, and z.
   (e) How do you evaluate these line integrals?

4. (a) Define the line integral of a vector field $\mathbf{F}$ along a smooth curve C given by a vector function $\mathbf{r}(t)$.
   (b) If $\mathbf{F}$ is a force field, what does the line integral represent?
   (c) If $\mathbf{F} = \langle P, Q, R \rangle$, what is the connection between the line integral of $\mathbf{F}$ and the line integrals of the component functions P, Q, and R?

5. State the Fundamental Theorem for Line Integrals.

6. (a) What does it mean to say that $\int \mathbf{F} \cdot d\mathbf{r}$ is independent of path?
   (b) If you know that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, what can you say about $\mathbf{F}$?

7. State Green's Theorem.

8. Write expressions for the area enclosed by a curve C in terms of line integrals around C.

9. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$.
   (a) Define curl $\mathbf{F}$.
   (b) Define div $\mathbf{F}$.
   (c) If $\mathbf{F}$ is a velocity field in fluid flow, what are the physical interpretations of curl $\mathbf{F}$ and div $\mathbf{F}$?

10. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, how do you test to determine whether $\mathbf{F}$ is conservative? What if $\mathbf{F}$ is a vector field on $\mathbb{R}^3$?

11. (a) Write a definition of the surface integral of a scalar function f over a surface S.
    (b) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?
    (c) What if S is given by an equation $z = g(x, y)$?
    (d) If a thin sheet has the shape of a surface S, and the density at (x, y, z) is $\rho(x, y, z)$, write expressions for the mass and center of mass of the sheet.

12. (a) What is an oriented surface? Give an example of a nonorientable surface.
    (b) Define the surface integral (or flux) of a vector field $\mathbf{F}$ over an oriented surface S with unit normal vector $\mathbf{n}$.
    (c) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?
    (d) What if S is given by an equation $z = g(x, y)$?

13. State Stokes' Theorem.

14. State the Divergence Theorem.

15. In what ways are the Fundamental Theorem for Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem similar?

True-False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that contradicts the statement.

1. If $\mathbf{F}$ is a vector field, then div $\mathbf{F}$ is a vector field.

2. If $\mathbf{F}$ is a vector field, then curl $\mathbf{F}$ is a vector field.

3. If f has continuous partial derivatives of all orders on $\mathbb{R}^3$, then $\operatorname{div}(\operatorname{curl} \nabla f) = 0$

4. If f has continuous partial derivatives on $\mathbb{R}^3$ and C is any circle, then $\int_C \nabla f \cdot d\mathbf{r} = 0$.

5. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ and $P_y = Q_x$ in an open region D, then $\mathbf{F}$ is conservative.

6. $\int_{-C} f(x, y)\, ds = -\int_C f(x, y)\, ds$

7. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields and div $\mathbf{F}$ = div $\mathbf{G}$, then $\mathbf{F} = \mathbf{G}$.

8. If S is a sphere and $\mathbf{F}$ is a constant vector field, then $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$.

9. The work done by a conservative force field in moving a particle around a closed path is zero.

10. There is a vector field $\mathbf{F}$ such that
$$\operatorname{curl} \mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

11. If **F** and **G** are vector fields, then

$$\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$$

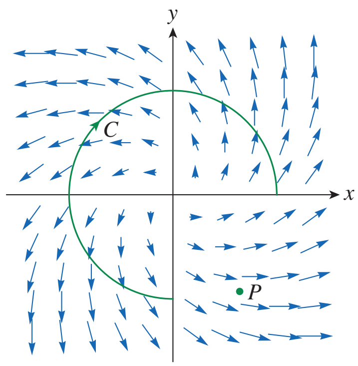
12. If **F** and **G** are vector fields, then

$$\operatorname{curl}(\mathbf{F} \cdot \mathbf{G}) = \operatorname{curl} \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$$

13. The area of the region bounded by the positively oriented, piecewise-smooth, simple closed curve C is $A = \oint_C y\, dx$.

Exercises

1. A vector field **F**, a curve C, and a point P are shown in the figure.



(a) Is $\int_C \mathbf{F} \cdot d\mathbf{r}$ positive, negative or zero? Explain your reasoning.

(b) Is div $\mathbf{F}(P)$ positive, negative, or zero? Explain your reasoning.

Evaluate the line integral.

2. $\int_C x\, ds$,

C is the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$

3. $\int_C yz \cos x\, ds$,

C: $x = t,\ y = 3 \cos t,\ z = 3 \sin t,\ 0 \le t \le \pi$

4. $\int_C y\, dx + (x + y^2)\, dy$, C is the ellipse $4x^2 + 9y^2 = 36$ with counterclockwise orientation.

5. $\int_C y^3\, dx + x^2\, dy$, C is the arc of the parabola $x = 1 - y^2$ from $(0, -1)$ to $(0, 1)$

6. $\int_C \sqrt{xy}\, dx + e^y\, dy + xz\, dz$,

C is given by $\mathbf{r}(t) = t^4\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}, \quad 0 \le t \le 1$

7. $\int_C xy\, dx + y^2\, dy + yz\, dz$,

C is the line segment from $(1, 0, -1)$ to $(3, 4, 2)$

8. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(\mathbf{x}, y) = xy\,\mathbf{i} + x^2\,\mathbf{j}$ and C is given by $\mathbf{r}(t) = \sin t\,\mathbf{i} + (1 + t)\,\mathbf{j},\ 0 \le t \le \pi$

9. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = e^z\,\mathbf{i} + xz\,\mathbf{j} + (x + y)\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} - t\,\mathbf{k},\ 0 \le t \le 1$

10. Find the work done by the force field

$$\mathbf{F}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$$

in moving a particle from the point $(3, 0, 0)$ to the point $(0, \pi/2, 3)$ along

(a) a straight line

(b) the helix $x = 3 \cos t,\ y = t,\ z = 3 \sin t$

Show that F is a conservative vector field. Then find a function f such that $\mathbf{F} = \nabla f$.

11. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + (e^y + x^2 e^{xy})\,\mathbf{j}$

12. $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + x \cos y\,\mathbf{j} - \sin z\,\mathbf{k}$

Show that **F** is conservative and use this fact to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve.

13. $\mathbf{F}(x, y) = (4x^3y^2 - 2xy^3)\,\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\,\mathbf{j}$,

C: $\mathbf{r}(t) = (t + \sin \pi t)\,\mathbf{i} + (2t + \cos \pi t)\,\mathbf{j},\ 0 \le t \le 1$

14. $\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + (xe^y + e^z)\,\mathbf{j} + ye^z\,\mathbf{k}$,

C is the line segment from $(0, 2, 0)$ to $(4, 0, 3)$

15. Verify that Green's Theorem is true for the line integral $\int_C xy^2\, dx - x^2y\, dy$, where C consists of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$ and the line segment from $(1, 1)$ to $(-1, 1)$.

16. Use Green's Theorem to evaluate

$$\int_C \sqrt{1 + x^3}\, dx + 2xy\, dy$$

where C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$.

17. Use Green's Theorem to evaluate $\int_C x^2y\,dx - xy^2\,dy$, where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

18. Find curl $\mathbf{F}$ and div $\mathbf{F}$ if

$$\mathbf{F}(x, y, z) = e^{-x}\sin y\,\mathbf{i} + e^{-y}\sin z\,\mathbf{j} + e^{-z}\sin x\,\mathbf{k}$$

19. Show that there is no vector field $\mathbf{G}$ such that

$$\text{curl } \mathbf{G} = 2x\,\mathbf{i} + 3yz\,\mathbf{j} - xz^2\,\mathbf{k}$$

20. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields whose component functions have continuous first partial derivatives, show that

$$\text{curl}(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\text{ div }\mathbf{G} - \mathbf{G}\text{ div }\mathbf{F} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$$

21. If C is any piecewise-smooth simple closed plane curve and f and g are differentiable functions, show that

$$\int_C f(x)\,dx + g(y)\,dy = 0$$

22. If f and g are twice differentiable functions, show that

$$\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f\cdot\nabla g$$

23. If f is a harmonic function, that is, $\nabla^2 f = 0$, show that the line integral $\int f_y\,dx - f_x\,dy$ is independent of path in any simple region D.

24. (a) Sketch the curve C with parametric equations

$$x = \cos t \quad y = \sin t \quad z = \sin t \quad 0 \le t \le 2\pi$$

(b) Find

$$\int_C 2xe^{2y}\,dx + (2x^2e^{2y} + 2y\cot z)\,dy - y^2\csc^2 z\,dz.$$

Evaluate the surface integral.

25. $\iint_S z\,dS$, where S is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$

26. $\iint_S (x^2z + y^2z)\,dS$, where S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$

27. $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} - 2y\,\mathbf{j} + 3x\,\mathbf{k}$ and S is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation

28. $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ with upward orientation

29. Verify that Stokes' Theorem is true for the vector field $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$, where S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane and S has upward orientation.

30. Use Stokes' Theorem to evaluate $\iint_S \text{curl }\mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + yz^2\,\mathbf{j} + z^3e^{xy}\,\mathbf{k}$, S is the part of the sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$, and S is oriented upward.

31. Use Stokes' Theorem to evaluate $\iint_S \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, and C is the triangle with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1), oriented counterclockwise as viewed from above.

32. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.

33. Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, where E is the unit ball $x^2 + y^2 + z^2 \le 1$.

34. Compute the outward flux of

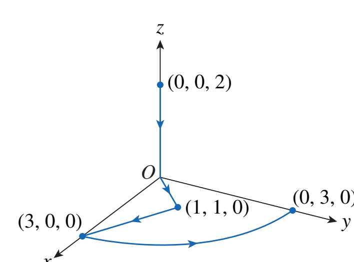
$$\mathbf{F}(x, y, z) = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

through the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$.

35. Let

$$\mathbf{F} = (x, y, z) = (3x^2yz - 3y)\,\mathbf{i} + (x^3z - 3x)\,\mathbf{j} + (x^3y + 2z)\,\mathbf{k}.$$

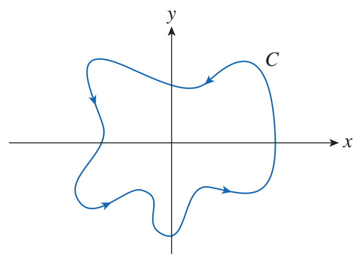
Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve with initial point (0, 0, 2) and terminal point (0, 3, 0) shown in the figure.



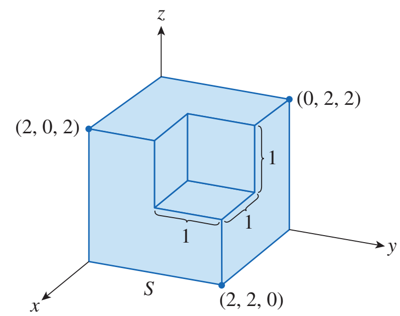
36. Let

$$\mathbf{F}(x, y) = \frac{(2x^3 + 2xy^2 - 2y)\,\mathbf{i} + (2y^3 + 2x^2y + 2x)\,\mathbf{j}}{x^2 + y^2}$$

Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is shown in the figure.



37. Find $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{j}$ and S is the outwardly oriented surface shown in the figure (the boundary surface of a cube with a unit corner cube removed).



38. If the components of $\mathbf{F}$ have continuous second partial derivatives and S is the boundary surface of a simple solid region, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

Focus on Problem Solving

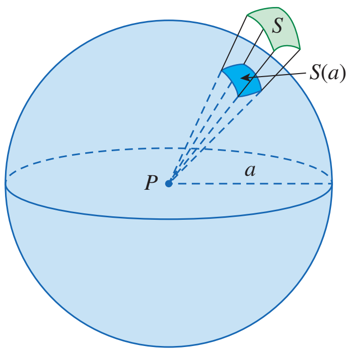


Figure 13.75
A visualization of the measure of the solid angle.

1. Let S be a smooth parametric surface and let P be a point such that each line that starts at P intersects S at most once. The **solid angle** $\Omega(S)$ subtended by S at P is the set of lines starting at P and passing through S. Let $S(a)$ be the intersection of $\Omega(S)$ with the surface of the sphere with center P and radius a. Then the measure of the solid angle (in *steradians*) is defined to be

$$|\Omega(S)| = \frac{\text{area of } S(a)}{a^2}$$

See Figure 13.75.

Apply the Divergence Theorem to the part of $\Omega(S)$ between $S(a)$ and S to show that

$$|\Omega(S)| = \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3}\, dS$$

where $\mathbf{r}$ is the radius vector from P to any point on S, $r = |\mathbf{r}|$, and the unit normal vector $\mathbf{n}$ is directed away from P.

This shows that the definition of the measure of a solid angle is independent of the radius a of the sphere. Thus, the measure of the solid angle is equal to the area subtended on a *unit* sphere. (Note the analogy with the definition of radian measure.) The total solid angle subtended by a sphere at its center is thus 4π steradians.

2. Prove the following identity:

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times \text{curl } \mathbf{G} + \mathbf{G} \times \text{curl } \mathbf{F}$$

3. If $\mathbf{a}$ is a constant vector, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and S is an oriented, smooth surface with a simple, closed, smooth positively oriented boundary curve C, show that

$$\iint_S 2\mathbf{a} \cdot d\mathbf{S} = \int_C (\mathbf{a} \times \mathbf{r}) \cdot d\mathbf{r}$$

4. Find the positively oriented simple closed curve C for which the value of the line integral

$$\int_C (y^3 - y)\, dx - 2x^3 dy$$

is a maximum.

5. Let C be a simple closed, piecewise-smooth space curve that lies in a plane with unit normal vector $\mathbf{n} = \langle a, b, c \rangle$ and has positive orientation with respect to $\mathbf{n}$. Show that the plane area enclosed by C is

$$\frac{1}{2}\int_C (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz$$

6. The figure depicts the sequence of events in each cylinder of a four-cylinder internal combustion engine. Each piston moves up and down and is connected by a pivoted arm to a rotating crankshaft. Let $P(t)$ and $V(t)$ be the pressure and volume within a cylinder at time t, where $a \le t \le b$ gives the time required for a complete cycle. Figure 13.76 shows how P and V vary through one cycle of a four-stroke engine.

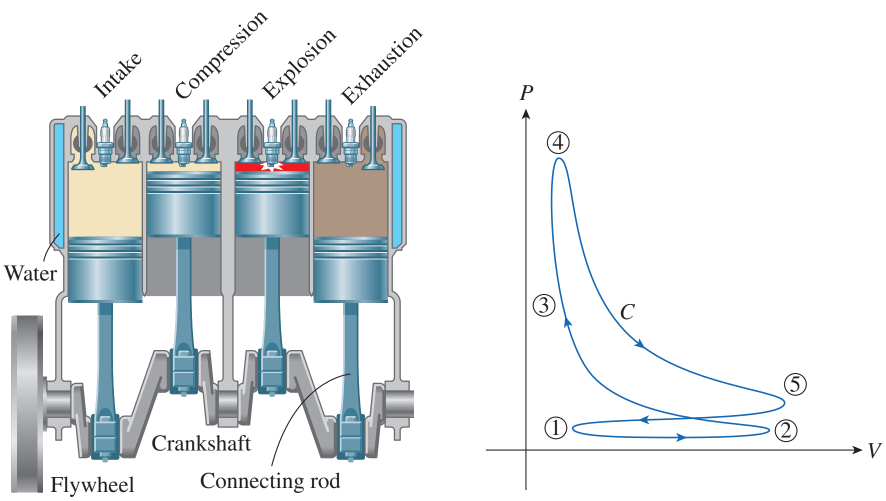


Figure 13.76
One cycle of a four-stroke engine.

During the intake stroke (from 1 to 2) a mixture of air and gasoline at atmospheric pressure is drawn into a cylinder through the intake valve as the piston moves downward. Then the piston rapidly compresses the mix with the valves closed in the compression stroke (from 2 to 3) during which the pressure rises and the volume decreases. At 3, the sparkplug ignites the fuel, raising the temperature and pressure at almost constant volume to 4. Then, with valves closed, the rapid expansion forces the piston downward during the power stroke (from 4 to 5). The exhaust valve opens, temperature and pressure drop, and mechanical energy stored in a rotating flywheel pushes the piston upward, forcing the waste products out of the exhaust valve in the exhaust stroke. The exhaust valve closes and the intake valve opens. We're now back at 1 and the cycle starts again.

(a) Show that the work done on the piston during one cycle of a four-stroke engine is

$W = \int_C P\, dV$, where C is the curve in the PV-plane shown in Figure 13.76.

Hint: Let $x(t)$ be the distance from the piston to the top of the cylinder and note that the force on the piston is $\mathbf{F} = AP(t)\,\mathbf{i}$, where A is the area of the top of the piston.

Then $W = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$, where C_1 is given by $\mathbf{r}(t) = x(t)\,\mathbf{i},\ a \le t \le b$.

An alternative approach is to work directly with Riemann sums.

(b) Use Equation 13.4.5 to show that the work is the difference of the areas enclosed by the two loops of C.

Appendixes

A Intervals, Inequalities, and Absolute Values

Figure 1
Geometric representation of the open interval (a, b).

Figure 2
Geometric representation of the closed interval $[a, b]$.

Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments. For example, if $a < b$, the **open interval** from a to b consists of all numbers between a and b and is denoted by the expression (a, b). Using set-builder notation, we can write

$$(a,\ b) = \{x \mid a < x < b\}$$

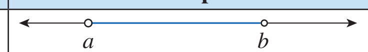
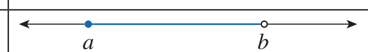
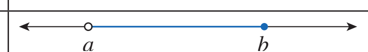
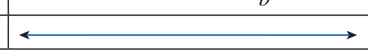
Notice that the endpoints of the interval—namely, a and b—are excluded. This is indicated by the round brackets, or parentheses, ( ) and by the open dots in Figure 1. The **closed interval** from a to b is the set

$$[a,\ b] = \{x \mid a \le x \le b\}$$

Here the endpoints of the interval are included. This is indicated by the square brackets [ ] and by the solid dots in Figure 2. It is also possible to include only one endpoint in an interval, as shown in Table 1.

Table of Intervals

This table lists the nine possible types of intervals. When these intervals are discussed, it is always assumed that $a < b$.

Notation	Set description	Geometric representation
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \le x \le b\}$	
$[a, b)$	$\{x \mid a \le x < b\}$	
$(a, b]$	$\{x \mid a < x \le b\}$	
$(a,\ \infty)$	$\{x \mid x > a\}$	
$[a,\ \infty)$	$\{x \mid x \ge a\}$	
$(-\infty,\ b]$	$\{x \mid x < b\}$	
$(-\infty,\ b]$	$\{x \mid x \le b\}$	
$(-\infty,\ \infty)$	$\mathbb{R}$ (set of all real numbers)	

We also need to consider infinite intervals such as

$$(a,\ \infty) = \{x | x > a\}$$

This does not mean that ∞ ("infinity") is a number. The notation (a, ∞) stands for the set of all numbers that are greater than a, so the symbol ∞ simply indicates that the interval extends indefinitely far in the positive direction.

Inequalities

The following rules are often helpful when working with inequalities.

Rules for Inequalities

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. If $a < b$ and $c < 0$, then $ac > bc$.
5. If $0 < a < b$, then $1/a > 1/b$.

Rule 1 says that we can add any number to both sides of an inequality, and Rule 2 says that two inequalities can be added. However, we have to be careful with multiplication. Rule 3 says that we can multiply both sides of an inequality by a *positive* number, but Rule 4 says that if we multiply both sides of an inequality by a negative number, then we reverse the direction of the inequality. For example, if we take the inequality $3 < 5$ and multiply by 2, we get $6 < 10$, but if we multiply by -2, we get $-6 > -10$. Finally, Rule 5 says that if we take reciprocals, then we reverse the direction of an inequality (provided the numbers are positive).

Example 1

Solve the inequality $1 + x < 7x + 5$.

Solution

The given inequality is satisfied by some values of x but not by others. To *solve* an inequality means to determine the set of numbers x for which the inequality is true. This is called the *solution set.*

First subtract 1 from each side of the inequality (using Rule 1 with $c = -1$):

$$x < 7x + 4$$

Then subtract $7x$ from both sides (Rule 1 with $c = -7x$):

$$-6x < 4$$

Finally, divide both sides by -6 $\left(\text{Rule 4 with } c = -\frac{1}{6}\right)$:

$$x > -\frac{4}{6} = -\frac{2}{3}$$

These steps can all be reversed, so the solution set consists of all numbers greater than $-\frac{2}{3}$. In other words, the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. ■

Example 2

Solve the inequality $x^2 - 5x + 6 \le 0$.

Solution

A visual method for solving Example 2 is to use a graphing device to graph the parabola $y = x^2 - 5x + 6$ (as in Figure 3) and observe that the curve lies on or below the x-axis when $2 \le x \le 3$.

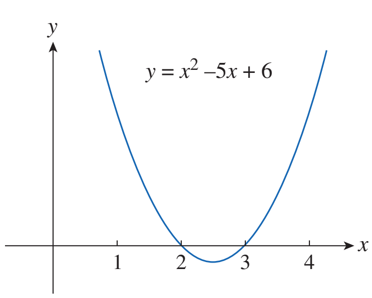


Figure 3
Graph of the parabola $y = x^2 - 5x + 6$.

First factor the left side:

$$(x - 2)(x - 3) \le 0$$

We know that the corresponding equation $(x - 2)(x - 3) = 0$ has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2) \quad (2, 3) \quad (3, \infty)$$

On each of these intervals we determine the signs of the factors. For instance,

$$x \in (-\infty, 2) \quad \Rightarrow \quad x < 2 \quad \Rightarrow \quad x - 2 < 0$$

Then we record these signs in the following chart:

Interval	$x - 2$	$x - 3$	$(x - 2)(x - 3)$
$x < 2$	$-$	$-$	$+$
$2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

Another method for obtaining the information in the chart is to use *test values.* For instance, if we use the test value $x = 1$ for the interval $(-\infty, 2)$, then substitution in $x^2 - 5x + 6$ gives

$$1^2 - 5(1) + 6 = 2$$

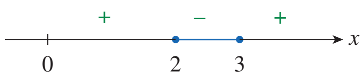


Figure 4
Geometric illustration of the solution to Example 2.

The polynomial $x^2 - 5x + 6$ doesn't change sign inside any of the three intervals, so we conclude that it is positive on $(-\infty, 2)$.

Then we read from the chart that $(x-2)(x-3)$ is negative when $2 < x < 3$. Thus the solution of the inequality $(x-2)(x-3) \le 0$ is

$$\{x \mid 2 \le x \le 3\} = [2, 3]$$

Notice that we have included the endpoints 2 and 3 because we are looking for values of x such that the product is either negative or zero. The solution is illustrated in Figure 4. ■

Example 3

Solve $x^3 + 3x^2 > 4x$.

Solution

First we take all nonzero terms to one side of the inequality sign and factor the resulting expression:

$$x^3 + 3x^2 - 4x > 0 \quad \text{or} \quad x(x-1)(x+4) > 0$$

As in Example 2, we solve the corresponding equation $x(x-1)(x+4) = 0$ and use the solutions $x = -4$, $x = 0$, and $x = 1$ to divide the real line into four intervals $(-\infty, -4)$, $(-4, 0)$, $(0, 1)$, and $(1, \infty)$. On each interval the product is a constant sign as shown in the following chart:

Interval	x	$x-1$	$x+4$	$x(x-1)(x+4)$
$x < -4$	−	−	−	−
$-4 < x < 0$	−	−	+	+
$0 < x < 1$	+	−	+	−
$x > 1$	+	+	+	+

Figure 5
Geometric illustration of the solution to Example 3.

Then we read from the chart that the solution set is

$$\{x \mid -4 < x < 0 \quad \text{or} \quad x > 1\} = (-4, 0) \cup (1, \infty)$$ ■

The solution is illustrated in Figure 5.

■ Absolute Value

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \ge 0 \quad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0 \qquad |\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

Remember that if a is negative, then $-a$ is positive.

In general, we have

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases} \tag{1}$$

Example 4

Express $|3x - 2|$ without using the absolute-value symbol.

Solution

$$|3x-2| = \begin{cases} 3x-2 & \text{if } 3x-2 \geq 0 \\ -(3x-2) & \text{if } 3x-2 < 0 \end{cases}$$

$$= \begin{cases} 3x-2 & \text{if } x \geq \dfrac{2}{3} \\[2ex] 2-3x & \text{if } x < \dfrac{2}{3} \end{cases}$$

■

Recall that the symbol $\sqrt{\ }$ means "the positive square root of." Thus $\sqrt{r} = s$ means $s^2 = r$ and $s \geq 0$. Therefore, the equation $\sqrt{a^2} = a$ is not always true. It is true only when $a \geq 0$. If $a < 0$, then $-a > 0$, so we have $\sqrt{a^2} = -a$. In view of Equation 2, we then have the equation

$$\sqrt{a^2} = |a| \tag{2}$$

which is true for all values of a.

Hints for the proofs of the following properties are given in the exercises.

Properties of Absolute Values Suppose a and b are any real numbers and n is an integer. Then

1. $|ab| = |a||b|$ **2.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$ $(b \neq 0)$ **3.** $|a^n| = |a|^n$

For solving equations or inequalities involving absolute values, it's often very helpful to use the following statements.

Suppose $a > 0$. Then

4. $|x| = a$ if and only if $x = \pm a$

5. $|x| < a$ if and only if $-a < x < a$

6. $|x| > a$ if and only if $x > a$ or $x < -a$

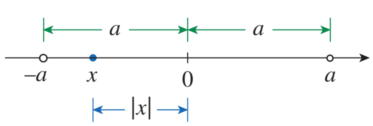


Figure 6
Geometric illustration of Property 5 of absolute values.

For instance, the inequality $|x| < a$ says that the distance from x to the origin is less than a, and you can see from Figure 6 that this is true if and only if x lies between $-a$ and a.

If a and b are any real numbers, then the distance between a and b is the absolute value of the difference, namely, $|a - b|$, which is also equal to $|b - a|$. (See Figure 7.)

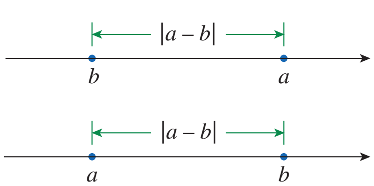


Figure 7
The length of the line segment between a and b is $|a - b|$.

Example 5

Solve $|2x - 5| = 3$.

Solution

By Property 4 of absolute values, $|2x - 5| = 3$ is equivalent to

$2x - 5 = 3$ or $2x - 5 = -3$

So $2x = 8$ or $2x = 2$. Thus $x = 4$ or $x = 1$. ■

Example 6

Solve $|x - 5| < 2$.

Solution 1

By Property 5 of absolute values, $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2$$

Therefore, adding 5 to all sides, we have

$$3 < x < 7$$

and the solution set is the open interval (3, 7).

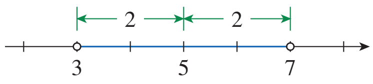


Figure 8
Geometric illustration of the solution to Example 6.

Solution 2

Geometrically, the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 8 we see that this is the interval (3, 7). ■

Example 7

Solve $|3x + 2| \geq 4$.

Solution

By Properties 4 and 6 of absolute values, $|3x + 2| \geq 4$ is equivalent to

$$3x + 2 \geq 4 \qquad \text{or} \qquad 3x + 2 \leq -4$$

In the first case, $3x \geq 2$, which gives $x \geq \frac{2}{3}$. In the second case, $3x \leq -6$, which gives $x \leq -2$. So the solution set is

$$\left\{x \middle| x \leq -2 \text{ or } x \geq \frac{2}{3}\right\} = (-\infty, -2] \cup \left[\frac{2}{3}, \infty\right)$$ ■

A Exercises

Rewrite the expression without using the absolute value symbol.

1. $|5 - 23|$

2. $|\pi - 2|$

3. $|\sqrt{5} - 5|$

4. $||-2| - |-3||$

5. $|x - 2|$ if $x < 2$

6. $|x - 2|$ if $x > 2$

7. $|x + 1|$

8. $|2x - 1|$

9. $|x^2 + 1|$

10. $|1 - 2x^2|$

Solve the inequality in terms of intervals and illustrate the solution set on the real number line.

11. $2x + 7 > 3$

12. $4 - 3x \geq 6$

13. $1 - x \leq 2$

14. $1 + 5x > 5 - 3x$

15. $0 \leq 1 - x < 1$

16. $1 < 3x + 4 \leq 16$

17. $(x - 1)(x - 2) > 0$

18. $x^2 < 2x + 8$

19. $x^2 < 3$

20. $x^2 \geq 5$

21. $x^3 - x^2 \leq 0$

22. $(x + 1)(x - 2)(x + 3) \geq 0$

23. $x^3 > x$

24. $x^3 + 3x < 4x^2$

25. $\frac{1}{x} < 4$

26. $-3 < \frac{1}{x} \leq 1$

27. The relationship between the Celsius and Fahrenheit temperature scales is given by $C = \frac{5}{9}(F - 32)$, where C is the temperature in degrees Celsius and F is the temperature in degrees Fahrenheit. What interval on the Celsius scale corresponds to the temperature range $50 \leq F \leq 95$?

28. Use the relationship between C and F given in Exercise 27 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \leq C \leq 30$.

29. As dry air moves upward, it expands and in so doing cools at a rate of about 1°C for each 100-m rise, up to about 12 km.

(a) If the ground temperature is 20°C, write a formula for the temperature at height h.

(b) What range of temperature can be expected if a plane takes off and reaches a maximum height of 5 km?

30. If a ball is thrown upward from the top of a building 128 ft high with an initial velocity of 16 ft/s, then the height h above the ground t seconds later will be

$$h = 128 + 16t - 16t^2$$

During what time interval will the ball be at least 32 ft above the ground?

Solve the equation for x.

31. $|x + 3| = |2x + 1|$ **32.** $|3x + 5| = 1$

Solve the inequality.

33. $|x| < 3$ **34.** $|x| \geq 3$

35. $|x - 4| < 1$ **36.** $|x - 6| < 0.1$

37. $|x + 5| \geq 2$ **38.** $|x + 1| \geq 3$

39. $|2x - 3| \leq 0.4$ **40.** $|5x - 2| < 6$

41. Solve the inequality $a(bx - c) \geq bc$ for x, assuming that a, b, and c are negative constants.

42. Solve the inequality $ax + b < c$ for x, assuming that a, b, and c are positive constants.

43. Prove that $|ab| = |a||b|$. Hint: Use Equation 3.

44. Show that if $0 < a < b$, then $a^2 < b^2$.

B Coordinate Geometry

The points in a plane can be identified with ordered pairs of real numbers. We start by drawing two perpendicular coordinate lines that intersect at the origin O on each line. Usually one line is horizontal with positive direction to the right and is called the x-axis; the other line is vertical with positive direction upward and is called the y-axis.

Any point P in the plane can be located by a unique ordered pair of numbers as follows. Draw lines through P perpendicular to the x- and y-axes. These lines intersect the axes in points with coordinates a and b as shown in Figure 1. Then the point P is assigned the ordered pair (a, b). The first number a is called the **x-coordinate** of P; the second number b is called the **y-coordinate** of P. We say that P is the point with coordinates (a, b), and we denote the point by the symbol $P(a, b)$. Several points are labeled with their coordinates in Figure 2.

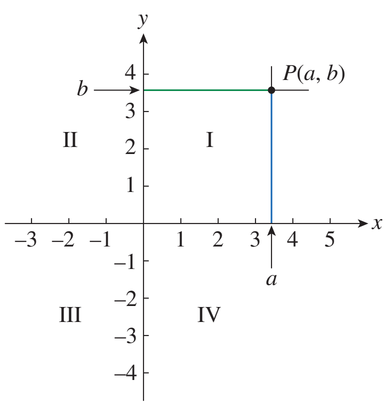


Figure 1
Lines through P perpendicular to the x- and y-axes, and the four quadrants.

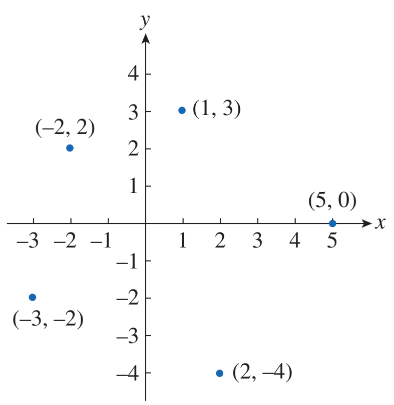


Figure 2
Example points in the plane.

By reversing the preceding process, we can start with an ordered pair (a, b) and arrive at the corresponding point P. Often we identify the point P with the ordered pair (a, b) and refer to "the point (a, b)." Although the notation used for an open interval (a, b) is the same as the notation used for a point (a, b), you will be able to tell from the context which meaning is intended.

This coordinate system is called the **rectangular coordinate system** or the **Cartesian coordinate system** in honor of the French mathematician René Descartes (1596–1650), even though another Frenchman, Pierre de Fermat (1601–1665), invented

the principles of analytic geometry at about the same time as Descartes. The plane, together with this coordinate system, is called the **coordinate plane** or the **Cartesian plane** and is denoted by $\mathbb{R}^2$.

The x- and y-axes are called the **coordinate axes** and divide the Cartesian plane into four quadrants, which are labeled I, II, III, and IV in Figure 1. Notice that the first quadrant consists of those points whose x- and y-coordinates are both positive.

Example 1

Describe and sketch the regions given by the following sets.

(a) $\{(x, y) \mid x \geq 0\}$ (b) $\{(x, y) \mid y = 1\}$ (c) $\{(x, y) \mid |y| < 1\}$

Solution

(a) The points whose x-coordinates are 0 or positive lie on the y-axis or to the right of it as indicated by the shaded region in Figure 3(a).

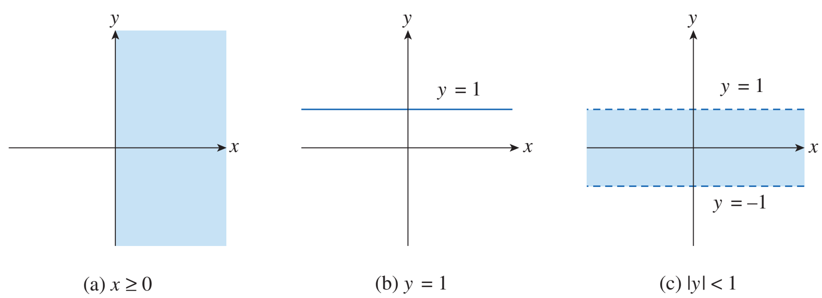


Figure 3
Geometric illustration of the regions described in Example 3.

(b) The set of all points with y-coordinate 1 is a horizontal line one unit above the x-axis. See Figure 3(b).

(c) Recall from Appendix A that

$$|y| < 1 \qquad \text{if and only if} \qquad -1 < y < 1$$

The given region consists of those points in the plane whose y-coordinates lie between -1 and 1. Thus the region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$. These lines are shown as dashed lines in Figure 3(c) to indicate that the points on these lines don't lie in the set. ■

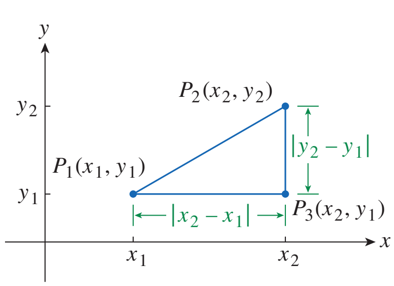


Figure 4
Geometric illustration of the Distance Formula.

Recall from Appendix A that the distance between points a and b on a number line is $|a - b| = |b - a|$. Thus the distance between points $P_1(x_1, y_1)$ and $P_3(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$ and the distance between $P_2(x_2, y_2)$ and $P_3(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$. See Figure 4.

To find the distance $|P_1P_2|$ between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, we note that triangle $P_1P_2P_3$ in Figure 4 is a right triangle, and so by the Pythagorean Theorem, we have

$$|P_1P_2| = \sqrt{|P_1P_3|^2 + |P_2P_3|^2} = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance Formula

The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For instance, the distance between $(1, -2)$ and $(5, 3)$ is

$$\sqrt{(5-1)^2 + [3-(-2)]^2} = \sqrt{4^2+5^2} = \sqrt{41}$$

Circles

An **equation of a curve** is an equation satisfied by the coordinates of the points on the curve and by no other points. Let's use the distance formula to find the equation of a circle with radius r and center (h, k). By definition, the circle is the set of all points $P(x, y)$ whose distance from the center $C(h, k)$ is r. (See Figure 5.) Thus P is on the circle if and only if $|PC| = r$. From the distance formula, we have

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

or equivalently, squaring both sides, we get

$$(x-h)^2 + (y-k)^2 = r^2$$

This is the desired equation.

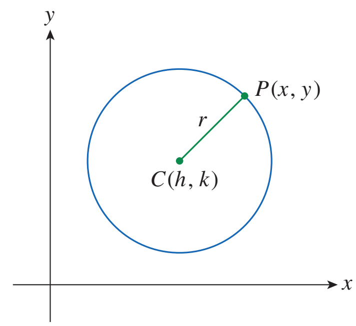


Figure 5
A point P is on the circle if and only if the distance to the center C is r.

Equation of a Circle

An equation of the circle with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

In particular, if the center is the origin $(0, 0)$, the equation is

$$x^2 + y^2 = r^2$$

For instance, an equation of the circle with radius 3 and center $(2, -5)$ is

$$(x-2)^2 + (y+5)^2 = 9$$

Example 2

Sketch the graph of the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ by first showing that it represents a circle and then finding its center and radius.

Solution

We first group the x-terms and y-terms as follows:

$$(x^2 + 2x) + (y^2 - 6y) = -7$$

Then we complete the square within each grouping, adding the appropriate constants (the squares of half the coefficients of x and y) to both sides of the equation:

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9$$

or $\quad (x+1)^2 + (y-3)^2 = 3$

Comparing this equation with the standard equation of a circle, we see that $h = -1$, $k = 3$, and $r = \sqrt{3}$, so the given equation represents a circle with center $(-1, 3)$ and radius $\sqrt{3}$. It is sketched in Figure 6. ■

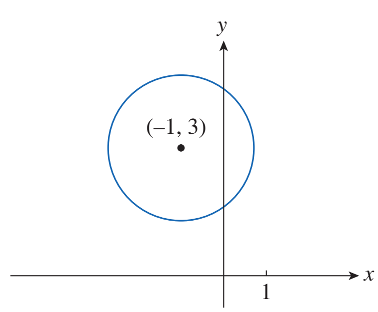


Figure 6
Graph of the circle described by $x^2 + y^2 + 2x - 6y + 7 = 0$.

Lines

To find the equation of a line L, we use its *slope*, which is a measure of the steepness of the line.

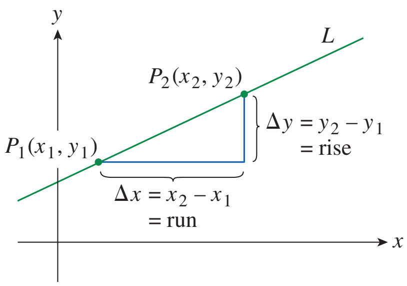


Figure 7
Geometric illustration of the slope of a line L.

Definition • Slope

The **slope** of a nonvertical line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

Thus the slope of a line is the ratio of the change in y, Δy, to the change in x, Δx. (See Figure 7.) The slope is therefore the rate of change of y with respect to x. The fact that the line is straight means that the rate of change is constant.

Figure 8 shows several lines labeled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. Notice also that the steepest lines are the ones for which the absolute value of the slope is largest and a horizontal line has slope 0.

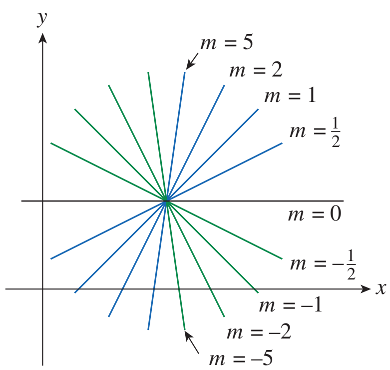


Figure 8
Graphs of lines with various slopes.

Now let's find an equation of the line that passes through a given point $P_1(x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through P_1 and P is equal to m; that is,

$$\frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

and we observe that this equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore, it is an equation of the given line.

Point-Slope Form of the Equation of a Line

An equation of the line passing through the point $P_1(x_1, y_1)$ and having slope m is

$$y - y_1 = m(x - x_1)$$

Example 3

Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

Solution

The slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{3}{2}$$

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\frac{3}{2}(x + 1)$$

which simplifies to $3x + 2y = 1$ ■

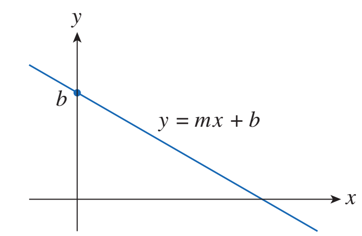


Figure 9
Graph of a nonvertical line with slope m and y-intercept b.

Suppose a nonvertical line has slope m and y-intercept b. (See Figure 9.) This means it intersects the y-axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x_1 = 0$ and $y_1 = b$, becomes

$$y - b = m(x - 0)$$

This simplifies as follows.

Slope-Intercept Form of the Equation of a Line

An equation of the line with slope m and y-intercept b is

$$y = mx + b$$

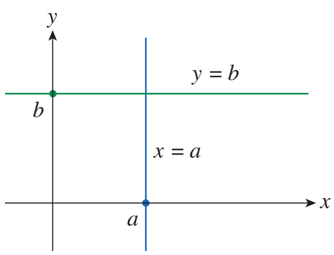


Figure 10
Graphs of a horizontal and vertical line, and their corresponding equations.

In particular, if a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where b is the y-intercept (see Figure 10). A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the x-coordinate of every point on the line is a.

Example 4

Graph the inequality $x + 2y > 5$.

Solution

We are asked to sketch the graph of the set $\{(x, y) \mid x + 2y > 5\}$, and we begin by solving the inequality for y:

$$\begin{aligned} x + 2y &> 5 \\ 2y &> -x + 5 \\ y &> -\frac{1}{2}x + \frac{5}{2} \end{aligned}$$

Compare this inequality with the equation $y = -\frac{1}{2}x + \frac{5}{2}$, which represents a line with slope $-\frac{1}{2}$ and y-intercept $\frac{5}{2}$. We see that the given graph consists of points whose y-coordinates are *larger* than those on the line $y = -\frac{1}{2}x + \frac{5}{2}$. Thus the graph is the region that lies *above* the line, as illustrated in Figure 11. ■

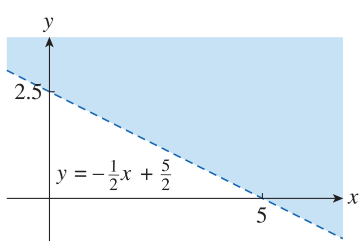


Figure 11
Graph of the region in the plane that satisfies the inequality $x + 2y > 5$.

Parallel and Perpendicular Lines

Slopes can be used to show that lines are parallel or perpendicular. The following facts are proved, for instance, in *Precalculus: Mathematics for Calculus, Fifth Edition* by Stewart, Redlin, and Watson (Belmont, CA, 2006).

Parallel and Perpendicular Lines

1. Two nonvertical lines are parallel if and only if they have the same slope.
2. Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$; that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

Example 5

Find an equation of the line through the point (5, 2) that is parallel to the line $4x + 6y + 5 = 0$.

Solution

The given line can be written in the form

$$y = -\frac{2}{3}x - \frac{5}{6}$$

which is in slope-intercept form with $m = -\frac{2}{3}$. Parallel lines have the same slope, so the required line has slope $-\frac{2}{3}$ and its equation in point-slope form is

$$y - 2 = -\frac{2}{3}(x - 5)$$

We can write this equation as $2x + 3y = 16$. ■

Example 6

Show that the lines $2x + 3y = 1$ and $6x - 4y - 1 = 0$ are perpendicular.

Solution

The equations can be written as

$$y = -\frac{2}{3}x + \frac{1}{3} \quad \text{and} \quad y = \frac{3}{2}x - \frac{1}{4}$$

from which we see that the slopes are

$$m_1 = -\frac{2}{3} \quad \text{and} \quad m_2 = \frac{3}{2}$$

Since $m_1 m_2 = -1$, the lines are perpendicular. ■

Conic Sections

Here we review the geometric definitions of parabolas, ellipses, and hyperbolas and their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 12.

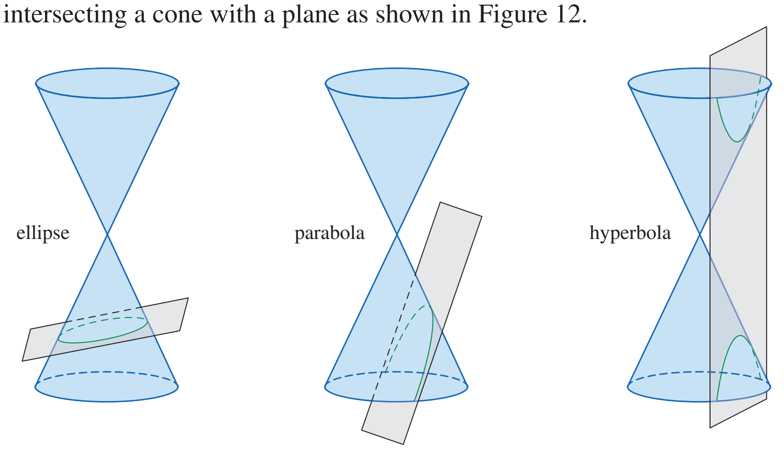


Figure 12
The intersection of a plane and a cone forms a conic section.

Parabolas

A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). This definition is illustrated by Figure 13. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**. The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

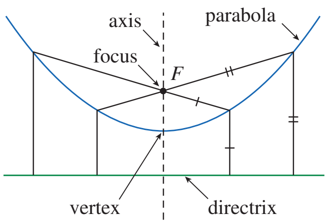


Figure 13
Geometric illustration of the definition of a parabola.

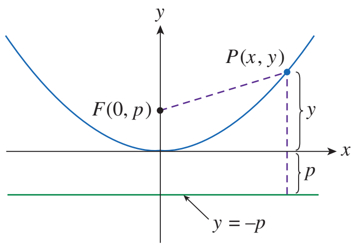


Figure 14
Parabola with vertex at the origin O and directrix parallel to the x-axis.

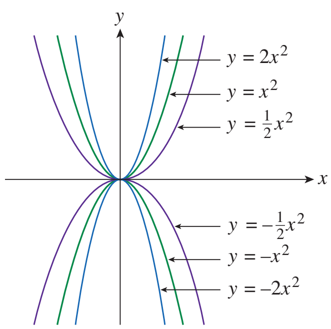


Figure 15
Graphs of parabolas of the form $y = ax^2$.

In the 16th century, Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges. (See Problem 18 on page 290 for the reflection property of parabolas that makes them so useful.)

We obtain a particularly simple equation for a parabola if we place its vertex at the origin O and its directrix parallel to the x-axis as in Figure 14. If the focus is the point $(0, p)$, then the directrix has the equation $y = -p$ and the parabola has the equation

$$x^2 = 4py$$

(See Exercise 47.)

If we write $a = 1/(4p)$, then the equation of the parabola becomes

$$y = ax^2$$

Figure 15 shows the graphs of several parabolas with equations of the form $y = ax^2$ for various values of the number a. We see that the parabola $y = ax^2$ opens upward if $a > 0$ and downward if $a < 0$ (as in Figure 16). The graph is symmetric with respect to the y-axis because its equation is unchanged when x is replaced by $-x$. This corresponds to the fact that the function $f(x) = ax^2$ is an even function.

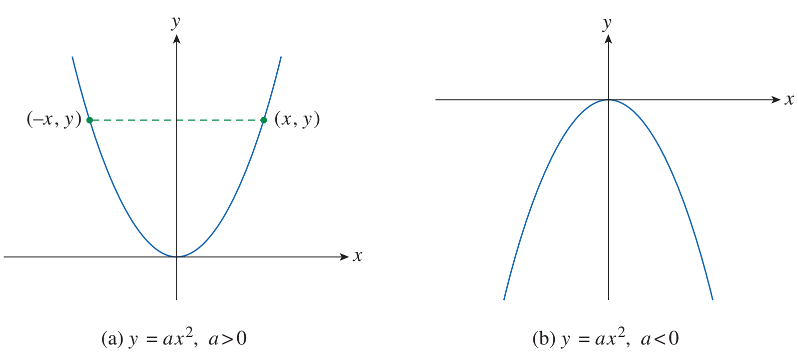


Figure 16
The graph of $y = ax^2$ opens upward if $a > 0$ and opens downward if $a < 0$ and is symmetric with respect to the y-axis.

If we interchange x and y in the equation $y = ax^2$, the result is $x = ay^2$, which also represents a parabola. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola $x = ay^2$ opens to the right if $a > 0$ and to the left if $a < 0$. (See Figure 17.) This time the parabola is symmetric with respect to the x-axis because the equation is unchanged when y is replaced by $-y$.

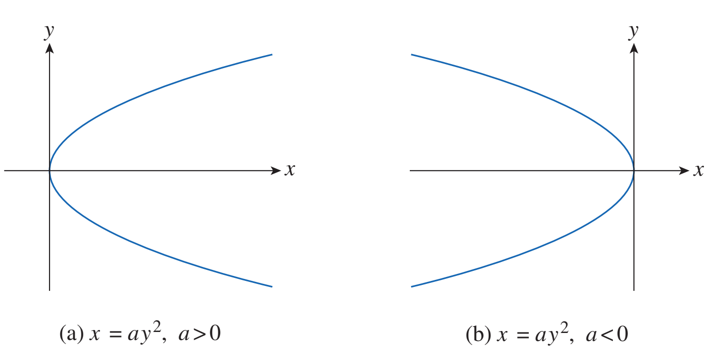


Figure 17
Graphs of parabolas described by $x = ay^2$.

Example 7

Sketch the region bounded by the parabola $x = 1 - y^2$ and the line $x + y + 1 = 0$.

Solution

First we find the points of intersection by solving the two equations. Substituting $x = -y - 1$ into the equation $x = 1 - y^2$, we get $-y - 1 = 1 - y^2$, which gives

$$0 = y^2 - y - 2 = (y - 2)(y + 1)$$

so $y = 2$ or -1. Thus the points of intersection are $(-3, 2)$ and $(0, -1)$, and we draw the line $x + y + 1 = 0$ passing through these points.

To sketch the parabola $x = 1 - y^2$, we start with the parabola $x = -y^2$ in Figure 17(b) and shift one unit to the right. We also make sure it passes through the points $(-3, 2)$ and $(0, -1)$. The region bounded by $x = 1 - y^2$ and $x + y + 1 = 0$ means the finite region whose boundaries are these curves. It is sketched in Figure 18. ■

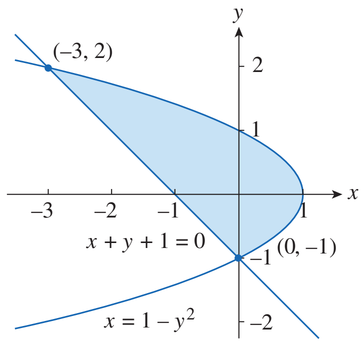


Figure 18
The region in the plane bounded by the graphs of $x = 1 - y^2$ and $x + y + 1 = 0$.

Ellipses

An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant (see Figure 19). These two fixed points are called the **foci** (plural of **focus**). One of Kepler's laws is that the orbits of the planets in the solar system are ellipses with the sun at one focus.

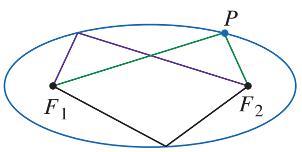


Figure 19
Geometric illustration of the definition of an ellipse.

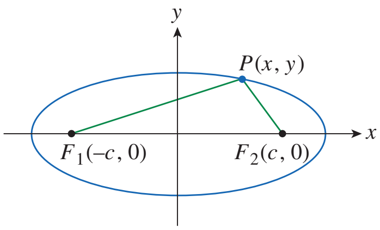


Figure 20
An ellipse with foci on the x-axis.

In order to obtain the simplest equation for an ellipse, we place the foci on the x-axis at the points $(-c, 0)$ and $(c, 0)$ as in Figure 20, so that the origin is halfway between the foci. If we let the sum of the distances from a point on the ellipse to the foci be $2a$, then we can write an equation of the ellipse as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

where $c^2 = a^2 - b^2$. (See Exercise 49 and Figure 21.) Notice that the x-intercepts are $\pm a$, the y-intercepts are $\pm b$, the foci are $(\pm c, 0)$, and the ellipse is symmetric with respect to both axes. If the foci of an ellipse are located on the y-axis at $(0, \pm c)$, then we can find its equation by interchanging x and y in (1).

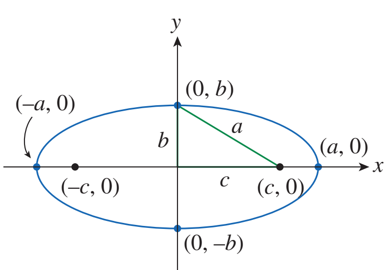


Figure 21
Geometric illustration of an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a \geq b$.

Example 8

Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

Solution

Divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse, so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$. The x-intercepts are ± 4 and the y-intercepts are ± 3. Also, $c^2 = a^2 - b^2 = 7$, so $c = \sqrt{7}$ and the foci are $(\pm\sqrt{7}, 0)$. The graph is sketched in Figure 22. ■

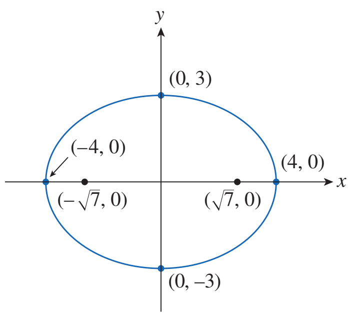


Figure 22
Graph of the ellipse with equation $9x^2 + 16y^2 = 144$.

Like parabolas, ellipses have an interesting reflection property that has practical consequences. If a source of light or sound is placed at one focus of a surface with elliptical cross-sections, then all the light or sound is reflected off the surface to the other focus (see Exercise 55). This principle is used in *lithotripsy,* a treatment for kidney stones. A reflector with elliptical cross-section is placed in such a way that the kidney stone is at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it without damaging surrounding tissue. The patient is spared the trauma of surgery and recovers within a few days.

Hyperbolas

A **hyperbola** is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is a constant. This definition is illustrated in Figure 23.

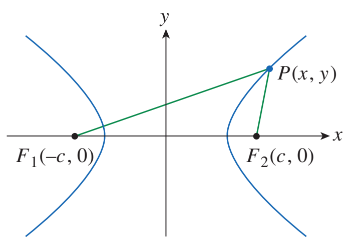


Figure 23
Geometric illustration of the definition of a hyperbola; P is on the hyperbola when $|PF_1| - |PF_2| = \pm 2a$.

Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances. It is left as Exercise 51 to show that when the foci are on the x-axis at $(\pm c, 0)$ and the difference of distances is $|PF_1| - |PF_2| = \pm 2a$, then the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{2}$$

where $c^2 = a^2 + b^2$. Notice that the x-intercepts are again $\pm a$. But if we put $x = 0$ in Equation 2, we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. The hyperbola is symmetric with respect to both axes.

To analyze the hyperbola further, we look at Equation 2 and obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$$

This shows that $x^2 \geq a^2$, so $|x| = \sqrt{x^2} \geq a$. Therefore, we have $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two parts, called its *branches.*

When we draw a hyperbola, it is useful to first draw its *asymptotes,* which are the lines $y = \left(\frac{b}{a}\right)x$ and $y = -\left(\frac{b}{a}\right)x$ shown in Figure 24. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. If the foci of a hyperbola are on the y-axis, we find its equation by reversing the roles of x and y.

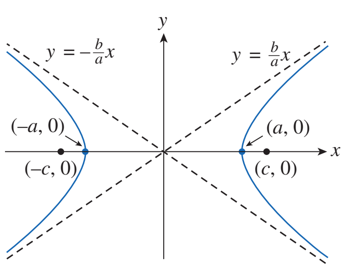


Figure 24
Graph of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its asymptotes.

Example 9

Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

Solution

If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is of the form given in Equation 2 with $a = 4$ and $b = 3$. Since $c^2 = 16 + 9 = 25$, the foci are $(\pm 5, 0)$. The asymptotes are the lines $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$. The graph is shown in Figure 25. ■

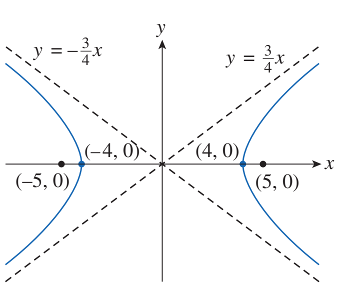


Figure 25
Graph of the hyperbola $9x^2 - 16y^2 = 144$.

B Exercises

Find the distance between the points.

1. $(1, 1), \quad (4, 5)$ **2.** $(1, -3), \quad (5, 7)$

Find the slope of the line through P and Q.

3. $P(-3, 3), Q(-1, -6)$ **4.** $P(-1, -4), Q(6, 0)$

5. Show that the points $(-2, 9)$, $(4, 6)$, $(1, 0)$, and $(-5, 3)$ are the vertices of a square.

6. (a) Show that the points $A(-1, 3)$, $B(3, 11)$, and $C(5, 15)$ are collinear (lie on the same line) by showing that $|AB| + |BC| = |AC|$.

(b) Use slopes to show that A, B, and C are collinear.

Sketch the graph of the equation.

7. $x = 3$ **8.** $y = -2$

9. $xy = 0$ **10.** $|y| = 1$

Find an equation of the line that satisfies the given conditions.

11. Through $(2, -3)$, slope 6

12. Through $(-3, -5)$, slope $-\frac{7}{2}$

13. Through $(2, 1)$ and $(1, 6)$

14. Through $(-1, -2)$ and $(4, 3)$

15. Slope 3, y-intercept -2

16. Slope $\frac{2}{5}$, intercept 4

17. x-intercept 1, y-intercept -3

18. x-intercept -8, y-intercept 6

19. Through $(4, 5)$, parallel to the x-axis

20. Through $(4, 5)$, parallel to the y-axis

21. Through $(1, -6)$, parallel to the line $x + 2y = 6$

22. y-intercept 6, parallel to the line $2x + 3y + 4 = 0$

23. Through $(-1, -2)$, perpendicular to the line $2x + 5y + 8 = 0$

24. Through $\left(\frac{1}{2}, -\frac{2}{3}\right)$, perpendicular to the line $4x - 8y = 1$

Find the slope and y-intercept of the line and draw its graph.

25. $x + 3y = 0$ **26.** $2x - 3y + 6 = 0$

27. $3x - 4y = 12$ **28.** $4x + 5y = 10$

Sketch the region in the xy-plane.

29. $\{(x, y) \mid x < 0\}$ **30.** $\{(x, y) \mid x \geq 1 \text{ and } y < 3\}$

31. $\{(x, y) \mid |x| \leq 2\}$

32. $\{(x, y) \mid |x| < 3 \text{ and } |y| < 2\}$

33. $\{(x, y) \mid 0 \leq y \leq 4 \text{ and } x \leq 2\}$

34. $\{(x, y) \mid y > 2x - 1\}$

35. $\{(x, y) \mid 1 + x \leq y \leq 1 - 2x\}$

36. $\{(x, y) - x \leq y < \frac{1}{2}(x + 3)\}$

Find an equation of a circle that satisfies the given conditions.

37. Center $(3, -1)$, radius 5

38. Center $(-1, 5)$, passes through $(-4, -6)$

Show that the equation represents a circle and find the center and radius.

39. $x^2 + y^2 - 4x + 10y + 13 = 0$

40. $x^2 + y^2 + 6y + 2 = 0$

41. Show that the lines $2x - y = 4$ and $6x - 2y = 10$ are not parallel and find their point of intersection.

42. Show that the lines $3x - 5y + 19 = 0$ and $10x + 6y - 50 = 0$ are perpendicular and find their point of intersection.

43. Show that the midpoint of the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

44. Find the midpoint of the line segment joining the points $(1, 3)$ and $(7, 15)$.

45. Find an equation of the perpendicular bisector of the line segment joining the points $A(1, 4)$ and $B(7, -2)$.

46. (a) Show that if the x- and y-intercepts of a line are nonzero numbers a and b, then the equation of the line can be put in the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

This equation is called the **two-intercept form** of an equation of a line.

(b) Use part (a) to find an equation of the line whose x-intercept is 6 and whose y-intercept is -8.

47. Suppose that $P(x, y)$ is any point on the parabola with focus $(0, p)$ and directrix $y = -p$. (See Figure 14.) Use the definition of a parabola to show that $x^2 = 4py$.

48. Find the focus and directrix of the parabola $y = x^2$. Illustrate with a diagram.

49. Suppose an ellipse has foci $(\pm c, 0)$ and the sum of the distances from any point $P(x, y)$ on the ellipse to the foci is $2a$. Show that the coordinates of P satisfy Equation 1.

50. Find the foci of the ellipse $x^2 + 4y^2 = 4$ and sketch its graph.

51. Use the definition of a hyperbola to derive Equation 2 for a hyperbola with foci $(\pm c, 0)$.

52. (a) Find the foci and asymptotes of the hyperbola $x^2 - y^2 = 1$ and sketch its graph.

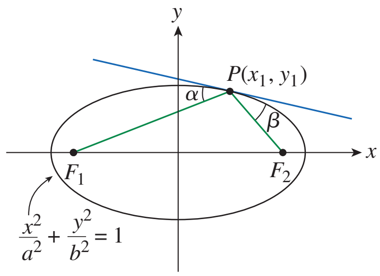
(b) Sketch the graph of $y^2 - x^2 = 1$.

Sketch the region bounded by the curves.

53. $x + 4y = 8$ and $x = 2y^2 - 8$

54. $y = 4 - x^2$ and $x - 2y = 2$

55. Let $P(x_1, y_1)$ be a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci F_1 and F_2 and let α and β be the angles between the lines PF_1, PF_2 and the ellipse as shown in the figure. Prove that $\alpha = \beta$. This explains how whispering galleries and lithotripsy work. Sound coming from one focus is reflected and passes through the other focus. [Hint: Use the formula in Problem 17 on page 290 to show that $\tan \alpha = \tan \beta$.]



C Trigonometry

Here we review the aspects of trigonometry that are used in calculus: radian measure, trigonometric functions, trigonometric identities, and inverse trigonometric functions.

Angles

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains 360°, which is the same as 2π rad. Therefore,

$$\pi \text{ rad} = 180° \tag{1}$$

and

$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \approx 57.3° \qquad 1° = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad} \tag{2}$$

Example 1

(a) Find the radian measure of 60°.
(b) Express $5\pi/4$ rad in degrees.

Solution

(a) From Equation 1 or 2, we see that to convert from degrees to radians, we multiply by $\pi/180$. Therefore,

$$60° = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ rad}$$

(b) To convert from radians to degrees, we multiply by $180/\pi$. Thus

$$\frac{5\pi}{4} \text{ rad} = \frac{5\pi}{4}\left(\frac{180}{\pi}\right) = 225°$$ ■

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

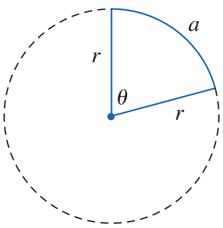


Figure 1
Sector of a circle.

Figure 1 shows a sector of a circle with central angle θ and radius r subtending an arc with length a. Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference $2\pi r$ and central angle 2π, we have

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for θ and for a, we obtain

$$\theta = \frac{a}{r} \qquad\qquad a = r\theta \tag{3}$$

Remember that the equations in (3) are valid only when θ is measured in radians.

In particular, putting $a = r$ in the equations in (3), we see that an angle of 1 rad is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle (see Figure 2).

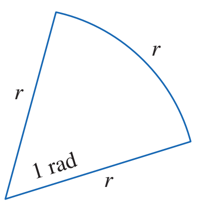


Figure 2
Angle of 1 rad.

Example 2

(a) If the radius of a circle is 5 cm, what angle is subtended by an arc of 6 cm?

(b) If a circle has radius 3 cm, what is the length of an arc subtended by a central angle of $3\pi/8$ rad?

Solution

(a) Using Equation 3 with $a = 6$ and $r = 5$, we see that the angle is

$$\theta = \frac{6}{5} = 1.2 \text{ rad}$$

(b) With $r = 3$ cm and $\theta = 3\pi/8$ rad, the arc length is

$$a = r\theta = 3\left(\frac{3\pi}{8}\right) = \frac{9\pi}{8} \text{ cm}$$

■

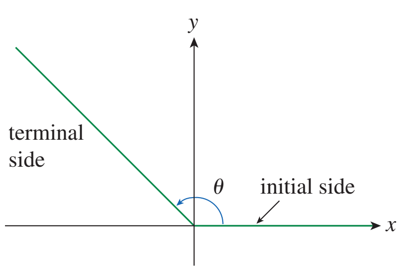


Figure 3
Angle in standard position; $\theta \geq 0$.

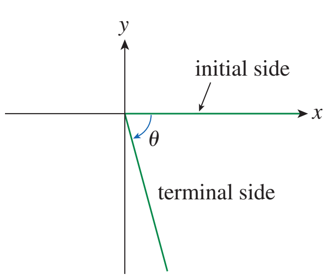


Figure 4
Angle in standard position; $\theta < 0$.

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x-axis as in Figure 3. A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation as in Figure 4.

Figure 5 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles $3\pi/4$, $-5\pi/4$, and $11\pi/4$ have the same initial and terminal sides because

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \qquad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and 2π rad represents a complete revolution.

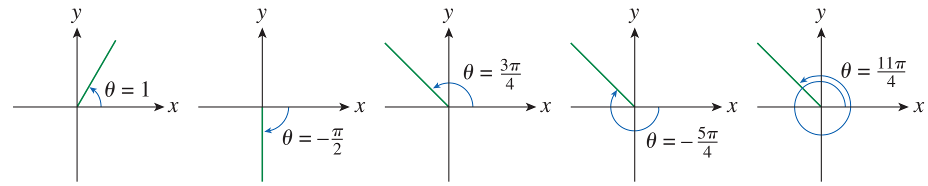


Figure 5
Angles in standard position.

The Trigonometric Functions

For an acute angle θ, the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 6).

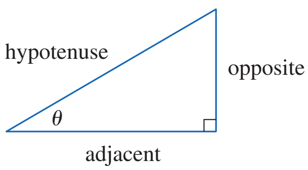


Figure 6
Reference right triangle for defining trigonometric functions.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \tag{4}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

This definition doesn't apply to obtuse or negative angles, so for a general angle θ in standard position, we let $P(x, y)$ be any point on the terminal side of θ and we let r be the distance $|OP|$ as in Figure 7. Then, we define

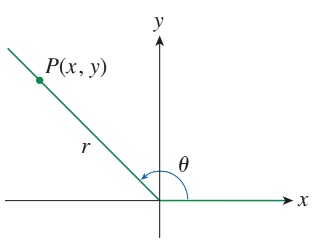


Figure 7
Geometric illustration for defining trigonometric functions for a general angle θ.

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x} \tag{5}$$
$$\tan\theta = \frac{y}{r} \qquad \cot\theta = \frac{x}{y}$$

Since division by 0 is not defined, $\tan\theta$ and $\sec\theta$ are undefined when $x = 0$ and $\csc\theta$ and $\cot\theta$ are undefined when $y = 0$. Notice that the equations in (4) and (5) are consistent when θ is an acute angle.

If θ is a number, the convention is that $\sin\theta$ means the sine of the angle whose *radian* measure is θ. For example, the expression sin 3 implies that we are dealing with an angle of 3 rad. When using technology to find an approximation to this value, make sure the calculator is set to radian mode, and then we obtain

$$\sin 3 \approx 0.14112$$

If we want to know the sine of the angle 3°, we would write $\sin 3°$ and, with using technology in degree mode, we find that

$$\sin 3° \approx 0.05234$$

The exact trigonometric ratios for certain angles can be read from the triangles in Figure 8. For instance,

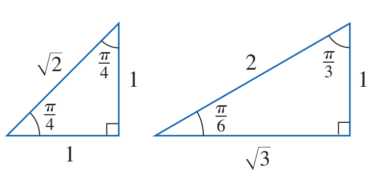


Figure 8
Common reference triangles.

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$
$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

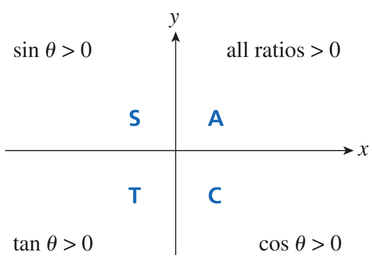


Figure 9
Memory technique for the signs of trigonometric functions.

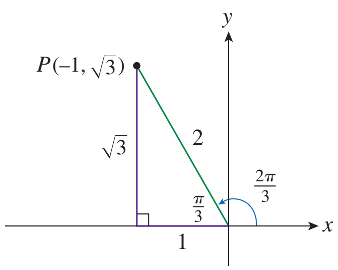


Figure 10
Geometric illustration for finding the trigonometric ratios for $\theta = \dfrac{2\pi}{3}$.

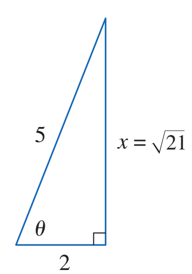


Figure 11
Reference triangle in which $\cos\ \theta = \dfrac{2}{5}$.

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule "**A**ll **S**tudents **T**ake **C**alculus" shown in Figure 9.

Example 3

Find the exact trigonometric ratios for $\theta = \dfrac{2\pi}{3}$.

Solution

From Figure 10 we see that a point on the terminal line for $\theta = \dfrac{2\pi}{3}$ is $P(-1, \sqrt{3})$. Therefore, taking

$$x = -1 \qquad y = \sqrt{3} \qquad r = 2$$

in the definitions of the trigonometric ratios, we have

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = -2 \qquad \cot\frac{2\pi}{3} = -\frac{1}{\sqrt{3}}$$

■

The following table gives some values of sin θ and cos θ found by the method of Example 3.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

Example 4

If $\cos\ \theta = \dfrac{2}{5}$ and $0 < \theta < \dfrac{\pi}{2}$, find the other five trigonometric functions of θ.

Solution

Since $\cos\ \theta = \dfrac{2}{5}$, we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in Figure 11. If the opposite side has length x, then the Pythagorean Theorem gives $x^2 + 4 = 25$ and so $x^2 = 21$, $x = \sqrt{21}$. We can now use Figure 11 to write the other five trigonometric functions:

$$\sin\ \theta = \frac{\sqrt{21}}{5} \qquad \tan\ \theta = \frac{\sqrt{21}}{2}$$

$$\csc\ \theta = \frac{5}{\sqrt{21}} \qquad \sec\ \theta = \frac{5}{2} \qquad \cot\ \theta = \frac{2}{\sqrt{21}}$$

■

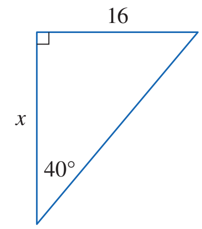


Figure 12
Reference triangle for Example 5.

Example 5

Use technology approximate the value of x in Figure 12.

Solution
From the figure we see that

$$\tan 40° = \frac{16}{x}$$

Therefore, $x = \dfrac{16}{\tan 40°} \approx 19.07$ ■

Trigonometric Identities

A trigonometric identity is a relationship among the trigonometric functions. The most elementary are the following, which are immediate consequences of the definitions of the trigonometric functions.

$$\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta} \qquad \cot \theta = \frac{1}{\tan \theta}$$
$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta} \tag{6}$$

For the next identity, we refer back to Figure 7. The distance formula (or, equivalently, the Pythagorean Theorem) tells us that $x^2 + y^2 = r^2$. Therefore,

$$\sin^2 \theta + \cos^2 \theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

We have therefore proved one of the most useful of all trigonometric identities:

$$\sin^2 \theta + \cos^2 \theta = 1 \tag{7}$$

If we now divide both sides of Equation 7 by $\cos^2 \theta$ and use Equations 6, we get

$$\tan^2 \theta + 1 = \sec^2 \theta \tag{8}$$

Similarly, if we divide both sides of Equation 7 by $\sin^2 \theta$, we get

$$1 + \cot^2 \theta = \csc^2 \theta \tag{9}$$

The identities

$$\sin(-\theta) = -\sin \theta \tag{10a}$$

$$\cos(-\theta) = \cos \theta \tag{10b}$$

Odd functions and even functions are discussed in Section 1.1.

show that sine is an odd function and cosine is an even function. They are easily proved by drawing a diagram showing θ and $-\theta$ in standard position (see Exercise 19).

Since the angles θ and $\theta + 2\pi$ have the same terminal side, we have

$$\sin(\theta + 2\pi) = \sin \theta \qquad \cos(\theta + 2\pi) = \cos \theta \tag{11}$$

These identities show that the sine and cosine functions are periodic with period 2π.

The remaining trigonometric identities are all consequences of two basic identities called the **addition formulas**:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \tag{12a}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \tag{12b}$$

The proofs of these addition formulas are outlined in Exercises 43, 44, and 45.

By substituting $-y$ for y in Equations 12a and 12b and using Equations 10a and 10b, we obtain the following **subtraction formulas**:

$$\sin(x - y) = \sin x \cos y - \cos x \sin y \tag{13a}$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y \tag{13b}$$

Then, by dividing the formulas in Equations 12 or Equations 13, we obtain the corresponding formulas for tan $(x \pm y)$:

Divide Equation 12a by Equation 12b and simplify to obtain Equation 14a, and similarly, divide Equation 13a by Equation 13b to obtain Equation 14b.

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \tag{14a}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \tag{14b}$$

If we let $y = x$ in the addition Equations 12a and 12b, we get the **double-angle formulas**:

$$\sin 2x = 2 \sin x \cos x \tag{15a}$$

$$\cos 2x = \cos^2 x - \sin^2 x \tag{15b}$$

Then, by using the identity $\sin^2 x + \cos^2 x = 1$, we obtain the following alternate forms of the double-angle formulas for cos $2x$:

$$\cos 2x = 2 \cos^2 x - 1 \tag{16a}$$

$$\cos 2x = 1 - 2 \sin^2 x \tag{16b}$$

If we now solve these equations for $\cos^2 x$ and $\sin^2 x$, we get the following **half-angle formulas**, which are useful in integral calculus:

$$\cos^2 x = \frac{1 + \cos 2x}{2} \tag{17a}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \tag{17b}$$

There are many other trigonometric identities, but those we have stated are the ones used most often in calculus. If you forget any of them, remember that they can all be deduced from Equations 12a and 12b.

Example 6

Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

Solution

Using the double-angle formula (15a), we rewrite the given equation as

$$\sin x = 2 \sin x \cos x \qquad \text{or} \qquad \sin x(1 - 2 \cos x) = 0$$

Therefore, there are two possibilities:

$$\sin x = 0 \qquad \text{or} \qquad 1 - 2\cos x = 0$$

$$x = 0,\ \pi,\ 2\pi \qquad\qquad \cos x = \frac{1}{2}$$

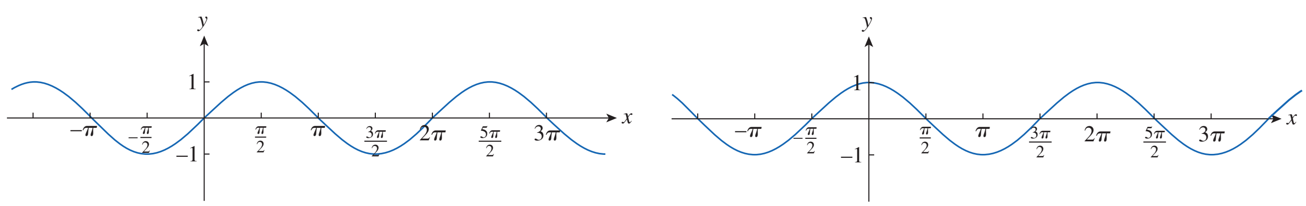
$$x = \frac{\pi}{3},\ \frac{5\pi}{3}$$

The given equation has five solutions: $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$. ■

Graphs of the Trigonometric Functions

The graph of the function $f(x) = \sin x$, shown in Figure 13(a), is obtained by plotting points for $0 \le x \le 2\pi$ and then using the periodic nature of the function (from Equation 11) to complete the graph. Notice that the zeros of the sine function occur at the integer multiples of π, that is,

$$\sin x = 0 \quad \text{whenever } x = n\pi, \quad n \text{ is an integer}$$



(a) $f(x) = \sin x$

(b) $g(x) = \cos x$

Figure 13
Graphs of the sine and cosine functions.

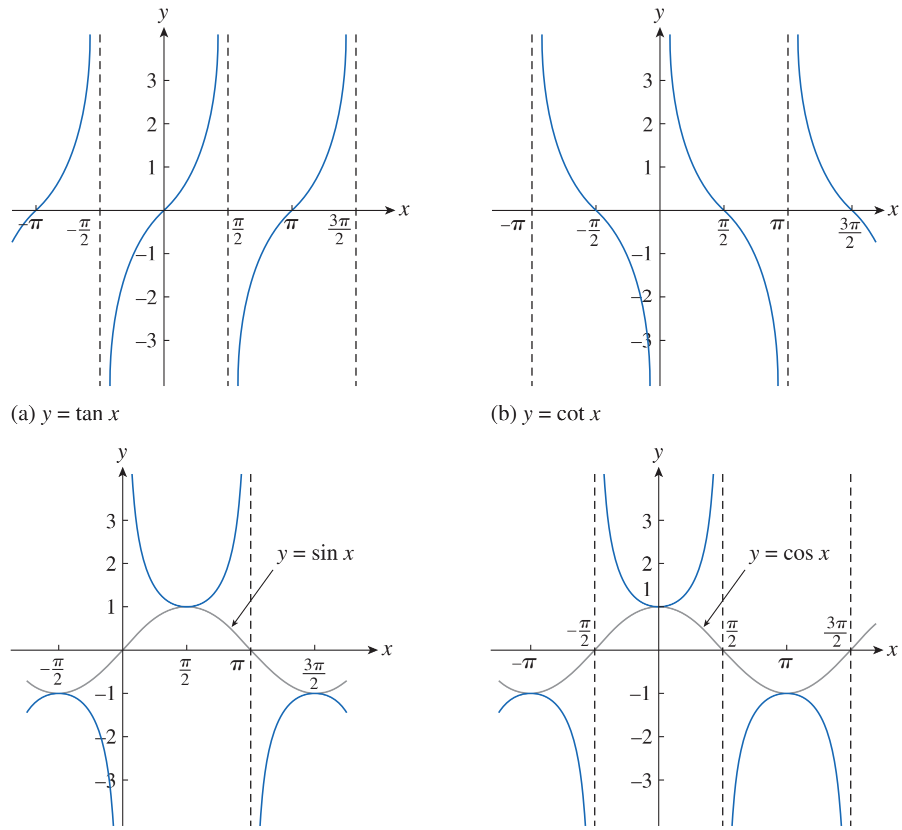
Because of the identity

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

(which can be verified using Equation 12a), the graph of cosine is obtained by shifting the graph of sine by an amount $\pi/2$ to the left [see Figure 13(b)]. Note that for both the sine and cosine functions, the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

$$-1 \le \sin x \le 1 \qquad -1 \le \cos x \le 1$$

The graphs of the remaining four trigonometric functions are shown in Figure 14 and their domains are indicated there. Notice that tangent and cotangent have range $(-\infty, \infty)$, whereas cosecant and secant have range $(-\infty, -1] \cup [1, \infty)$. All four functions are periodic: tangent and cotangent have period π, whereas cosecant and secant have period 2π.



(a) $y = \tan x$ (b) $y = \cot x$

(c) $y = \csc x$ (d) $y = \sec x$

Figure 14
Graphs of the remaining trigonometric functions.

C Exercises

Convert from degrees to radians.

1. (a) $210°$ (b) $9°$

2. (a) $-315°$ (b) $36°$

Convert from radians to degrees.

3. (a) 4π (b) $-\dfrac{3\pi}{8}$

4. (a) $-\dfrac{7\pi}{2}$ (b) $\dfrac{8\pi}{3}$

5. Find the length of a circular arc subtended by an angle of $\pi/12$ rad if the radius of the circle is 36 cm.

6. If a circle has radius 10 cm, find the length of the arc subtended by a central angle of 72°.

7. A circle has radius 1.5 m. What angle is subtended at the center of the circle by an arc 1 m long?

8. Find the radius of a circular sector with angle $3\pi/4$ and arc length 6 cm.

Draw, in standard position, the angle whose measure is given.

9. (a) $315°$ (b) $-\dfrac{3\pi}{4}$ rad

10. (a) $\dfrac{7\pi}{3}$ rad (b) -3 rad

Find the exact trigonometric ratios for the angle whose radian measure is given.

11. $\dfrac{3\pi}{4}$ **12.** $\dfrac{4\pi}{3}$

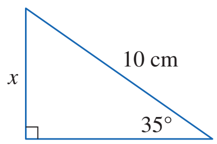
Find the remaining trigonometric ratios.

13. $\sin\theta = \dfrac{3}{5}, \quad 0 < \theta < \dfrac{\pi}{2}$

14. $\tan\alpha = 2, \quad 0 < \alpha < \dfrac{\pi}{2}$

Find, correct to five decimal places, the length of the side labeled x.

15.



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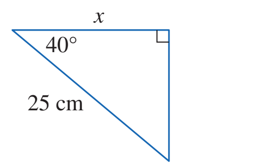
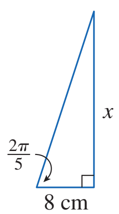
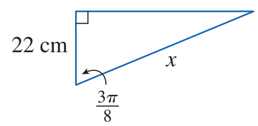


17.



18.



Prove each equation.

19. (a) Equation 10a (b) Equation 10b

20. (a) Equation 14a (b) Equation 14b

21–26 Prove the identity.

21. $\sin\left(\dfrac{\pi}{2} + x\right) = \cos x$

22. $\sin(\pi - x) = \sin x$

23. $\sin\theta \cot\theta = \cos\theta$

24. $(\sin x + \cos x)^2 = 1 + \sin 2x$

25. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

26. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

If $\sin x = \dfrac{1}{3}$ and $\sec y = \dfrac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate the expression.

27. $\sin(x + y)$

28. $\cos 2y$

Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

29. $2\cos x - 1 = 0$

30. $2\sin^2 x = 1$

31. $\sin 2x = \cos x$

32. $|\tan x| = 1$

Find all values of x in the interval $[0, 2\pi]$ that satisfy the inequality.

33. $\sin x \leq \dfrac{1}{2}$

34. $2\cos x + 1 > 0$

35. $-1 < \tan x < 1$

36. $\sin x > \cos x$

Graph the function by starting with the graphs in Figures 13 and 14 and applying the transformations of Section 1.3 where appropriate.

37. $y = \cos\left(x - \dfrac{\pi}{3}\right)$

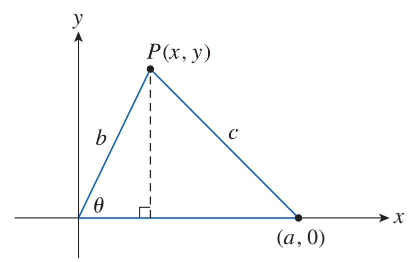
38. $y = \tan 2x$

39. $y = \dfrac{1}{3}\tan\left(x - \dfrac{\pi}{2}\right)$

40. $y = |\sin x|$

41. Prove the **Law of Cosines**: If a triangle has sides with lengths a, b, and c, and θ is the angle between the sides with lengths a and b, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

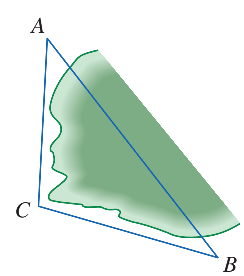


Hint: Introduce a coordinate system so that θ is in standard position, as in the figure. Express x and y in terms of θ and then use the distance formula to compute c.

42. In order to find the distance $|AB|$ across a small inlet, a point C was located as in the figure and the following measurements were recorded:

$$\angle C = 103° \quad |AC| = 820 \text{ m} \quad |BC| = 910 \text{ m}$$

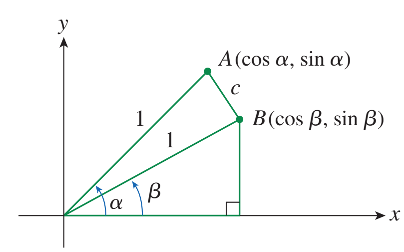
Use the Law of Cosines from Exercise 41 to find the required distance.



43. Use the figure to prove the subtraction formula

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Hint: Compute c^2 in two ways (using the Law of Cosines from Exercise 41 and also using the distance formula) and compare the two expressions.



44. Use the formula in Exercise 43 to prove the addition formula for cosine (Equation 12b).

45. Use the addition formula for cosine and the identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \quad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

to prove the subtraction formula (13a) for the sine function.

46. (a) Show that the area of a triangle with sides of lengths a and b and with included angle θ is

$$A = \frac{1}{2}ab\sin\theta$$

(b) Find the area of triangle ABC, correct to five decimal places, if

$$|AB| = 10 \text{ cm} \quad |BC| = 3 \text{ cm} \quad \angle ABC = 107°$$

D Precise Definitions of Limits

The definitions of limits that have been given in this book are appropriate for intuitive understanding of the basic concepts of calculus. For the purposes of deeper understanding and rigorous proofs, however, the precise definitions included in this appendix are necessary. In particular, the definition of a limit given here is used in Appendix E to prove that the limit of a sum is the sum of the limits.

When we say that $f(x)$ has a limit L as x approaches a, we mean, according to the intuitive definition in Section 2.2, that we can make $f(x)$ arbitrarily close to L by taking x close enough to a (but not equal to a). A more precise definition is based on the idea of specifying just how small we need to make the distance $|x - a|$ in order to make the distance $|f(x) - L|$ less than some given number. The following example illustrates the idea.

It is traditional to use the Greek letter δ (delta) in this situation.

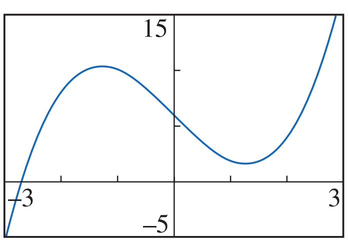


Figure 1
Graph of $y = x^3 - 5x + 6$.

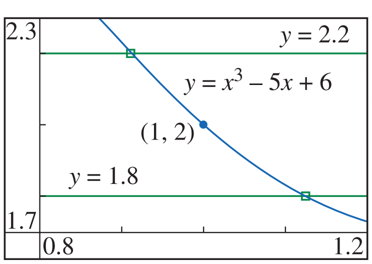


Figure 2
Graph of $y = x^3 - 5x + 6$, $y = 1.8$, and $y = 2.2$.

Example 1

Use a graph to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

Solution

A graph of $f(x) = x^3 - 5x + 6$ is shown in Figure 1; we are interested in the region near the point (1, 2). Notice that we can rewrite the inequality

$$|(x^3 - 5x + 6) - 2| < 0.2$$

as $1.8 < x^3 - 5x + 6 < 2.2$

So we need to determine the values of x for which the curve $y = x^3 - 5x + 6$ lies between the horizontal lines $y = 1.8$ and $y = 2.2$. Therefore, we graph the curves $y = x^3 - 5x + 6$, $y = 1.8$, and $y = 2.2$ near the point (1, 2) in Figure 2. Then we use the cursor, trace function, or built-in intersection function to estimate that the x-coordinate of the point of intersection of the line $y = 2.2$ and the curve $y = x^3 - 5x + 6$ is about 0.911. Similarly, $y = x^3 - 5x + 6$ intersects the line $y = 1.8$ when $x \approx 1.124$. So, rounding to be safe, we can say that

$$\text{if} \quad 0.92 < x < 1.12 \quad \text{then} \quad 1.8 < x^3 - 5x + 6 < 2.2$$

This interval (0.92, 1.12) is not symmetric about $x = 1$. The distance from $x = 1$ to the left endpoint is $1 - 0.92 = 0.08$, and the distance to the right endpoint is 0.12. We can

choose δ to be the smaller of these numbers, that is, $\delta = 0.08$. Then we can rewrite our inequalities in terms of distances as follows:

$$\text{if} \quad |x - 1| < 0.08 \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

This just says that by keeping x within 0.08 of 1, we are able to keep $f(x)$ within 0.2 of 2.

Although we chose $\delta = 0.08$, any smaller positive value of δ would also have worked. ■

Using the same graphical procedure as in Example 1, but replacing the number 0.2 by smaller numbers, we find that

$$\text{if} \quad |x - 1| < 0.046 \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.1$$
$$\text{if} \quad |x - 1| < 0.024 \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.05$$
$$\text{if} \quad |x - 1| < 0.004 \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.01$$

In each case we have found a number δ such that the values of the function $f(x) = x^3 - 5x + 6$ lie in successively smaller intervals centered at 2 if the distance from x to 1 is less than δ. It turns out that it is always possible to find such a number δ, no matter how small the interval is. In other words, for *any* positive number ε, no matter how small, there exists a positive number δ such that

$$\text{if} \quad |x - 1| < \delta \qquad \text{then} \qquad |(x^3 - 5x + 6) - 2| < \varepsilon$$

This indicates that

$$\lim_{x \to 1} (x^3 - 5x + 6) = 2$$

and suggests a more precise way of defining the limit of a general function.

Definition • Limit of a Function

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

The condition $0 < |x - a|$ is just another way of saying that $x \neq a$.

This definition is illustrated in Figures 3–5. If a number $\varepsilon > 0$ is given, then we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of f. (See Figure 3.) If $\lim_{x \to a} f(x) = L$, then we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ and take $x \neq a$, then the curve $y = f(x)$ lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$. (See Figure 4.) You can see that if such a δ has been found, then any smaller δ will also work.

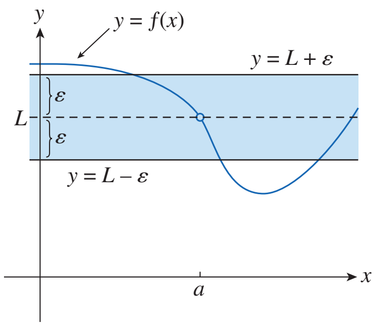


Figure 3
For any $\varepsilon > 0$, draw the lines $y = L - \varepsilon$ and $y = L + \varepsilon$.

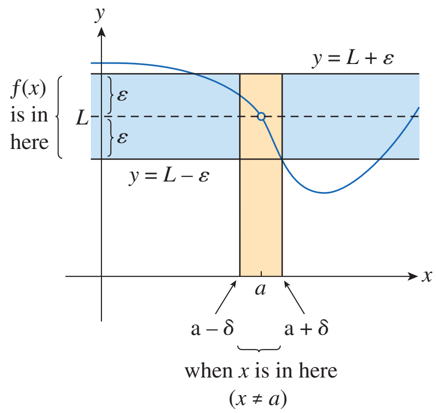


Figure 4
If x is in the interval $(a - \delta,\ a + \delta)$, then $f(x)$ is in $(L - \varepsilon,\ L + \varepsilon)$.

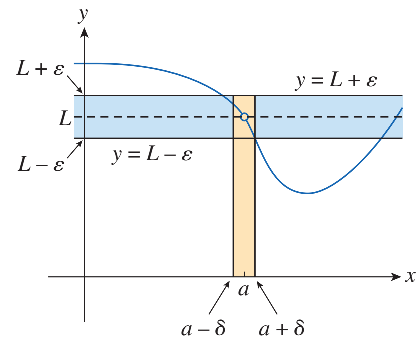


Figure 5
If we choose a smaller ε, a smaller δ may be required.

It's important to realize that the process illustrated in Figures 3 and 4 must work for *every* positive number ε no matter how small it is chosen. Figure 5 shows that if a smaller ε is chosen, then a smaller δ may be required.

Example 2

Use the ε, δ definition to prove that $\lim_{x \to 0} x^2 = 0$.

Solution

Let ε be a given positive number. According to Definition 1, with $a = 0$ and $L = 0$, we need to find a number δ such that

$$\text{if} \quad 0 < |x - 0| < \delta \qquad \text{then} \qquad |x^2 - 0| < \varepsilon$$

$$\text{that is, if} \quad 0 < |x| < \delta \qquad \text{then} \qquad x^2 < \varepsilon$$

But, since the square root function is an increasing function, we know that

$$x^2 < \varepsilon \quad \Leftrightarrow \quad \sqrt{x^2} < \sqrt{\varepsilon} \quad \Leftrightarrow \quad |x| < \sqrt{\varepsilon}$$

So if we choose $\delta = \sqrt{\varepsilon}$, then $x^2 < \varepsilon \quad \Leftrightarrow \quad |x| < \delta$. (See Figure 6.) This shows that $\lim_{x \to 0} x^2 = 0$. ■

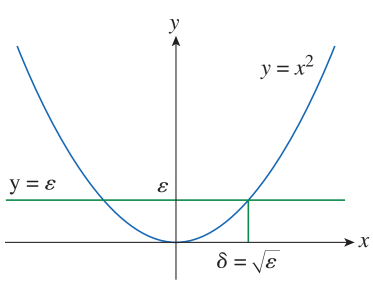


Figure 6
Visualization that shows that for any $\varepsilon > 0$, choose $\delta = \sqrt{\varepsilon}$.

In proving limit statements, it may be helpful to think of the definition of a limit as a challenge. First it challenges you with a number ε. Then you must be able to produce a suitable δ. You have to be able to do this for *every* $\varepsilon > 0$, not just a particular ε.

Imagine a contest between two people, A and B, and imagine yourself to be B. Person A stipulates that the fixed number L should be approximated by the values of $f(x)$ to within a degree of accuracy ε (say, 0.01). Person B then responds by finding a number δ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$. Then A may become more exacting and challenge B with a smaller value of ε (say, 0.0001). Again B has to respond by finding a

corresponding δ. Usually the smaller the value of ε, the smaller the corresponding value of δ must be. If B always wins, no matter how small A makes ε, then $\lim_{x \to a} f(x) = L$.

Example 3

Prove that $\lim_{x \to 3}(4x - 5) = 7$.

Solution

1. *Preliminary analysis of the problem* (*guessing a value for* δ). Let ε positive number. We want to find a number δ such that

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |(4x - 5) - 7| < \varepsilon$$

But $|(4x - 5) - 7| = |4x - 12| = |4(x - 3)| = 4|x - 3|$. Therefore, we want δ such that

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad 4|x - 3| < \varepsilon$$

$$\text{that is, if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |x - 3| < \frac{\varepsilon}{4}$$

This suggests that we should choose $\delta = \varepsilon/4$.

2. *Proof* (*showing that this* δ *works*). Given $\varepsilon > 0$, choose $\delta = \varepsilon/4$. If $0 < |x - 3| < \delta$, then

$$|(4x - 5) - 7| = |4x - 12| = 4|x - 3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

$$\text{Thus if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit,

$$\lim_{x \to 3}(4x - 5) = 7$$

This example is illustrated by Figure 7. ■

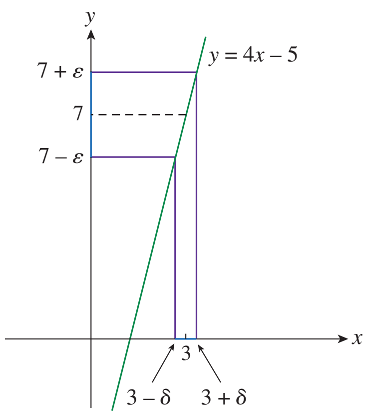


Figure 7
Visualization of the proof that $\lim_{x \to 3}(4x - 5) = 7$.

Note that in the solution of Example 3 there were two stages—guessing and proving. We made a preliminary analysis that enabled us to guess a value for δ. But then in the second stage, we had to go back and prove in a careful, logical fashion that we had made a correct guess. This procedure is typical of much of mathematics. Sometimes it is necessary to first make an intelligent guess about the answer to a problem and then prove that the guess is correct.

It's not always easy to prove that limit statements are true using the ε, δ definition. For a more complicated function such as $f(x) = (6x^2 - 8x + 9)/(2x^2 - 1)$, a proof would require a great deal of ingenuity. Fortunately, this is not necessary because the Limit Laws stated in Section 2.3 can be proved using the previous definition, and then the limits of complicated functions can be found rigorously from the Limit Laws without resorting to the definition directly.

■ Limits at Infinity

Infinite limits and limits at infinity can also be defined in a precise way. The following is a precise version of Definition 4 in Section 2.5.

Definition • Limit at Infinity

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$, there is a corresponding number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε, where ε is any positive number) by taking x sufficiently large (larger than N, where N depends on ε). Graphically it says that by choosing x large enough (larger than some number N), we can make the graph of f lie between the given horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ as in Figure 8. This must be true no matter how small we choose ε. If a smaller value of ε is chosen, then a larger value of N may be required.

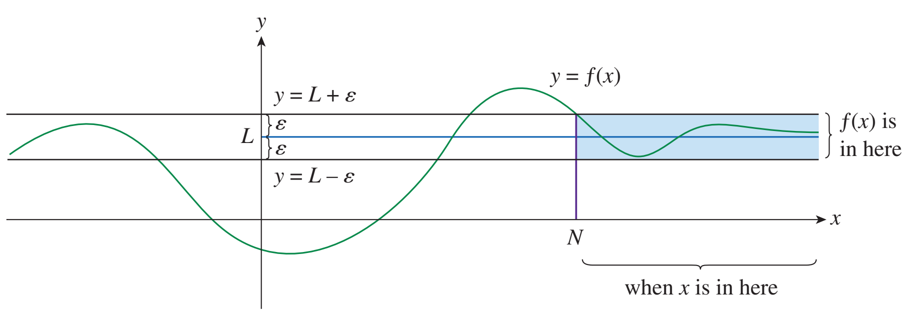


Figure 8
Illustration of a limit at infinity: $\lim_{x \to \infty} f(x) = L$.

In Example 5 in Section 2.5, we evaluated the limit

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

In the next example, we use technology to relate this statement to Definition 2 with $L = \frac{3}{5}$ and $\varepsilon = 0.1$.

Example 4

Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1$$

Solution

We rewrite the given inequality as

$$0.5 < \frac{3x^2 - x - 2}{5x^2 + 4x + 1} < 0.7$$

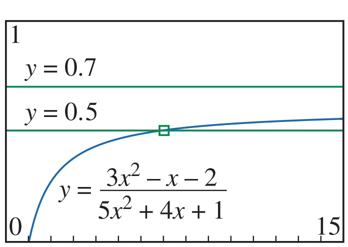


Figure 9
Estimate the values of x at which the curve intersects the two horizontal lines.

We need to determine the values of x for which the given curve lies between the horizontal lines $y = 0.5$ and $y = 0.7$. So we graph the curve and these lines in Figure 9. Then we use the cursor, trace function, or built-in intersection function to estimate that the curve crosses the line $y = 0.5$ when $x \approx 6.7$. To the right of this number, it seems

that the curve stays between the lines $y = 0.5$ and $y = 0.7$. Rounding to be safe, we can say that

$$\text{if} \quad x > 7 \quad \text{then} \quad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1$$

In other words, for $\varepsilon = 0.1$ we can choose $N = 7$ (or any larger number) in Definition 2. ■

Example 5

Use Definition 2 to prove that $\displaystyle\lim_{x \to \infty} \frac{1}{x} = 0$.

Solution

Given $\varepsilon > 0$, we want to find N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| < \varepsilon$$

In computing the limit, we may assume that $x > 0$. Then $1/x < \varepsilon \quad \Leftrightarrow \quad x > 1/\varepsilon$. Let's choose $N = 1/\varepsilon$. So

$$\text{if} \quad x > N = \frac{1}{\varepsilon} \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| = \frac{1}{x} < \varepsilon$$

Therefore, by Definition 2,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Figure 10 illustrates the proof by showing some values of ε and the corresponding values of N.

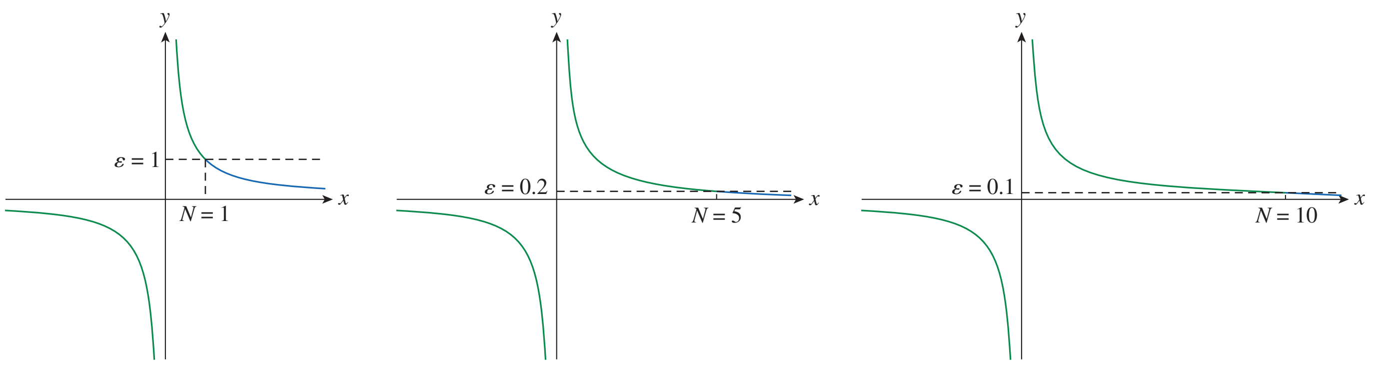


Figure 10
Illustration that shows the value of N that corresponds to specific values of ε. ■

Infinite limits can also be formulated precisely. See Exercise 20.

Definite Integrals

In Section 5.2 we defined the definite integral of a function f on an interval $[a, b]$ as

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

where, for any value of n, divide $[a, b]$ into n subintervals of equal width, $\Delta x = (b - a)/n$, and x_i^* is any sample point in the ith subinterval. The precise meaning of this limit that defines the integral is as follows:

For every number $\varepsilon > 0$, there is an integer N such that

$$\left| \int_a^b f(x)\, dx - \sum_{i=1}^{n} f(x_i^*)\ \Delta x \right| < \varepsilon$$

for every integer $n > N$ and for every choice of x_i^* in the ith subinterval.

This means that a definite integral can be approximated to within any desired degree of accuracy by a Riemann sum.

Sequences

In Section 8.1 we used the notation

$$\lim_{n \to \infty} a_n = L$$

to mean that the terms of the sequence $\{a_n\}$ approach L as n becomes large. Notice that the following precise definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity.

Definition • Limit of a sequence

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$, there is a corresponding integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad |a_n - L| < \varepsilon$$

This definition is illustrated by Figure 11, in which the terms $a_1, a_2, a_3, \ldots$ are plotted on a number line. No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.

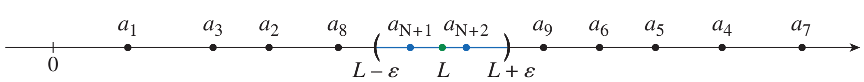


Figure 11
A number line illustration of a convergent sequence.

Another illustration of this definition is given in Figure 12. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.

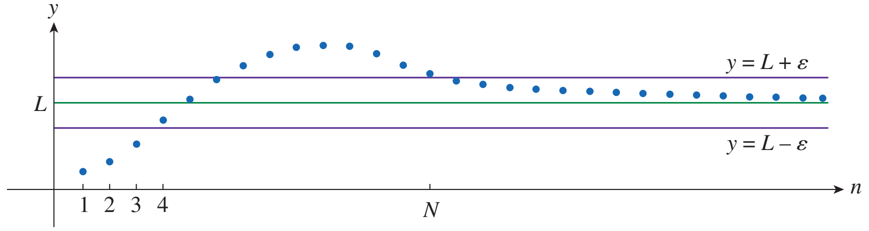


Figure 12
A graphical illustration of a convergent sequence.

If you compare the two previous definitions, you will see that the only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is required to be an integer. The following definition shows how to make precise the idea that $\{a_n\}$ becomes infinite as n becomes infinite.

Definition • A Sequence That increases Without Bound

The notation $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M, there is an integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad a_n > M$$

Example 6

Prove that $\lim_{n\to\infty} \sqrt{n} = \infty$.

Solution

Let M be any positive number. (Think of it as being very large.) Then

$$\sqrt{n} > M \quad \Leftrightarrow \quad n > M^2$$

So if we take $N = M^2$, then Definition 4 shows that $\lim_{n\to\infty} \sqrt{n} = \infty$. ■

Functions of Two Variables

Here is a precise version of the definition of a limit in Section 11.2:

Definition • Limit of a function of Two Variables

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y)\to(a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Notice that $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L, and $\sqrt{(x-a)^2 + (y-b)^2}$ is the distance between the point (x, y) and the point (a, b). Thus Definition 5 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). An illustration of Definition 5 is given in Figure 13 where the surface S is the graph of f. If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x, y) is restricted to lie in the disk D_δ with center (a, b) and radius δ, and if $(x, y) \neq (a, b)$, then the corresponding part of S lies between the horizontal planes $z = L - \varepsilon$ and $z = L + \varepsilon$.

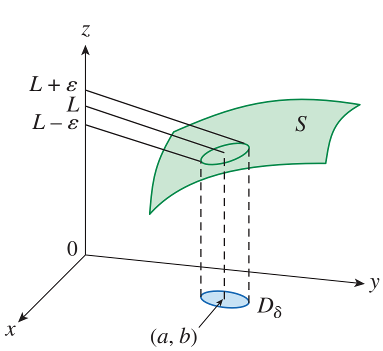


Figure 13
A graphical illustration of $\lim_{(x, y)\to(a, b)} f(x, y) = L$.

Example 7

Prove that $\lim_{(x, y)\to(0, 0)} \dfrac{3x^2y}{x^2 + y^2} = 0$.

Solution

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \varepsilon$$

that is, if $\quad 0 < \sqrt{x^2 + y^2} < \delta \quad$ then $\quad \dfrac{3x^2|y|}{x^2 + y^2} < \varepsilon$

But $x^2 \leq x^2 + y^2$ since $y^2 \geq 0$, so $\dfrac{x^2}{(x^2 + y^2)} \leq 1$ and therefore

$$\frac{3x^2|y|}{x^2 + y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2 + y^2}$$

Thus if we choose $\delta = \dfrac{\varepsilon}{3}$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\left|\frac{3x^2y}{x^2 + y^2} - 0\right| \leq 3\sqrt{x^2 + y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

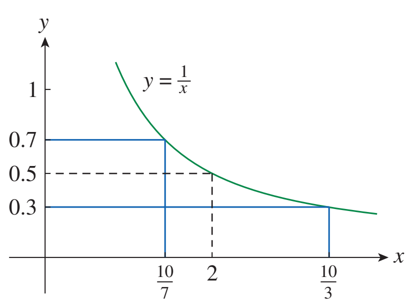
Hence, by the previous definition,

$$\lim_{(x,\,y)\to(0,\,0)} \frac{3x^2y}{x^2 + y^2} = 0$$

D Exercises

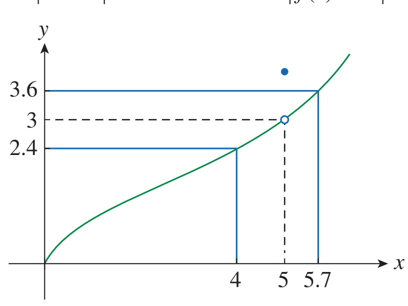
1. Use the given graph of $f(x) = 1/x$ to find a number δ such that

if $\quad |x - 2| < \delta \quad$ then $\quad \left|\dfrac{1}{x} - 0.5\right| < 0.2$



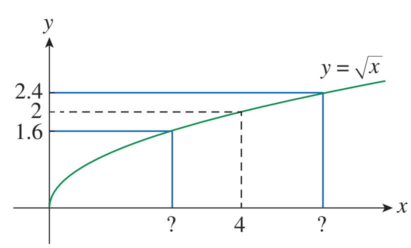
2. Use the given graph of f to find a number δ such that

if $\quad 0 < |x - 5| < \delta \quad$ then $\quad |f(x) - 3| < 0.6$



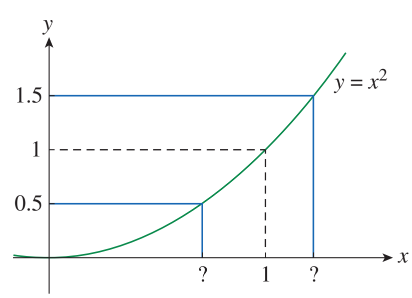
3. Use the given graph of $f(x) = \sqrt{x}$ to find a number δ such that

if $\quad |x - 4| < \delta \quad$ then $\quad |\sqrt{x} - 2| < 0.4$



4. Use the given graph of $f(x) = x^2$ to find a number δ such that

if $\quad |x - 1| < \delta \quad$ then $\quad |x^2 - 1| < \dfrac{1}{2}$



5. Use a graph to find a number δ such that

if $\quad \left|x - \dfrac{\pi}{4}\right| < \delta \quad$ then $\quad |\tan x - 1| < 0.2$

6. Use a graph to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad \left| \frac{2x}{x^2 + 4} - 0.4 \right| < 0.1$$

7. For the limit

$$\lim_{x \to 1} (4 + x - 3x^3) = 2$$

illustrate Definition 1 by finding values of δ that correspond to $\varepsilon = 1$ and $\varepsilon = 0.1$.

8. For the limit

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

illustrate Definition 1 by finding values of δ that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

9. Use Definition 1 to prove that $\lim_{x \to 0} x^3 = 0$.

10. (a) How would you formulate an ε, δ definition of the one-sided $\lim_{x \to a^+} f(x) = L$?

(b) Use your definition in part (a) to prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

11. A machinist is required to manufacture a circular metal disk with area 1000 cm^2.

(a) What radius produces such a disk?

(b) If the machinist is allowed an error tolerance of $\pm 5 \text{ cm}^2$ in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?

(c) In terms of the ε, δ definition of $\lim_{x \to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

12. A crystal growth furnace is used in research to determine how best to manufacture crystals used in electronic components for the space shuttle. For proper growth of the crystal, the temperature must be controlled accurately by adjusting the input power. Suppose the relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts.

(a) How much power is needed to maintain the temperature at 200°C?

(b) If the temperature is allowed to vary from 200°C by up to ± 1°C, what range of wattage is allowed for the input power?

(c) In terms of the ε, δ definition of $\lim_{x \to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

13. (a) Find a number δ such that if $|x - 2| < \delta$, then $|4x - 8| < \varepsilon$, where $\varepsilon = 0.1$.

(b) Repeat part (a) with $\varepsilon = 0.01$.

14. Given that $\lim_{x \to 2} (5x - 7) = 3$, illustrate Definition 1 by finding values of δ that correspond to $\varepsilon = 0.1$, $\varepsilon = 0.05$, and $\varepsilon = 0.01$.

Prove the statement using the ε, δ definition of a limit and illustrate with a diagram like Figure 7.

15. $\lim_{x \to -3} (1 - 4x) = 13$

16. $\lim_{x \to -2} \left(\frac{1}{2}x + 3 \right) = 2$

---

17. Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{6x^2 + 5x - 3}{2x^2 - 1} - 3 \right| < 0.2$$

18. For the limit

$$\lim_{x \to \infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = 2$$

illustrate Definition 2 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

19. (a) Determine how large we have to take x so that

$$\frac{1}{x^2} < 0.0001$$

(b) Use Definition 2 to prove that

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

20. (a) For what values of x is it true that

$$\frac{1}{x^2} > 1{,}000{,}000$$

(b) The precise definition of $\lim_{x \to a} f(x) = \infty$ states that for every positive number M (no matter how large), there is a corresponding positive number δ such that if $0 < |x - a| < \delta$, then $f(x) > M$. Use this definition to prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

21. (a) Use a graph to guess the value of the limit

$$\lim_{n \to \infty} \frac{n^5}{n!}$$

(b) Use a graph of the sequence in part (a) to find the smallest values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.001$ in Definition 3.

22. Use Definition 3 to prove that $\lim_{n \to \infty} r^n = 0$ when $|r| < 1$.

23. Use Definition 3 to prove that if $\lim_{n \to \infty} |a_n| = 0$, then $\lim_{n \to \infty} a_n = 0$.

24. Use Definition 4 to prove that $\lim_{n \to \infty} n^3 = \infty$.

25. Use Definition 5 to prove that $\lim_{(x, y) \to (0, 0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0$.

E A Few Proofs

In this appendix we present proofs of some theorems that were stated in the main body of the text. We start by proving the Triangle Inequality, which is an important property of absolute value.

The Triangle Inequality

If a and b are any real numbers, then

$$|a + b| \leq |a| + |b|$$

Observe that if the numbers a and b are both positive or both negative, then the two sides in the Triangle Inequality are actually equal. But if a and b have opposite signs, the left side involves a subtraction and the right side does not. This makes the Triangle Inequality seem reasonable, but we can prove it as follows.

Notice that

$$-|a| \leq a \leq |a|$$

is always true because a equals either $|a|$ or $-|a|$. The corresponding statement for b is

$$-|b| \leq b \leq |b|$$

Adding these inequalities, we get

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

When combined, Properties 4 and 5 of absolute value (see Appendix A) say that

$$|x| \leq a \quad \Leftrightarrow \quad -a \leq x \leq a$$

If we now apply Properties 4 and 5 of absolute value from Appendix A (with x replaced by $a + b$ and a by $|a| + |b|$), we obtain

$$|a + b| \leq |a| + |b|$$

which is what we wanted to show.

Next we use the Triangle Inequality to prove the Sum Law for limits.

The Sum Law was first stated in Section 2.3.

Sum Law

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ both exist, then

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

Proof

Let $\varepsilon > 0$ be given. According to Definition 1 in Appendix D, we must find $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) + g(x) - (L + M)| < \varepsilon$$

Using the Triangle Inequality, we can write

$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \end{aligned} \tag{1}$$

We will make $|f(x) + g(x) - (L + M)|$ less than ε by making each of the terms $|f(x) - L|$ and $|g(x) - M|$ less than $\varepsilon/2$.

Since $\varepsilon/2 > 0$ and $\lim_{x \to a} f(x) = L$, there exists a number $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, since $\lim_{x \to a} g(x) = M$, there exists a number $\delta_2 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_2 \quad \text{then} \quad |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of the numbers δ_1 and δ_2. Notice that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad 0 < |x - a| < \delta_1 \quad \text{and} \quad 0 < |x - a| < \delta_2$$

$$\text{and so} \quad |f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - M| < \frac{\varepsilon}{2}$$

Therefore, by (1),

$$\begin{aligned} |f(x) + g(x) - (L + M)| &\le |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

To summarize,

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) + g(x) - (L + M)| < \varepsilon$$

Thus, by the definition of a limit,

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$ ■

Fermat's Theorem was discussed in Section 4.2.

Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof

Suppose, for the sake of definiteness, that f has a local maximum at c. Then, $f(c) \ge f(x)$ if x is sufficiently close to c. This implies that if h is sufficiently close to 0, with h being positive or negative, then

$$f(c) \ge f(c + h)$$

and therefore

$$f(c + h) - f(c) \le 0 \tag{2}$$

We can divide both sides of an inequality by a positive number. Thus, if $h > 0$ and h is sufficiently small, we have

$$\frac{f(c + h) - f(c)}{h} \le 0$$

Taking the right-hand limit of both sides of this inequality (using Theorem 2.3.2), we get

$$\lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h} \le \lim_{h \to 0^+} 0 = 0$$

But since $f'(c)$ exists, we have

$$f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$$

and so we have shown that $f'(c) \le 0$.

If $h < 0$, then the direction of the inequality (2) is reversed when we divide by h:

$$\frac{f(c+h)-f(c)}{h} \ge 0 \qquad h < 0$$

So, taking the left-hand limit, we have

$$f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h} \ge 0$$

We have shown that $f'(c) \ge 0$ and also that $f'(c) \le 0$. Since both of these inequalities must be true, the only possibility is that $f'(c) = 0$.

We have proved Fermat's Theorem for the case of a local maximum. The case of a local minimum can be proved in a similar manner. ■

This theorem was stated and used in Section 8.1.

Theorem • Limit of a continuous Function Applied to a Sequence

If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Proof

We must show that, given a number $\varepsilon > 0$, there is an integer N such that if $n > N$, then $|f(a_n) - f(L)| < \varepsilon$.

Suppose $\varepsilon > 0$. Since f is continuous at L, there is a number $\delta > 0$ such that if $|x - L| < \delta$, then $|f(x) - f(L)| < \varepsilon$. Because $\lim_{n\to\infty} a_n = L$, there is an integer N such that if $n > N$, then $|a_n - L| < \delta$. Suppose $n > N$. Then $|a_n - L| < \delta$ and so $|f(a_n) - f(L)| < \varepsilon$.

This shows that $\lim_{n\to\infty} f(a_n) = f(L)$. ■

Clairaut's Theorem was discussed in Section 11.3.

Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Proof

For small values of h, $h \ne 0$, consider the difference

$$\Delta(h) = [f(a+h, b+h) - f(a+h, b)] - [f(a, b+h) - f(a, b)]$$

Notice that if we let $g(x) = f(x, b+h) - f(x, b)$, then

$$\Delta(h) = g(a+h) - g(a)$$

By the Mean Value Theorem, there is a number c between a and $a + h$ such that

$$g(a + h) - g(a) = g'(c)h = h[f_x(c, b + h) - f_x(c, b)]$$

Applying the Mean Value Theorem again, this time to f_x, we get a number d between b and $b + h$ such that

$$f_x(c, b + h) - f_x(c, b) = f_{xy}(c, d)h$$

Combining these equations, we obtain

$$\Delta(h) = h^2 f_{xy}(c, d)$$

If $h \to 0$, then $(c, d) \to (a, b)$, so the continuity of f_{xy} at (a, b) gives

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = \lim_{(c, d)\to(a, b)} f_{xy}(c, d) = f_{xy}(a, b)$$

Similarly, by writing

$$\Delta(h) = [f(a + h, b + h) - f(a, b + h)] - [f(a + h, b) - f(a, b)]$$

and using the Mean Value Theorem twice and the continuity of f_{yx} at (a, b), we obtain

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a, b)$$

It follows that $f_{xy}(a, b) = f_{yx}(a, b)$. ■

This was stated as Theorem 8 in Section 11.4.

Theorem • Partial Derivatives and Differentiability at a Point

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Proof

Let

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

According to Definition 11.4.7, to prove that f is differentiable at (a, b), we have to show that we can write Δz in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

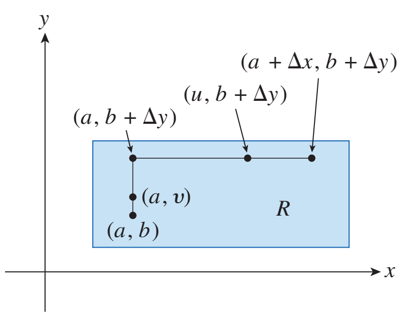


Figure 1
A graphical illustration to help write an expression for Δz.

Referring to Figure 1, we write

$$\Delta z = [f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y)] + [f(a, b + \Delta y) - f(a, b)] \tag{3}$$

Observe that the function of a single variable

$$g(x) = f(x, b + \Delta y)$$

is defined on the interval $[a, a + \Delta x]$ and $g'(x) = f_x(x, b + \Delta y)$. If we apply the Mean Value Theorem to g, we get

$$g(a + \Delta x) - g(a) = g'(u)\Delta x$$

where u is some number between a and $a + \Delta x$. In terms of f, this equation becomes

$$f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y) = f_x(u, b + \Delta y)\Delta x$$

This gives us an expression for the first part of the right side of Equation 3. For the second part, we let $h(y) = f(a, y)$. Then h is a function of a single variable defined on the interval $[b, b + \Delta y]$ and $h'(y) = f_y(a, y)$. A second application of the Mean Value Theorem then gives

$$h(b + \Delta y) - h(b) = h'(v)\Delta y$$

where v is some number between b and $b + \Delta y$. In terms of f, this becomes

$$f(a, b + \Delta y) - f(a, b) = f_y(a, v)\Delta y$$

We now substitute these expressions into Equation 3 and obtain

$$\begin{aligned}\Delta z &= f_x(u, b + \Delta y)\Delta x + f_y(a, v)\Delta y \\ &= f_x(a, b)\Delta x + [f_x(u, b + \Delta y) - f_x(a, b)]\Delta x + f_y(a, b)\Delta y \\ &\qquad + [f_y(a, v) - f_y(a, b)]\Delta y \\ &= f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\,\Delta y\end{aligned}$$

$$\begin{aligned}\text{where } \varepsilon_1 &= f_x(u, b + \Delta y) - f_x(a, b) \\ \varepsilon_2 &= f_y(a, v) - f_y(a, b)\end{aligned}$$

Since $(u, b + \Delta y) \to (a, b)$ and $(a, v) \to (a, b)$ as $(\Delta x, \Delta y) \to (0, 0)$ and since f_x and f_y are continuous at (a, b), we see that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Therefore, f is differentiable at (a, b). ■

The Second Derivatives Test was discussed in Section 11.7. Parts (b) and (c) have similar proofs.

Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Proof of Part (a)

We compute the second-order directional derivative of f in the direction of $\mathbf{u} = \langle h, k \rangle$. The first-order derivative is given by Theorem 11.6.3:

$$D_{\mathrm{u}} f = f_x h + f_y k$$

Applying this theorem a second time, we have

$$\begin{aligned} D_{\mathrm{u}}^2 f = D_{\mathrm{u}}(D_{\mathrm{u}} f) &= \frac{\partial}{\partial x}(D_{\mathrm{u}} f)h + \frac{\partial}{\partial y}(D_{\mathrm{u}} f)k \\ &= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy}k)k \\ &= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \qquad \text{(by Clairaut's Theorem)} \end{aligned}$$

If we complete the square in this expression, we obtain

$$D_{\mathrm{u}}^2 f = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}(f_{xx}f_{yy} - f_{xy}^2) \tag{4}$$

We are given that $f_{xx}(a, b) > 0$ and $D(a, b) > 0$. But f_{xx} and $D = f_{xx}f_{yy} - f_{xy}^2$ are continuous functions, so there is a disk B with center (a, b) and radius $\delta > 0$ such $f_{xx}(x, y) > 0$ and $D(x, y) > 0$ whenever (x, y) is in B. Therefore, by looking at Equation 4, we see that $D_{\mathbf{u}}^2 f(x, y) > 0$ whenever (x, y) is in B. This means that if C is curve obtained by intersecting the graph of f with the vertical plane through $P(a, b, f(a, b))$ in the direction of $\mathbf{u}$, then C is concave upward on an interval of length 2δ. This is true in the direction of every vector $\mathbf{u}$, so if we restrict (x, y) to lie in B, the graph of f lies above its horizontal tangent plane at P. Thus $f(x, y) \geq f(a, b)$ whenever (x, y) is in B. This shows that $f(a, b)$ is a local minimum. ■

F Sigma Notation

A convenient way of writing sums uses the Greek letter Σ (capital sigma, corresponding to our letter S) and is called **sigma notation**.

This tells us to end with $i = n$.

This tells us to add. $\displaystyle\sum_{i=m}^{n} a_i$

This tells us to start with $i = m$.

Definition • Sigma Notation

If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leq n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

Using function notation, Definition 1 can be written as

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

Thus the symbol $\displaystyle\sum_{i=m}^{n}$ indicates a summation in which the letter i (called the **index of summation**) takes on consecutive integer values beginning with m and ending with n, that is, $m, m+1, \ldots, n$. Other letters can also be used as the index of summation.

Example 1

(a) $\displaystyle\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

(b) $\displaystyle\sum_{i=3}^{n} i = 3 + 4 + 5 + \cdots + (n-1) + n$

(c) $\displaystyle\sum_{j=0}^{5} 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

(d) $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

(e) $\displaystyle\sum_{i=1}^{3} \frac{i-1}{i^2+3} = \frac{1-1}{1^2+3} + \frac{2-1}{2^2+3} + \frac{3-1}{3^2+3} = 0 + \frac{1}{7} + \frac{1}{6} = \frac{13}{42}$

(f) $\displaystyle\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 8$ ■

Example 2

Write the sum $2^3 + 3^3 + \cdots + n^3$ in sigma notation.

Solution

There is no unique way of writing a sum in sigma notation. We could write

$$2^3 + 3^3 + \cdots + n^3 = \sum_{i=2}^{n} i^3 \qquad \text{or}$$

$$2^3 + 3^3 + \cdots + n^3 = \sum_{j=1}^{n-1} (j+1)^3 \qquad \text{or}$$

$$2^3 + 3^3 + \cdots + n^3 = \sum_{k=0}^{n-2} (k+2)^3$$ ■

The following theorem gives three simple rules for working with sigma notation.

Theorem • Properties Involving Summations

If c is any constant (that is, it does not depend on i), then

(a) $\displaystyle\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$

(b) $\displaystyle\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

(c) $\displaystyle\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$

Proof

To see why these rules are true, all we have to do is write both sides in expanded form. Rule (a) is just the distributive property of real numbers:

$$ca_m + ca_{m+1} + \cdots + ca_n = c(a_m + a_{m+1} + \cdots + a_n)$$

Rule (b) follows from the associative and commutative properties:

$$(a_m + b_m) + (a_{m+1} + b_{m+1}) + \cdots + (a_n + b_n)$$

$$= (a_m + a_{m+1} + \cdots + a_n) + (b_m + b_{m+1} + \cdots + b_n)$$

Rule (c) is proved similarly. ■

Example 3

Find $\sum_{i=1}^{n} 1$.

Solution

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$

Example 4

Prove the formula for the sum of the first n positive integers:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Solution

This formula can be proved by mathematical induction (see page 84) or by the following method used by the German mathematician Carl Friedrich Gauss (1777–1855) when he was ten years old.

Write the sum S twice, once in the usual order and once in reverse order:

$$S = 1 + 2 + 3 + \cdots + (n-1) + n$$

$$S = n + (n-1) + (n-2) + \cdots + 2 + 1$$

Adding all columns vertically, we get

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

On the right side, there are n terms, each of which is $n + 1$, so

$$2S = n(n+1) \quad \text{or} \quad S = \frac{n(n+1)}{2}$$

Example 5

Prove the formula for the sum of the squares of the first n positive integers:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution 1

Let S be the desired sum. We start with the *telescoping sum* (or collapsing sum):

Most terms cancel in pairs.

$$\sum_{i=1}^{n} [(1+i)^3 - i^3] = (\cancel{2^3} - 1^3) + (\cancel{3^3} - \cancel{2^3}) + (\cancel{4^3} - \cancel{3^3}) + \cdots + [(n+1)^3 - \cancel{n^3}]$$

$$= (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n$$

On the other hand, using the properties of summations and Examples 3 and 4, we have

$$\sum_{i=1}^{n} [(1+i)^3 - i^3] = \sum_{i=1}^{n} [3i^2 + 3i + 1] = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$= 3S + 3\frac{n(n+1)}{2} + n = 3S + \frac{3}{2}n^2 + \frac{5}{2}n$$

Thus we have

$$n^3 + 3n^2 + 3n = 3S + \frac{3}{2}n^2 + \frac{5}{2}n$$

Solving this equation for S, we obtain

$$3S = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \qquad \text{or}$$

$$S = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

Solution 2

Principle of Mathematical Induction

Let S_n be a statement involving the positive integer n. Suppose that

1. S_1 is true.

2. If S_k is true, then S_{k+1} is true.

Then S_n is true for all positive integers n.

See pages 84 and 87 for a more thorough discussion of mathematical induction.

Let S_n be the given formula.

1. S_1 is true because $\qquad 1^2 = \dfrac{1(1+1)(2 \cdot 1 + 1)}{6}$

2. Assume that S_k is true; that is,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Then

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 &= (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1)\frac{k(2k+1) + 6(k+1)}{6} \\ &= (k+1)\frac{2k^2 + 7k + 6}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} \end{aligned}$$

So S_{k+1} is true.

By the Principle of Mathematical Induction, S_n is true for all n. ■

We list the results of Examples 3, 4, and 5 together with a similar result for cubes (see Exercises 37–40) as a theorem. These formulas are needed for finding areas and evaluating integrals in Chapter 5.

Theorem • Summation Formulas

Let c be a constant and n a positive integer. Then

(a) $\displaystyle\sum_{i=1}^{n} 1 = n$ $\qquad$ (b) $\displaystyle\sum_{i=1}^{n} c = nc$

(c) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\qquad$ (d) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(e) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example 6

Evaluate $\displaystyle\sum_{i=1}^{n} i(4i^2 - 3)$.

Solution

Using Theorems 2 and 3, we have

$$\begin{aligned}
\sum_{i=1}^{n} i(4i^2 - 3) &= \sum_{i=1}^{n} (4i^3 - 3i) = 4\sum_{i=1}^{n} i^3 - 3\sum_{i=1}^{n} i \\
&= 4\left[\frac{n(n+1)}{2}\right]^2 - 3\frac{n(n+1)}{2} \\
&= \frac{n(n+1)[2n(n+1) - 3]}{2} \\
&= \frac{n(n+1)(2n^2 + 2n - 3)}{2}
\end{aligned}$$

The type of calculation in Example 7 arises in Chapter 5 when we compute areas.

Example 7

Find $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right]$.

Solution

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right] &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{3}{n^3} i^2 + \frac{3}{n}\right] \\
&= \lim_{n\to\infty} \left[\frac{3}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1\right] \\
&= \lim_{n\to\infty} \left[\frac{3}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\cdot n\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{2}\cdot\frac{n}{n}\cdot\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + 3\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{2}\cdot 1\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 3\right] \\
&= \frac{1}{2}\cdot 1\cdot 1\cdot 2 + 3 = 4
\end{aligned}$$

F Exercises

Write the sum in expanded form.

1. $\displaystyle\sum_{i=1}^{5} \sqrt{i}$

2. $\displaystyle\sum_{i=1}^{6} \frac{1}{i+1}$

3. $\displaystyle\sum_{i=4}^{6} 3^i$

4. $\displaystyle\sum_{i=4}^{6} i^3$

5. $\displaystyle\sum_{k=0}^{4} \frac{2k-1}{2k+1}$

6. $\displaystyle\sum_{k=5}^{8} x^k$

7. $\displaystyle\sum_{i=1}^{n} i^{10}$

8. $\displaystyle\sum_{j=n}^{n+3} j^2$

9. $\displaystyle\sum_{j=0}^{n-1} (-1)^j$

10. $\displaystyle\sum_{i=1}^{n} f(x_i)\,\Delta x_i$

Write the sum in sigma notation.

11. $1 + 2 + 3 + 4 + \cdots + 10$

12. $\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7}$

13. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{19}{20}$

14. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27}$

15. $2 + 4 + 6 + 8 + \cdots + 2n$

16. $1 + 3 + 5 + 7 + \cdots + (2n - 1)$

17. $1 + 2 + 4 + 8 + 16 + 32$

18. $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$

19. $x + x^2 + x^3 + \cdots + x^n$

20. $1 - x + x^2 - x^3 + \cdots + (-1)^n x^n$

Find the value of the sum.

21. $\sum_{i=4}^{8} (3i - 2)$

22. $\sum_{i=3}^{6} i(i + 2)$

23. $\sum_{j=1}^{6} 3^{j+1}$

24. $\sum_{k=0}^{8} \cos k\pi$

25. $\sum_{n=1}^{20} (-1)^n$

26. $\sum_{i=1}^{100} 4$

27. $\sum_{i=0}^{4} (2^i + i^2)$

28. $\sum_{i=-2}^{4} 2^{3-i}$

29. $\sum_{i=1}^{n} 2i$

30. $\sum_{i=1}^{n} (2 - 5i)$

31. $\sum_{i=1}^{n} (i^2 + 3i + 4)$

32. $\sum_{i=1}^{n} (3 + 2i)^2$

33. $\sum_{i=1}^{n} (i + 1)(i + 2)$

34. $\sum_{i=1}^{n} i(i + 1)(i + 2)$

35. $\sum_{i=1}^{n} (i^3 - i - 2)$

---

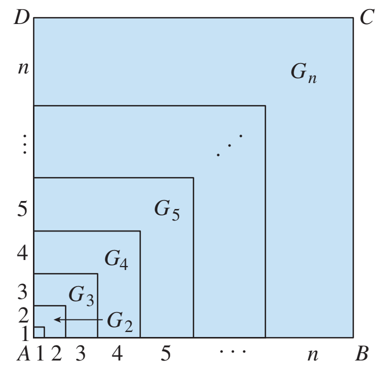
36. Find the number n such that $\sum_{i=1}^{n} i = 78$.

37. Prove summation formula (b).

38. Prove summation formula (e) using mathematical induction.

39. Prove summation formula (e) using a method similar to that of Example 5, Solution 1 [start with $(1 + i)^4 - i^4$].

40. Prove summation formula (e) using the following method published by Abu Bekr Mohammed ibn Alhusain Alkarchi in about ad 1010. The figure shows a square $ABCD$ in which sides AB and AD have been divided into segments of lengths $1, 2, 3, \ldots, n$. Thus the side of the square has length $\frac{n(n+1)}{2}$ so the area is $\left[\frac{n(n+1)}{2}\right]^2$. But the area is also the sum of the areas of the n "gnomons" $G_1, G_2, \ldots, G_n$ shown in the figure. Show that the area of G_i is i^3 and conclude that formula (e) is true.



41. Evaluate each telescoping sum.

(a) $\sum_{i=1}^{n} [i^4 - (i - 1)^4]$

(b) $\sum_{i=1}^{100} (5^i - 5^{i-1})$

(c) $\sum_{i=3}^{99} \left(\frac{1}{i} - \frac{1}{i+1}\right)$c

(d) $\sum_{i=1}^{n} (a_i - a_{i-1})$

42. Prove the generalized triangle inequality:

$$\left| \sum_{i=1}^{n} a_i \right| \leq \sum_{i=1}^{n} |a_i|$$

Find the limit.

43. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2$

44. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right]$

45. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^3 + 5\left(\frac{2i}{n}\right)\right]$

46. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(1 + \frac{3i}{n}\right)^3 - 2\left(1 + \frac{3i}{n}\right)\right]$

---

47. Prove the formula for the sum of a finite geometric series with first term a and common ratio $r \neq 1$:

$$\sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

48. Evaluate $\sum_{i=1}^{n} \frac{3}{2^{i-1}}$.

49. Evaluate $\sum_{i=1}^{n} (2i + 2^i)$.

50. Evaluate $\sum_{i=1}^{m} \left[\sum_{j=1}^{n} (i + j)\right]$.

G | Integration of Rational Functions by Partial Fractions

In this appendix we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions,* that we already know how to integrate. To illustrate the method, observe that by writing the sum of the fractions $\dfrac{2}{(x-1)}$ and $\dfrac{1}{(x+2)}$ with a common denominator, we obtain

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

If we now reverse the procedure, we see how to integrate the function on the right side of this equation:

$$\begin{aligned}\int \frac{x+5}{x^2+x-2}\,dx &= \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right)dx \\ &= 2\ln|x-1| - \ln|x+2| + C\end{aligned}$$

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. It's possible to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called *proper.* Recall that if

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, then the degree of P is n and we write $\deg(P) = n$.

If f is improper, that is, $\deg(P) \geq \deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \tag{1}$$

where S and R are also polynomials.

As the following example illustrates, sometimes this preliminary step is all that is required.

Example 1

Find $\displaystyle\int \frac{x^3+x}{x-1}\,dx.$

$$\begin{array}{r@{}l} & \underline{\;x^2 + x + 2\;} \\ x-1\,\big) & x^3 \qquad\quad + 3 \\ & \underline{x^3 - x^2} \\ & \qquad x^2 + x \\ & \qquad \underline{x^2 - x} \\ & \qquad\qquad 2x \\ & \qquad\qquad \underline{2x - 2} \\ & \qquad\qquad\qquad 2 \end{array}$$

Solution

Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division. This enables us to write

$$\begin{aligned}\int \frac{x^3+x}{x-1}\,dx &= \int \left(x^2 + x + 2 + \frac{2}{x-1}\right)dx \\ &= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C\end{aligned}$$

■

The next step is to factor the denominator $Q(x)$ as far as possible. It can be shown that any polynomial Q can be factored as a product of linear factors (of the form $ax + b$) and irreducible quadratic factors (of the form $ax^2 + bx + c$, where $b^2 - 4ax < 0$). For instance, if $Q(x) = x^4 - 16$, we could factor it as

$$Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$

The third step is to express the proper rational function $R(x)/Q(x)$ (from Equation 1) as a sum of **partial fractions** of the form

$$\frac{A}{(ax + b)^i} \quad \text{or} \quad \frac{Ax + B}{(ax^2 + bx + c)^j}$$

A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

Case I The denominator $Q(x)$ is a product of distinct linear factors.

This means that we can write

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_k x + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_k}{a_k x + b_k} \tag{2}$$

These constants can be determined as in the following example.

Example 2

Evaluate $\displaystyle\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx.$

Solution

Since the degree of the numerator is less than the degree of the denominator, we don't need to divide. We factor the denominator as

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand (2) has the form

$$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2} \tag{3}$$

Another method for finding A, B, and C is given in the note after this example.

To determine the values of A, B, and C, we multiply both sides of this equation by the product of the denominators, $x(2x - 1)(x + 2)$, obtaining

$$x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1) \tag{4}$$

Expanding the right side of Equation 4 and writing it in the standard form for polynomials, we get

$$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A \tag{5}$$

The polynomials in Equation 5 are identical, so their coefficients must be equal. The coefficient of x^2 on the right side, $2A + B + 2C$, must equal the coefficient of x^2 on the left side—namely, 1. Likewise, the coefficients of x are equal and the constant terms are equal. This gives the following system of equations for A, B, and C:

$$\begin{aligned} 2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ -2A \qquad\qquad &= -1 \end{aligned}$$

Solving, we get $A = \frac{1}{2}$, $B = \frac{1}{5}$, and $C = -\frac{1}{10}$, and so

We could check our work by taking the terms to a common denominator and adding them.

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx = \int \left[\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}\right] dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + K$$

In integrating the middle term we have made the substitution $u = 2x - 1$, which gives $du = 2\,dx$ and $dx = du/2$. ■

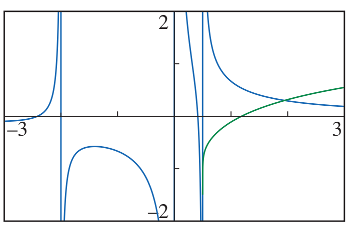


Figure 1
The graph of the integrand in Example 2 and its indefinite integral (with $K = 0$). Can you tell which is which?

Note: We can use an alternate method to find the coefficients A, B, and C in Example 2. Equation 4 is an identity; it is true for every value of x. Let's choose values of x that simplify the equation. If we put $x = 0$ in Equation 4, then the second and third terms on the right side vanish and the equation then becomes $-2A = -1$, or $A = \frac{1}{2}$. Likewise, $x = \frac{1}{2}$ gives $5B/4 = \frac{1}{5}$ and $x = -2$ gives $10C = -1$, so $B = \frac{1}{5}$ and $C = -\frac{1}{10}$. (You may object that Equation 3 is not valid for $x = 0$, $\frac{1}{2}$, or -2, so why should Equation 4 be valid for those values? In fact, Equation 4 is true for all values of x, even $x = 0$, $\frac{1}{2}$, and -2. See Exercise 45 for the reason.)

Example 3

Find $\displaystyle\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

Solution

The method of partial fractions gives

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

and therefore

$$A(x + a) + B(x - a) = 1$$

Using the method of the preceding note, we put $x = a$ in this equation and get $A(2a) = 1$, so $A = 1/(2a)$. If we put $x = -a$, we get $B(-2a) = 1$, so $B = -1/(2a)$. Thus

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\int \left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx$$

$$= \frac{1}{2a}\big[\ln|x-a| - \ln|x+a|\big] + C$$

Since $\ln x - \ln y = \ln(x/y)$, we can write the integral as

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$$

Case II $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r} \tag{6}$$

For example, here is a partial fraction decomposition in which there are repeated linear factors.

$$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

In the next Example, we will include the details for finding the constants.

Example 4

Find $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$.

Solution

The first step is to divide. The result of long division is

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

The second step is to factor the denominator $Q(x) = x^3 - x^2 - x + 1$. Since $Q(1) = 0$, we know that $x - 1$ is a factor, and we obtain

$$\begin{aligned} x^3 - x^2 - x + 1 &= (x-1)(x^2-1) = (x-1)(x-1)(x+1) \\ &= (x-1)^2(x+1) \end{aligned}$$

Since the linear factor $x - 1$ occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiplying by the least common denominator, $(x-1)^2(x+1)$, we get

$$\begin{aligned} 4x &= A(x-1)(x+1) + B(x+1) + C(x-1)^2 \\ &= (A + C)x^2 + (B - 2C)x + (-A + B + C) \end{aligned} \tag{7}$$

Another method for finding the coefficients:
Let $x = 1$ in Equation 7: $B = 2$.
Let $x = -1$: $C = -1$.
Let $x = 0$: $A = B + C = 1$.

Now we equate coefficients:

$$\begin{aligned} A \qquad + C &= 0 \\ B - 2C &= 4 \\ -A + B + C &= 0 \end{aligned}$$

Solving, we obtain $A = 1$, $B = 2$, and $C = -1$, so

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right]dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + K$$

Case III *Q*(*x*) contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 6, the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c} \tag{8}$$

where A and B are constants to be determined. For instance, the function given by $f(x) = \dfrac{x}{(x-2)(x^2+1)(x^2+4)}$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{x^2+4}$$

When integrating the term given in (8), it will often be necessary to use the formula

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \tag{9}$$

Example 5

Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

Solution

Since $x^3 + 4x = x(x^2 + 4)$ can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying by $x(x^2 + 4)$, we have

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$= (A + B)x^2 + Cx + 4A$$

Equating coefficients, we obtain

$$A + B = 2 \quad C = -1 \quad 4A = 4$$

Thus $A = 1$, $B = 1$, and $C = -1$ and so

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right)dx$$

In order to integrate the second term, we split it into two parts:

$$\int \frac{x-1}{x^2+4}dx = \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$$

We make the substitution $u = x^2 + 4$ in the first of these integrals so that $du = 2x\ dx$. We evaluate the second integral by means of Formula 9 with $a = 2$:

$$\int \frac{2x^2 - x + 4}{x(x^2+4)}dx = \int \frac{1}{x}dx + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$$

$$= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + K$$

Example 6

Evaluate $\displaystyle\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}dx$.

Solution

Since the degree of the numerator is *not less than* the degree of the denominator, we first divide and obtain

$$\frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} = 1 + \frac{x-1}{4x^2 - 4x + 3}$$

Notice that the quadratic $4x^2 - 4x + 3$ is irreducible because its discriminant is $b^2 - 4ac = -32 < 0$. This means it can't be factored, so we don't need to use the partial fraction technique.

To integrate the given function, we complete the square in the denominator:

$$4x^2 - 4x + 3 = (2x-1)^2 + 2$$

This suggests that we make the substitution $u = 2x - 1$. Then, $du = 2\ dx$ and $x = \frac{1}{2}(u+1)$, so

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}dx = \int \left(1 + \frac{x-1}{4x^2 - 4x + 3}\right)dx$$

$$= x + \frac{1}{2}\int \frac{\frac{1}{2}(u+1) - 1}{u^2+2}du = x + \frac{1}{4}\int \frac{u-1}{u^2+2}du$$

$$= x + \frac{1}{4}\int \frac{u}{u^2+2}du - \frac{1}{4}\int \frac{1}{u^2+2}du$$

$$= x + \frac{1}{8}\ln(u^2+2) - \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= x + \frac{1}{8}\ln(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$$

Note: Example 6 illustrates the general procedure for integrating a partial fraction of the form

$$\frac{Ax+B}{ax^2+bx+c} \quad \text{where} \quad b^2 - 4ac < 0$$

We complete the square in the denominator and then make a substitution that brings the integral into the form

$$\int \frac{Cu + D}{u^2 + a^2}\,du = C\int \frac{u}{u^2 + a^2}\,du + D\int \frac{1}{u^2 + a^2}\,du$$

Then the first integral is a logarithm and the second is expressed in terms of $\tan^{-1}$.

Case IV $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction (8), the sum

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r} \tag{10}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the terms in (10) can be integrated by using a substitution or by first completing the square.

It would be extremely tedious to work out by hand the numerical values of the coefficients in Example 7. Most computer algebra systems, however, can find the numerical values very quickly. For instance, the Maple command

```
convert(f, parfrac, x)
```

or the Mathematica command

```
Apart[f[x]]
```

gives the following values:

$A = -1, \quad B = \frac{1}{8}, \quad C = D = -1,$

$E = \frac{15}{8}, \quad F = -\frac{1}{8}, \quad G = H = \frac{3}{4},$

$I = -\frac{1}{2}, \quad J = \frac{1}{2}$

Example 7

Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2 + x + 1)(x^2 + 1)^3}$$

Solution

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2 + x + 1)(x^2 + 1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx + D}{x^2 + x + 1} + \frac{Ex + F}{x^2 + 1} + \frac{Gx + H}{(x^2+1)^2} + \frac{Ix + J}{(x^2+1)^3}$$

Example 8

Evaluate $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2}\,dx$.

Solution

The form of the partial fraction decomposition is

$$\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2+1)^2}$$

Multiplying by $x(x^2 + 1)^2$, we have

$$\begin{aligned} -x^3 + 2x^2 - x + 1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0 \quad C = -1 \quad 2A + B + D = 2 \quad C + E = -1 \quad A = 1$$

which has the solution $A = 1$, $B = -1$, $C = -1$, $D = 1$, and $E = 0$.

Thus

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\,dx = \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2} \right) dx$$

$$= \int \frac{dx}{x} - \int \frac{x}{x^2+1}\,dx - \int \frac{dx}{x^2+1} + \int \frac{x\,dx}{(x^2+1)^2}$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + K$$

In the second and fourth terms, we made the substitution $u = x^2 + 1$.

G Exercises

Write out the form of the partial fraction decomposition of the function (as in Example 7). Do not determine the numerical values of the coefficients.

1. (a) $\dfrac{2x}{(x+3)(3x+1)}$ (b) $\dfrac{1}{x^3 + 2x^2 + x}$

2. (a) $\dfrac{x}{x^2 + x - 2}$ (b) $\dfrac{x^2}{x^2 + x + 2}$

3. (a) $\dfrac{x^4 + 1}{x^5 + 4x^3}$ (b) $\dfrac{1}{(x^2 - 9)^2}$

4. (a) $\dfrac{x^3}{x^2 + 4x + 3}$ (b) $\dfrac{2x + 1}{(x+1)^3(x^2+4)^2}$

5. (a) $\dfrac{x^4}{x^4 - 1}$ (b) $\dfrac{t^4 + t^2 + 1}{(t^2+1)(t^2+4)^2}$

6. (a) $\dfrac{x^4}{(x^3 + x)(x^2 - x + 3)}$ (b) $\dfrac{1}{x^6 - x^3}$

Evaluate the integral.

7. $\displaystyle\int \frac{x}{x-6}\,dx$

8. $\displaystyle\int \frac{r^2}{r+4}\,dr$

9. $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx$

10. $\displaystyle\int \frac{1}{(t+4)(t-1)}\,dt$

11. $\displaystyle\int_2^3 \frac{1}{x^2-1}\,dx$

12. $\displaystyle\int_0^1 \frac{x-1}{x^2+3x+2}\,dx$

13. $\displaystyle\int \frac{ax}{x^2 - bx}\,dx$

14. $\displaystyle\int \frac{1}{(x+a)(x+b)}\,dx$

15. $\displaystyle\int_3^4 \frac{x^3 - 2x^2 - 4}{x^3 - 2x^2}\,dx$

16. $\displaystyle\int_0^1 \frac{x^3 - 4x - 10}{x^2 - x - 6}\,dx$

17. $\displaystyle\int_1^2 \frac{4y^2 - 7y - 12}{y(y+2)(y-3)}\,dy$

18. $\displaystyle\int \frac{x^2 + 2x - 1}{x^3 - x}\,dx$

19. $\displaystyle\int \frac{1}{(x+5)^2(x-1)}\,dx$

20. $\displaystyle\int \frac{x^2 - 5x + 16}{(2x+1)(x-2)^2}\,dx$

21. $\displaystyle\int \frac{5x^2 + 3x - 2}{x^3 + 2x^2}\,dx$

22. $\displaystyle\int \frac{x^2 - x + 6}{x^3 + 3x}\,dx$

23. $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

24. $\displaystyle\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}\,dx$

25. $\displaystyle\int \frac{x^3 + x^2 + 2x + 1}{(x^2+1)(x^2+2)}\,dx$

26. $\displaystyle\int \frac{x^2 + x + 1}{(x^2+1)^2}\,dx$

27. $\displaystyle\int \frac{x+4}{x^2 + 2x + 5}\,dx$

28. $\displaystyle\int_0^1 \frac{x}{x^2 + 4x + 13}\,dx$

29. $\displaystyle\int \frac{1}{x^3 - 1}\,dx$

30. $\displaystyle\int \frac{x^3}{x^3 + 1}\,dx$

31. $\displaystyle\int \frac{dx}{x(x^2+4)^2}$

32. $\displaystyle\int \frac{x^4 + 3x^2 + 1}{x^5 + 5x^3 + 5x}\,dx$

33. $\displaystyle\int \frac{x-3}{(x^2 + 2x + 4)^2}\,dx$

34. $\displaystyle\int \frac{3x^2 + x + 4}{x^4 + 3x^2 + 2}\,dx$

Make a substitution to express the integrand as a rational function and then evaluate the integral.

35. $\displaystyle\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx$

36. $\displaystyle\int \frac{dx}{2\sqrt{x+3} + x}$

37. $\displaystyle\int \frac{e^{2x}}{e^{2x} + 3e^x + 2}\,dx$

38. $\displaystyle\int \frac{\cos x}{\sin^2 x + \sin x}\,dx$

39. Use a graph of $f(x) = \dfrac{1}{(x^2 - 2x - 3)}$ to decide whether $\int_0^2 f(x)\,dx$ is positive or negative. Use the graph to give a rough estimate of the value of the integral and then use partial fractions to find the exact value.

40. Graph both $y = \dfrac{1}{(x^3 - 2x^2)}$ and an antiderivative on the same screen.

41. One method of slowing the growth of an insect population without using pesticides is to introduce into the population a number of sterile males that mate with fertile females but

produce no offspring. If P represents the number of female insects in a population, S the number of sterile males introduced each generation, and r the population's natural growth rate, then the female population is related to time t by

$$t = \int \frac{P + S}{P[(r - 1)P - S]} dP$$

Suppose an insect population with 10,000 females grows at a rate of $r = 0.10$ and 900 sterile males are added. Evaluate the integral to give an equation relating the female population to time. (Note that the resulting equation can't be solved explicitly for P.)

42. The region under the curve

$$y = \frac{1}{x^2 + 3x + 2}$$

from $x = 0$ to $x = 1$ is rotated about the x-axis. Find the volume of the resulting solid.

43. (a) Use a computer algebra system to find the partial fraction decomposition of the function

$$f(x) = \frac{4x^3 - 27x^2 + 5x - 32}{30x^5 - 13x^4 + 50x^3 - 286x^2 - 299x - 70}$$

(b) Use part (a) to find $\int f(x)\, dx$ (by hand) and compare with the result of using the CAS to integrate f directly. Comment on any discrepancy.

44. (a) Find the partial fraction decomposition of the function

$$f(x) = \frac{12x^5 - 7x^3 - 13x^2 + 8}{100x^6 - 80x^5 + 116x^4 - 80x^3 + 41x^2 - 20x + 4}$$

(b) Use part (a) to find $\int f(x)\, dx$ and graph f and its indefinite integral on the same screen.

(c) Use the graph of f to discover the main features of the graph of $\int f(x)\, dx$.

45. Suppose that F, G, and Q are polynomials and

$$\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$$

for all x except when $Q(x) = 0$. Prove that $F(x) = G(x)$ for all x. Hint: Use continuity.

46. If f is a quadratic function such that $f(0) = 1$ and

$$\int \frac{f(x)}{x^2(x + 1)^3} dx$$

is a rational function, find the value of $f'(0)$.

H Polar Coordinates

Polar coordinates offer an alternative way of locating points in a plane. They are useful because, for certain types of regions and curves, polar coordinates provide very simple descriptions and equations. The principal applications of this idea occur in multivariable calculus: the evaluation of double integrals and the derivation of Kepler's laws of planetary motion.

H.1 Curves in Polar Coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the **polar coordinate system**, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

If P is any other point in the plane, let r be the distance from O to P, and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$, and we agree that $(0, \theta)$ represents the pole for any value of θ.

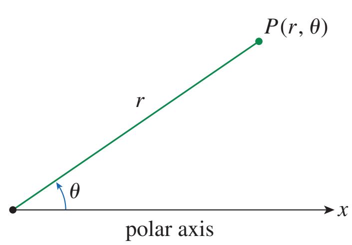


Figure 1
The polar coordinate system.

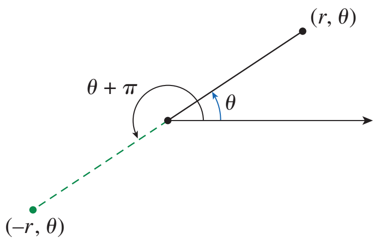


Figure 2
The points (r, θ) and $(-r, \theta)$.

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that, as in Figure 2, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r > 0$, the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

Example 1 Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

Solution

The points are plotted in Figure 3. In part (d) the point $(-3, 3\pi/4)$ is located three units from the pole in the fourth quadrant because the angle $3\pi/4$ is in the second quadrant and $r = -3$ is negative. ■

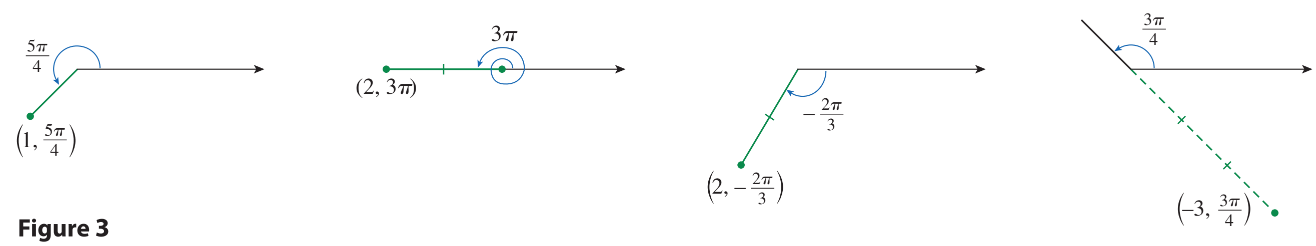


Figure 3
Polar coordinates.

In the Cartesian coordinate system, every point has only one representation, but in the polar coordinate system, each point has many representations. For instance, the point $(1, 5\pi/4)$ in Example 1(a) could be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(1, -\pi/4)$. (See Figure 4.)

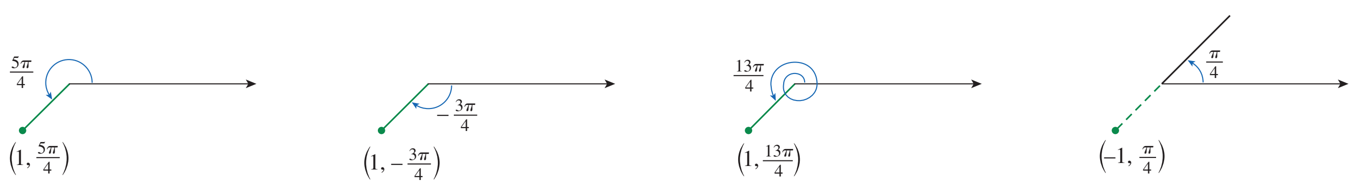


Figure 4
In the polar coordinate system, each point has many representations.

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \quad \text{and} \quad (-r, \theta + (2n + 1)\pi)$$

where n is any integer.

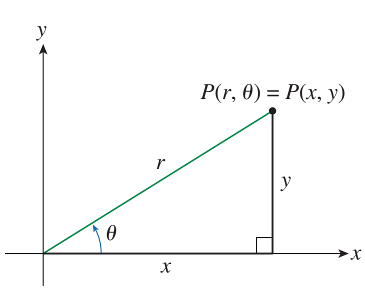


Figure 5
The connection between polar and Cartesian coordinates.

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive x-axis. If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure, we have

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

and so

$$x = r\cos\theta \qquad y = r\sin\theta \tag{1}$$

Although the equations in (1) were deduced from Figure 5, which illustrates the case where $r > 0$ and $0 < \theta < \pi/2$, these equations are valid for all values of r and θ. (See the general definition of $\sin\theta$ and $\cos\theta$ in Appendix C.)

The equations in (1) allow us to find the Cartesian coordinates of a point when the polar coordinates are known. To find r and θ when x and y are known, we use the equations

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \tag{2}$$

which can be deduced from the equations in (1) or simply read from Figure 5.

Example 2 Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Solution

Since $r = 2$ and $\theta = \pi/3$, the equations in (1) give

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore, the point is $(1, \sqrt{3})$ in Cartesian coordinates. ■

Example 3

Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Solution

If we choose r to be positive, then the equations in (2) give

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

Since the point $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\frac{\pi}{4}$ or $\theta = \frac{7\pi}{4}$. Thus one possible answer is $\left(\sqrt{2}, -\frac{\pi}{4}\right)$ another is $\left(\sqrt{2}, -\frac{7\pi}{4}\right)$. ■

Note: The equations in (2) do not uniquely determine θ when x and y are given because, as θ increases through the interval $0 \le \theta < 2\pi$, each value of $\tan\theta$ occurs twice. Therefore, in converting from Cartesian to polar coordinates, it's not good enough just to find r and θ that satisfy the equations in (2). As in Example 3, we must choose θ so that the point (r, θ) lies in the correct quadrant.

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

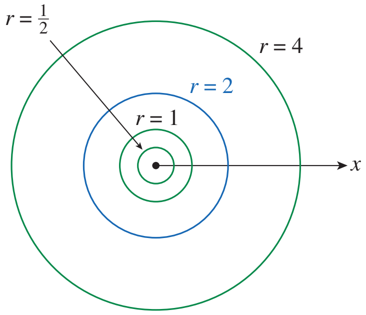


Figure 6
The polar equation $r = a$ represents a circle.

Example 4

What curve is represented by the polar equation $r = 2$?

Solution

The curve consists of all points (r, θ) with $r = 2$. Since r represents the distance from the point to the pole, the curve $r = 2$ represents the circle with center O and radius 2. In general, the equation $r = a$ represents a circle with center O and radius $|a|$. (See Figure 6.) ■

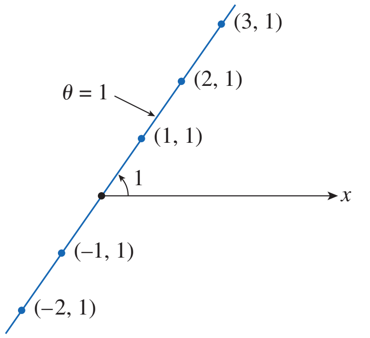


Figure 7
The polar equation $\theta = 1$ represents a straight line.

Example 5

Sketch the polar curve $\theta = 1$.

Solution

This curve consists of all points (r, θ) such that the polar angle θ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis (see Figure 7). Notice that the points $(r, 1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant. ■

Example 6

(a) Sketch the curve with polar equation $r = 2\cos\theta$.

(b) Find a Cartesian equation for this curve.

Solution

(a) In Figure 8 we find the values of r for some convenient values of θ and plot the corresponding points (r, θ). Then we join these points to sketch the curve, which appears to be a circle. We have used only values of θ between 0 and π, since if we let θ increase beyond π, we obtain the same points again.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

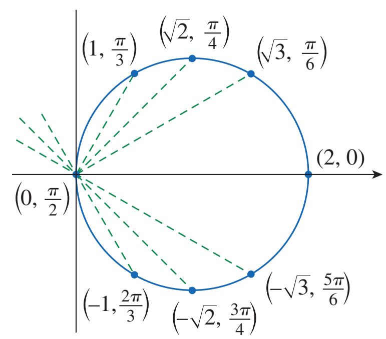


Figure 8
Table of values and graph of $r = 2\cos\theta$.

(b) To convert the given equation into a Cartesian equation, we use Equations 1 and 2. From $x = r\cos\theta$ we have $\cos\theta = \dfrac{x}{r}$, so the equation $r = 2\cos\theta$ becomes $r = \dfrac{2x}{r}$, which gives

$$2x = r^2 = x^2 + y^2 \quad \text{or} \quad x^2 + y^2 - 2x = 0$$

Completing the square, we obtain

$$(x - 1)^2 + y^2 = 1$$

which is an equation of a circle with center $(1, 0)$ and radius 1. ■

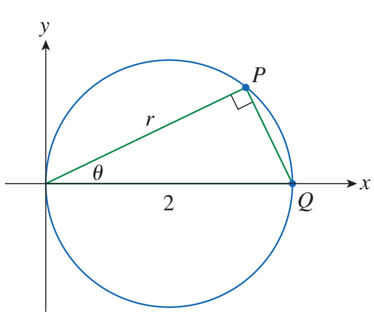


Figure 9
The graph of the polar equation $r = 2\cos\theta$ is a circle.

Figure 9 shows a geometrical illustration that the circle in Example 6 has the equation $r = 2\cos\theta$. The angle OPQ is a right angle and so $\dfrac{r}{2} = \cos\theta$.

Example 7

Sketch the curve $r = 1 + \sin\theta$.

Solution

Instead of plotting points as in Example 6, we first sketch the graph of $r = 1 + \sin\theta$ in *Cartesian* coordinates in Figure 10 by shifting the sine curve up one unit. This enables us to read at a glance the values of r that correspond to increasing values of θ.

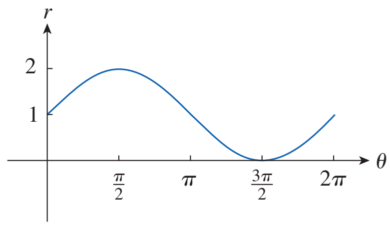


Figure 10
The graph of $r = 1 + \sin\,\theta$ in Cartesian coordinates for $0 \le \theta \le 2\pi$.

For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2, so we sketch the corresponding part of the polar curve in Figure 11(a). As θ increases from $\pi/2$ to π, Figure 10 shows that r decreases from 2 to 1, so we sketch the next part of the curve as in Figure 11(b). As θ increases from π to $3\pi/2$, r decreases from 1 to 0 as shown in part (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 0 to 1 as shown in part (d). If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace our path. Putting together the parts of the curve from Figure 11(a)–(d), we sketch the complete curve in part (e). It is called a **cardioid** because it's shaped like a heart.

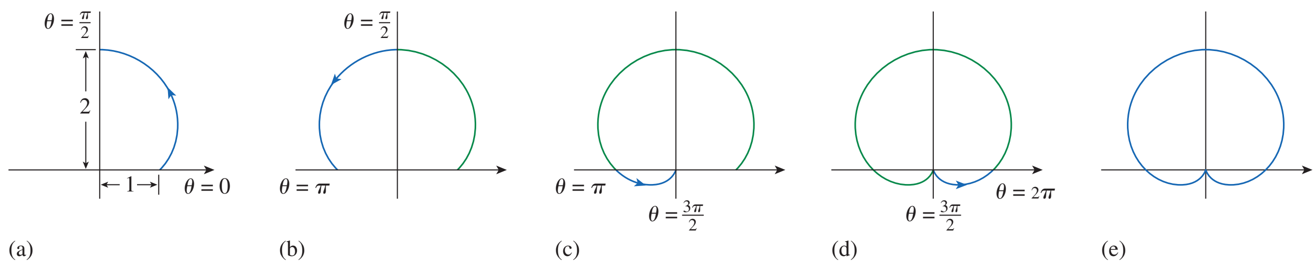


Figure 11
Stages in sketching the cardioid $r = 1 + \sin\,\theta$.

Example 8

Sketch the curve $r = \cos\,2\theta$.

Solution

As in Example 7, we first sketch $r = \cos\,2\theta$, $0 \le \theta \le 2\pi$, in Cartesian coordinates in Figure 12. As θ increases from 0 to $\pi/4$, Figure 12 shows that r decreases from 1 to 0 and so we draw the corresponding portion of the polar curve in Figure 13 (indicated by ①). As θ increases from $\pi/4$ to $\pi/2$, r goes from 0 to -1. This means that the distance from O increases from 0 to 1, but instead of being in the first quadrant, this portion of the polar curve (indicated by ②) lies on the opposite side of the pole in the third quadrant. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four loops and is called a **four-leaved rose**.

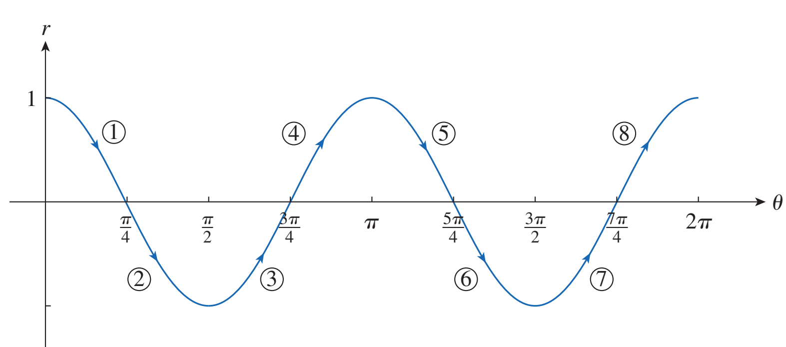


Figure 12
The graph of the polar equation $r = \cos\,2\theta$ in Cartesian coordinates.

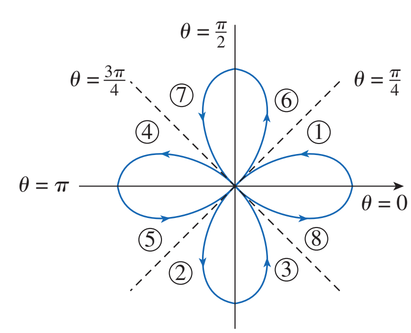


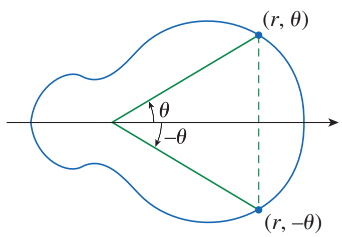
Figure 13
Graph of the four-leaved rose described by $r = \cos\,2\theta$.

When we sketch polar curves, it is sometimes helpful to take advantage of symmetry. The following three rules are explained by Figure 14.

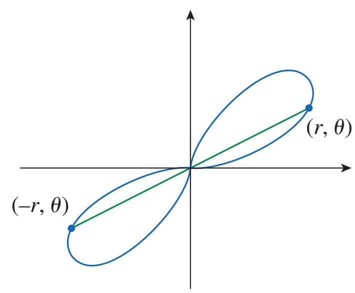
(a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.

(b) If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)

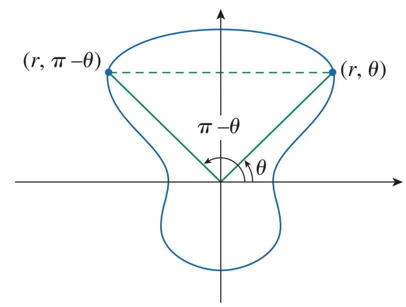
(c) If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \pi/2$.



(a) Symmetric about the polar axis



(b) Symmetric about the pole



(c) Symmetric about the vertical line $\theta = \pi/2$

Figure 14
Examples of symmetry in polar curves.

The curves sketched in Examples 6 and 8 are symmetric about the polar axis, since $\cos(-\theta) = \cos\theta$. The curves in Examples 7 and 8 are symmetric about $\theta = \pi/2$ because $\sin(\pi - \theta) = \sin\theta$ and $\cos 2(\pi - \theta) = \cos 2\theta$. The four-leaved rose is also symmetric about the pole. These symmetry properties could have been used in sketching the curves. For instance, in Example 6 we need only have plotted points for $0 \le \theta \le \pi/2$ and then reflected about the polar axis to obtain the complete circle.

Tangents to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Then, using the method for finding slopes of parametric curves (Equation 3.4.7) and the Product Rule, we have

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta} \tag{3}$$

We locate horizontal tangents by finding the points where $\dfrac{dy}{d\theta} = 0$ (provided that $\dfrac{dx}{d\theta} \neq 0$).

Likewise, we locate vertical tangents at the points where $\dfrac{dx}{d\theta} = 0$ (provided that $\dfrac{dy}{d\theta} \neq 0$).

Notice that if we are looking for tangent lines at the pole, then $r = 0$, and Equation 3 simplifies to

$$\frac{dy}{dx} = \tan\theta \qquad \text{if } \frac{dr}{d\theta} \neq 0$$

For instance, in Example 8 we found that $r = \cos 2\theta$ when $\theta = \pi/4$ or $3\pi/4$. This means that the lines $\theta = \pi/4$ and $\theta = 3\pi/4$ (or $y = x$ and $y = -x$) are tangent lines to $r = \cos\ 2\theta$ at the origin.

Example 9

(a) For the cardioid $r = 1 + \sin\ \theta$ of Example 7, find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

Solution

Using Equation 3 with $r = 1 + \sin\ \theta$, we have

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\ \theta + r\cos\ \theta}{\dfrac{dr}{d\theta}\cos\ \theta - r\sin\ \theta} = \frac{\cos\ \theta\ \sin\ \theta + (1 + \sin\ \theta)\cos\ \theta}{\cos\ \theta\ \cos\ \theta - (1 + \sin\ \theta)\sin\ \theta}$$

$$= \frac{\cos\ \theta(1 + 2\sin\ \theta)}{1 - 2\sin^2\theta - \sin\ \theta} = \frac{\cos\ \theta(1 + 2\sin\ \theta)}{(1 + \sin\ \theta)(1 - 2\sin\ \theta)}$$

(a) The slope of the tangent at the point where $\theta = \pi/3$ is

$$\left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{3}} = \frac{\cos\left(\dfrac{\pi}{3}\right)\left(1 + 2\sin\left(\dfrac{\pi}{3}\right)\right)}{\left(1 + \sin\left(\dfrac{\pi}{3}\right)\right)\left(1 - 2\sin\left(\dfrac{\pi}{3}\right)\right)} = \frac{\dfrac{1}{2}(1 + \sqrt{3})}{\left(1 = \dfrac{\sqrt{3}}{2}\right)(1 - \sqrt{3})}$$

$$= \frac{1 + \sqrt{3}}{(2 + \sqrt{3})(1 - \sqrt{3})} = \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = -1$$

(b) Observe that

$$\frac{dy}{d\theta} = \cos\ \theta(1 + 2\sin\ \theta) = 0 \qquad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1 + \sin\ \theta)(1 - 2\sin\ \theta) = 0 \qquad \text{when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Therefore, there are horizontal tangents at the points $\left(2, \dfrac{\pi}{2}\right)$, $\left(\dfrac{1}{2}, \dfrac{7\pi}{6}\right)$, and $\left(\dfrac{1}{2}, \dfrac{11\pi}{6}\right)$ and vertical tangents at $\left(\dfrac{3}{2}, \dfrac{\pi}{6}\right)$ and $\left(\dfrac{3}{2}, \dfrac{5\pi}{6}\right)$. When $\theta = \dfrac{3\pi}{2}$, both $dy/d\theta$ and $dx/d\theta$ are 0, so we must be careful. Using l'Hospital's Rule, we have

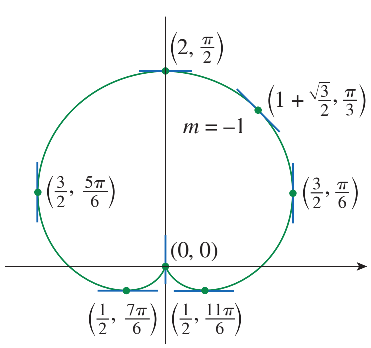


Figure 15
Graph of the polar equation, horizontal, and vertical tangent lines, and tangent line at the point where $\theta = \dfrac{\pi}{3}$.

$$\lim_{\theta \to (3\pi/2)^-} \frac{dy}{dx} = \left(\lim_{\theta \to (3\pi/2)^-} \frac{1 + 2\sin\ \theta}{1 - 2\sin\ \theta}\right)\left(\lim_{\theta \to (3\pi/2)^-} \frac{\cos\ \theta}{1 + \sin\ \theta}\right)$$

$$= -\frac{1}{3}\lim_{\theta \to (3\pi/2)^-} \frac{\cos\ \theta}{1 + \sin\ \theta} = -\frac{1}{3}\lim_{\theta \to (3\pi/2)^-} \frac{-\sin\ \theta}{\cos\ \theta} = \infty$$

By symmetry, $\displaystyle\lim_{\theta \to (3\pi/2)^+} \frac{dy}{dx} = -\infty$

Thus there is a vertical tangent line at the pole (see Figure 15). ■

Note: Instead of having to remember Equation 3, we could employ the method used to derive it. For instance, in Example 9 we could have written

$$x = r\cos\theta = (1 + \sin\theta)\cos\theta = \cos\theta + \frac{1}{2}\sin 2\theta$$
$$y = r\sin\theta = (1 + \sin\theta)\sin\theta = \sin\theta + \sin^2\theta$$

Then we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta}$$

which is equivalent to our previous expression.

Graphing Polar Curves with Graphing Devices

Although it's useful to be able to sketch simple polar curves by hand, we need to use a graphing calculator or computer when we are faced with a curve as complicated as the ones shown in Figures 16 and 17.

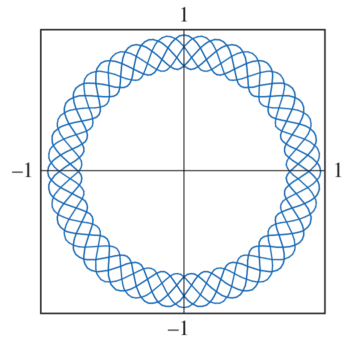


Figure 16
Graph of $r = \sin^2(2.4\theta) + \cos^4(2.4\theta)$.

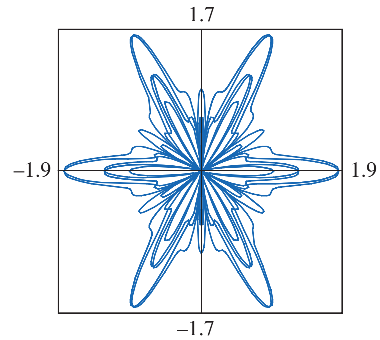


Figure 17
Graph of $r = \sin^2(1.2\theta) + \cos^3(6\theta)$.

Some graphing devices have commands that enable us to graph polar curves directly. With other machines we need to convert to parametric equations first. In this case we take the polar equation $r = f(\theta)$ and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Some machines require that the parameter be called t rather than θ.

Example 10

Graph the curve $r = \sin\left(\dfrac{8\theta}{5}\right)$.

Solution

Let's assume that our graphing device doesn't have a built-in polar graphing command. In this case we need to work with the corresponding parametric equations, which are

$$x = r\cos\theta = \sin\left(\frac{8\theta}{5}\right)\cos\theta \qquad y = r\sin\theta = \sin\left(\frac{8\theta}{5}\right)\sin\theta$$

In any case we need to determine the domain for θ. So we ask ourselves: how many complete rotations are required until the curve starts to repeat itself? If the answer is n, then

$$\sin\frac{8(\theta + 2n\pi)}{5} = \sin\left(\frac{8\theta}{5} + \frac{16n\pi}{5}\right) = \sin\frac{8\theta}{5}$$

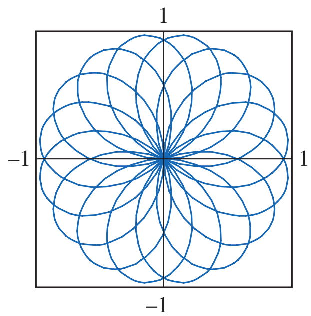


Figure 18
Graph of $r = \sin(8\theta/5)$.

and so we require that $16n\dfrac{\pi}{5}$ be an even multiple of π. This will first occur when $n = 5$. Therefore, we will graph the entire curve if we specify that $0 \le \theta \le 10\pi$. Switching from θ to t, we have the equations

$$x = \sin\left(\frac{8t}{5}\right)\cos t \quad y = \sin\left(\frac{8t}{5}\right)\sin t \quad 0 \le t \le 10\pi$$

and Figure 18 shows the resulting curve. Notice that this rose has 16 loops. ■

Example 11

Investigate the family of polar curves given by $r = 1 + c\ \sin\ \theta$. How does the shape change as c changes? (These curves are called **limaçons**, after a French word for snail, because of the shape of the curves for certain values of c.)

Solution

In Exercise 47 you are asked to prove analytically what we have discovered from the graphs in Figure 19.

Figure 19 shows several graphs for various values of c. For $c > 1$ there is a loop that decreases in size as c decreases. When $c = 1$ the loop disappears and the curve becomes the cardioid that we sketched in Example 7. For c between 1 and $\frac{1}{2}$, the cardioid's cusp is smoothed out and becomes a "dimple." When c decreases from $\frac{1}{2}$ to 0, the limaçon is shaped like an oval. This oval becomes more circular as $c \to 0$, and when $c = 0$ the curve is just the circle $r = 1$.

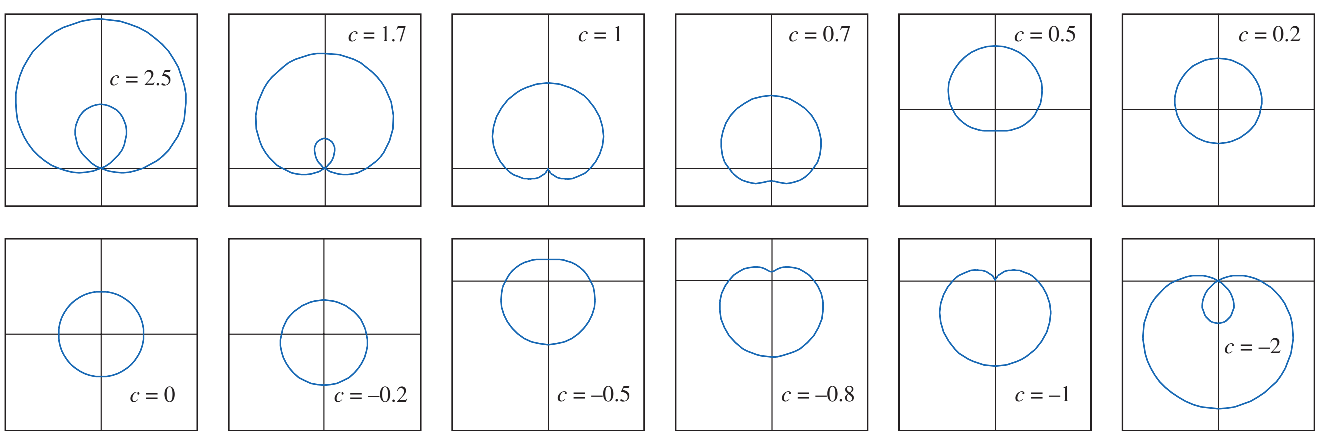


Figure 19
Graphs of limaçons $r = 1 + c\ \sin\ \theta$ for various values of c.

The remaining parts of Figure 19 show that as c becomes negative, the shapes change in reverse order. In fact, these curves are reflections about the horizontal axis of the corresponding curves with positive c. ■

H.1 Exercises

Plot the point whose polar coordinates are given. Then find two other pairs of polar coordinates of this point, one with $r > 0$.

1. (a) $\left(2, \dfrac{\pi}{3}\right)$ (b) $\left(1, -\dfrac{3\pi}{4}\right)$ (c) $\left(-1, \dfrac{\pi}{2}\right)$

2. (a) $\left(1, \dfrac{7\pi}{4}\right)$ (b) $\left(-3, \dfrac{\pi}{6}\right)$ (c) $(1, -1)$

Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the point.

3. (a) $(1, \pi)$ (b) $\left(2, -\dfrac{2\pi}{3}\right)$ (c) $\left(-2, \dfrac{3\pi}{4}\right)$

4. (a) $\left(-\sqrt{2}, \dfrac{5\pi}{4}\right)$ (b) $\left(1, \dfrac{5\pi}{2}\right)$ (c) $\left(2, -\dfrac{7\pi}{6}\right)$

The Cartesian coordinates of a point are given.

(i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \le \theta < 2\pi$.

(ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \le \theta < 2\pi$.

5. (a) $(2, -2)$ (b) $(-1, \sqrt{3})$

6. (a) $(3\sqrt{3}, 3)$ (b) $(1, -2)$

Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions.

7. $1 \le r \le 2$

8. $r \ge 0, \quad \dfrac{\pi}{3} \le \theta \le \dfrac{2\pi}{3}$

9. $0 \le r < 4, \quad \dfrac{-\pi}{2} \le \theta < \dfrac{\pi}{6}$

10. $2 < r \le 5, \quad \dfrac{3\pi}{4} < \theta < \dfrac{5\pi}{4}$

11. $2 < r < 3, \quad \dfrac{5\pi}{3} \le \theta \le \dfrac{7\pi}{3}$

12. $r \ge 1, \quad \pi \le \theta \le 2\pi$

Identify the curve by finding a Cartesian equation for the curve.

13. $r = 3 \sin \theta$ **14.** $r = 2 \sin \theta + 2 \cos \theta$

15. $r = \csc \theta$ **16.** $r = \tan \theta \sec \theta$

Find a polar equation for the curve represented by the given Cartesian equation.

17. $x = -y^2$ **18.** $x + y = 9$

19. $x^2 + y^2 = 2cx$ **20.** $xy = 4$

For each of the described curves, decide if the curve would be more easily given by a polar equation or a Cartesian equation. Then write an equation for the curve.

21. (a) A line through the origin that makes an angle of $\dfrac{\pi}{6}$ with the positive x-axis

(b) A vertical line through the point $(3, 3)$

22. (a) A circle with radius 5 and center $(2, 3)$

(b) A circle centered at the origin with radius 4

Sketch the curve with the given polar equation.

23. $\theta = -\dfrac{\pi}{6}$ **24.** $r^2 - 3r + 2 = 0$

25. $r = \sin \theta$ **26.** $r = -3 \cos \theta$

27. $r = 2(1 - \sin \theta), \quad \theta \ge 0$ **28.** $r = 1 - 3 \cos \theta$

29. $r = \theta, \quad \theta \ge 0$ **30.** $r = \ln \theta, \quad \theta \ge 1$

31. $r = 4 \sin 3\theta$ **32.** $r = \cos 5\theta$

33. $r = 2 \cos 4\theta$ **34.** $r = 3 \cos 6\theta$

35. $r = 1 - 2 \sin \theta$ **36.** $r = 2 + \sin \theta$

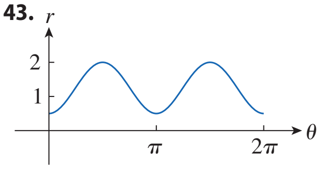
37. $r^2 = 9 \sin 2\theta$ **38.** $r^2 = \cos 4\theta$

39. $r = 2 \cos\left(\dfrac{3\theta}{2}\right)$ **40.** $r^2\theta = 1$

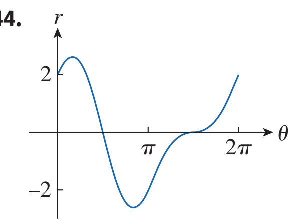
41. $r = 1 + 2 \cos 2\theta$ **42.** $r = 1 + 2 \cos\left(\dfrac{\theta}{2}\right)$

The figure shows a graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.

43.



44.



45. Show that the polar curve $r = 4 + 2 \sec \theta$ (called a **conchoid**) has the line $x = 2$ as a vertical asymptote by showing that $\lim_{r \to \pm\infty} x = 2$. Use this fact to help sketch the conchoid.

46. Show that the curve $r = \sin \theta \tan \theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote. Show also that the curve lies entirely within the vertical strip $0 \le x < 1$. Use these facts to help sketch the cissoid.

47. (a) In Example 11 the graphs suggest that the limaçon $r = 1 + c \sin \theta$ has an inner loop when $|c| > 1$. Prove that this is true, and find the values of θ that correspond to the inner loop.

(b) From Figure 19 it appears that the limaçon loses its dimple when $c = \dfrac{1}{2}$. Prove this.

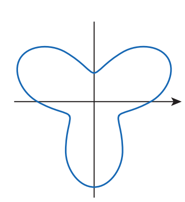
48. Match the polar equations with the graphs labeled I–VI. Give reasons for your choices. (Don't use a graphing device.)

(a) $r = \sqrt{\theta}, \quad 0 \le \theta \le 16\pi$ (b) $r = \theta^2, \quad 0 \le \theta \le 16\pi$

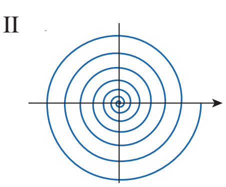
(c) $r = \cos\left(\dfrac{\theta}{3}\right)$ (d) $r = 1 + 2 \cos \theta$

(e) $r = 2 + \sin 3\theta$ (f) $r = 1 + 2 \sin 3\theta$

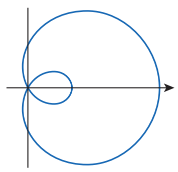
I



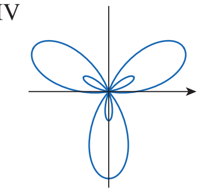
II



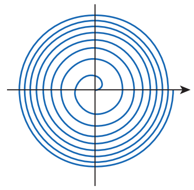
III



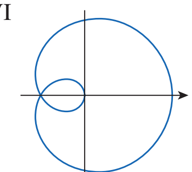
IV



V



VI



Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

49. $r = \dfrac{1}{\theta}, \quad \theta = \pi$

50. $r = 2 - \sin\theta, \quad \theta = \dfrac{\pi}{3}$

51. $r = \cos 2\theta, \quad \theta = \dfrac{\pi}{4}$

52. $r = \cos\left(\dfrac{\theta}{3}\right), \quad \theta = \pi$

Find the points on the given curve where the tangent line is horizontal or vertical.

53. $r = 3\cos\theta$

54. $r = e^{\theta}$

55. $r = 1 + \cos\theta$

56. $r = 1 - \sin\theta$

57. Show that the polar equation $r = a\sin\theta + b\cos\theta$, where $ab \neq 0$, represents a circle, and find its center and radius.

58. Show that the curves $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles.

Use a graphing device to graph the polar curve. Choose the parameter interval carefully to make sure that you produce an appropriate curve.

59. $r = e^{\sin\theta} - 2\cos(4\theta)$ (butterfly curve)

60. $r = |\tan\theta|^{|\cot\theta|}$ (valentine curve)

61. $r = 2 - 5\sin\left(\dfrac{\theta}{6}\right)$

62. $r = \cos\left(\dfrac{\theta}{2}\right) + \cos\left(\dfrac{\theta}{3}\right)$

---

63. How are the graphs of $r = 1 + \sin\left(\theta - \dfrac{\pi}{6}\right)$ and $r = 1 + \sin\left(\theta - \dfrac{\pi}{3}\right)$ related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

64. Use a graph to estimate the y-coordinate of the highest points on the curve $r = \sin 2\theta$. Then use calculus to find the exact value.

65. (a) Investigate the family of curves defined by the polar equations $r = \sin n\theta$, where n is a positive integer. How is the number of loops related to n?

(b) What happens if the equation in part (a) is replaced by $r = |\sin n\theta|$?

66. A family of curves is given by the equations $r = 1 + c\sin n\theta$, where c is a real number and n is a positive integer. How does the graph change as n increases? How does it change as c changes? Illustrate by graphing enough members of the family to support your conclusions.

67. A family of curves has polar equations

$$r = \frac{1 - a\cos\theta}{1 + a\cos\theta}$$

Investigate how the graph changes as the number a changes. In particular, you should identify the transitional values of a for which the basic shape of the curve changes.

68. The astronomer Giovanni Cassini (1625–1712) studied the family of curves with polar equations

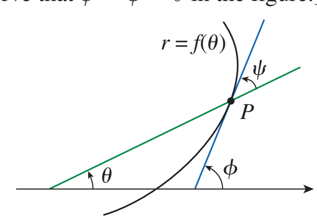
$$r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$$

where a and c are positive real numbers. These curves are called the **ovals of Cassini** even though they are oval shaped only for certain values of a and c. (Cassini thought that these curves might represent planetary orbits better than Kepler's ellipses.) Investigate the variety of shapes that these curves may have. In particular, how are a and c related to each other when the curve splits into two parts?

69. Let P be any point (except the origin) on the curve $r = f(\theta)$. If ψ is the angle between the tangent line at P and the radial line OP, show that

$$\tan\psi = \frac{r}{dr/d\theta}$$

[Hint: Observe that $\psi = \phi - \theta$ in the figure.]



70. (a) Use Exercise 69 to show that the angle between the tangent line and the radial line is $\psi = \dfrac{\pi}{4}$ at every point on the curve $r = e^{\theta}$.

(b) Illustrate part (a) by graphing the curve and the tangent lines at the points where $\theta = 0$ and $\dfrac{\pi}{2}$.

(c) Prove that any polar curve $r = f(\theta)$ with the property that the angle ψ between the radial line and the tangent line is a constant must be of the form $r = Ce^{k\theta}$, where C and k are constants.

H.2 Areas and Lengths in Polar Coordinates

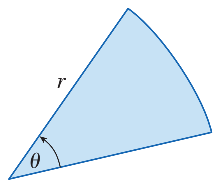


Figure 1
Sector of a circle with radius r.

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle

$$A = \frac{1}{2} r^2 \theta \tag{1}$$

where, as in Figure 1, r is the radius and θ is the radian measure of the central angle. Formula 1 follows from the fact that the area of a sector is proportional to its central angle: $A = \left(\frac{\theta}{2}\pi\right)\pi r^2 = \frac{1}{2}r^2\theta$.

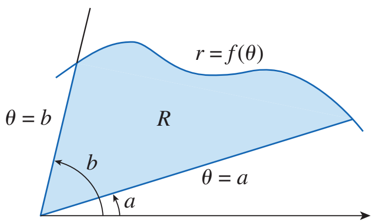


Figure 2
The region $\Re$ is bounded by the graph of $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$.

Let R be the region, illustrated in Figure 2, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leq 2\pi$. We divide the interval $[a, b]$ into subintervals with endpoints $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ and equal width $\Delta\theta$. The rays $\theta = \theta_i$ then divide R into n smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$. If we choose θ_i^* in the ith subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the ith region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$. (See Figure 3.)

Thus from Formula 1, we have

$$\Delta A_i \approx \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta$$

and so an approximation to the total area A of $\Re$ is

$$A \approx \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta \tag{2}$$

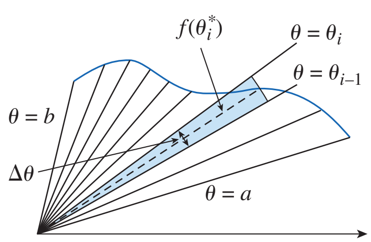


Figure 3
ΔA_i is approximated by the area of the sector of a circle.

It appears from Figure 3 that the approximation in (2) improves as $n \to \infty$. But the sums in (2) are Riemann sums for the function $g(\theta) = \frac{1}{2}[f(\theta)]^2$, so

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta$$

It therefore appears plausible (and can in fact be proved) that the formula for the area of A of the polar region R is

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta \tag{3}$$

Equation 3 is often written as

$$A = \int_a^b \frac{1}{2} r^2 d\theta \tag{4}$$

with the understanding that $r = f(\theta)$. Note the similarity between Equations 1 and 4.

When we apply Equations 3 or 4, it is helpful to think of the area as being swept out by a rotating ray through O that starts with angle a and ends with angle b.

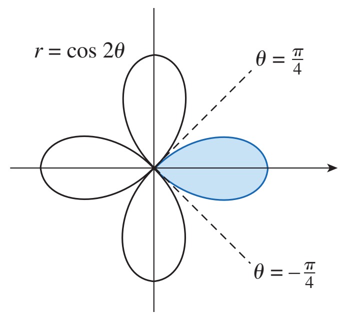


Figure 4
Four-leaved rose with one loop shaded.

Example 1 Find the area enclosed by one loop of the four-leaved rose $r = \cos\ 2\theta$.

Solution

The curve $r = \cos\ 2\theta$ was sketched in Example 8 in Section H.1. Notice from Figure 4 that the region enclosed by the right loop is swept out by a ray that rotates from $\theta = -\pi/4$ to $\theta = \pi/4$. Therefore, Formula 4 gives

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2\, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta = \int_0^{\pi/4} \cos^2 2\theta\, d\theta$$

We could evaluate the integral using Formula 64 in the Table of Integrals. Or, as in Section 5.7, we could use the identity $\cos^2 x = \frac{1}{2}(1 + \cos\ 2x)$ to write

$$A = \int_0^{\pi/4} \frac{1}{2}(1 + \cos\ 4\theta)\ d\theta = \frac{1}{2}\Big[\theta + \frac{1}{4}\sin\ 4\theta\Big]_0^{\pi/4} = \frac{\pi}{8}$$

Example 2

Find the area of the region that lies inside the circle $r = 3\ \sin\ \theta$ and outside the cardioid $r = 1 + \sin\ \theta$.

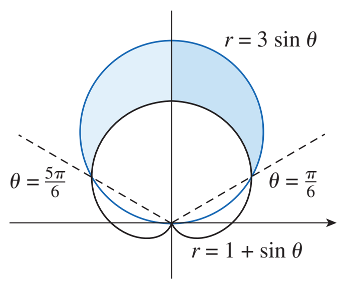


Figure 5
The region of interest is outside the cardioid and inside the circle.

Solution

The cardioid (see Example 7 in Section H.1) and the circle are sketched in Figure 5 and the desired region is shaded. The values of a and b in Formula 4 are determined by finding the points of intersection of the two curves. They intersect when $3\ \sin\ \theta = 1 + \sin\ \theta$, which gives $\sin\ \theta = \frac{1}{2}$, so $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. The desired area can be found by subtracting the area inside the cardioid between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ from the area inside the circle from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$. Thus

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\ \sin\ \theta)^2\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 + \sin\ \theta)^2\, d\theta$$

Since the region is symmetric about the vertical axis $\theta = \pi/2$, we can write

$$\begin{aligned} A &= 2\Big[\frac{1}{2}\int_{\pi/6}^{\pi/2} 9\ \sin^2\theta\, d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2} (1 + 2\ \sin\ \theta + \sin^2\theta)\ d\theta\Big] \\ &= \int_{\pi/6}^{\pi/2} (8\ \sin^2\theta - 1 - 2\ \sin\ \theta)\ d\theta \\ &= \int_{\pi/6}^{\pi/2} (3 - 4\ \cos\ 2\theta - \sin\ \theta)\ d\theta \qquad \Big[\text{because } \sin^2\theta = \tfrac{1}{2}(1 - \cos\ 2\theta)\Big] \\ &= 3\theta - 2\sin 2\theta + 2\cos\theta\Big]_{\pi/6}^{\pi/2} = \pi \end{aligned}$$

Example 2 illustrates the procedure for finding the area of the region bounded by two polar curves. In general, let R be a region, as illustrated in Figure 6, that is bounded by curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$, and $\theta = b$, where $f(\theta) \geq g(\theta) \geq 0$ and $0 < b - a \leq 2\pi$. The area A of R is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$, so using Formula 3 we have

$$\begin{aligned} A &= \int_a^b \frac{1}{2}[f(\theta)]^2\ d\theta - \int_a^b \frac{1}{2}[g(\theta)]^2\ d\theta \\ &= \frac{1}{2}\int_a^b \big([f(\theta)]^2 - [g(\theta)]^2\big)\ d\theta \end{aligned}$$

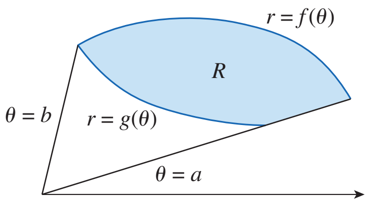


Figure 6
The area of the region R is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$.

CAUTION The fact that a single point has many representations in polar coordinates sometimes makes it difficult to find all the points of intersection of two polar curves. For instance, it is obvious from Figure 5 that the circle and the cardioid have three points of intersection; however, in Example 2 we solved the equations $r = 3 \sin \theta$ and $r = 1 + \sin \theta$ and found only two such points, $\left(\frac{3}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$. The origin is also a point of intersection, but we can't find it by solving the equations of the curves because the origin has no single representation in polar coordinates that satisfies both equations. Notice that, when represented as $(0, 0)$ or $(0, \pi)$, the origin satisfies $r = 3 \sin \theta$ and so it lies on the circle; when represented as $\left(0, \frac{3\pi}{2}\right)$, it satisfies $r = 3 \sin \theta$ and so it lies on the cardioid. Think of two points moving along the curves as the parameter value θ increases from 0 to 2π. On one curve, the origin is reached at $\theta = 0$ and $\theta = \pi$; on the other curve, is reached at $\theta = \frac{3\pi}{2}$. The points don't collide at the origin because they reach the origin at different times, but the curves intersect there nonetheless.

Thus, to find *all* points of intersection of two polar curves, it is recommended that you draw the graphs of both curves. It is especially convenient to use a graphing calculator or computer to help with this task.

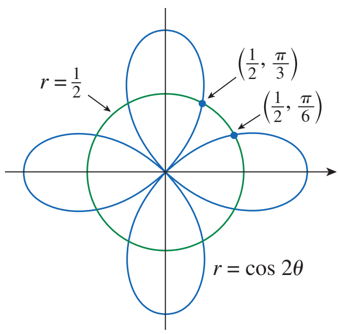


Figure 7

Graphs of $r = \cos 2\theta$ and $r = \frac{1}{2}$.

Example 3

Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

Solution

If we solve the equations $r = \cos 2\theta$ and $r = \frac{1}{2}$, we get $\cos 2\theta = \frac{1}{2}$ and therefore $2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$. Thus the values of θ between 0 and 2π that satisfy both equations are $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. We have found four points of intersection: $\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right), \left(\frac{1}{2}, \frac{7\pi}{6}\right)$, and $\left(\frac{1}{2}, \frac{11\pi}{6}\right)$.

However, you can see from Figure 7 that the curves have four other points of intersection—namely, $\left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{2\pi}{3}\right), \left(\frac{1}{2}, \frac{4\pi}{3}\right)$, and $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$. These can be found using symmetry or by noticing that another equation of the circle is $r = -\frac{1}{2}$ and then solving the equations $r = \cos 2\theta$ and $r = -\frac{1}{2}$. ■

Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \leq \theta$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r \cos \theta = f(\theta) \cos \theta \qquad y = r \sin \theta = f(\theta) \sin \theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$$

so, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\,\sin\theta + r^2\sin^2\theta$$

$$+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\,\cos\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Assuming that f' is continuous, we can use Formula 6.4.1 to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore, the length of a curve with polar equation $r = f(\theta)$, $a \le \theta \le b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{5}$$

Example 4

Find the length of the cardioid $r = 1 + \sin\theta$.

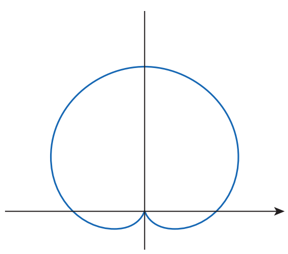


Figure 8
Graph of the cardioid $r = 1 + \sin\theta$.

Solution

The cardioid is shown in Figure 8. (We sketched it in Example 7 in Section H.1.) Its full length is given by the parameter interval $0 \le \theta \le 2\pi$, so Equation 5 gives

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

We could evaluate this integral by multiplying and dividing the integrand by $\sqrt{2 - 2\sin\theta}$, or we could use a computer algebra system. In any event, we find that the length of the cardioid is $L = 8$. ■

H.2 Exercises

Find the area of the region that is bounded by the given curve and lies in the specified sector.

1. $r = \theta^2, \quad 0 \le \theta \le \dfrac{\pi}{4}$

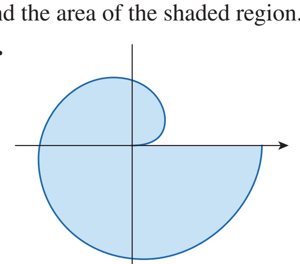
2. $r = e^{\theta/2}, \quad \pi \le \theta \le 2\pi$

3. $r = \sin\theta, \quad \dfrac{\pi}{3} \le \theta \le \dfrac{2\pi}{3}$

4. $r = \sqrt{\sin\theta}, \quad 0 \le \theta \le \pi$

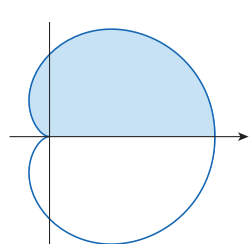
Find the area of the shaded region.

5.



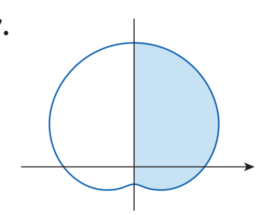
$r = \sqrt{\theta}$

6.



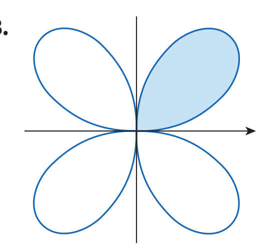
$r = 1 + \cos\theta$

7.



$r = 4 + 3\sin\theta$

8.



$r = \sin 2\theta$

Sketch the curve and find the area that it encloses.

9. $r^2 = 4\cos 2\theta$

10. $r = 2 - \sin\theta$

11. $r = 2\cos 3\theta$

12. $r = 2 + \cos 2\theta$

Graph the curve and find the area that it encloses.

13. $r = 1 + 2\sin 6\theta$

14. $r = 2\sin\theta + 3\sin 9\theta$

Find the area of the region enclosed by one loop of the curve.

15. $r = \sin 2\theta$ **16.** $r = 4 \sin 3\theta$

17. $r = 1 + 2 \sin \theta$ (inner loop)

18. $r = 2 \cos \theta - \sec \theta$

Find the area of the region that lies inside the first curve and outside the second curve.

19. $r = 2 \cos \theta, \quad r = 1$ **20.** $r = 1 - \sin \theta, \quad r = 1$

21. $r = 3 \cos \theta, \quad r = 1 + \cos \theta$

22. $r = 3 \sin \theta, \quad r = 2 - \sin \theta$

Find the area of the region that lies inside both curves.

23. $r = \sqrt{3} \cos \theta, \quad r = \sin \theta$

24. $r = 1 + \cos \theta, \quad r = 1 - \cos \theta$

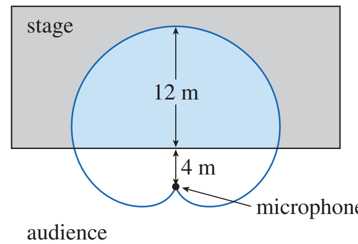
25. $r = \sin 2\theta, \quad r = \cos 2\theta$

26. $r = 3 + 2 \cos \theta, \quad r = 3 + 2 \sin \theta$

---

27. Find the area inside the larger loop and outside the smaller loop of the limaçon $r = \frac{1}{2} + \cos \theta$.

28. When recording live performances, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage (as in the figure) and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8 \sin \theta$, where r is measured in meters and the microphone is at the pole. The musicians want to know the area they will have on stage within the optimal pickup range of the microphone. Answer their question.



Find all points of intersection of the given curves.

29. $r = 2 \sin 2\theta, \quad r = 1$ **30.** $r = \cos 3\theta, \quad r = \sin 3\theta$

31. $r = \sin \theta, \quad r = \sin 2\theta$ **32.** $r^2 = \sin 2\theta, \quad r^2 = \cos 2\theta$

---

33. The points of intersection of the cardioid $r = 1 + \sin \theta$ and the spiral loop $r = 2\theta$, $-\pi/2 \le \theta \le \pi/2$, can't be found exactly. Use a graphing device to find the approximate values of θ at which they intersect. Then use these values to estimate the area that lies inside both curves.

34. Use a graph to estimate the values of θ for which the curves $r = 3 + \sin 5\theta$ and $r = 6 \sin \theta$ intersect. Then estimate the area that lies inside both curves.

Find the exact length of the polar curve.

35. $r = 3 \sin \theta, \quad 0 \le \theta \le \pi/3$

36. $r = e^{2\theta}, \quad 0 \le \theta \le 2\pi$

37. $r = \theta^2, \quad 0 \le \theta \le 2\pi$

38. $r = \theta, \quad 0 \le \theta \le 2\pi$

Use a calculator to find the length of the curve correct to four decimal places.

39. $r = 3 \sin 2\theta$ **40.** $r = 4 \sin 3\theta$

Discovery Project | Conic Sections in Polar Coordinates

In this project we give a unified treatment of all three types of conic sections in terms of a focus and directrix. We will see that if we place the focus at the origin, then a conic section has a simple polar equation. In Chapter 10 we will use the polar equation of an ellipse to derive Kepler's laws of planetary motion.

Let F be a fixed point (called the **focus**) and l be a fixed line (called the **directrix**) in a plane. Let e be a fixed positive number (called the **eccentricity**). Let C be the set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from F to the distance from l is the constant e). Notice that if the eccentricity is $e = 1$, then $|PF| = |Pl|$ and so the given condition simply becomes the definition of a parabola as given in Appendix B.

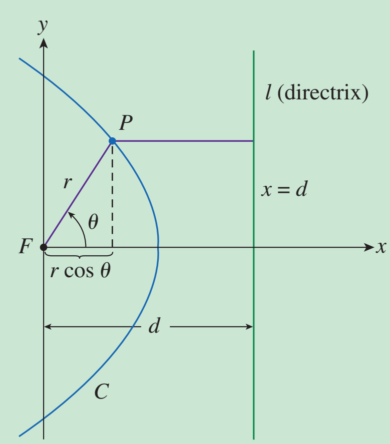


Figure 1
The conic sections can be described in terms of a focus and directrix.

1. If we place the focus F at the origin and the directrix parallel to the y-axis and d units to the right, then the directrix has equation $x = d$ and is perpendicular to the polar axis. If the point P has polar coordinates (r, θ), use Figure 1 to show that

$$r = e(d - r \cos \theta)$$

2. By converting the polar equation in Problem 1 to rectangular coordinates, show that the curve C is an ellipse if $e < 1$. (See Appendix B for a discussion of ellipses.)
3. Show that C is a hyperbola if $e > 1$.
4. Show that the polar equation

$$r = \frac{ed}{1 + e \cos \theta}$$

represents an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

5. For each of the following conics, find the eccentricity and directrix. Then identify and sketch the conic.

(a) $r = \dfrac{4}{1 + 3 \cos \theta}$ (b) $r = \dfrac{8}{3 + 3 \cos \theta}$ (c) $r = \dfrac{2}{2 + \cos \theta}$

6. Graph the conics $r = \dfrac{e}{(1 - e \cos \theta)}$ with $e = 0.4, 0.6, 0.8$, and 1.0 on a common screen. How does the value of e affect the shape of the curve?
7. (a) Show that the polar equation of an ellipse with directrix $x = d$ can be written in the form

$$r = \frac{a(1 - e^2)}{1 - e \cos \theta}$$

(b) Find an approximate polar equation for the elliptical orbit of the planet Earth around the sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about 2.99×10^8 km.

8. (a) The planets move around the sun in elliptical orbits with the sun at one focus. The positions of a planet that are closest to and farthest from the sun are called its *perihelion* and *aphelion,* respectively. (See Figure 2.) Use Problem 7(a) to show that the perihelion distance from a planet to the sun is $a(1 - e)$ and the aphelion distance is $a(1 + e)$.

(b) Use the data of Problem 7(b) to find the distances from the planet Earth to the sun at perihelion and at aphelion.

9. (a) The planet Mercury travels in an elliptical orbit with eccentricity 0.206. Its minimum distance from the sun is 4.6×10^7 km. Use the results of Problem 8(a) to find its maximum distance from the sun.

(b) Find the distance traveled by the planet Mercury during one complete orbit around the sun. (Use technology to evaluate the definite integral.)

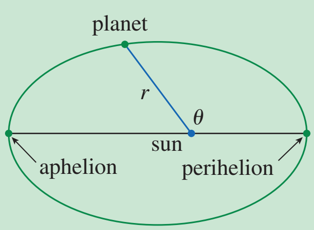


Figure 2
The positions of a planet closest to and farthest from the sun.

I Complex Numbers

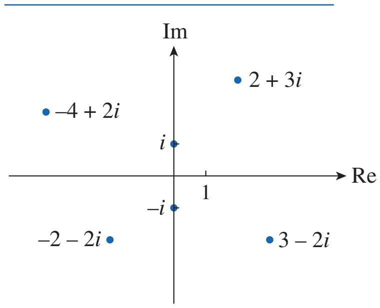


Figure 1
Complex numbers as points in the Argand plane.

A **complex number** can be represented by an expression of the form $a + bi$, where a and b are real numbers and i is a symbol with the property that $i^2 = -1$. The complex number $a + bi$ can also be represented by the ordered pair (a, b) and plotted as a point in a plane (called the Argand plane) as in Figure 1. Thus the complex number $i = 0 + 1 \cdot i$ is identified with the point $(0, 1)$.

The **real part** of the complex number $a + bi$ is the real number a and the **imaginary part** is the real number b. Thus the real part of $4 - 3i$ is 4 and the imaginary part is -3. Two complex numbers $a + bi$ and $c + di$ are **equal** if $a = c$ and $b = d$, that is, their real parts are equal and their imaginary parts are equal. In the Argand plane, the horizontal axis is called the real axis and the vertical axis is called the imaginary axis.

The sum and difference of two complex numbers are defined by adding or subtracting their real parts and their imaginary parts:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

For instance,

$$(1 - i) + (4 + 7i) = (1 + 4) + (-1 + 7)i = 5 + 6i$$

The product of complex numbers is defined so that the usual commutative and distributive laws hold:

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + (bi)(c + di)\\ &= ac + adi + bci + bdi^2\end{aligned}$$

Since $i^2 = -1$, this becomes

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Example 1

$$\begin{aligned}(-1 + 3i)(2 - 5i) &= (-1)(2 - 5i) + 3i(2 - 5i)\\ &= -2 + 5i + 6i - 15(-1) = 13 + 11i\end{aligned}$$

Division of complex numbers is much like rationalizing the denominator of a rational expression. For the complex number $z = a + bi$, we define its **complex conjugate** to be $\bar{z} = a - bi$. To find the quotient of two complex numbers, we multiply numerator and denominator by the complex conjugate of the denominator.

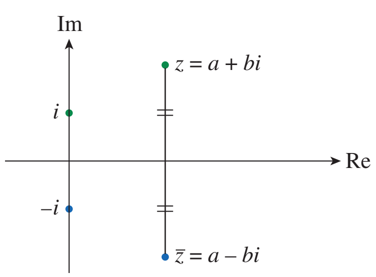


Figure 2
Geometric representation of a complex conjugate.

Example 2

Express the number $\dfrac{-1 + 3i}{2 + 5i}$ in the form $a + bi$.

Solution

We multiply numerator and denominator by the complex conjugate of $2 + 5i$, namely $2 - 5i$, and we take advantage of the result of Example 1:

$$\frac{-1 + 3i}{2 + 5i} = \frac{-1 + 3i}{2 + 5i} \cdot \frac{2 - 5i}{2 - 5i} = \frac{13 + 11i}{2^2 + 5^2} = \frac{13}{29} + \frac{11}{29}i$$

The geometric interpretation of the complex conjugate is shown in Figure 2: $\bar{z}$ is the reflection of z in the real axis. We list some of the properties of the complex conjugate in the following box. The proofs follow from the definition and are requested in Exercise 18.

Properties of Conjugates

$$\overline{z + w} = \bar{z} + \bar{w} \qquad \overline{zw} = \bar{z}\,\bar{w} \qquad \overline{z^n} = \bar{z}^n$$

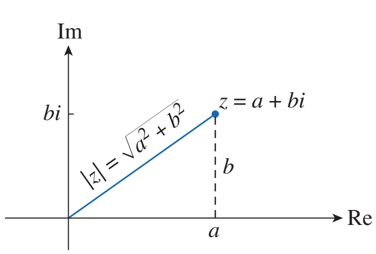


Figure 3
Geometric illustration of the absolute value of a complex number.

The **modulus**, or **absolute value**, $|z|$ of a complex number $z = a + bi$ is its distance from the origin. From Figure 3 we see that if $z = a + bi$, then

$$|z| = \sqrt{a^2 + b^2}$$

Notice that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2$$

and so

$$z\bar{z} = |z|^2$$

This explains why the division procedure in Example 2 works in general:

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}$$

Since $i^2 = -1$, we can think of i as a square root of -1. But notice that we also have $(-i)^2 = i^2 = -1$ and so $-i$ is also a square root of -1. We say that i is the **principal square root** of -1 and write $\sqrt{-1} = i$. In general, if c is any positive number, we write

$$\sqrt{-c} = \sqrt{c}\, i$$

With this convention, the usual derivation and formula for the roots of the quadratic equation $ax^2 + bx + c = 0$ are valid even when $b^2 - 4ac < 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 3

Find the roots of the equation $x^2 + x + 1 = 0$.

Solution

Using the quadratic formula, we have

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$ ■

We observe that the solutions of the equation in Example 3 are complex conjugates of each other. In general, the solutions of any quadratic equation $ax^2 + bx + c = 0$ with real coefficients a, b, and c are always complex conjugates. (If z is real, $\overline{z} = z$, so z is its own conjugate.)

We have seen that if we allow complex numbers as solutions, then every quadratic equation has a solution. More generally, it is true that every polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree at least one has a solution among the complex numbers. This fact is known as the Fundamental Theorem of Algebra and was proved by Gauss.

Polar Form

We know that any complex number $z = a + bi$ can be considered as a point (a, b) and that any such point can be represented by polar coordinates (r, θ) with $r \geq 0$. In fact,

$$a = r\cos\theta \qquad b = r\sin\theta$$

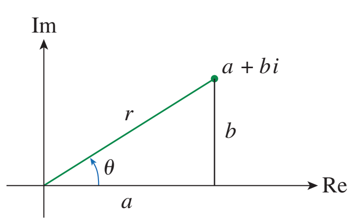


Figure 4
Geometric illustration to represent a complex number by polar coordinates.

as in Figure 4. Therefore, we have

$$z = a + bi = (r\cos\theta) + (r\sin\theta)i$$

Thus we can write any complex number z in the form

$$z = r(\cos\theta + i\sin\theta)$$

where $$r = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan\theta = \frac{b}{a}$$

The angle θ is called the **argument** of z and we write $\theta = \arg(z)$. Note that $\arg(z)$ is not unique; any two arguments of z differ by an integer multiple of 2π.

Example 4

Write the following numbers in polar form.

(a) $z = 1 + i$ (b) $w = \sqrt{3} - i$

Solution

(a) We have $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\tan\theta = 1$, so we can take $\theta = \pi/4$. Therefore, the polar form is

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(b) Here we have $r = |w| = \sqrt{3+1} = 2$ and $\tan\theta = -\dfrac{1}{\sqrt{3}}$. Since w lies in the fourth quadrant, we take $\theta = -\dfrac{\pi}{6}$ and

$$w = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

The numbers z and w are shown in Figure 5. ■

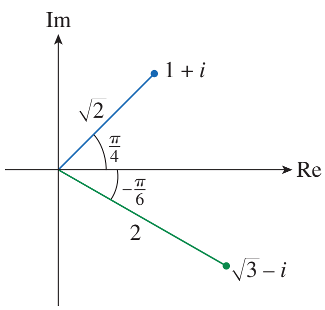


Figure 5
The complex numbers z and w in Example 4.

The polar form of complex numbers gives insight into multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be two complex numbers written in polar form. Then

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \end{aligned}$$

Therefore, using the addition formulas for cosine and sine, we have

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \tag{1}$$

This formula says that *to multiply two complex numbers, we multiply the moduli and add the arguments.* (See Figure 6.)

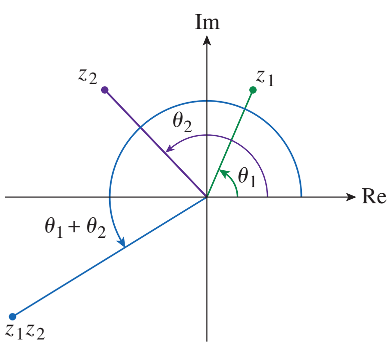


Figure 6
Geometric illustration for the multiplication of two complex numbers.

A similar argument using the subtraction formulas for sine and cosine shows that *to divide two complex numbers, we divide the moduli and subtract the arguments.*

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \quad z_2 \neq 0$$

In particular, taking $z_1 = 1$ and $z_2 = z$ (and therefore $\theta_1 = 0$ and $\theta_2 = \theta$), we have the following, which is illustrated in Figure 7.

$$\text{If} \quad z = r(\cos\theta + i\sin\theta), \quad \text{then} \quad \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta).$$

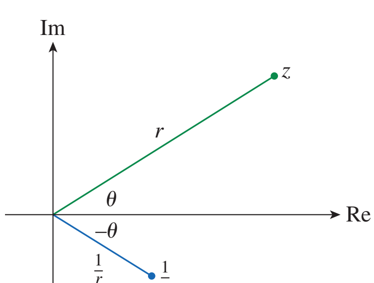


Figure 7
Geometric illustration for the reciprocal of a complex number.

Example 5

Find the product of the complex numbers $1 + i$ and $\sqrt{3} - i$ in polar form.

Solution

From Example 4 we have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

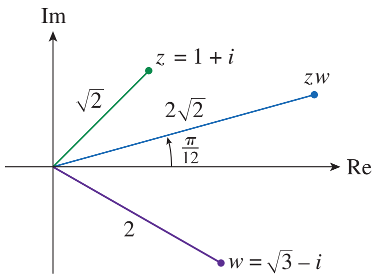


Figure 8
Geometric illustration for the product of z and w.

and $\sqrt{3} - i = 2\left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right]$

So, by Equation 1,

$$(1 + i)(\sqrt{3} - i) = 2\sqrt{2}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right]$$
$$= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i \sin\frac{\pi}{12}\right)$$

■

This is illustrated in Figure 8.

Repeated use of Formula 1 shows how to compute powers of a complex number. If

$$z = r(\cos\theta + i \sin\theta)$$

then $$z^2 = r^2(\cos 2\theta + i \sin 2\theta)$$

and $$z^3 = zz^2 = r^3(\cos 3\theta + i \sin 3\theta)$$

In general, we obtain the following result, which is named after the French mathematician Abraham De Moivre (1667–1754).

De Moivre's Theorem

If $z = r(\cos\theta + i \sin\theta)$ and n is a positive integer, then

$$z^n = [r(\cos\theta + i \sin\theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

This says that *to take the nth power of a complex number, we take the nth power of the modulus and multiply the argument by n.*

Example 6

Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$.

Solution

Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 4(a) that $\frac{1}{2} + \frac{1}{2}i$ has the polar form

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$$

So by De Moivre's Theorem,

$$\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i \sin\frac{10\pi}{4}\right)$$
$$= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i \sin\frac{5\pi}{2}\right) = \frac{1}{32}i$$

■

De Moivre's Theorem can also be used to find the nth roots of complex numbers. An nth root of the complex number z is a complex number w such that

$$w^n = z$$

Writing these two numbers in trigonometric form as

$$w = s(\cos\phi + i \sin\phi) \quad \text{and} \quad z = r(\cos\theta + i \sin\theta)$$

and using De Moivre's Theorem, we get

$$s^n(\cos n\phi + i \sin n\phi) = r(\cos \theta + i \sin \theta)$$

The equality of these two complex numbers shows that

$$s^n = r \quad \text{or} \quad s = r^{1/n}$$

and $$\cos n\phi = \cos \theta \quad \text{and} \quad \sin n\phi = \sin \theta$$

From the fact that sine and cosine have period 2π, it follows that

$$n\phi = \theta + 2k\pi \quad \text{or} \quad \phi = \frac{\theta + 2k\pi}{n}$$

Thus $$w = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Since this expression gives a different value of w for $k = 0, 1, 2, \ldots, n - 1$, we have the following.

Roots of a Complex Number

Let $z = r(\cos \theta + i \sin \theta)$ and let n be a positive integer. Then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$.

Notice that each of the nth roots of z has modulus $|w_k| = r^{1/n}$. Thus all the nth roots of z lie on the circle of radius $r^{1/n}$ in the complex plane. Also, since the argument of each successive nth root exceeds the argument of the previous root by $2\pi/n$, we see that the nth roots of z are equally spaced on this circle.

Example 7

Find the six sixth roots of $z = -8$ and graph these roots in the complex plane.

Solution

In trigonometric form, $z = 8(\cos \pi + i \sin \pi)$. Applying Equation 3 with $n = 6$, we get

$$w_k = 8^{1/6}\left(\cos\frac{\pi + 2k\pi}{6} + i \sin\frac{\pi + 2k\pi}{6}\right)$$

We get the six sixth roots of -8 by taking $k = 0, 1, 2, 3, 4, 5$ in this formula:

$$w_0 = 8^{1/6}\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$w_1 = 8^{1/6}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = \sqrt{2}i$$

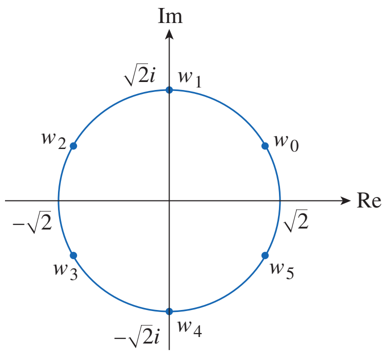


Figure 9
The six sixth roots of $z = -8$.

$$w_2 = 8^{1/6}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$w_3 = 8^{1/6}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$w_4 = 8^{1/6}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -\sqrt{2}i$$

$$w_5 = 8^{1/6}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

All these points lie on the circle of radius $\sqrt{2}$ as shown in Figure 9. ■

Complex Exponentials

We also need to give a meaning to the expression e^z when $z = x + iy$ is a complex number. The theory of infinite series as developed in Chapter 8 can be extended to the case where the terms are complex numbers. Using the Taylor series for e^x (8.7.9) as our guide, we define

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \tag{2}$$

and it turns out that this complex exponential function has the same properties as the real exponential function. In particular, it is true that

$$e^{z_1 + z_2} = e^{z_1}e^{z_2} \tag{3}$$

If we put $z = iy$, where y is a real number, in Equation 4, and use the facts that

$$i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = 1, \quad i^5 = i, \ldots$$

we get

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots \\ &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i \sin y \end{aligned}$$

Here we have used the Taylor series for $\cos y$ and $\sin y$ (Equations 8.7.16 and 8.7.15). The result is a famous formula called **Euler's formula**:

$$e^{iy} = \cos y + i \sin y \tag{4}$$

Combining Euler's formula with Equation 5, we get

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) \tag{5}$$

We could write the result of Example 8(a) as

$$e^{i\pi} + 1 = 0$$

This equation relates the five most famous numbers in all of mathematics: 0, 1, e, i, and π.

Example 8

Evaluate: (a) $e^{i\pi}$ (b) $e^{-1+i\pi/2}$

Solution

(a) From Euler's equation (4), we have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$$

(b) Using Equation 5 we get

$$e^{-1+i\pi/2} = e^{-1}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = \frac{1}{e}[0 + i(1)] = \frac{i}{e}$$ ■

Finally, we note that Euler's equation provides us with an easier method of proving De Moivre's Theorem:

$$[r(\cos \theta + i \sin \theta)]^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

I Exercises

Evaluate the expression and write your answer in the form $a + bi$.

1. $(5 - 6i) + (3 + 2i)$ **2.** $\left(4 - \frac{1}{2}i\right) - \left(9 + \frac{5}{2}i\right)$

3. $(2 + 5i)(4 - i)$ **4.** $(1 - 2i)(8 - 3i)$

5. $\overline{12 + 7i}$ **6.** $\overline{2i\left(\frac{1}{2} - i\right)}$

7. $\dfrac{1 + 4i}{3 + 2i}$ **8.** $\dfrac{3 + 2i}{1 - 4i}$

9. $\dfrac{1}{1 + i}$ **10.** $\dfrac{3}{4 - 3i}$

11. i^3 **12.** i^{100}

13. $\sqrt{-25}$ **14.** $\sqrt{-3}\sqrt{-12}$

Find the complex conjugate and the modulus of the number.

15. $12 - 5i$ **16.** $-1 + 2\sqrt{2}\,i$

17. $-4i$

18. Prove the following properties of complex numbers.
(a) $\overline{z + w} = \bar{z} + \bar{w}$ (b) $\overline{zw} = \bar{z}\,\bar{w}$
(c) $\overline{z^n} = \bar{z}^n$, where n is a positive integer

Hint: Write $z = a + bi$, $w = c + di$.

Find all solutions of the equation.

19. $4x^2 + 9 = 0$ **20.** $x^4 = 1$

21. $x^2 + 2x + 5 = 0$ **22.** $2x^2 - 2x + 1 = 0$

23. $z^2 + z + 2 = 0$ **24.** $z^2 + \frac{1}{2}z + \frac{1}{4} = 0$

Write the number in polar form with argument between 0 and 2π.

25. $-3 + 3i$ **26.** $1 - \sqrt{3}\,i$

27. $3 + 4i$ **28.** $8i$

Find polar forms for zw, z/w, and $1/z$ by first putting z and w into polar form.

29. $z = \sqrt{3} + i$, $w = 1 + \sqrt{3}\,i$

30. $z = 4\sqrt{3} - 4i$, $w = 8i$

31. $z = 2\sqrt{3} - 2i$, $w = -1 + i$

32. $z = 4(\sqrt{3} + i)$, $w = -3 - 3i$

Find the indicated power using De Moivre's Theorem.

33. $(1 + i)^{20}$ **34.** $(1 - \sqrt{3}i)^5$

35. $(2\sqrt{3} + 2i)^5$ **36.** $(1 - i)^8$

Find the indicated roots. Sketch the roots in the complex plane.

37. The eighth roots of 1 **38.** The fifth roots of 32

39. The cube roots of i **40.** The cube roots of $1 + i$

Write the number in the form $a + bi$.

41. $e^{i\pi/2}$

42. $e^{2\pi i}$

43. $e^{i\pi/3}$

44. $e^{-i\pi}$

45. $e^{2+i\pi}$

46. $e^{\pi+i}$

---

47. Use De Moivre's Theorem with $n = 3$ to express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos \theta$ and $\sin \theta$.

48. Use Euler's formula to prove the following formulas for $\cos x$ and $\sin x$:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

49. If $u(x) = f(x) + ig(x)$ is a complex-valued function of a real variable x and the real and imaginary parts $f(x)$ and $g(x)$ are differentiable functions of x, then the derivative of u is defined to be $u'(x) = f'(x) + ig'(x)$. Use this together with Equation 7 to prove that if $F(x) = e^{rx}$, then $F'(x) = re^{rx}$ when $r = a + bi$ is a complex number.

50. (a) If u is a complex-valued function of a real variable, its indefinite integral $\int u(x)\, dx$ is an antiderivative of u. Evaluate

$$\int e^{(1+i)x}\, dx$$

(b) By considering the real and imaginary parts of the integral in part (a), evaluate the real integrals

$$\int e^x \cos x\, dx \quad \text{and} \quad \int e^x \sin x\, dx$$

(c) Compare with the method used in Example 4 in Section 5.6.

Answers to Odd-Numbered Exercises

Chapter 1

Exercises 1.1 ■ Page 20

1. (a) 3 (b) -0.2 (c) $x = 0, 3$ (d) -0.8
(e) Domain: $\{-2 \le x \le 4\}$, Range: $\{-1 \le y \le 3\}$ (f) $[-2, 1]$

3. (a) 12 (b) 16 (c) $3a^2 - a + 2$ (d) $3a^2 + a + 2$
(e) $3a^2 + 5a + 4$ (f) $6a^2 - 2a + 4$ (g) $12a^2 - 2a + 2$
(h) $3a^4 - a^2 + 2$ (i) $9a^4 - 6a^3 + 13a^2 - 4a + 4$
(j) $3a^2 + 6ah + 3h^2 - a - h + 2$

5. $3a^2 + 3ah + h^2$

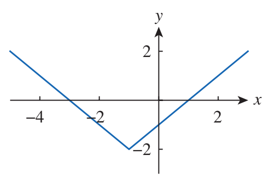
7. $-1/(x + 1)$

9. Domain: $\{x \in \mathbb{R} \mid x \ne -3, 2\}$

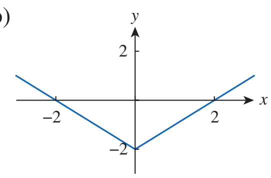
11. $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$

13. $[0, 4]$

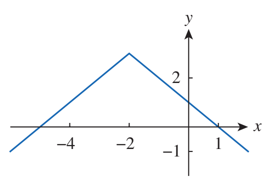
15. (a)



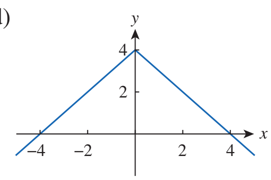
(b)



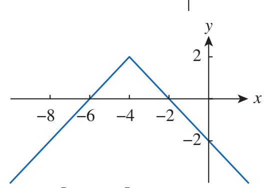
(c)



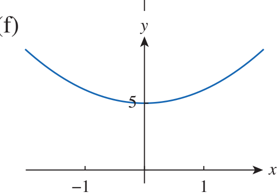
(d)



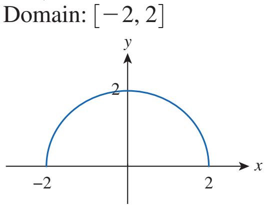
(e)



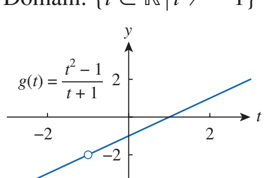
(f)



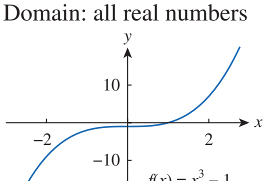
17. Domain: $[-2, 2]$



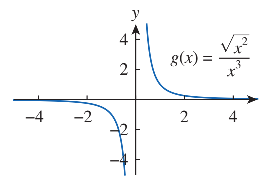
19. Domain: $\{t \in \mathbb{R} \mid t \ne -1\}$



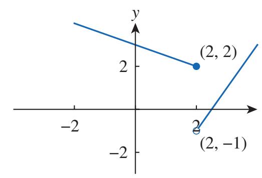
21. Domain: all real numbers



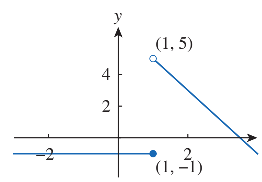
23. Domain: $\{x \in \mathbb{R} \mid x \ne 0\}$



25. $\frac{9}{2}$, 3, -1



27. $-1, -1, 3$



29. $f(x) = -\frac{5}{3}x + \frac{5}{3}$, $-5 \le x \le 7$

31. $f(x) = 2 + \sqrt{4 - x^2}$, $-2 \le x \le 2$

33. $f(x) = \begin{cases} -\frac{3}{2}x - 3 & \text{if } -4 \le x \le -2 \\ \sqrt{4 - x^2} & \text{if } -2 < x < 2 \\ \frac{3}{2}x - 3 & \text{if } 2 \le x \le 4 \end{cases}$

35. Answers will vary.

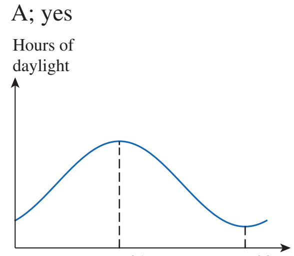
37. The curve is the graph of a function.

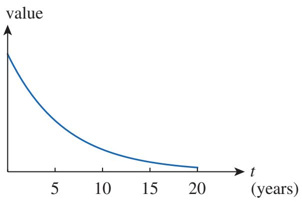
39. The curve is not the graph of a function.

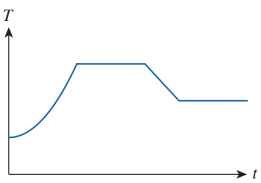
41. (a) $[0, 1.6]$ (b) According to the graph, the earth gradually cooled from 1550 to 1700, warmed into the late 1700s, cooled again into the late 1800s, and has been steadily warming since then. In the mid-19th century, there was variation that could have been associated with volcanic eruptions.

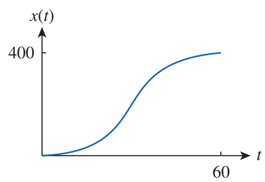
43. A; yes

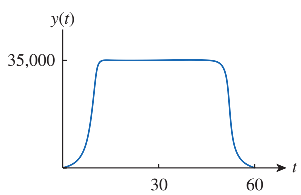
45.

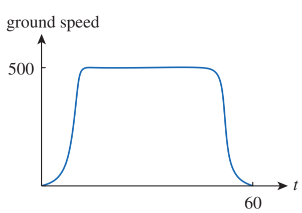


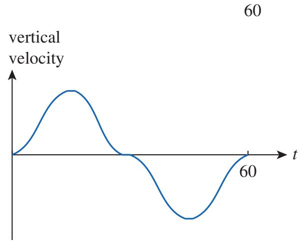
47. 

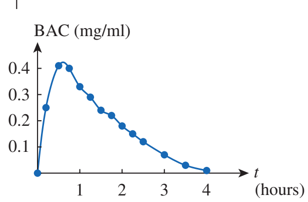
49. 

51. (a) 

(b) 

(c) 

(d) 

53. (a) 

(b) The rate or BAC jumps rapidly from zero to the maximum of 0.41 mg/mL. The concentration then gradually decreases over the next four hours to near zero.

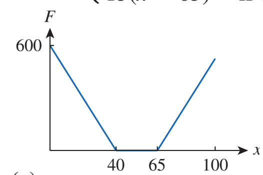
55. $A(L) = L(10 - L) = 10L - L^2$

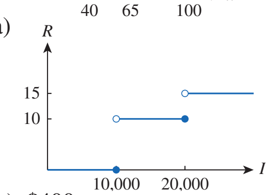
57. $A(x) = \dfrac{\sqrt{3}}{4}x^2$

59. $S(x) = x^2 + (8/x)$

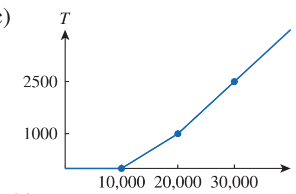
61. $V(x) = (20 - 2x)(12 - 2x)(x) = 4x^3 - 64x^2 + 240x$

63. $F(x) = \begin{cases} 15(40 - x) & \text{if } 0 \le x < 40 \\ 0 & \text{if } 40 \le x \le 65 \\ 15(x - 65) & \text{if } 65 < x \le 100 \end{cases}$



65. (a) 

(b) \$400

(c) 

67. f odd; g even

69. (a) $(-5, 3)$ (b) $(-5, -3)$

71. Odd

73. Neither

75. Even

77. Even; odd; neither (unless $f = 0$ or $g = 0$)

Exercises 1.2 ■ Page 35

1. (a) Logarithmic (b) Rational (c) Polynomial, degree 2 (d) Polynomial, degree 2 (e) Power (f) Polynomial, degree 6 (g) Exponential (h) Trigonometric

3. (a) $y = x^2$; blue (b) $y = x^5$; red (c) $y = x^8$; green

5. Domain and range: all real numbers

7. Domain and range: all real numbers

9. Domain and range: all real numbers

11. Domain: $\mathbb{R}$, range: $[2, \infty)$

13. Vertical asymptote: $x = -4$; horizontal asymptote: $y = 1$

15. Vertical asymptote: $x = -2$; horizontal asymptote: $y = 0$

17. Vertical asymptote: $x = 2$; horizontal asymptote: $y = 0$

19. Graph is of a function.

21. Graph is not of a function.

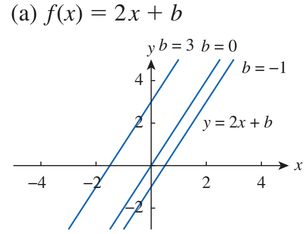
23. $y = 2x + 5$

25. 5

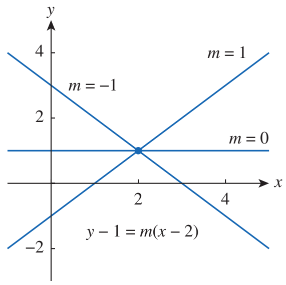
27. $f(x) = -x^2 + 8x - 10$

29. $f(x) = \dfrac{x - 4}{x - 3}$

31. (a) $f(x) = 2x + b$

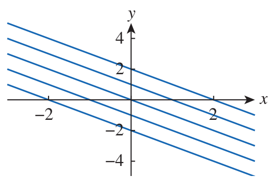


(b) $f(x) = mx + (1 - 2m)$



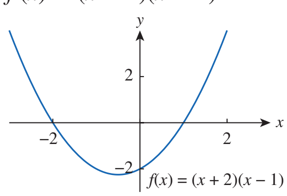
(c) $f(x) = 2x - 3$

33. Lines; slope -1. y-intercept c



35. $f(x) = -3x(x + 1)(x - 1)$

37. $f(x) = (x + 2)(x - 1)$

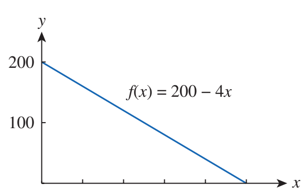


39. (a) The slope is 0.02: the average surface temperature of the world is increasing at a rate of 0.02°C per year. The T intercept is 8.50: the average surface temperature in °C in the year 1900.

(b) 12.50°C

41. 112

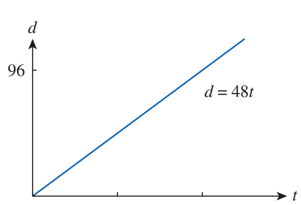
43. (a)



(b) Slope $= -4$: For each increase of 1 dollar for a rental space, the number of spaces rented decreases by 4. y-intercept $= 200$: the number of spaces occupied if there was no charge for each space. x-intercept $= 50$: the smallest rental fee that results in no spaces rented.

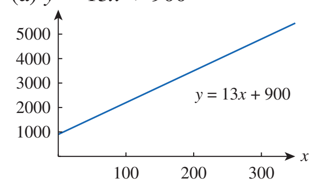
45. (a) $d = 48t$

(b)



(c) slope $= 48$: the car's speed in mi/h

47. (a) $y = 13x + 900$



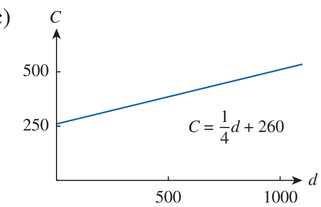
(b) Slope $= 13$: the cost (in dollars) of producing each additional chair

(c) y-intercept $= 900$: the fixed daily costs of operating the factory

49. (a) $C = \frac{1}{4}d + 260$

(b) \$635

(c)



Slope $= 0.25$: the cost per mile

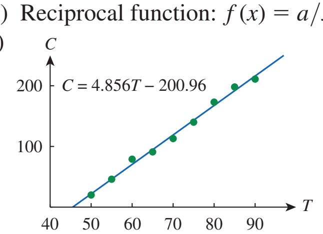
(d) y-intercept $= 260$: the fixed cost

(e) There are fixed monthly costs as well as costs for each additional mile driven.

51. (a) Exponential model: $f(x) = ab^x$ or $f(x) = ab^x + c$

(b) Reciprocal function: $f(x) = a/x$

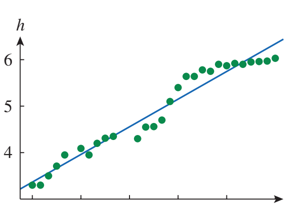
53. (a)



(b) $C = 4.856T - 220.96$

(c) 265

55. (a)



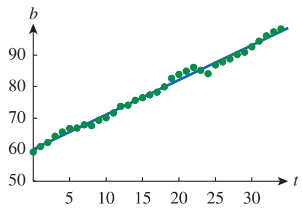
A linear model seems appropriate over the time interval considered.

(b) $h = 0.0247965t - 43.6486$

(c) 6.44

(d) No. The height appears to be leveling off.

57. (a)



A linear model seems appropriate over the time interval considered.

(b) $b = 1.099t + 60.109$

(c) 78.804

(d) 98.599

59. 4

61. (a) $T = 1.000431227d^{1.499528750}$

(b) $T \approx d^{1.5} \quad \Rightarrow \quad T^2 \approx d^3$; matches Kepler's Third Law

Exercises 1.3 ■ Page 47

1. (a) $y = f(x) + 3$ (b) $y = f(x) - 3$ (c) $y = f(x - 3)$

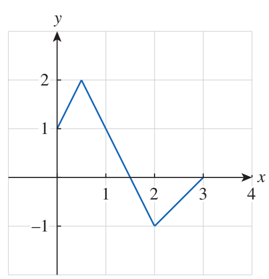
(d) $y = f(x + 3)$ (e) $y = -f(x)$ (f) $y = f(-x)$

(g) $y = 3f(x)$ (h) $y = \dfrac{1}{3}f(x)$

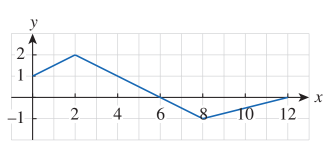
3. (a) Graph 3 (b) Graph 1 (c) Graph 4 (d) Graph 5

(e) Graph 2

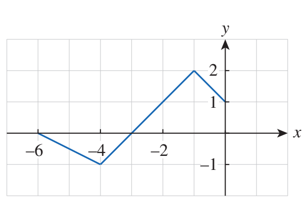
5. (a)



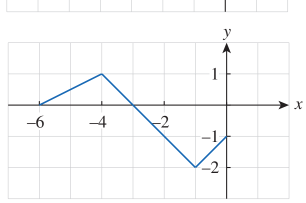
(b)



(c)

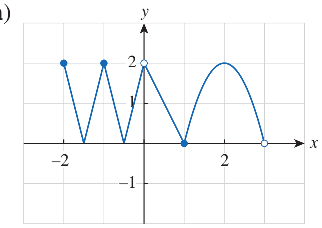


(d)

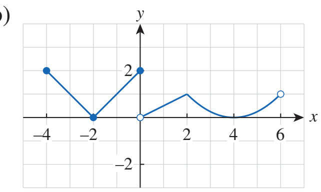


7. $g \circ f = \left\{ \left(1, \dfrac{1}{4}\right), \left(0, \dfrac{1}{2}\right), \left(-4, \dfrac{1}{3}\right), \left(2, \dfrac{1}{9}\right) \right\}$

9. (a)



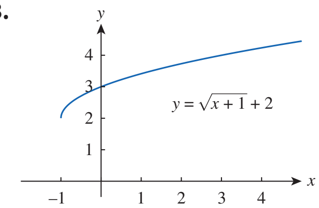
(b)



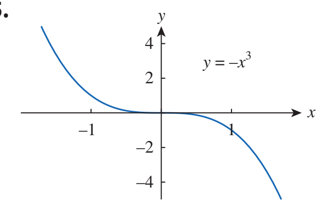
11. (a) $[-1, 1) \cup (1, 3]$ (b) $[-7, -6) \cup [-3, 3)$

(c) $[-8, -6) \cup [-6, 6) \cup (6, 8)$ (d) $[-4, 3) \cup (3, 4)$

13.



15.



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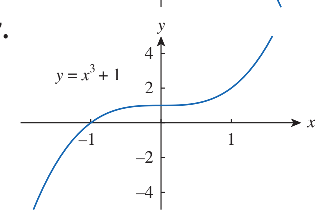

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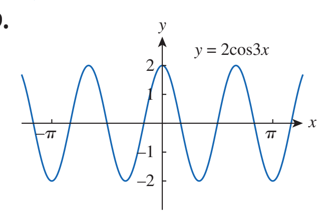

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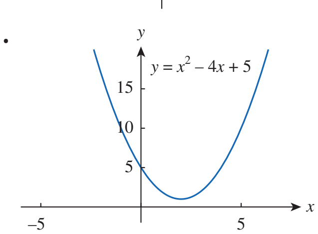

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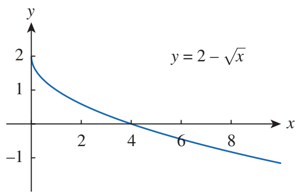

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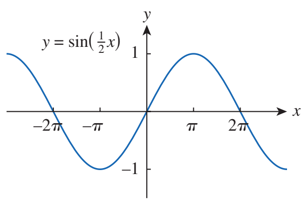

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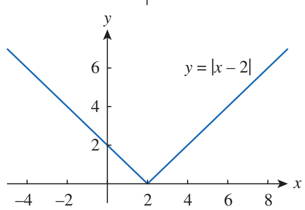

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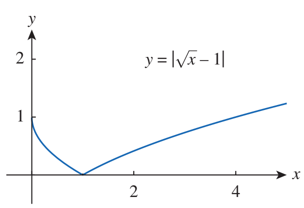
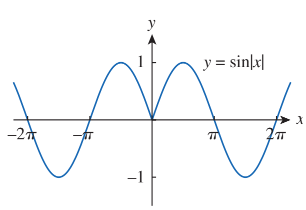


31. $L(t) = 12 + 2\sin\left(\dfrac{2\pi}{365}(t - 80)\right)$; $L(90) \approx 12.34$

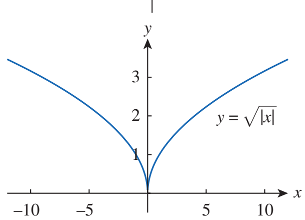
33. $D(t) = 5\cos\left(\dfrac{2\pi}{12}(t - 13.033)\right) + 7$

35. (a) Portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

(b)



(c)



37. (a) $(f + g)(x) = x^3 + 5x^2 - 1$; all reals

(b) $(f - g)(x) = x^3 - x^2 + 1$; all reals

(c) $(fg)(x) = 3x^5 + 6x^4 - x^3 - 2x^2$; all reals

(d) $(f/g)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1}$; $\left\{x \in \mathbb{R} \,\middle|\, x \neq \pm\dfrac{\sqrt{3}}{3}\right\}$

39. (a) $(f \circ g)(x) = 3x^2 + 3x + 5$; all reals

(b) $(g \circ f)(x) = 9x^2 + 32x + 30$; all reals

(c) $(f \circ f)(x) = 9x + 20$; all reals

(d) $(g \circ g)(x) = x^4 + 2x^3 + 2x^2 + x$; all reals

41. (a) $(f \circ g)(x) = \sqrt{4x - 2}$; $\left[-\dfrac{1}{2}, \infty\right)$

(b) $(g \circ f)(x) = 4\sqrt{x + 1} - 3$; $[-1, \infty)$

(c) $(f \circ f)(x) = \sqrt{\sqrt{x + 1} + 1}$; $[-1, \infty)$

(d) $(g \circ g)(x) = 16x - 15$; all reals

43. (a) $(f \circ g)(x) = \dfrac{2x^2 + 6x + 5}{(x + 1)(x + 2)}$; $\{x \in \mathbb{R} \mid x \neq -2, -1\}$

(b) $(g \circ f)(x) = \dfrac{x^2 + x + 1}{x^2 + 2x + 1}$; $\{x \in \mathbb{R} \mid x \neq -1, 0\}$

(c) $(f \circ f)(x) = \dfrac{x^4 + 3x^2 + 1}{x(x^2 + 1)}$; $\{x \in \mathbb{R} \mid x \neq 0\}$

(d) $(g \circ g)(x) = \dfrac{2x + 3}{3x + 5}$; $\left\{x \in \mathbb{R} \,\middle|\, x \neq -2, -\dfrac{5}{3}\right\}$

45. $(f \circ g \circ h)(x) = 3\sin x^2 - 2$

47. $(f \circ g \circ h)(x) = \sqrt{x^6 + 4x^3 + 1}$

49. $f(x) = x^4$; $g(x) = 2x + x^2$

51. $f(x) = \dfrac{x}{1 + x}$; $g(x) = \sqrt[3]{x}$

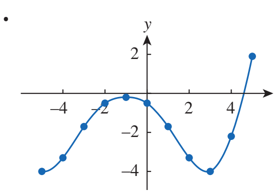
53. $f(t) = \sec t \tan t$; $g(t) = t^2$

55. $f(x) = \sqrt{x}$; $g(x) = x - 1$; $h(x) = \sqrt{x}$

57. $f(t) = t^2$; $g(t) = \sin t$; $h(t) = \cos t$

59. (a) 5 (b) 2 (c) 4 (d) 3 (e) 1 (f) 4

61.



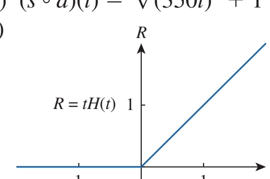
63. (a) $r = 2t$

(b) $V = \dfrac{32}{3}\pi t^3$; volume as a function of time

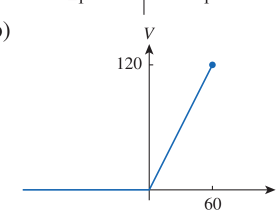
65. (a) $d(t) = 350t$ (b) $s(d) = \sqrt{d^2 + 1}$

(c) $(s \circ d)(t) = \sqrt{(350t)^2 + 1}$

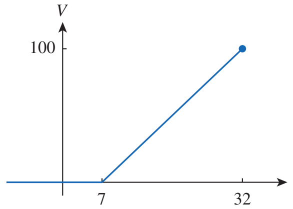
67. (a)



(b)



(c)



69. Investment amounts after 2, 3, and 4 years; $1.04^n x$

71. $g(x) = 4x - 17$

73. No; odd; even

Exercises 1.4 ■ Page 58

1. (a) 4 (b) $x^{-4/3}$

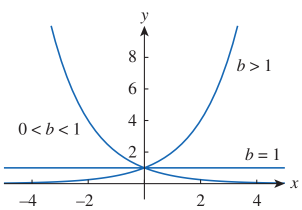
3. (a) $16b^{12}$ (b) $648y^7$

5. (a) $7\sqrt{5}$ (b) $2^{5/2} = 4\sqrt{2}$

7. (a) x^{4n-3} (b) $a^{1/6}b^{-1/12}$

9. (a) $f(x) = b^x$ (b) $\mathbb{R}$ (c) $(0, \infty)$

(d)



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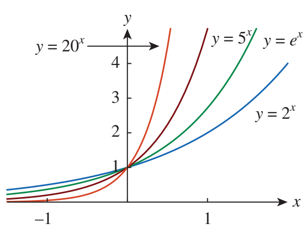

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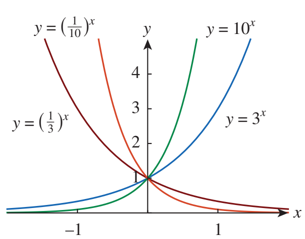

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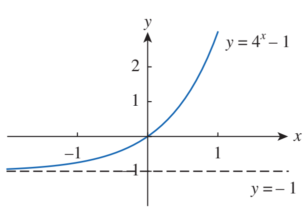

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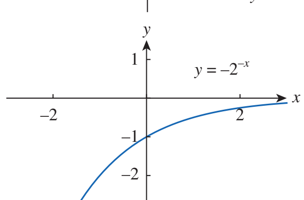

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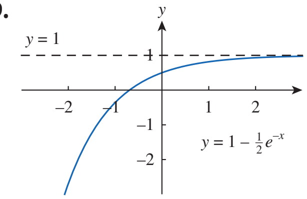

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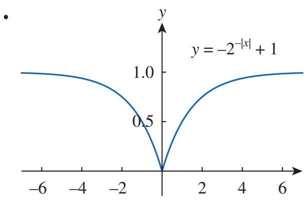


23. (a) $y = e^x - 2$ (b) $y = e^{x-2}$ (c) $y = -e^x$ (d) $y = e^{-x}$
(e) $y = -e^{-x}$

25. $\{x \mid x / \pm 1\}$

27. $[2, \infty)$

29. $x = \dfrac{6}{5}$

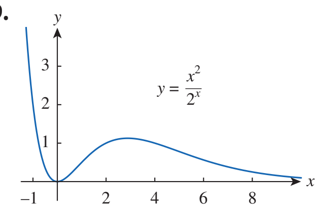
31. 1, 2

33. $\dfrac{1}{3^{1/3}}$

35. $f(x) = 3 \cdot 2^x$

37. $f(x) = 3(1 - e^{-x})$

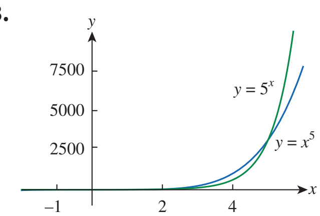
39.



As x increases, the function approaches 0; 2^x grows faster than x^2.

41. Second method

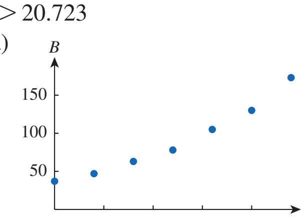
43.



$x = 1.765, 5; 5^x$

45. $x > 20.723$

47. (a)



(b) $f(x) = 36.7826 \cdot 1.06633^t$

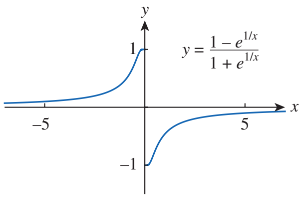
(c) 10.884

49. (a) 25 mg (b) $y = 200 \cdot 2^{-t/5}$ (c) 10.882 (d) 38.219

51. 5.817 days

53. 5637.17, 8135.39

55.



Exercises 1.5 ■ Page 68

1. (a) $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

(b) Horizontal line test

3. No

5. No

7. Yes

9. Yes

11. No

13. No

15. (a) 6 (b) 3

17. 0

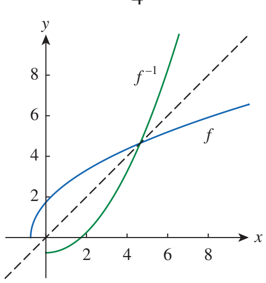
19. $F = \frac{9}{5}C + 32; [-273.15, \infty)$

21. $f^{-1}(x) = \frac{1}{3}(x-1)^2 - \frac{2}{3}$

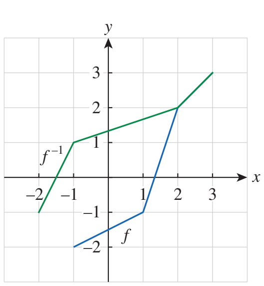
23. $f^{-1}(x) = \frac{1}{2}(1 + \ln x)$

25. $f^{-1}(x) = e^x - 3$

27. $f^{-1}(x) = \dfrac{x^2 - 3}{4}$



29.

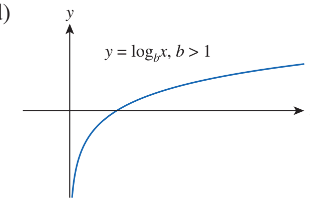


31. (a) f and f^{-1} are the same.

(b) Quarter circle; reflection about $y = x$ is the same.

33. (a) $\log_b x = y \Leftrightarrow b^y = x$ (b) $(0, \infty)$ (c) $\mathbb{R}$

(d)



35. (a) 5 (b) $\frac{1}{3}$

37. (a) 2 (b) $\frac{2}{3}$

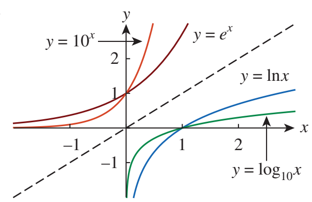
39. ln 250

41. $\ln \dfrac{\sqrt{x}}{x+1}$

43. $\sqrt{p}$

45. 3.680144

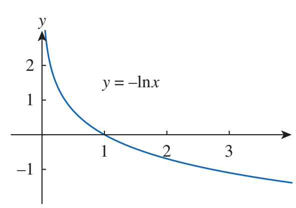
47.



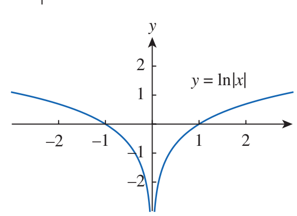
49. 1,084,587.7

51. 3.059

53.

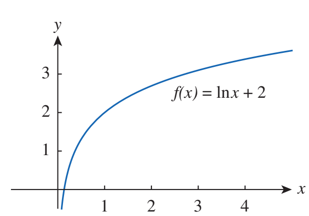


55.



57. (a) $x > 0$; $\mathbb{R}$ (b) $x = e^{-1}$

(c)



59. $x = \frac{1}{4}(7 - \ln 6)$

61. $x = \pm\sqrt{e^3 + 1}$

63. $x = 5 + \dfrac{\ln 3}{\ln 2}$

65. $x = e^e$

67. $x > 5 + 10^{\ln 6}$

69. $x = e^{-1/2}, e$

71. $x \in (e^{-1}, e)$

73. $x > \ln 5$

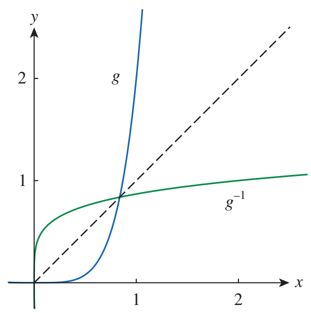
75. $x > e^{-1}$

77. (a) 300 (b) Answers will vary.

79. (a) $g^{-1}(x) = \dfrac{\sqrt{6}}{6} \dfrac{\sqrt{C^{1/3}(C^{2/3} - 2C^{1/3} + 4)}}{C^{1/3}}$;

$C = 108x + 12\sqrt{3}\sqrt{x(27x - 4)}$

(b)



81. (a) $t = -a \ln(1 - Q/Q_0)$; time necessary to obtain a given charge.

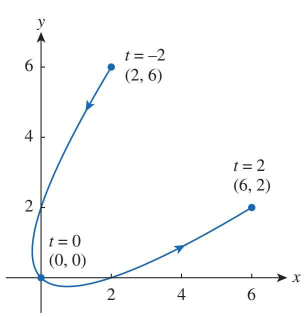
(b) 4.6 seconds

83. (a) Reflection shifted down the same number of units as the curve itself is shifted to the left; $g^{-1}(x) = f^{-1}x - c$

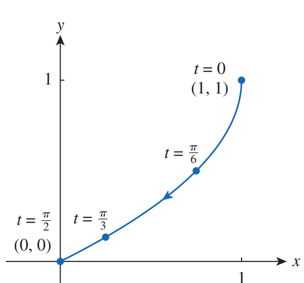
(b) $h^{-1}(x) = (1/c) f^{-1}(x)$

Exercises 1.6 ■ Page 76

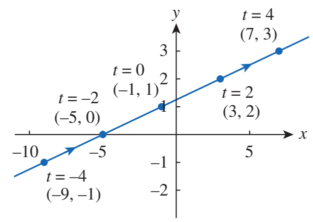
1.



3.

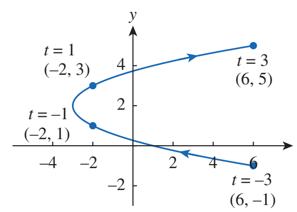


5. (a)



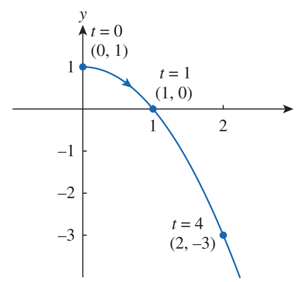
(b) $y = \dfrac{1}{4}x + \dfrac{5}{4}$

7. (a)



(b) $x = y^2 - 4y + 1, \ -1 \le y \le 5$

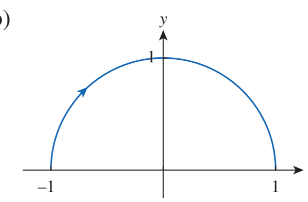
9. (a)



(b) $y = 1 - x^2, x \ge 0$

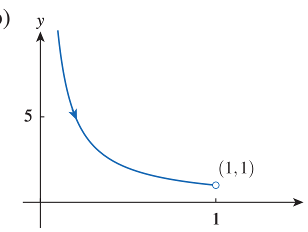
11. (a) $x^2 + y^2 = 1, y \ge 0$

(b)



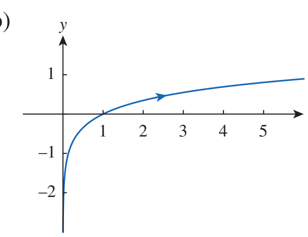
13. (a) $y = \dfrac{1}{x}, y > 1$

(b)



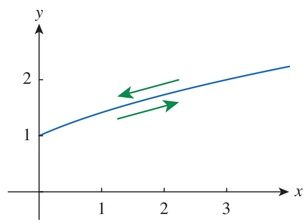
15. (a) $x = e^{2y}$

(b)



17. (a) $x = y^2 - 1$, first quadrant

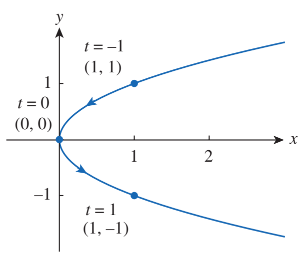
(b)



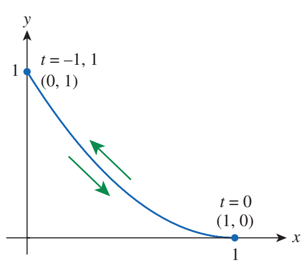
19. Moves on an ellipse centered at (0, 4). As t goes from 0 to $3\pi/2$, the particle starts at the point (0, 5) and moves clockwise to $(-2, 4)$.

21. Moves on the parabola $y = 1 - x^2$. As t goes from -2π to $-\pi$, the particle starts at the point (0, 1), moves to (1, 0), and goes back to (0, 1). As t goes from $-\pi$ to 0, the particle moves to $(-1, 0)$ and goes back to (0, 1). The particle repeats this motion as t goes from 0 to 2π.

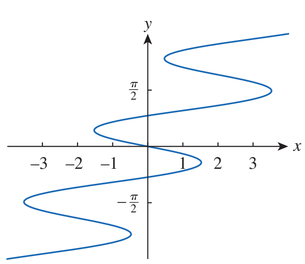
23.



25.



27.



29. (b) $x = -2 + 5t$, $y = 7 - 8t$, $0 \le t \le 1$

31. (a) $x = 2\cos t$, $y = 1 - 2\sin t$, $0 \le t \le 2\pi$

(b) $x = 2\cos t$, $y = 1 + 2\sin t$, $0 \le t \le 6\pi$

(c) $x = 2\cos t$, $y = 1 + 2\sin t$, $\dfrac{\pi}{2} \le t \le \dfrac{3\pi}{2}$

33. $x = 2 + 2\cos t$, $y = 2 + 2\sin t$, $0 \le t \le 2\pi$;

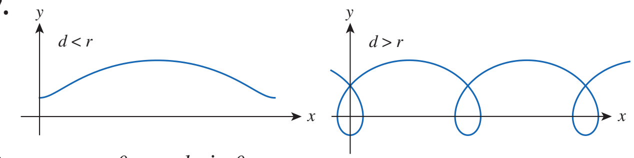
$x = 1 + 0.1\cos t$, $y = 3 + 0.1\sin t$, $0 \le t \le 2\pi$;

$x = 3 + 0.1\cos t$, $y = 3 + 0.1\sin t$, $0 \le t \le 2\pi$;

$x = 2 + \cos t$, $y = 2 + \sin t$, $\pi \le t \le 2\pi$

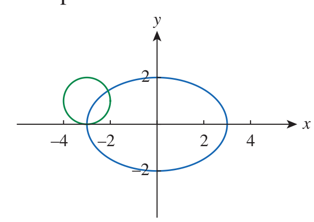
35. The curve $y = x^{2/3}$ is generated in (a). In (b), only the portion with $x \ge 0$ is generated, and in (c) we get only the portion with $x > 0$.

37.



39. $x = a\sec\theta$, $y = b\sin\theta$

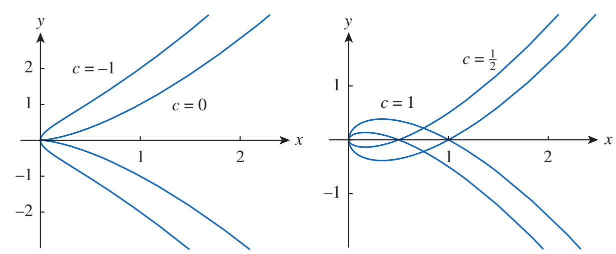
41. (a) Two points of intersection



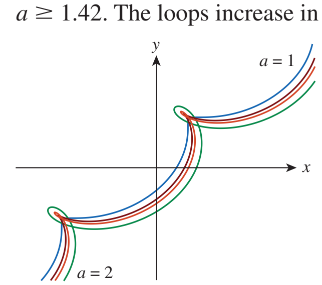
(b) One collision point at $(-3, 0)$ when $t = 3\pi/2$.

(c) There are still two intersection points, but no collision points.

43. For $c = 0$, there is a cusp; for $c > 0$, there is a loop whose size increases as c increases.



45. The curves roughly follow the line $y = x$, with loops for $a \ge 1.42$. The loops increase in size as a increases.



47. As n increases, the number of oscillations increases; a and b determine the width and height.

Chapter 1 Review ■ Page 81

True-False Quiz

1. False **3.** False **5.** False **7.** True **9.** False
11. False

Exercises

1. (a) 2.7 (b) 2.3, 5.6 (c) $[-6, 6]$ (d) $[-4, 4]$
(e) $[-4, 4]$ (f) No; fails the Horizontal Line Test (g) Odd

3. $2a + h - 2$

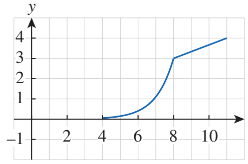
5. $[-2, 2]$; $[0, 4]$

7. $\mathbb{R}$; [2, 4]

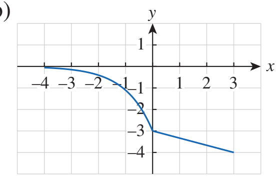
9. $\mathbb{R}$; $(2, \infty)$

11. $\left\{x \,|\, x \neq (2k+1)\dfrac{\pi}{2} - 1 \text{ for } k \in \mathbb{Z}\right\}$; $\mathbb{R}$

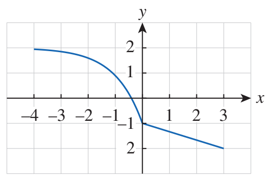
13. (a)



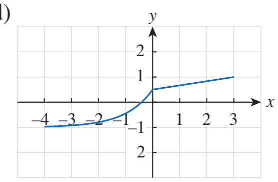
(b)



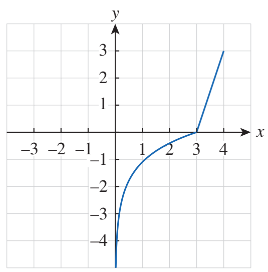
(c)



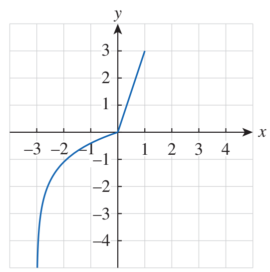
(d)



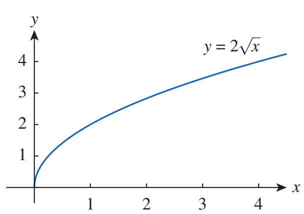
(e)



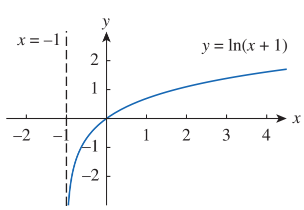
(f)



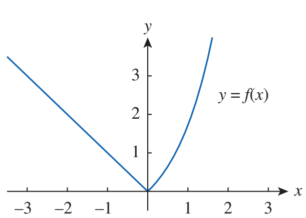
15.



17.



19.



21. (a) $3^a 2^{2a+b}$ (b) $\dfrac{x}{y}$ (c) $\ln\left(\dfrac{a^2b^3}{c^4}\right)$

23. (a) $2a+3$ (b) $2a+2$ (c) $2a+2$ (d) $2a+2$

25. (a) $\ln(x^2-9)$; $(-\infty, -3) \cup (3, \infty)$
(b) $(\ln x)^2 - 9$; $(0, \infty)$
(c) $\ln(\ln x)$; $(1, \infty)$
(d) $(x^2-9)^2 - 9$; $\mathbb{R}$

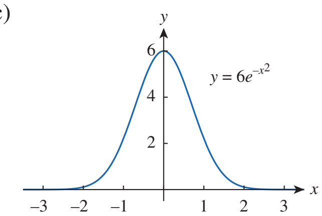
27. $y = 0.2357x - 397.262$; 81.2

29. 1

31. (a) 9 (b) 2

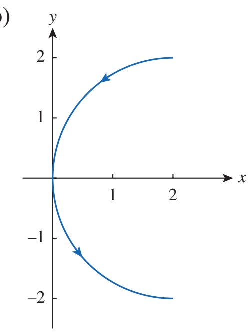
33. (a) $\dfrac{1}{16}$ g (b) $m(t) = 2^{-t/4}$ (c) $t(m) = -4\log_2 m$; time elapsed when there are m grams of ^{100}Pd

35. (a) Even (b) -1.103
(c)

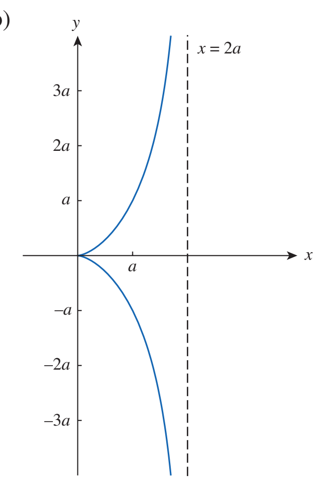


37. As x increases without bound, $y = a^x$ has the largest y-values and $y = \log_a x$ has the smallest y-values.

39. (a) $x = 2 - 2\sin t$; $y = 2\cos t$, $0 \leq t \leq \pi$
(b)



41. (a) $x = 2a\sin^2\theta$, $y = 2a\sin^2\theta\tan\theta$
(b)

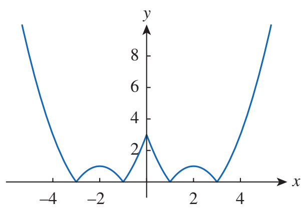


Principles of Problem Solving ■ Page 90

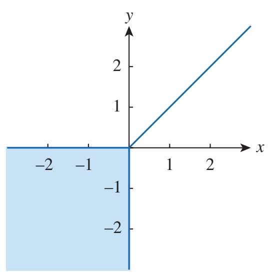
1. $a = \dfrac{4}{\sqrt{h^2 - 16}}\, h$, where a is the length of the altitude and h is the length of the hypotenuse

3. $-\frac{7}{3}, 9$

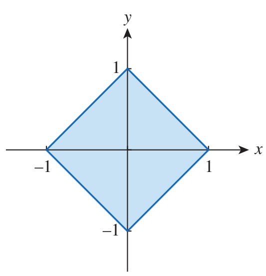
5.



7.



9.



11. 5

13. $x \in \left[-1, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, 3\right]$

15. 40 mi/h

19. $f_n(x) = x^{2n+1}$

Chapter 2

Exercises 2.1 ■ Page 97

1. (a) $m_{\text{sec}} = \dfrac{f(b) - f(a)}{b - a}$ (b) $m_{\text{sec}} = f(a+1) - f(a)$

(c) $m_{\text{sec}} = \dfrac{f(a+h) - f(a)}{h}$

(d) $m_{\text{sec}} = \dfrac{f(a+h) - f(a-h)}{2h}$

3. (a) $m_{PQ_1} = 6.3$, $m_{PQ_2} = 6.03$, $m_{PQ_3} = 6.003$, $m_{\text{sec}} = 6$
(b) $m_{PQ_1} = 11.4$, $m_{PQ_2} = 11.04$, $m_{PQ_3} = 11.004$, $m_{\text{sec}} = 11$
(c) $m_{PQ_1} = 2.31$, $m_{PQ_2} = 2.0301$, $m_{PQ_3} = 203{,}001$, $m_{\text{sec}} = 2$

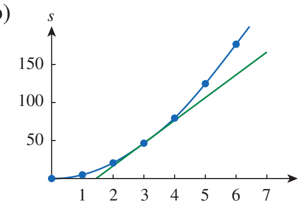
5. (a) 69.67 (b) 71.75 (c) 71 (d) 66; 68

7. (a) 1.3863, 1.0536, 1.005, 1.0005, 0.8109, 0.9531, 0.995, 0.9995
(b) 1 (c) $y = x - 1$

9. (a) $v_{\text{ave}} = -24 - 16h$; -32, -25.6, -24.8, -24.16 (b) -24

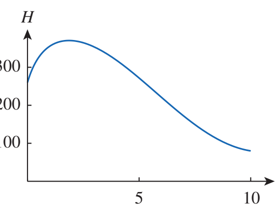
11. (a) 29.45, 33, 45.3, 48.6

(b)



instantaneous velocity ≈ 30

13. (a)



(b) -5.7, -4.39292, -4.2596, -4.246, -4.24
(c) -4.24

Exercises 2.2 ■ Page 107

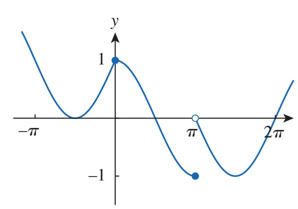
1. As x approaches 2, $f(x)$ approaches 5. Yes, if f has a hole at the point (2, 3).

3. (a) 3 (b) 1 (c) Does not exist (d) 3 (e) 4 (f) 4

5. (a) 4 (b) 4 (c) 4 (d) Does not exist (e) 1 (f) -1
(g) Does not exist (h) 1 (i) 2 (j) Does not exist (k) 3
(l) Does not exist

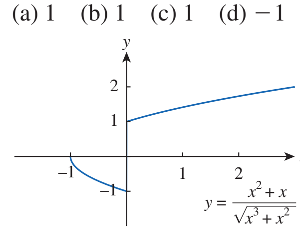
7. 150, 300. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

9.



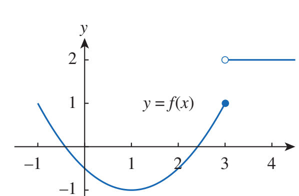
(a) 1 (b) 1 (c) 1 (d) -1 (e) 0 (f) Does not exist

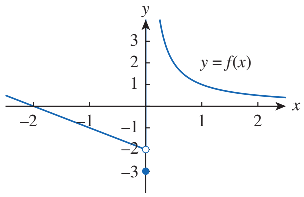
11.

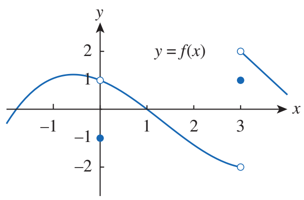


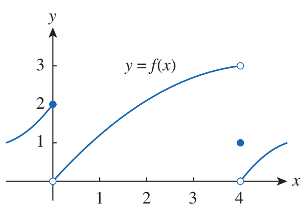
(a) -1 (b) 1 (c) Does not exist

13.



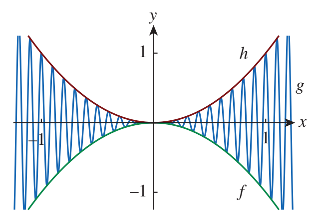
15.


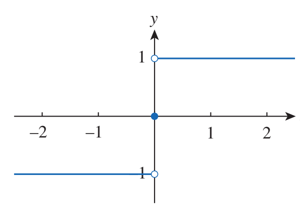
17.


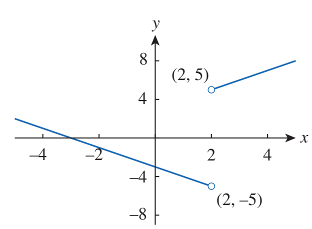
19.


21. Does not exist
23. 0
25. 80
27. 0.6
29. 1.6094
31. 0.59
33. 0
35. 0.32
37. 0.693
39. (a) 0.55740773, 0.37041992, 0.33467209, 0.33366700, 0.33334667, 0.33333667; $1/3$
(b) Technology will eventually return an incorrect value.
(c) When we consider a small enough viewing rectangle, we see an incorrect graph.
41. Within 0.021; within 0.011
43. (c) Does not exist

Exercises 2.3 ■ Page 119

1. (a) -6 (b) -8 (c) 2 (d) -6 (e) Does not exist (f) 0
3. 105
5. $\frac{7}{8}$
7. 390
9. $\frac{3}{2}$
11. -4
13. $-\frac{1}{2}$
15. 0
17. $\frac{3}{7}$
19. $\frac{1}{4}$
21. 12
23. $\frac{4}{3}$
25. $\frac{2}{3}$
27. $-\frac{1}{9}$
29. 1
31. 0
33. $-\frac{4}{5}$
35. $3x^2$
37. (a), (b), (c) $\frac{2}{3}$
39.


41. 7
45. 6
47. Does not exist
49. Does not exist
51. (a)


(b) 1, -1, does not exist, 1
53. (a) 5, -5 (b) Does not exist
(c)


55. 7
57. (a) -2, does not exist, -3 (b) $n-1$, n
61. 1
65. 8
69. $f(x) = H(x)$, $g(x) = 1 - H(x)$, where H is the Heaviside function. $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ does not exist. $\lim_{x\to 0}\left[f(x)g(x)\right] = 0$
71. 15; -1

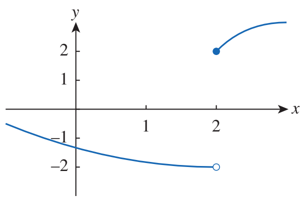
Exercises 2.4 ■ Page 133

1. The function f is continuous at $x = 4$ provided that $\lim_{x \to 4} f(x) = f(4)$.

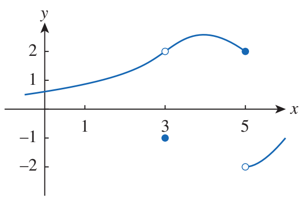
3. (a) -4: $f(-4)$ not defined; $a = -2, 2, 4$: $\lim_{x \to a} f(x)$ does not exist

(b) -4: neither; -2: left; 2: right; 4: right

5.



7.



9. $f(x) = \begin{cases} 2 & \text{if } x \neq 3 \\ 4 & \text{if } x = 3 \end{cases}$

11. (a) Continuous; at each location, the temperature changes smoothly as time passes, without any jumps from one temperature to another.

(b) Continuous; temperature at a specific time changes smoothly as the distance due west of New York City increases, without any jumps.

(c) Discontinuous; as the distance due west from New York City increases, the altitude may jump from one height to another.

(d) Discontinuous; as distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when a light is switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all the intermediate values. (This is debatable, depending on the definition of current.)

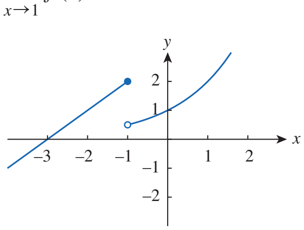
(f) Continuous; speed changes smoothly as time increases.

13. $\lim_{x \to 2} g(x) = \dfrac{14}{5} = g(2)$

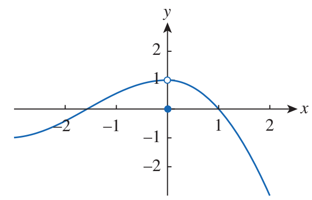
15. $\lim_{x \to 2} f(x) = 40 = f(2)$

17. $\lim_{x \to a} g(x) = g(a),\ a \in (-\infty, -2)$

19. $\lim_{x \to 1} f(x)$ does not exist



21. $\lim_{x \to 0} f(x) \neq f(0)$



23. $f(2) = 3$

25. $f(2) = -\dfrac{5}{8}$

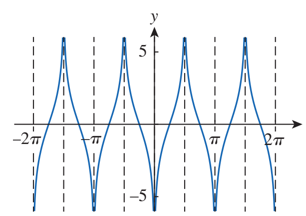
27. $(-\infty, \infty)$

29. $(-\infty, \infty)$

31. $(-2, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, 2)$

33. $(-\infty, \infty)$

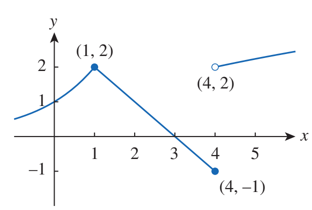
35. $x = \dfrac{\pi}{2} n$, n is any integer



37. 0

39. 1

43. 4, left



45. $\lim_{r \to R} F(r) = F(R)$; F is a continuous function of r.

47. $3a^2$

49. No; $\lim_{x \to a} f(x)$ does not exist, therefore the function is not continuous at $x = a$.

51. (a) Removable; $g(x) = x^3 + x^2 + x + 1$

(b) Removable; $g(x) = x^2 + x$

(c) Jump discontinuity

53. f is continuous on $[31, 32]$; $f(31) \approx 957 < 1000 < 1030 \approx f(32)$; by the IVT there is a number c in $(31, 32)$ such that $f(c) = 1000$.

55. $f(x) = x^4 + x - 3$ is continuous on $[1, 2]$; $f(1) = -1 < 0 < 15 = f(2)$; by the IVT there is a number c in $(1, 2)$ such that $f(c) = 0$.

57. $f(x) = e^x + 2x - 3$ is continuous on $[0, 1]$; $f(0) = -1 < 0 < 1 = f(1)$; by the IVT there is a number c in $(0, 1)$ such that $f(c) = 0$ or $e^c = 3 - 2c$.

59. (a) $f(x) = \cos x - x^3$ is continuous on $[0, 1]$;
$f(1) \approx -0.46 < 0 < 1 = f(0)$; by the IVT there is a number c in $(0, 1)$ such that $f(c) = 0$.
(b) $(0.86, 0.87)$

61. (a) $f(x) = 100e^{-x/100} - 0.01x^2$ is continuous on $[0, 100]$;
$f(100) \approx -63.2 < 0 < 100 = f(0)$; by the IVT there is a number c in $(0, 100)$ such that $f(c) = 0$.
(b) 70.347

63. (a) $f(x) = \ln x - e^{-2x}$ is continuous on $[1, 2]$;
$f(1) \approx -0.135 < 0 < 0.675 \approx f(2)$; by the IVT there is a number c in $(1, 2)$ such that $f(c) = 0$.
(b) 1.114

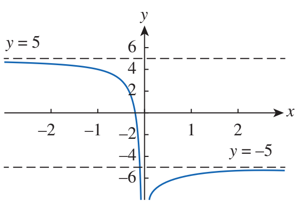
71. Continuous nowhere

Exercises 2.5 ■ Page 149

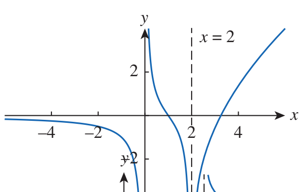
1. (a) As x approaches 2, the values of $f(x)$ become large.
(b) As x approaches 1 from the right, the values of $f(x)$ become large negative.
(c) As x becomes large, the values of $f(x)$ approach 5.
(d) As x becomes large negative, the values of $f(x)$ approach 3.

3. (a) ∞ (b) ∞ (c) $-\infty$ (d) 1 (e) 2 (f) $x = -1, x = 2$; $y = 1, y = 2$

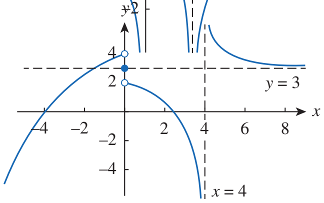
5.



7.



9.



11. 0

13. $\lim\limits_{x\to 1^-} \dfrac{1}{x^3 - 1} = -\infty$, $\lim\limits_{x\to 1^+} \dfrac{1}{x^3 - 1} = \infty$

15. ∞

17. ∞

19. $-\infty$

21. $-\infty$

23. $-\infty$

25. $-\infty$

27. $\dfrac{3}{2}$

29. 0

31. -1

33. 4

35. -2

37. $\dfrac{\sqrt{3}}{4}$

39. $\dfrac{1}{6}$

41. $\dfrac{a - b}{2}$

43. ∞

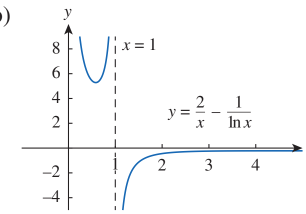
45. 0

47. 1

49. 0

51. 0

53. (a) $0, \infty, \infty, -\infty$
(b)



55. $\dfrac{5\sqrt{3}}{6} \approx 1.4434$

57. $y = \dfrac{2}{3}$; $x = \dfrac{1}{3}$, $x = -1$

59. $y = -1$; $x = 0$, $x = 1$, $x = -1$

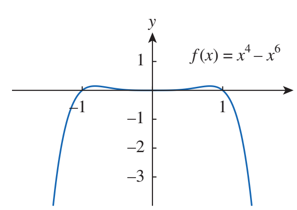
61. $y = 2$, $y = 0$, $x = \ln 5$

63. $y = \pm\dfrac{\sqrt{2}}{3} \approx \pm 0.471404$

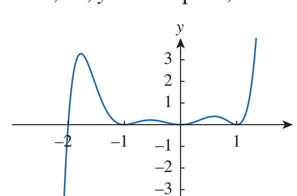
65. $f(x) = \dfrac{2 - x}{x^2(x - 3)}$

67. (a) $\dfrac{5}{4}$ (b) 5

69. $-\infty$; $-\infty$; y-intercept: 0; x-intercepts: $0, -1, 1$

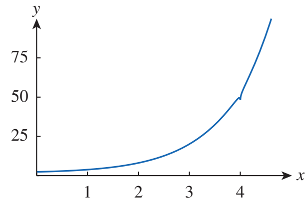


71. $-\infty$; ∞; y-intercept: 3; x-intercepts: $3, -1, 1$



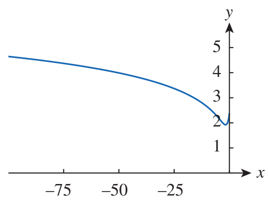
73. (a) 0 (b) Infinitely many times

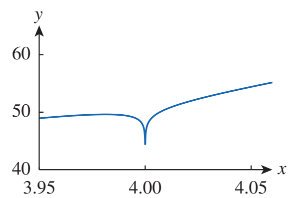
75. (a)



This is not an accurate representation of the graph of f. The graph of f has an infinite discontinuity at $x = 4$.

(b) Several graphs are needed: f looks like $\ln|x - 4|$ for large negative values of x and like e^x for $x > 5$ but has an infinite discontinuity at $x = 4$.





77. 5

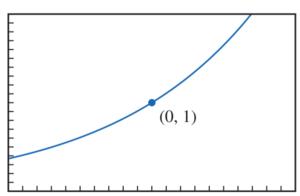
79. $m \to \infty$

81. (a) 23.03 (b) $10 \ln 10$

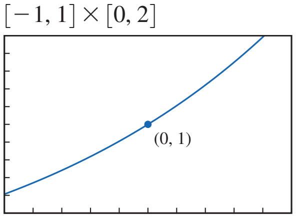
Exercises 2.6 ■ Page 159

1. (a) $\dfrac{f(x) - f(3)}{x - 3}$ (b) $\displaystyle\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$

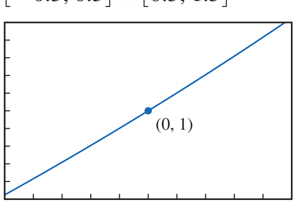
3.



$[-1, 1] \times [0, 2]$



$[-0.5, 0.5] \times [0.5, 1.5]$



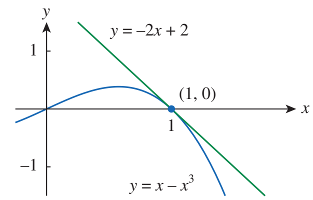
$[-0.1, 0.1] \times [0.9, 1.1]$

As we zoom in near (0, 1), the graph looks more and more like a straight line with slope 1.

5. (a) -2

(b) $y = -2x + 2$

(c)



As we zoom in near (1, 0), the graph and the tangent line become indistinguishable.

7. $y = 9x - 15$

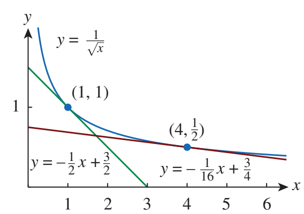
9. $y = \dfrac{1}{3}x + \dfrac{2}{3}$

11. (a) $-\dfrac{1}{2a^{3/2}}$

(b) $y = -\dfrac{1}{2}x + \dfrac{3}{2}$

(c) $y = -\dfrac{1}{16}x + \dfrac{3}{4}$

(d)



13. (a) A: runs the entire race at the same velocity; B: starts at a slower velocity than runner A, but finishes at a faster velocity.

(b) Between 9 and 10 seconds

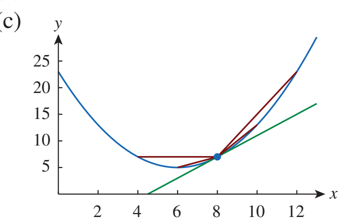
(c) When the slopes of their position functions are the same, around 9.5 seconds

15. (a) 6.28 m/s (b) $10 - 3.72a$ m/s (c) $10/1.86 \approx 5.4$ s (d) -10 m/s

17. (a) 0, 1, 3, 4

(b) 2

(c)



19. $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$

21. (a) 20

(b) No; the tangent line at $x = 10$ is steeper than the tangent line at $x = 30$.

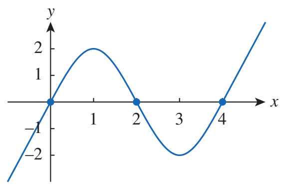
(c) Yes; the slope of the tangent line at $x = 60$ is greater than the line through $(40, f(40))$ and $(80, f(80))$.

23. 3

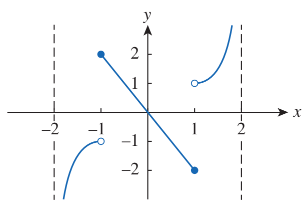
25. (a) k (b) $2ka$ (c) $3ka^2$ (d) $4ka^3$ (e) nka^{n-1}

27. $3, \frac{1}{4}$

29.



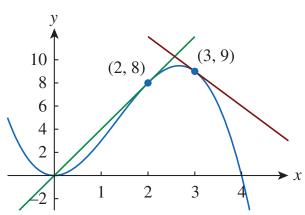
31.



33. 4; $y = 4x - 5$

35. (a) $G'(a) = 8a - 3a^2$; $G'(2) = 4$, $G'(3) = -3$; $y = 4x$; $y = -3x + 18$

(b)



37. $6a^2 + 1$

39. $-\dfrac{2}{a^3}$

41. $\dfrac{2}{(1-a)^{3/2}}$

43. $2a - 4$

45. $f(x) = e^x$; $a = -2$

47. $f(x) = \dfrac{1}{x}$; $a = \dfrac{1}{4}$

49. $f(x) = \sin x$; $a = \dfrac{\pi}{6}$

51. $f(x) = \tan^{-1} x$; $a = 1$

53. $-9/5$; $9/5$

55. -0.7

57. (a) 2148, 1958.5, locations/year; growth of locations is slowing.

(b) 1932 locations/year

59. (a) -5700; -5400; -4617; -4374 dollars/year

(b) -4364; The car is changing in value at a rate of -4364 dollars/year.

61. $V'(t) = \dfrac{500}{9}(t - 60)$; gal/min;

t	Flow rate (gal/min)	Water remaining (gal)
0	-3333.3	100000.0
10	-2777.8	69444.4
20	-2222.2	44444.4
30	-1666.7	25000.0
40	-1111.1	11111.1
50	-555.6	2777.8
60	0.0	0.0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

63. (a) Rate of growth of the bacteria population at $t = 5$ hours; bacteria/hour

(b) $f'(5) < f'(10)$; growth rate slows at some point and the opposite inequality may be true.

65. (a) Rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound; pounds/(dollars/pound)

(b) Negative; quantity of coffee sold will decrease as the price charged increases.

67. (a) Rate of change of the average amount of milk per cow at time t

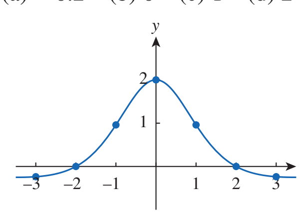
(b) 25; the rate of change in the average amount of milk per cow at $t = 5$ (2019) is 25 gallons per cow/year.

69. Does not exist

71. (a) Slope ≈ 1 (b) Slope ≈ 1 (c) Slope ≈ 0 (d) Answers will vary.

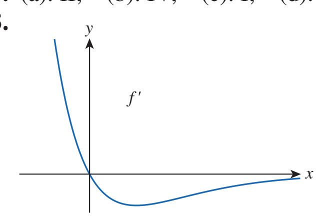
Exercises 2.7 ■ Page 174

1. (a) -0.2 (b) 0 (c) 1 (d) 2 (e) 1 (f) 0 (g) -0.2

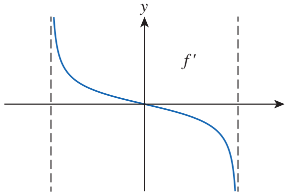


3. (a): II; (b): IV; (c): I; (d): III

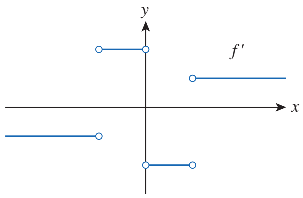
5.



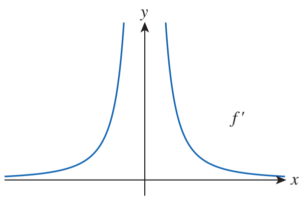
7.



9.

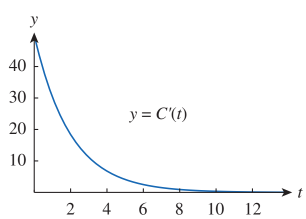


11.



13. (a) Instantaneous rate of change of percentage of full capacity with respect to elapsed time in hours

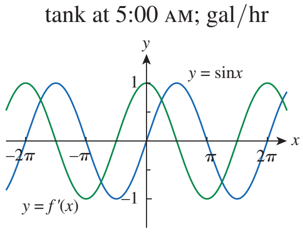
(b)



15. (a) -30

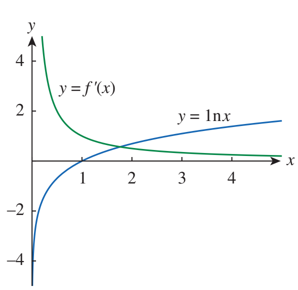
(b) Instantaneous rate of change in the amount of oil in the tank at 5:00 AM; gal/hr

17.



Guess: $f'(x) = \cos x$

19.



Guess: $f'(x) = \dfrac{1}{x}$

21. (a) 0, 0.75, 3, 12, 27 (b) 0.75, 3, 12, 27 (c) $3x^2$ (d) $3x^2$

23. $f'(x) = m$; $\mathbb{R}$; $\mathbb{R}$

25. $f'(x) = 8 - 10x$; $\mathbb{R}$; $\mathbb{R}$

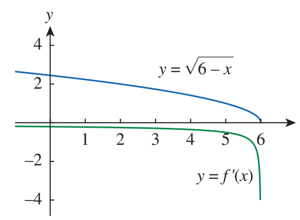
27. $g'(t) = -\dfrac{1}{2t^{3/2}}$; $(0, \infty)$; $(0, \infty)$

29. $f'(x) = \dfrac{2x^2 - 6x + 2}{(2x - 3)^2}$;
domain of f = domain of $f' = \left(-\infty, \dfrac{3}{2}\right) \cup \left(\dfrac{3}{2}, \infty\right)$

31. $f'(x) = \dfrac{3}{2}x^{1/2}$; domain of f = domain of $f' = [0, \infty)$

33. $f''(x) = 12x - 6$

35. (a), (b)

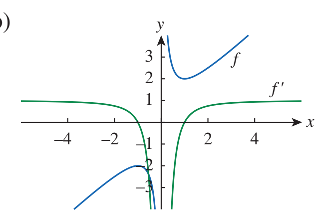


(c) $f'(x) = -\dfrac{1}{2\sqrt{6 - x}}$;
domain of $f = (-\infty, 6]$; domain of $f' = (-\infty, 6)$

(d) $f'(x)$ is never 0, so f has no horizontal tangent lines. All tangent lines to f have negative slope so the graph of f' is always below the x-axis. The graph of f' approaches $-\infty$ as $x \to 6^-$. As $x \to -\infty$, the graph of f' approaches 0.

37. (a) $f'(x) = 1 - \dfrac{1}{x^2}$

(b)



$f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope. Both functions are discontinuous at $x = 0$.

39. (a) The rate at which the unemployment rate is changing with respect to time; pc/yr

(b)

t	$U'(t)$	t	$U'(t)$
2004	-0.90	2013	-0.95
2005	-0.70	2014	-1.05
2006	-0.25	2015	-0.65
2007	0.60	2016	-0.50
2008	2.35	2017	-0.50
2009	1.90	2018	-0.30
2010	-0.20	2019	1.40
2011	-0.75	2020	3.00
2012	-0.75		

41.

t	14	21	28	35	42	49
$H'(t)$	13/7	23/14	18/14	1	11/14	5/7

43. (a) Rate at which the percentage of the city's electrical power produced by solar panels changes with respect to time, t, measured in percentage points per year.

(b) Two years after January 1, 2000 (January 1, 2002), the percentage of electrical power produced by solar panels was increasing at a rate of 3.5 percentage points per year.

45. $x = -4$, corner; $x = 0$, discontinuous; $x = 2$, vertical tangent

47. $x = 1$, f not defined there

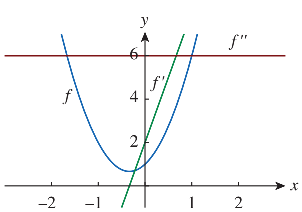
49. As we zoom in near $(-1, 0)$, the curve looks like a straight line, so f is differentiable at $x = -1$. As we zoom in near the origin, the curve does not straighten out. f is not differentiable at $x = 0$.

51. $a = f$, $b = f'$; $f'(-1) < f''(1)$

53. $a = f$, $b = f'$, $c = f''$

55. a: acceleration, b: velocity, c: position

57. $f'(x) = 6x + 2$; $f''(x) = 6$



59. -1

61. (a) False (b) False (c) False (d) False

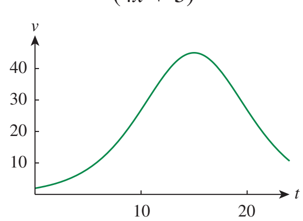
63. (a) $f'(x) = \dfrac{a-b}{(bx+1)^2}$

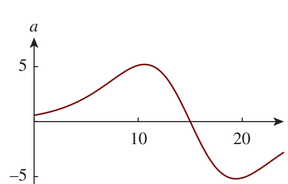
(b) $g'(x) = \dfrac{2}{(5x+1)^2}$

(c) $f'(x) = \dfrac{(a-b)c}{(bx+c)^2}$

(d) $g'(x) = -\dfrac{5}{(4x+5)^2}$

65. (a)





$a(15) = 0$

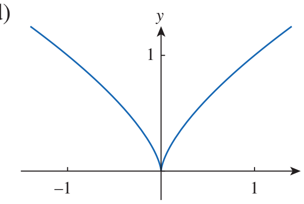
(b) -2

67. (a) $g'(0) = \lim_{x\to 0} \dfrac{1}{x^{1/3}}$ which does not exist

(b) $g'(a) = \dfrac{2}{3a^{1/3}}$

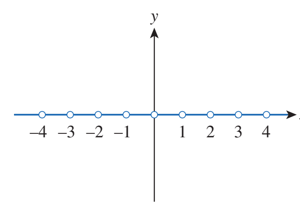
(c) g is continuous at $x = 0$ and $\lim_{x\to 0} |g'(x)| = 0$

(d)

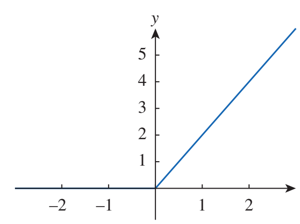


69. (a) f is not continuous at any integer n, so f is not differentiable at n.

(b) $f'(x) = 0$, x not an integer.



71. (a)

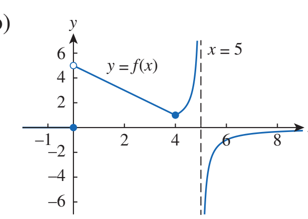


(b) $(-\infty, 0) \cup (0, \infty)$

(c) $g'(x) = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

73. (a) $f'_-(4) = -1$; $f'_+(4) = 1$

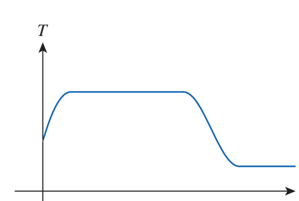
(b)



(c) 0, 5

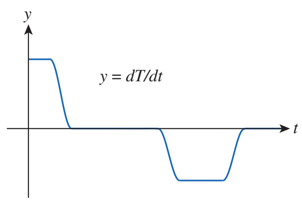
(d) 0

75. (a)

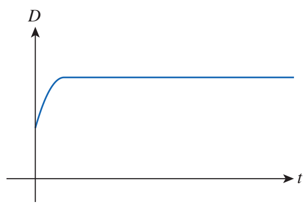


(b) When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. Then $dT/dt = 0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

(c)



(d)



When the water from the hot water tank starts coming out, dD/dt is large and positive as D increases to a set temperature. Then $dD/dt = 0$ as the water remains at a constant, set temperature.

Chapter 2 Review ■ Page 180

True-False Quiz

1. False **3.** True **5.** True **7.** False **9.** True
11. False **13.** True **15.** True **17.** True **19.** False
21. True **23.** False **25.** False **27.** False **29.** False

Exercises

1. (a) 3, 0, does not exist, 2, ∞, $-\infty$, 4, -1
(b) $y = -1$, $y = 4$
(c) $x = 0$, $x = 2$
(d) -3, jump; 0, infinite; 3, infinite; 4, removable

3. $f(x) = \dfrac{3(x^2+8)(x+2)}{x(x+4)(x+2)}$

5. $f(x) = \dfrac{(2x^2-8)(x-3)}{(x-3)(x-2)^2}$

7. 1

9. $\dfrac{3}{2}$

11. 3

13. 0

15. $\dfrac{4}{7}$

17. 1

19. $\dfrac{1}{2}$

21. $-\dfrac{1}{2}$

23. $\dfrac{1}{3}$

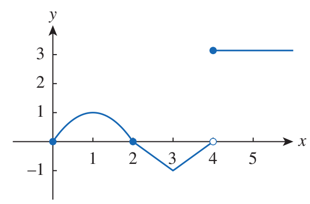
25. 0

27. $y = 0$; $x = 0$

29. 1

31. -3

33. (a) 2: continuous; 3: continuous; 4: right
(b)



35. $(-\infty, -3) \cup (3, \infty)$

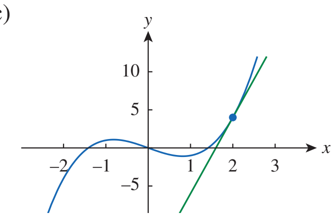
37. $k = 10$; 20

39. (3, 1)

41. $f(x) = \cos\sqrt{x} - e^x + 2$ is continuous on [0, 1]; $f(1) \approx -0.2 < 0 < 2 = f(0)$; by the IVT there is a number c in (0, 1) such that $f(c) = 0$

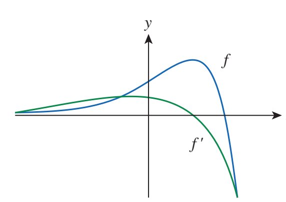
43. (a) -0.016
(b) $V = \dfrac{800}{P} \quad \Rightarrow \quad \dfrac{dV}{dP} = -\dfrac{800}{P^2}$

45. (a) 10
(b) $y = 10x - 16$
(c)

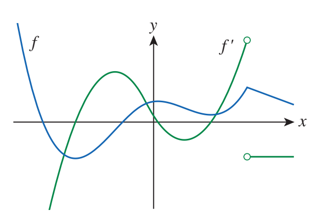


47. $a = 2$

49.



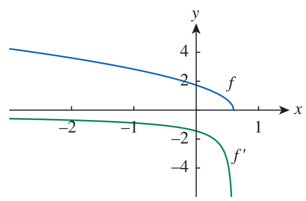
51.



53. (a) $f'(x) = -\dfrac{5}{2\sqrt{3-5x}}$

(b) $\left(-\infty, \dfrac{3}{5}\right]; \left(-\infty, \dfrac{3}{5}\right)$

(c)



55. $x = -4$: f is not continuous; $x = -1$: f has a corner; $x = 2$: f is not continuous; $x = 5$: f has a vertical tangent.

57. $C'(2017) \approx 104.25$; the rate at which the total value of U.S. currency in circulation is changing in billions of dollars per year at $t = 2017$.

Chapter 3

Exercises 3.1 ■ Page 200

1. (a) e is the number such that $\displaystyle\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

(b) $\displaystyle\lim_{h\to 0} \frac{2.7^h - 1}{h} \approx 0.993$, $\displaystyle\lim_{h\to 0} \frac{2.8^h - 1}{h} \approx 1.03$

$2.7 < e < 2.8$

3. $f'(x) = 0$

5. $f'(x) = 5.2$

7. $f'(t) = 6t^2 - 6t - 4$

9. $g'(x) = 2x - 6x^2$

11. $g'(t) = -\dfrac{3}{2}t^{-7/4}$

13. $F'(r) = -\dfrac{15}{r^4}$

15. $R'(a) = 18a + 6$

17. $S'(p) = \dfrac{1}{2}p^{-1/2} - 1$

19. $y' = 3e^x - \dfrac{4}{3}x^{-4/3}$

21. $h'(u) = 3Au^2 + 2Bu + C$

23. $y' = \dfrac{3x^2 + 4x - 3}{2x\sqrt{x}}$

25. $f'(x) = 2.4x^{1.4}$

27. $G'(q) = -2q^{-2} - 2q^{-3}$

29. $f'(v) = -\dfrac{2}{3}v^{-5/3} - 2e^v$

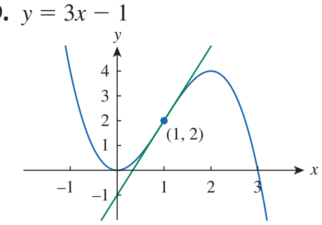
31. $z' = -10Ay^{-11} + Be^y$

33. $y = 4x - 1$

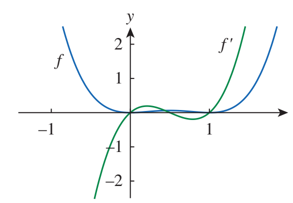
35. $y = \dfrac{1}{2}x + 2$

37. $y = 2x + 2$; $y = -\dfrac{1}{2}x + 2$

39. $y = 3x - 1$



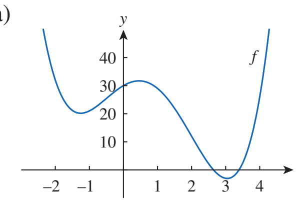
41. $f'(x) = 4x^3 - 6x^2 + 2x$



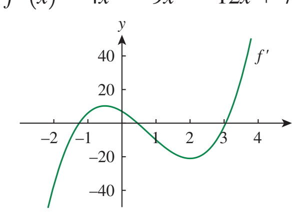
$f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing. When f changes direction, f' crosses the x-axis.

43. $f'(1) = 3$

45. (a)

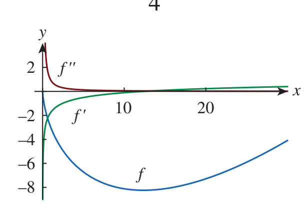


(c) $f'(x) = 4x^3 - 9x^2 - 12x + 7$



47. $f'(x) = 0.005x^4 - 0.06x^2$; $f''(x) = 0.02x^3 - 1.2x$

49. $f'(x) = 2 - \dfrac{15}{4}x^{-1/4}$; $f''(x) = \dfrac{15}{16}x^{-5/4}$



51. (a) $v(t) = 3t^2 - 3$; $a(t) = 6t$

(b) 12 m/s^2

(c) 6 m/s^2

53. -3 in/s^2

55. $y = 3x + 2$

57. 7; rate at which the length is changing at age 5 in mm/yr

59. (a) $L = -0.275P^2 + 19.749P - 273.552$

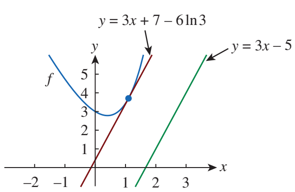
(b) $\left.\dfrac{dL}{dP}\right|_{P=30} \approx 3.2$; $\left.\dfrac{dL}{dP}\right|_{P=40} \approx -2.3$

The derivative is the instantaneous rate of change of tire life with respect to pressure. The units are (1000 miles)/(lb/in^2). When $\dfrac{dL}{dP}$ is positive, the life of the tire is increasing, when $\dfrac{dL}{dP}$ is negative, the life of the tire is decreasing.

61. $x = \ln 2$

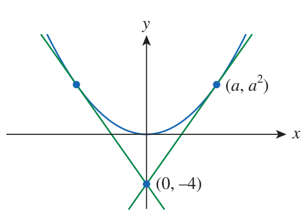
63. $y = 32x - 47$

65. $(\ln 3, 7 - \ln 3) \approx (1.986, 3.704)$



67. $y = -\dfrac{1}{6}x + \dfrac{3}{2}$

69.



$(2, 4)$, $(-2, 4)$

73. $P(x) = x^2 - x + 3$

75. $(0, 3)$

77. $a = 5$, $b = -3$

79. $y = 2x^2 - x$

81. $a = -\dfrac{1}{2}$, $b = 2$

83. $c = -\dfrac{1}{3}$

85. $m = \dfrac{1}{4}, \dfrac{1}{8}$

91. 1000

93. $c > \dfrac{1}{2}$: 3; $c \le \dfrac{1}{2}$:1

Exercises 3.2 ■ Page 210

1. $f'(x) = 1 - 2x + 6x^2 - 8x^3$

3. $h'(x) = \dfrac{x(x-2)}{(x-1)^2}$

5. $g'(x) = e^x\left(2 + 2\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)$

7. $y' = \dfrac{e^x}{(1 - e^x)^2}$

9. $G'(x) = \dfrac{2x^2 + 2x + 4}{(2x+1)^2}$

11. $J'(v) = 1 + v^{-2} + 6v^{-4}$

13. $f'(z) = 1 - ze^z - 2e^{2z}$

15. $y' = \dfrac{2 - x}{2\sqrt{x}(2+x)^2}$

17. $y' = -\dfrac{3t^2 + 4t}{(t^3 + 2t^2 - 1)^2}$

19. $h'(r) = \dfrac{abe^r}{(b + e^r)^2}$

21. $y' = \dfrac{5z^2 + e^z + 2ze^z}{2\sqrt{z}}$

23. $V'(t) = \dfrac{(t+2)^2}{t^2e^t}$

25. $F'(t) = -\dfrac{A(B + 2Ct)}{t^2(B + Ct)^2}$

27. $f'(x) = \dfrac{ad - bc}{(cx + d)^2}$

29. $f'(x) = \dfrac{2x+1}{2\sqrt{x}}e^x$; $f''(x) = \dfrac{4x^2 + 4x - 1}{4x^{3/2}}e^x$

31. $f'(x) = \dfrac{-x^2 - 1}{(x^2 - 1)^2}$; $f''(x) = \dfrac{2x^3 + 6x}{(x^2 - 1)^3}$

33. $y = \dfrac{1}{4}x + \dfrac{1}{2}$

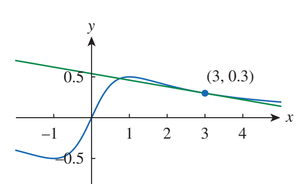
35. $y = 1, x = 1$

37. $x = -1, 1$

39. $\dfrac{\sqrt{5}}{10}$

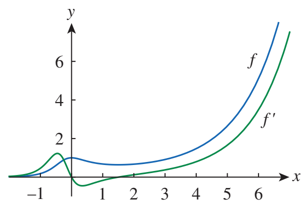
41. (a) $y = -0.08x + 0.54$

(b)



43. (a) $f'(x) = \dfrac{e^x(2x^2 - 3x)}{(2x^2 + x + 1)^2}$

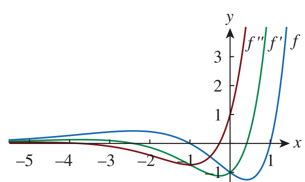
(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

45. (a) $f'(x) = e^x(x^2 + 2x - 1)$; $f''(x) = e^x(x^2 + 4x - 1)$

(b)



f has horizontal tangents where $f'(x) = 0$; f' has horizontal tangents where $f''(x) = 0$.

47. $g^{(n)}(x) = \dfrac{(x - n)(-1)^n}{e^x}$

49. (a) -6 (b) 24 (c) $\dfrac{36}{25}$ (d) $-\dfrac{36}{49}$

51. (a) 3 (b) $\dfrac{37}{9}$

53. -2.5

55. $y = -2x + 18$

57. (a) 0 (b) $-\dfrac{2}{3}$

59. (a) $y' = xg'(x) + g(x)$ (b) $y' = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y' = \dfrac{xg'(x) - g(x)}{[g(x)]^2}$ (d) $y' = \dfrac{e^x g(x) - e^x g'(x)}{[g(x)]^2}$

61. $x \approx 0.964$

63. $y = \dfrac{1}{2}x - \dfrac{1}{2}$, $y = \dfrac{1}{2}x + \dfrac{7}{2}$

65. 4

67. 1,235,725,056 dol/yr; portion of the rate of change of total income due to the existing population's increasing income; portion of the rate of change of total income due to increasing population

69. $\dfrac{dv}{d[S]} = \dfrac{0.0021}{(0.15 + [S])^2}$

rate of change of the rate of an enzymatic reaction with respect to the concentration of a substrate S

71. (c) $f'(x) = 3e^{3x}$

73. $f'(x) = e^x(x^2 + 2x)$; $f''(x) = e^x(x^2 + 4x + 2)$;
$f'''(x) = e^x(x^2 + 6x + 6)$; $f^{(4)}(x) = e^x(x^2 + 8x + 12)$;
$f^{(5)}(x) = e^x(x^2 + 10x + 20)$;
$f^{(n)}(x) = e^x(x^2 + 2nx + n(n - 1))$

Exercises 3.3 ■ Page 219

1. $f'(x) = x^2 \cos x + 2x \sin x$

3. $f'(x) + e^x (\cos x - \sin x)$

5. $g'(x) = \sec x(\sec^2 x + \tan^2 x)$

7. $f'(t) = t \sin t + \cos t + t^2 \cos t$

9. $f'(x) = \dfrac{2 - \tan x + x \sec^2 x}{(2 - \tan x)^2}$

11. $f'(\theta) = \dfrac{1}{1 + \cos\theta}$

13. $f'(t) = \dfrac{(t^2 + t)\cos t + \sin t}{(1 + t)^2}$

15. $f'(\theta) = \dfrac{1}{2}\sin 2\theta + \cos 2\theta$

17. $f'(x) = 2\dfrac{\sin x - x(\sec x + \cos x)}{\sin^2 x \tan x}$

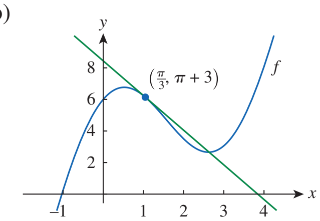
23. $y = x + 1$

25. $y = x - \pi - 1$

27. $y = \dfrac{\sqrt{2}}{4}(\pi + 4)x - \dfrac{\sqrt{2}}{16}\pi^2$

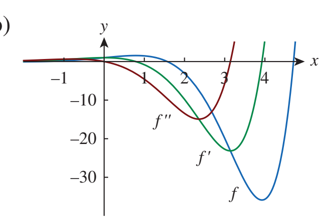
29. (a) $y = (3 - 3\sqrt{3})x + 3 + \pi\sqrt{3}$

(b)



31. (a) $f'(x) = e^x(\cos x - \sin x)$; $f''(x) = -2e^x \sin x$

(b)



$f' = 0$ where f has a minimum and $f'' = 0$ where f' has a minimum. f' is negative when f is decreasing and f'' is negative when f' is decreasing.

33. $f'(x) = \cos^2 x - \sin^2 x$; $f''(x) = -4 \cos x \sin x$

35. (a) $f'(x) = \dfrac{1 + \tan x}{\sec x}$

(b) $f'(x) = \cos x + \sin x$

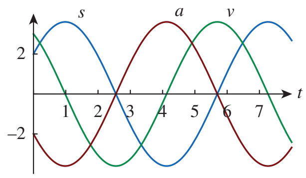
37. (a) $2 - \sqrt{3}$ (b) $\dfrac{1 - 2\sqrt{3}}{16}$

39. $x = \dfrac{\pi}{4} + n\pi$, n an integer

41. $t = \dfrac{7\pi}{6}$

43. (a) $v(t) = -2 \sin t + 3 \cos t$; $a(t) = -2 \cos t - 3 \sin t$

(b)



(c) $t \approx 2.55$ s

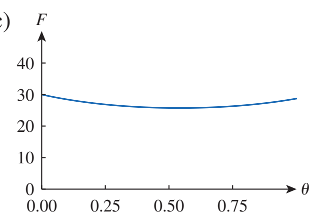
(d) 3.6 cm

(e) $2.55 + n\pi$, n a positive integer

45. (a) $\dfrac{dF}{d\theta} = \dfrac{\mu W(\sin \theta - \mu \cos \theta)}{(\mu \sin \theta + \cos \theta)^2}$

(b) $\theta = \tan^{-1} \mu$

(c)



$\theta \approx 0.54 \approx \tan^{-1} 0.6$

47. $\dfrac{d^{35}}{dx^{35}}(x \sin x) = -35 \sin x - x \cos x$

49. (a) 5 (b) $5 \cos 5x$

51. (a) $\sec^2 x = \dfrac{1}{\cos^2 x}$

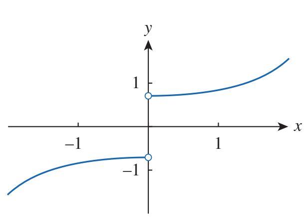
(b) $\sec x \tan x = \dfrac{\sin x}{\cos^2 x}$

(c) $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$

53. 15

55. 0

57. (a)



It appears that the graph of f has a jump discontinuity at $x = 0$.

(b) $\displaystyle\lim_{x\to 0^-} f(x) = -\frac{\sqrt{2}}{2}$, $\displaystyle\lim_{x\to 0^+} f(x) = \frac{\sqrt{2}}{2}$; These limits confirm the answer in part (a).

Exercises 3.4 ■ Page 231

1. $y' = \dfrac{4}{3\sqrt[3]{(1 + 4x)^2}}$

3. $y' = \pi \sec^2 \pi x$

5. $y' = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

7. $F'(x) = 24x^{11}(5x^3 + 2)^3(5x^3 + 1)$

9. $f'(x) = \dfrac{5}{2\sqrt{5x + 1}}$

11. $f'(\theta) = -2\theta \sin(\theta^2)$

13. $y' = xe^{-3x}(2 - 3x)$

15. $f'(t) = e^{at}(b \cos bt + a \sin bt)$

17. $f'(x) = (2x - 3)^3(x^2 + x + 1)^4(28x^2 - 12x - 7)$

19. $h'(t) = \dfrac{2}{3}(t + 1)^{-1/3}(2t^2 - 1)^2(20t^2 + 18t - 1)$

21. $y' = \dfrac{1}{2\sqrt{x}(x + 1)^{3/2}}$

23. $y' = (\sec^2 \theta)e^{\tan \theta}$

25. $g'(u) = \dfrac{48u^2(u^3 - 1)^7}{(u^3 + 1)^9}$

27. $r'(t) = \dfrac{10^{2\sqrt{t}}(\ln 10)}{\sqrt{t}}$

29. $H'(r) = \dfrac{2(r^2 - 1)(r^2 + 3r + 5)}{(2r + 1)^6}$

31. $F'(t) = e^{t \sin 2t}(2t \cos 2t + \sin 2t)$

33. $G'(x) = -c \ln 4 \dfrac{4^{c/x}}{x^2}$

35. $y' = \dfrac{4e^{2x}}{(1 + e^{2x})^2} \sin\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$

37. $y' = -2 \cos \theta \cot(\sin \theta) \csc^2(\sin \theta)$

39. $f'(t) = -\sec^2(\sec(\cos t)) \sec(\cos t) \tan(\cos t) \sin t$

41. $f'(t) = 4 \sin(e^{\sin^2 t}) \cos(e^{\sin^2 t})e^{\sin^2 t}\sin t \cos t$

43. $g'(x) = 2r^2 p (\ln a)(2ra^{rx} + n)^{p-1}a^{rx}$

45. $y' = \dfrac{-\pi \cos(\tan \pi x) \sec^2(\pi x) \sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}}$

47. $y' = -3 \cos 3\theta \sin(\sin 3\theta)$

$y'' = -9 \cos^2(3\theta) \cos(\sin 3\theta) + 9(\sin 3\theta) \sin(\sin 3\theta)$

49. $y' = \dfrac{-\sec t \tan t}{2\sqrt{1 - \sec t}}$

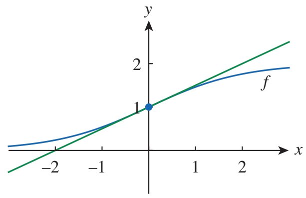
$y'' = \dfrac{\sec t(3\sec^3 t - 4\sec^2 t - \sec t + 2)}{4(1 - \sec t)^{3/2}}$

51. $y = (\ln 2)x + 1$

53. $y = -x + \pi$

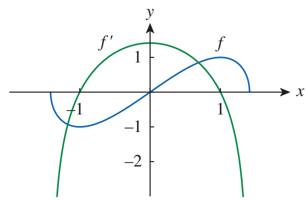
55. (a) $y = \dfrac{1}{2}x + 1$

(b)



57. (a) $f'(x) = \dfrac{2 - 2x^2}{\sqrt{2 - x^2}}$

(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

59. $\left(\dfrac{\pi}{2} + 2n\pi, 3\right)$ and $\left(\dfrac{3\pi}{2} + 2n\pi, -1\right)$, n any integer

61. (4, 3)

63. $h'(1) = \dfrac{6}{5}$

65. (a) 20 (b) 63

67. (a) $-\dfrac{1}{25}$ (b) 35 (c) $\dfrac{7}{4}$

69. $t = \pi$

71. (1, 3)

73. 12

75. $f^{(n)}(x) = a^n n!$

77. (a) 0.75 (b) 10

79. $h'(0.5) \approx -17.4$

81. (a) $F'(x) = f'(x^\alpha)\alpha x^{\alpha - 1}$

(b) $G'(x) = \alpha[f(x)]^{\alpha - 1}f'(x)$

83. (a) $g'(0) = c + 5$, $g''(0) = c^2 - 2$

(b) $y = (5 + 3k)x + 3$

85. $f''(x) = 4x^3 g''(x^2) + 6xg'(x^2)$

87. $F'(1) = 198$

89. $r = 2 \pm \sqrt{3}$

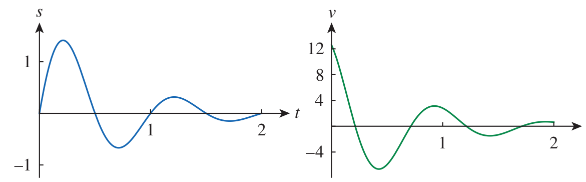
91. $f^{(1000)}(x) = (x - 1000)e^{-x}$

93. $v(t) = \dfrac{5\pi}{2}\cos(10\pi t)$ cm/s

95. (a) $B'(t) = \dfrac{7\pi}{54}\cos\dfrac{2\pi t}{5.4}$

(b) $B'(1) \approx 0.161$: the brightness on day 1 into the 5.4-day cycle is increasing at a rate of 0.161 on the brightness scale.

97. $v(t) = 2e^{-1.5t}(2\pi \cos 2\pi t - 1.5 \sin 2\pi t)$



99. (a) $C'(10) \approx 0.00752$: the BAC was increasing at approximately 0.00752 (mg/mL)/min after 10 minutes.

(b) $C'(40) \approx 0.003002$: the BAC was increasing at approximately 0.003002 (mg/ml)/min after 40 minutes.

101. $\dfrac{dv}{dt}$ is the rate of change of velocity with respect to time (acceleration). $\dfrac{dv}{ds}$ the rate of change of velocity with respect to displacement.

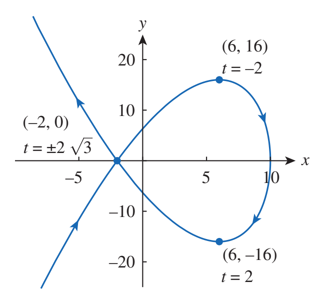
103. (a) $Q(t) = ab^t$: $a \approx 100.0124369$, $b \approx 0.000045145933$

(b) $Q'(t) \approx -670.629$

105. $y = -x$

107. $y = -\dfrac{2}{e}x + 3$

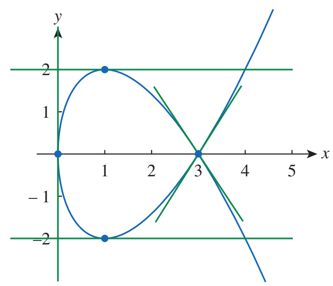
109. Horizontal tangent line: (6, 16), (6, −16); Vertical tangent line: (10, 0)



111. (a) $y = \sqrt{3}x - 3\sqrt{3}$, $y = -\sqrt{3}x + 3\sqrt{3}$

(b) Horizontal tangent: (1, −2), (1, 2); Vertical tangent: (0, 0)

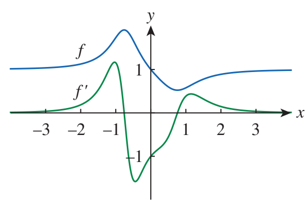
(c)



113. (a) $f'(x) = \dfrac{3x^4 - 1}{(x^4 - x + 1)(x^4 + x + 1)^{3/2}}$

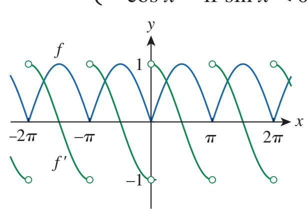
(b) $x \approx -0.7598, 0.7598$

(c)



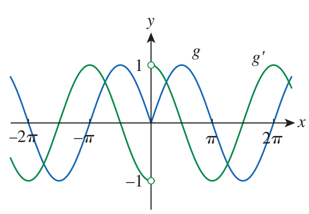
117. $y = 2$

119. (b) $f'(x) = \begin{cases} \cos x & \text{if } \sin x > 0 \\ -\cos x & \text{if } \sin x < 0 \end{cases}$



f is not differentiable when $x = n\pi$, n an integer

(c) $g'(x) = \begin{cases} \cos x & \text{if } x > 0 \\ -\cos x & \text{if } x < 0 \end{cases}$



g is not differentiable when $x = 0$

121. $\dfrac{d^3y}{dx^3} = \dfrac{d^3y}{du^3} + \left(\dfrac{dy}{dx}\right)^3 + 3\dfrac{du}{dx}\dfrac{d^2u}{dx^2}\dfrac{d^2y}{du^2} + \dfrac{dy}{du}\dfrac{d^3u}{dx^3}$

Exercises 3.5 ■ Page 243

1. (a) $y' = \dfrac{9x}{y}$

(b) $y = \pm\sqrt{9x^2 - 1}$; $y' = \pm\dfrac{9x}{\sqrt{9x^2 - 1}}$

(c) $y' = \dfrac{9x}{y} = \dfrac{9x}{\pm\sqrt{9x^2 - 1}}$

3. (a) $y' = -\dfrac{\sqrt{y}}{\sqrt{x}}$

(b) $y = 1 - 2\sqrt{x} + x$; $y' = 1 - \dfrac{1}{\sqrt{x}}$

(c) $y' = -\dfrac{\sqrt{y}}{\sqrt{x}} = -\dfrac{1 - \sqrt{x}}{\sqrt{x}} = -\dfrac{1}{\sqrt{x}} + 1$

5. $-\dfrac{1}{4}$

7. $y'' = -\dfrac{25}{y^3}$

9. $y' = \dfrac{-4x - y}{x - 2y}$

11. $y' = -\dfrac{y^2 - 3x^2}{y(3y - 2x)}$

13. $y' = \dfrac{1 - e^y}{xe^y + 1}$

15. $y' = -\dfrac{y\sin(xy)}{x\sin(xy) + \cos y}$

17. $y' = \dfrac{1 + y - e^y\cos x}{e^y\sin x - x}$

19. $y' = \dfrac{x - y\sqrt{x^2 + y^2}}{x\sqrt{x^2 + y^2} - y}$

21. $y' = \dfrac{-\sin y - y\cos x}{x\cos y + \sin x}$

23. $y' = \dfrac{(1 + x^2)\sec^2(x - y) + 2x\tan(x - y)}{1 + (1 + x^2)\sec^2(x - y)}$

25. $y' = \dfrac{-(xe^x + e^x)}{ye^y + e^y}$

27. $g'(0) = 0$

29. $x' = \dfrac{x\sec^2 y - \sec x}{y\sec x\tan x - \tan y}$

31. $y = \dfrac{1}{2}x$

33. $y = \dfrac{3}{4}x - \dfrac{1}{2}$

35. $y = x + \dfrac{1}{2}$

37. $y = -\dfrac{9}{13}x + \dfrac{40}{13}$

39. $y' = \dfrac{\cos^2(xy) - y}{x}$

41. $x = 1$

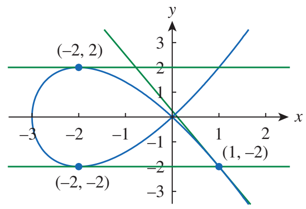
43. $(-2, -1)$

45. -3

47. (a) $y = -\frac{9}{4}x + \frac{1}{4}$

(b) $(-2, -2), (-2, 2)$

(c)

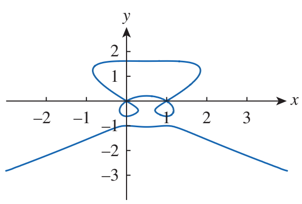


49. $y'' = -\dfrac{18}{(x + 2y)^3}$

51. $y'' = -\dfrac{14x}{y^5}$

53. $y''' = 42$

55. (a)

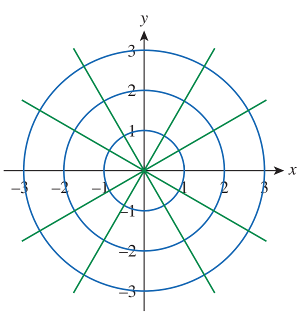


(b) 9 points with horizontal tangents: 3 at $x = 0$, 3 at $x = \frac{1}{2}$, 3 at $x = 1$

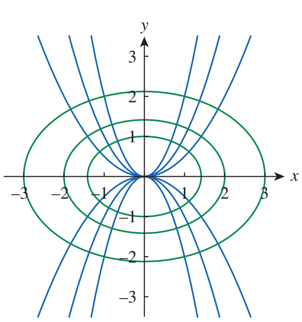
57. $\left(\pm\dfrac{5\sqrt{3}}{4}, \pm\dfrac{5}{4}\right)$

59. $\dfrac{x_0 x}{a^2} - \dfrac{y_0 y}{b^2} = 1$

63.



65.



69. (a) $\dfrac{dV}{dP} = \dfrac{V^3(nb - V)}{PV^3 - n^2aV + 2n^3ab}$

(b) $\dfrac{dV}{dP} \approx -4.04\ \mathrm{L/atm}$

71. $(\pm\sqrt{3}, 0)$; $y' = \dfrac{y - 2x}{2y - x}$, $y'(\sqrt{3}, 0) = y'(-\sqrt{3}, 0) = 2$

73. $(-1, -1), (1, 1)$

75. (a) $J'(0) = 0$ (b) $J''(0) = -\frac{1}{2}$

Exercises 3.6 ■ Page 250

1. (a) $\dfrac{\pi}{3}$ (b) π

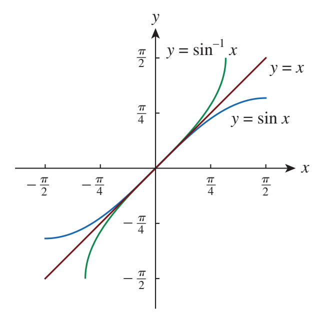
3. (a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{4}$

5. $\dfrac{2}{\sqrt{5}}$

7. $\dfrac{2\sqrt{2}}{3}$

11. $\dfrac{x}{\sqrt{1 + x^2}}$

13.



The graph of $y = \sin^{-1} x$ is the reflection of the graph of $y = \sin x$ about the line $y = x$.

17. $y' = \dfrac{2\tan^{-1} x}{1 + x^2}$

19. $y' = \dfrac{1}{\sqrt{-x^2 - x}}$

21. $G'(x) = -1 - \dfrac{x \arccos x}{\sqrt{1 - x^2}}$

23. $y' = \dfrac{1}{2(1 + x^2)}$

25. $y' = -\dfrac{1}{\sqrt{1 - (\sin^{-1} t)^2}} \cdot \dfrac{1}{\sqrt{1 - t^2}}$

27. $y' = \sin^{-1} x$

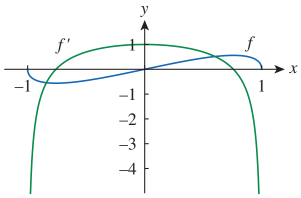
29. $y' = \dfrac{\sqrt{a^2 - b^2}}{|a + b\cos x|}$

31. $g'(x) = \dfrac{2}{\sqrt{1 - (3 - 2x)^2}}$

Domain of g: $[1, 2]$, Domain of g': $(1, 2)$

33. $g'(2) = \dfrac{\pi}{6}$

35. $f'(x) = 1 - \dfrac{x \arcsin x}{\sqrt{1 - x^2}}$



$f' = 0$ where the graph of f has a horizontal tangent; f' is negative when f is decreasing: f' is positive when f is increasing.

37. $-\dfrac{\pi}{2}$

39. $\dfrac{\pi}{2}$

41. (b) $\dfrac{3}{2}$

Exercises 3.7 ■ Page 258

1. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$ is simplest when $a = e$ because $\ln e = 1$

3. $f'(x) = \dfrac{1 - 2 \ln x}{x^3}$

5. $f'(x) = 2 \cot x$

7. $g'(x) = -\dfrac{1}{x(\ln x)^2}$

9. $f'(x) = \dfrac{1}{2(\ln 10)x}$

11. $g'(t) = \dfrac{1}{2t\sqrt{1 + \ln t}}$

13. $h'(x) = \dfrac{1}{\sqrt{x^2 - 1}}$

15. $P'(v) = \dfrac{1 - v + v \ln v}{v(1 - v)^2}$

17. $y' = \dfrac{1 - 3t^2}{1 + t - t^3}$

19. $y' = \csc x$

21. $H'(z) = \dfrac{2a^2 z}{z^4 - a^4}$

23. $y' = \dfrac{1 + \ln x}{x \ln x \ln 2}$

25. $f'(x) = 3^x x^2(x \ln 3 + 3)$

27. $g'(x) = e^{2x}\left(\dfrac{x}{x^2 + 4} + \ln(x^2 + 4)\right)$

29. $y' = \dfrac{e^{-x}\cos^2 x}{x^2 + x + 1}\left(1 + 2 \tan x + \dfrac{2x + 1}{x^2 + x + 1}\right)$

31. $y' = \sqrt{x}\, e^{x^2 - x}(x + 1)^{2/3}\left(\dfrac{1}{2x} + 2x - 1 + \dfrac{2}{3x + 3}\right)$

33. $y' = x^{\cos x}\left(\dfrac{\cos x}{x} - \ln x \sin x\right)$

35. $y' = \dfrac{1}{2}\sqrt{x}^{\,x}(1 + \ln x)$

37. $y' = (\sin x)^{\ln x}\left(\ln x \cot x + \dfrac{\ln \sin x}{x}\right)$

39. $y' = (\ln x)^{\cos x}\left(\dfrac{\cos x}{x \ln x} - \sin x \ln(\ln x)\right)$

41. $y' = \dfrac{1 - 2 \ln x}{x^3},\ y'' = \dfrac{6 \ln x - 5}{x^4}$

43. $y' = \dfrac{1}{x(1 + \ln x)^2},\ y'' = -\dfrac{3 + \ln x}{x^2(1 + \ln x)^3}$

45. $y' = \dfrac{1}{x(1 + \ln x)},\ y'' = -\dfrac{2 + \ln x}{x^2(1 + \ln x)^2}$

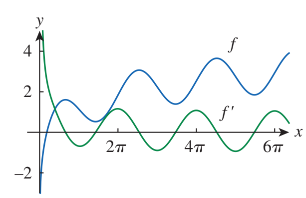
47. Domain of f: $[e^{-2}, \infty)$, $f'(x) = \dfrac{1}{2x\sqrt{2 + \ln x}}$

49. Domain of f: (e, ∞), $f'(x) = \dfrac{1}{\ln(\ln x)} \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x}$

51. $f'(1) = 0$

53. $y = x - 1$

55. $f'(x) = \cos x + \dfrac{1}{x}$



$f' = 0$ where the graph of f has a horizontal tangent; f' is negative when f is decreasing: f' is positive when f is increasing.

57. $\dfrac{2}{e^3}$

59. $\left(e, \dfrac{1}{e}\right)$

61. $c = 7$

63. $s(1) = \ln 2$

65. $y' = \dfrac{2x}{x^2 + y^2 - 2y}$

67. $f^{(n)}(x) = (-1)^{n-1}\dfrac{(n-1)!}{(x-1)^n}$

Exercises 3.8 ■ Page 270

1. (a) $v(t) = 3t^2 - 16t + 24$

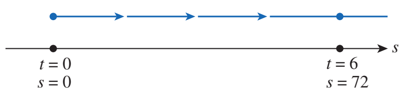
(b) 11 ft/s

(c) Particle is never at rest.

(d) $v(t) > 0$ for all t; particle is always moving in the positive direction.

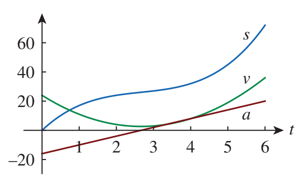
(e) 72 ft

(f)



(g) $a(1) = -10$ ft/s^2

(h)



(i) Speeding up: $t > \dfrac{8}{3}$; slowing down: $0 \le t < \dfrac{8}{3}$

3. (a) $v(t) = \dfrac{\pi}{2}\cos\left(\dfrac{\pi t}{2}\right)$

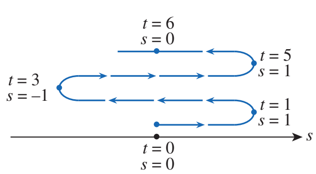
(b) 0 ft/s

(c) $t = 1 + 2n$, n a nonnegative integer

(d) (0, 1), (3, 5), (7, 9), etc.

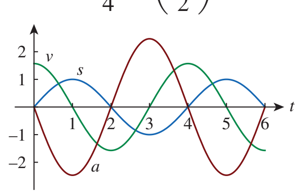
(e) 6 ft

(f)



(g) $a(t) = -\dfrac{\pi^2}{4}\sin\left(\dfrac{\pi t}{2}\right)$; $a(1) = -\dfrac{\pi^2}{4}$ ft/s^2

(h)



(i) Speeding up: (1, 2), (3, 4), (5, 6)
slowing down: (0, 1), (2, 3), (4, 5)

5. (a) Speeding up: (0, 1), (2, 3); slowing down: (1, 2)

(b) Speeding up: (0, 1), (3, 4); slowing down: (1, 3)

7. (a) $v(2) = 4.9$; $v(4) = -14.7$ (b) $t = 2.5$ (c) 32.625
(d) $t \approx 5.080$ (e) -25.3

9. (a) 7.56 (b) 6.245, -6.245

11. (a) $A'(15) = 30$; rate at which the area is increasing with respect to the side length as x reaches 15 mm

(b) $\Delta A \approx 2x\,\Delta x$

13. (a) 5π, 4.5π, 4.1π

(b) $A'(2) = 4\pi$

(c) $\Delta A \approx 2\pi r\,\Delta r$

15. (a) 8π (b) 16π (c) 24π; As time goes by, the area grows at an increasing rate.

17. (a) 6 (b) 12 (c) 18; highest: right end of the rod; lowest: left end of the rod.

19. (a) 4.75 (b) 5; $t = \dfrac{2}{3}$ s

23. (a) $\dfrac{dV}{dP} = -\dfrac{C}{P^2}$

(b) Volume decreasing more rapidly at the beginning

25. $n(t) = 400 \cdot 3^t$; $n'(2.5) = 400 \cdot 3^{2.5} \cdot \ln 3 \approx 6850$

27. (a) 81.5; 83.75

(b)
$P(t) = -0.00101836t^3 + 0.629308t^2 - 10.2201t + 1741.33$

(c) $P'(t) = -0.00305507t^2 + 1.25862t - 10.2201$

(d) 78.31, 91.26

(e) $f'(t) = 1358.03 \cdot \ln(1.01478) \cdot (1.01478)^t$

(f) 76.64, 100.10

(g) 94.12, 107.72

29. (a) 0.926, 0.694, 0

(b) 0, -92.593, -185.185

(c) $r = 0$, $r = R = 0.01$

31. (a) $C'(x) = 3 + 0.02x + 0.0006x^2$

(b) 11; rate at which the cost is increasing as the100$^{\text{th}}$ pair of jeans is produced, the approximate cost of the 101$^{\text{st}}$ pair.

(c) 11.07: close to the marginal cost from (b)

33. (a) $A'(x) = \dfrac{xp'(x) - p(x)}{x^2}$; the average productivity increases as the size of the workforce increases.

35. $\dfrac{dt}{dc} = \dfrac{3\sqrt{9c^2 - 8c} + 9c - 4}{\sqrt{9c^2 - 8c}\left(3c + \sqrt{9c^2 - 8c}\right)}$
the rate of change of duration of dialysis with respect to the initial urea concentration

37. -0.2436

39. (a) $\dfrac{dC}{dt} = 0$ and $\dfrac{dW}{dt} = 0$

(b) $C = 0$

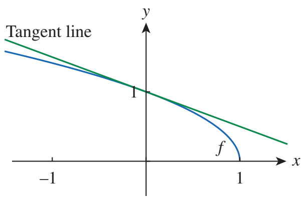
(c) (0, 0), (500, 50), it is possible for the two species to live in harmony.

Exercises 3.9 ■ Page 280

1. $L(x) = 16x + 23$

3. $L(x) = \frac{1}{4}x + 1$

5. $f(x) \approx 1 - \frac{1}{2}x$; 0.95; 0.995



7. $-1.204 < x < 0.706$

9. $-0.045 < x < 0.055$

11. $dy = \dfrac{-2}{(x-1)^2}\,dx$

13. $dy = \pi \sec^2(\pi x) e^{\tan \pi x}\,dx$

15. 15.986

17. 10.003

19. 1.1

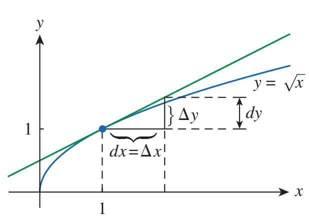
21. The linear approximation of $f(x) = \sec x$ at 0 is $L(x) = 1$. Since 0.08 is close to 0, the approximation is reasonable.

23. The linear approximation of $f(x) = \ln x$ at 1 is $L(x) = x - 1$. $L(1.05) = 0.05$: the approximation is reasonable.

25. (a) $dy = \dfrac{1}{2\sqrt{x}}\,dx$

(b) $dy = \frac{1}{2}$, $\Delta y \approx 0.414$

(c)



27. 148°F; underestimate

29. Answers will vary: 20.75, 24.87; too high

31. (a) 270, 0.01, 1%

(b) 36, 0.0067, 0.67%

33. (a) $\dfrac{84}{\pi} \approx 27$, 0.012

(b) $\dfrac{1764}{\pi^2} \approx 179$, 0.018

35. (a) $\Delta V \approx dV = 2\pi rh\,\Delta r$

(b) $\Delta V - dV = \pi(\Delta r)^2 h$

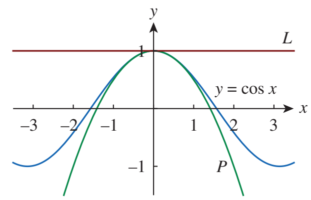
37. $\dfrac{dF}{F} = 4\left(\dfrac{dR}{R}\right)$; 20% increase in blood flow

39. (a) $L(x) = 2x + 3$; $L(0.9) = 4.8$ $L(1.1) = 5.2$

(b) Overestimates

Laboratory Project ■ Page 281

1. $P(x) = 1 - \frac{1}{2}x^2$



The quadratic approximation is much better than the linear approximation.

Chapter 3 Review ■ Page 283

True-False Quiz

1. True **3.** True **5.** False **7.** False **9.** True
11. True **13.** True **15.** True

Exercises

1. $y' = 4x^7(x+1)^3(3x+2)$

3. $y' = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} - x^{-3/2}$

5. $y' = x(\pi x \cos \pi x + 2 \sin \pi x)$

7. $y' = \dfrac{8t^3}{(t^4+1)^2}$

9. $y' = \dfrac{1 + \ln x}{x \ln x}$

11. $y' = \dfrac{\cos\sqrt{x} - \sqrt{x}\sin\sqrt{x}}{2\sqrt{x}}$

13. $y' = \dfrac{-e^{1/x}(1+2x)}{x^4}$

15. $y' = \dfrac{2xy - \cos y}{1 - x\sin y - x^2}$

17. $y' = \dfrac{1}{2\sqrt{\arctan x}\,(1+x^2)}$

19. $y' = \dfrac{1-t^2}{(1+t^2)^2}\sec^2\left(\dfrac{t}{1+t^2}\right)$

21. $y' = 3^{x \ln x}(\ln 3)(1 + \ln x)$

23. $y' = -(x-1)^{-2}$

25. $y' = \dfrac{2x - y\cos(xy)}{x\cos(xy) + 1}$

27. $y' = \dfrac{2}{(1+2x)\ln 5}$

29. $y' = \cot x - \sin x \cos x$

31. $y' = \dfrac{4x}{1+16x^2} + \tan^{-1}(4x)$

33. $y' = 5 \sec 5x$

35. $y' = -6x\csc^2(3x^2+5)$

37. $y' = \cos(\tan\sqrt{1+x^3})(\sec^2\sqrt{1+x^3})\dfrac{3x^2}{2\sqrt{1+x^3}}$

39. $y' = 2[\tan(\sin\theta)]\cdot\sec^2(\sin\theta)\cdot\cos\theta$

41. $y' = \dfrac{(2-x)^4(3x^2-55x-52)}{2\sqrt{x+1}(x+3)^8}$

43. $y' = \dfrac{mx\cos mx - \sin mx}{x^2}$

45. $y' = \dfrac{-3\sin(e^{\sqrt{\tan 3x}})e^{\sqrt{\tan 3x}}\sec^2(3x)}{2\sqrt{\tan 3x}}$

47. $-\dfrac{4}{27}$

49. $y'' = -\dfrac{5x^4}{y^{11}}$

51. $f^{(n)}(x) = (x+n)e^x$

53. $y = -1$

55. $y = 2x+2$

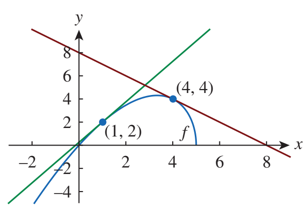
57. $y = -\dfrac{4}{5}x + \dfrac{13}{5}, y = \dfrac{5}{4}x - \dfrac{3}{2}$

59. $y = -\dfrac{1}{25e^{4/5}}(x-29), y = 25e^{4/5}x - 100e^{4/5} + e^{-4/5}$

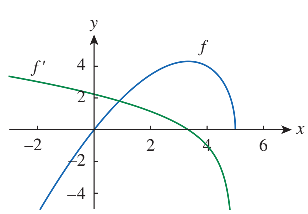
61. (a) $f'(x) = \dfrac{10-3x}{2\sqrt{5-x}}$

(b) $y = \dfrac{7}{4}x + \dfrac{1}{4}, y = -x+8$

(c)



(d)



f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

63. $\left(\dfrac{\pi}{4}, \sqrt{2}\right), \left(\dfrac{5\pi}{4}, -\sqrt{2}\right)$

67. (a) 4 (b) 6 (c) $\dfrac{7}{9}$ (d) 12

69. $f'(x) = x(xg'(x) + 2g(x))$

71. $f'(x) = 2g(x)g'(x)$

73. $f'(x) = g'(e^x)e^x$

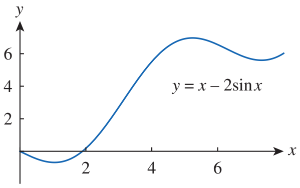
75. $f'(x) = \dfrac{g'(x)}{g(x)}$

77. $f'(x) = \dfrac{g'(\arctan x)}{1+x^2}$

79. $h'(x) = \dfrac{f'(x)[g(x)]^2 + g'(x)[f(x)]^2}{[f(x)+g(x)]^2}$

81. $h'(x) = 4f'(g(\sin 4x))g'(\sin 4x)\cos 4x$

83. (a)



(b) [2, 3]

(c) $x = 2$

(d) $f'(2) \approx 1.8323 > 0.4327 \approx f'(5)$

85. (a) $y = \dfrac{1}{4}x + \dfrac{1}{4}(\ln 4 + 1)$

(b) $y = ex$

87. $y = -\dfrac{2}{3}x^2 + \dfrac{14}{3}x$

89. $v(t) = -Ae^{-ct}[\omega\sin(\omega t+\delta) + c\cos(\omega t+\delta)]$
$a(t) = Ae^{-ct}[(c^2-\omega^2)\cos(\omega t+\delta) + 2c\omega\sin(\omega t+\delta)]$

91. (a) $e^{-10} \approx 0.0000454$

(b) $t > 5$, 10

(c) $-\dfrac{1}{5e} \approx -0.0736$

(d) 3.677

(e) Decreasing

93. 71

95. 3

97. (a) $v(t) = \dfrac{c^2t}{\sqrt{b^2+c^2t^2}}, a(t) = \dfrac{b^2c^2}{(b^2+c^2t^2)^{3/2}}$

(b) $v(t) > 0$ for $t > 0$

99. 4

101. (a) $C'(x) = 2 - 0.04x + 0.00021x^2$

(b) 0.10; rate at which costs are increasing as the hundredth unit is produced, and the approximate cost of producing the 101st unit

(c) $c(101) - C(100) = 0.10107$, slightly larger than $C'(100)$

103. $12 + \dfrac{3\pi}{2} \approx 16.7$

105. $\frac{1}{32}$

107. $-\frac{3}{e^9}$

Problem Solving

1. $\left(0, \frac{5}{4}\right)$

5. $3\sqrt{2}$

7. (a) $\frac{4\pi\sqrt{3}}{\sqrt{11}} \approx 6.56$

(b) $40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$

(c) $-480\pi \sin\theta\left(1 + \frac{\cos\theta}{\sqrt{8 + \cos^2\theta}}\right)$

11. $x_T \in (3, \infty)$, $y_T \in (2, \infty)$, $x_N \in \left(0, \frac{5}{3}\right)$, $y_N \in \left(-\frac{5}{2}, 0\right)$

15. $k = 2\sqrt{e} \approx 3.297$

17. (b) $\alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53°$ [or 127°], $\alpha \approx 63°$ [or 117°]

19. The point R approaches the midpoint of the radius AO.

21. $(1, -2)$, $(-1, 0)$

23. $r = \frac{\sqrt{29}}{58} \approx 0.093$

Chapter 4

Exercises 4.1 ■ Page 299

1. $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

3. $48\ \text{cm}^2/\text{s}$

5. $\frac{3}{25\pi}$ m/min

7. $128\pi\ \text{cm}^2/\text{min}$

9. (a) $0.3\ \text{cm}^2/\text{min}$

(b) $0.3 + \frac{3}{4}\sqrt{3} \approx 1.6\ \text{cm}^2/\text{min}$

(c) $\frac{21}{8}\sqrt{3} + 0.3 \approx 4.85\ \text{cm}^2/\text{min}$

11. (a) $\frac{dx}{dt} = -\frac{1}{4}\sqrt{5}$ (b) $\frac{dy}{dt} = \frac{4}{\sqrt{5}}$

13. $\frac{dy}{dt} = 4$

15. $\frac{dx}{dt} = 6$ cm/s

17. $\frac{1}{20\pi}$ cm/min

19. $\frac{215}{\sqrt{101}} \approx 21.393$ km/h

21. 0.6 m/s

23. (a) $\frac{dh}{dt} = \frac{9}{49\pi} \approx 0.058$ ft/min

(b) The radius is constant.

25. (a) $r = 6$ in

(b) $21.6\pi\ \text{in}^3/\text{s}$

(c) $\frac{-0.025}{\pi} \approx -0.00797$ in/s

27. (a) $\frac{24}{\sqrt{5}} \approx 10.733$ ft/s (b) $\frac{24}{\sqrt{5}} \approx 10.733$ ft/s

29. $\frac{\sqrt{65}}{8} \approx 1.008$ m/s

31. $1 + \frac{3\sqrt{3}\pi}{2}$ cm/s

33. $\frac{4}{5}$ ft/m

35. $\frac{3}{2275} \approx 0.00132$ ft/m

37. $\frac{d\theta}{dt} = -\frac{1}{50}$ rad/s

39. $\frac{d\theta}{dt} = -\frac{1}{8}$ rad/s

41. $\frac{dx}{dt} \to -\infty$ as $y \to 0$

43. 0.804 cm/min

45. $\frac{dV}{dt} = \frac{250}{7} \approx 35.714\ \text{cm}^3/\text{min}$

47. $\frac{dB}{dt} \approx 1.045 \times 10^{-8}$ g/yr

49. $-\frac{10}{\sqrt{133}} \approx -0.867$ ft/s

51. $\frac{80\pi}{3} \approx 83.776$ km/min

53. 8π m/min

55. $\sqrt{13 - 6\sqrt{2}} \approx 2.125$ mi/h

57. $-\frac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx -18.5896$ mm/s

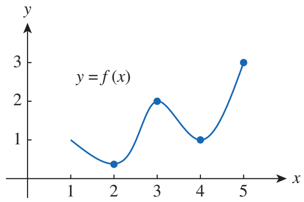
Exercises 4.2 ■ Page 310

1. A function f has an absolute minimum at $x = c$ if $f(c)$ is the smallest function value on the entire domain of f; f has a local minimum at $x = c$ if $f(c)$ is the smallest function value in a neighborhood of $x = c$.

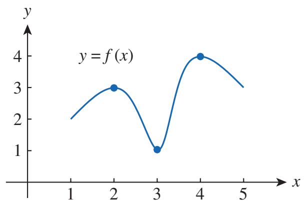
3. Absolute maximum at s, absolute minimum at r, local maximum at c, local minima at b and r, neither a maximum nor a minimum at a and d.

5. Absolute maximum: $f(4) = 5$; no absolute minimum value; local maximum values: $f(4) = 5$, $f(6) = 4$; local minimum values: $f(2) = 2$, $f(1) = f(5) = 3$

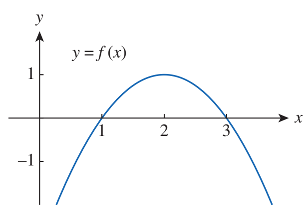
7.



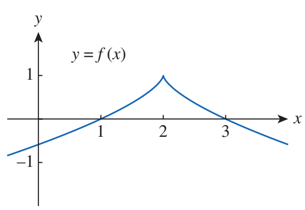
9.



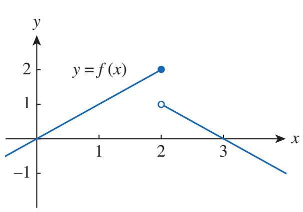
11. (a)



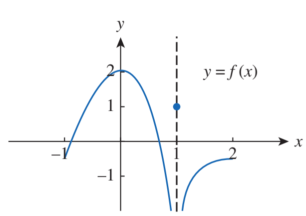
(b)



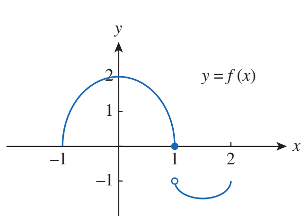
(c)



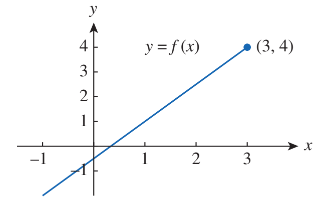
13. (a)



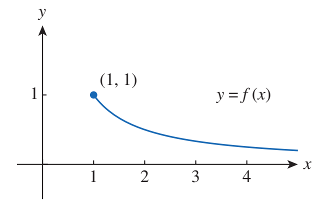
(b)



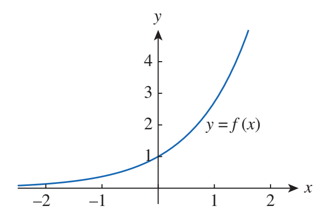
15. Absolute maximum: $f(3) = 4$; no local maximum; no absolute or local minimum



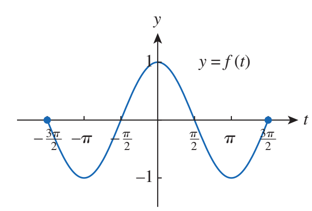
17. Absolute maximum: $f(1) = 1$; no local maximum; no absolute or local minimum



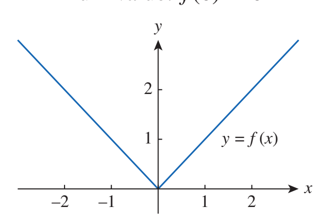
19. No absolute or local maximum or minimum values



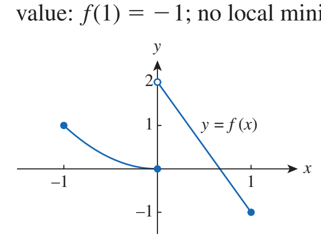
21. Absolute and local maximum value: $f(0) = 1$; absolute and local minimum value: $f(\pm\pi) = -1$



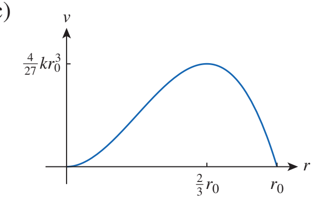
23. No absolute or local maximum values; absolute and local minimum value: $f(0) = 0$



25. No absolute or local maximum values; absolute minimum value: $f(1) = -1$; no local minimum

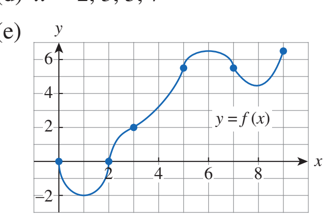


27. $x = \frac{1}{3}$
29. $x = -2, 3$
31. $t = 0$
33. $y = 0, 2$
35. $t = 0, \frac{4}{9}$
37. $x = 0, \frac{8}{7}, 4$
39. $\theta = n\pi$
41. $x = 0, \frac{2}{3}$
43. 10
45. Absolute maximum value: $f(2) = 16$; absolute minimum value: $f(5) = 7$
47. Absolute maximum value: $f(-1) = 8$; absolute minimum value: $f(2) = -19$
49. Absolute maximum value: $f(-2) = 33$; absolute minimum value: $f(2) = -31$
51. Absolute maximum value: $f(0.2) = 5.2$; absolute minimum value: $f(1) = 2$
53. Absolute maximum value: $f(4) = 4 - \sqrt[3]{4}$; absolute minimum value: $f\left(\frac{1}{3\sqrt{3}}\right) = -\frac{2\sqrt{3}}{9}$
55. Absolute maximum value: $f\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2}$; absolute minimum value: $f\left(\frac{\pi}{2}\right) = 0$
57. Absolute maximum value: $f(e^{1/2}) = \frac{1}{2e}$; absolute minimum value: $f\left(\frac{1}{2}\right) = -4 \ln 2$
59. Absolute maximum value: $f(4) = -4\sqrt{e}$; absolute minimum value: $f(-1) = -e^{1/8}$
61. Absolute maximum value: $f(4) = 4 - 2 \tan^{-1} 4$; absolute minimum value: $f(1) = 1 - \frac{\pi}{2}$
63. e
65. $-\frac{16}{3}$ ft/s
67. $f\left(\frac{a}{a+b}\right) = \frac{a^a b^b}{(a+b)^{a+b}}$
69. Absolute maximum value: $f(-0.77) = 2.19$, $f\left(-\sqrt{\frac{3}{5}}\right) = \frac{6}{25}\sqrt{\frac{3}{5}} + 2$; Absolute minimum value: $f(-0.77) = 1.81$, $f\left(\sqrt{\frac{3}{5}}\right) = -\frac{6}{25}\sqrt{\frac{3}{5}} + 2$
71. Absolute maximum value: $f(0.75) = 0.32$, $f\left(\frac{3}{4}\right) = \frac{3\sqrt{3}}{16}$; absolute minimum value: $f(0) = f(1) = 0$
73. (a) $x = -1, 1, 3$
(b) $f(-1) = f(3) = -14$
(c) $f(1) = 2$
(d) $f(-1) = f(3) = -14$
75. 0.177 mg/mL at 0.357 hours (21.4 minutes)
77. 3.9665°C
79. Cheapest: $t \approx 0.855$ (June 1994); most expensive: $t \approx 4.618$ (March 1998)
81. (a) $r = \frac{2}{3}r_0$
(b) $\frac{4}{27}kr_0^3$
(c)



Exercises 4.3 ■ Page 329

1. $f'(1) = f'(5) = 0$
3. $c = 2$
5. (b) $c \approx 1.8, 6.2$ (c) $c \approx 3.2, 5.2$
7. $\approx 0.8, 3.2, 4.4, 6.1$
9. (a) $[0, 1], [3, 7]$
(b) $[1, 3]$
(c) $(2, 4), (5, 7)$
(d) $(0, 2), (4, 5)$
(e) $(2, 2), (4, 3), (5, 4)$
13. (a) Increasing: $[0, 1], [5, 7]$; decreasing: $[1, 5], [7, 8]$
(b) Local maximum at $x = 1, 7$; local minimum at $x = 5$
15. (a) Increasing: $[1, 6], [8, 9]$; decreasing: $[0, 1], [6, 8]$
(b) Local maximum at $x = 6$; local minimum at $x = 1, 8$
(c) Concave up: $(0, 2), (3, 5), (7, 9)$; concave down: $(2, 3), (5, 7)$
(d) $x = 2, 3, 5, 7$
(e)



17. (a) $[0, 4], [6, 8]$
(b) Local maximum at $x = 4, 8$; local minimum at $x = 6$
(c) Concave up: $(0, 1), (2, 3), (5, 7)$; concave down: $(1, 2), (3, 5), (7, 9)$
(d) $x = 1, 2, 3, 5, 7$

19. (a) Increasing: $(-\infty, -1]$, $[3, \infty)$; decreasing: $[-1, 3]$
(b) Local maximum value: $f(-1) = 9$;
local minimum value: $f(3) = -23$
(c) Concave up: $(1, \infty)$; concave down: $(-\infty, +1)$;
IP: $(1, -7)$

21. (a) Increasing: $[-1, 0]$, $[1, \infty)$; decreasing: $(-\infty, -1]$, $[0, 1]$
(b) Local maximum value: $f(0) = 3$;
local minimum value: $f(\pm 1) = 2$
(c) Concave up: $\left(-\infty, -\frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{3}}{3}, \infty\right)$;
concave down: $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$; IP: $\left(\pm\frac{\sqrt{3}}{3}, \frac{22}{9}\right)$

23. (a) Increasing: $[-1, 1]$; decreasing: $(-\infty, -1]$, $[1, \infty)$
(b) Local minimum value: $f(1) = \frac{1}{2}$;
local minimum value: $f(-1) = -\frac{1}{2}$
(c) Concave up: $(-\sqrt{3}, 0)$, $(\sqrt{3}, \infty)$;
concave down: $(-\infty, -\sqrt{3})$, $(0, \sqrt{3})$
IP: $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$, $(0, 0)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$

25. (a) Increasing: $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$;
decreasing: $\left[0, \frac{\pi}{2}\right], \left[\frac{3\pi}{2}, 2\pi\right]$
(b) Local maximum value: $f\left(\frac{3\pi}{2}\right) = 2$;
local minimum value: $f\left(\frac{\pi}{2}\right) = -2$
(c) Concave up: $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$;
concave down: $\left(0, \frac{\pi}{6}\right), \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right)$;
IP: $\left(\frac{\pi}{6}, -\frac{1}{4}\right), \left(\frac{5\pi}{6}, -\frac{1}{4}\right)$

27. (a) Increasing: $[e^{-1/2}, \infty)$; decreasing: $(0, e^{-1/2}]$
(b) Local minimum value: $f(e^{-1/2}) = -\frac{1}{2e}$
(c) Concave up: $(e^{-3/2}, \infty)$; concave down: $(0, e^{-3/2})$;
IP: $\left(e^{-3/2}, -\frac{3}{2e^3}\right)$

29. (a) Increasing: $\left[0, \frac{1}{2}\right]$; decreasing: $\left[\frac{1}{2}, \infty\right)$
(b) Local maximum value: $f\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2e}}$
(c) Let $a = \frac{1}{2} + \frac{1}{2}\sqrt{2}$;
Concave up: (a, ∞); concave down: $(0, a)$;
IP: $(a, f(a)) \approx (1.21, 0.33)$

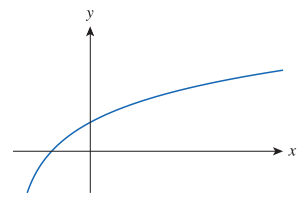
31. Local maximum value at $x = 0$; no local maxima; IP at $x = -1$

33. Local maximum value: $f(2) = \frac{1}{4}$;
local minimum value: $f(-2) = -\frac{1}{4}$

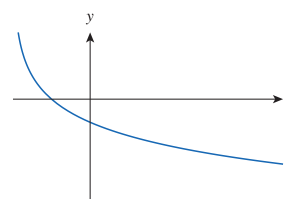
35. Local minimum value: $f\left(\frac{1}{16}\right) = -\frac{1}{4}$

37. (a) f has a local maximum at $x = 2$
(b) f has a horizontal tangent at $x = 6$

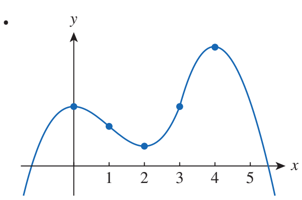
39. (a)



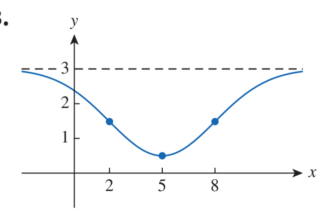
(b)



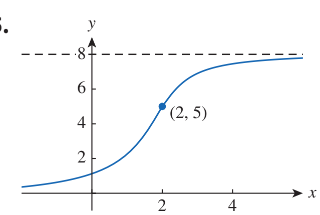
41.



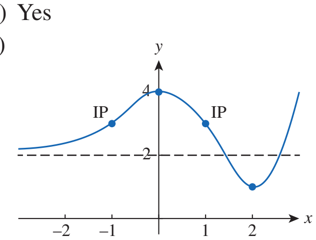
43.



45.



47. (a) No
(b) Yes
(c)



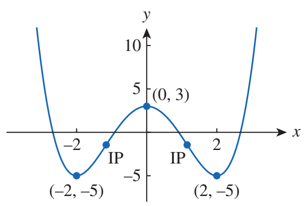
49. 1

51. (a) Increasing: $[-2, 0]$, $[2, \infty)$; decreasing: $(-\infty, -2]$, $[0, 2]$

(b) Local maximum value: $f(0) = 3$; local minimum value: $f(\pm 2) = -5$

(c) Concave up: $\left(-\infty, -\dfrac{2}{\sqrt{3}}\right), \left(\dfrac{2}{\sqrt{3}}, \infty\right)$; concave down $\left(-\dfrac{2}{\sqrt{3}}, \dfrac{2}{\sqrt{3}}\right)$; IP: $\left(\pm\dfrac{2}{\sqrt{3}}, -\dfrac{13}{9}\right)$

(d)

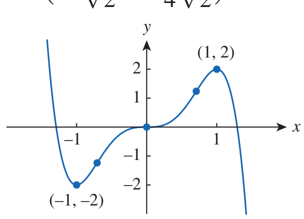


53. (a) Increasing: $[-1, 1]$; decreasing: $(-\infty, -1]$, $[1, \infty)$

(b) Local maximum value: $f(1) = 2$; local minimum value: $f(-1) = -2$

(c) Concave up: $\left(-\infty, -\dfrac{1}{\sqrt{2}}\right), \left(0, \dfrac{1}{\sqrt{2}}\right)$; concave down: $\left(-\dfrac{1}{\sqrt{2}}, 0\right), \left(\dfrac{1}{\sqrt{2}}, \infty\right)$; IP: $\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{7}{4\sqrt{2}}\right)$, $(0, 0)$ $\left(\dfrac{1}{\sqrt{2}}, \dfrac{7}{4\sqrt{2}}\right)$

(d)

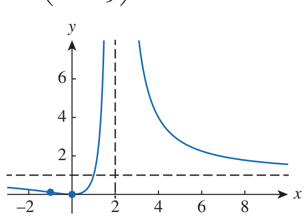


55. (a) Increasing: $[0, 2)$; decreasing: $(-\infty, 0]$, $(2, \infty)$

(b) Local minimum value: $f(0) = 0$

(c) Concave up: $(-1, 2)$, $(2, \infty)$; concave down: $(-\infty, -1)$; IP: $\left(-1, \dfrac{1}{9}\right)$

(d)

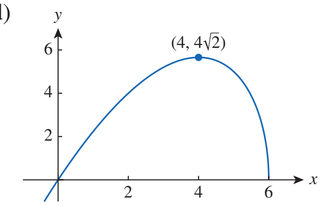


57. (a) Increasing: $(-\infty, 4]$; decreasing: $[4, 6)$

(b) Local maximum value: $f(4) = 4\sqrt{2}$

(c) Concave down: $(-\infty, 6)$

(d)

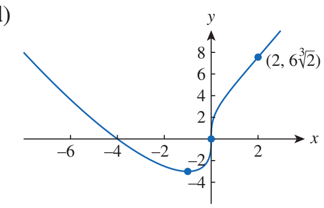


59. (a) Increasing: $[-1, \infty)$; decreasing: $(-\infty, -1]$

(b) Local minimum value: $f(-1) = -3$

(c) Concave up: $(-\infty, 0)$, $(2, \infty)$; concave down: $(0. 2)$; IP: $(0, 0)$, $(2, 6\sqrt[3]{2})$

(d)

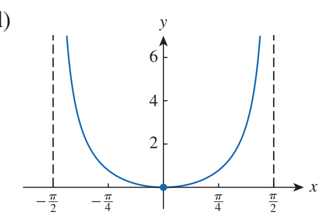


61. (a) Increasing: $\left[0, \dfrac{\pi}{2}\right)$; decreasing: $\left(-\dfrac{\pi}{2}, 0\right]$

(b) Local minimum value: $f(0) = 0$

(c) Concave up: $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$; no IP

(d)

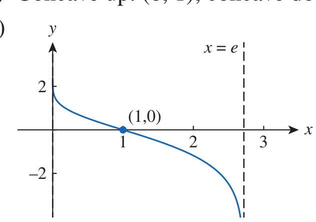


63. (a) Decreasing: $(0, e)$

(b) No local maximum or minimum value

(c) Concave up: $(0, 1)$; concave down: $(1, e)$; IP: $(1, 0)$

(d)

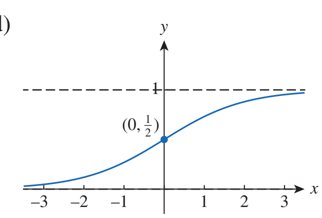


65. (a) Increasing: $(-\infty, \infty)$

(b) No local maximum or minimum

(c) Concave up: $(-\infty, 0)$; concave down: $(0, \infty)$; IP: $\left(0, \dfrac{1}{2}\right)$

(d)

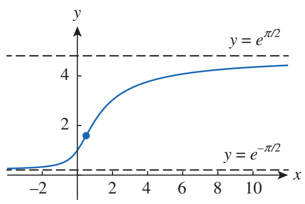


67. (a) Increasing: $(-\infty, \infty)$

(b) No local maximum or minimum

(c) Concave up: $\left(-\infty, \frac{1}{2}\right)$; concave down: $\left(\frac{1}{2}, \infty\right)$;
IP: $\left(\frac{1}{2}, e^{\arctan(1/2)}\right) \approx (.5, 1.59)$

(d)



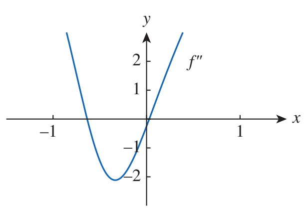
69. $[3, \infty)$

71. (a) Local maximum value: $f(1) = \sqrt{2}$; no local minimum

(b) $\dfrac{3 - \sqrt{17}}{4} \approx -0.281$

73. Concave up: $(0.848, 1.571)$, $(2.293, \pi)$;
concave down: $(0, 0.848)$, $(1.571, 2.293)$;
IP: $(0.848, .744)$, $(1.571, 0)$, $(2.293, -0.744)$

75.



Concave up: $(-\infty, -0.636)$, $(0.0288, \infty)$;
concave down: $(-0.636, 0.0288)$

77. (b) $\dfrac{d^2y}{dx^2} = \dfrac{6x^2 - 6xy + 6y^2}{(x - 2y)^3}$

(c) Local maximum

79. If $S(t)$ is the average SAT score as a function of time t, then $S'(t) < 0$ and $S''(t) > 0$.

81. (a) Very unhappy. It's uncomfortably hot, and $f'(3) = 2$ indicates that the temperature is increasing, but $f''(3) = 4$ indicates that the rate of increase is increasing. (The temperature is rapidly getting warmer.)

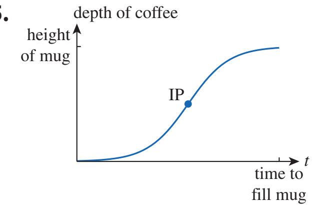
(b) Still unhappy, but not as unhappy as in part (a). It's uncomfortably hot, and $f'(3) = 2$ indicates that the temperature is increasing, but $f''(3) = -4$ indicates that the rate of increase is decreasing. (The temperature is slowly getting warmer.)

(c) Somewhat happy. It's uncomfortably hot, and $f(3) = -2$ indicates that the temperature is decreasing, but $f''(3) = 4$ indicates that the rate of change is increasing. (The rate of change is negative, but it's becoming less negative. The temperature is slowly getting cooler.)

(d) Very happy. It's uncomfortably hot, and $f(3) = -2$ indicates that the temperature is decreasing, and $f''(3) = -4$ indicates that the rate of change is decreasing, that is, becoming more negative. (The temperature is rapidly getting cooler.)

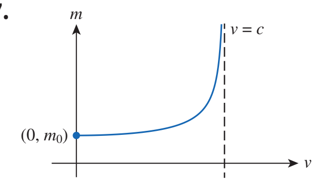
83. $\dfrac{dy}{dx} = \dfrac{1}{4}\sec t$; $\dfrac{d^2y}{dx^2} = -\dfrac{1}{16}\sec^3 t$; concave up: $\dfrac{\pi}{2} < t < \pi$

85.

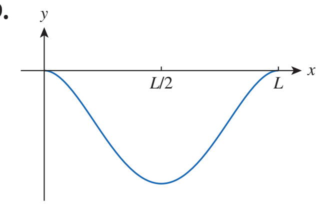


The inflection point occurs when the mug is half full; where the rate of increase of the depth has a maximum, where the mug is narrowest.

87.



89.



91. $t_1 = \dfrac{200}{7} \approx 28.571$, $t_2 = \dfrac{600}{7} \approx 85.714$

At t_1 minutes, the rate of increase of the level of medication in the bloodstream is at its greatest, and at t_2 minutes, the rate of decrease is the greatest.

93. $f(x) = \dfrac{1}{9}(2x^3 + 3x^2 - 12x + 7)$

95. (a) $a = 0$, $b = -1$

(b) $y = -x$

97. IP: $(1, 1)$, $\left(-2 - \sqrt{3}, \frac{1}{4}(1 - \sqrt{3})\right)$,
$\left(-2 + \sqrt{3}, \frac{1}{4}(1 + \sqrt{3})\right)$

107. Two inflection points when $|c| > \dfrac{2\sqrt{6}}{3} \approx 1.63$;

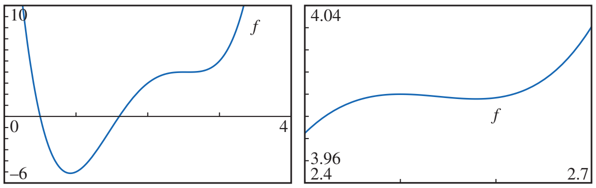
no inflection points

when $|c| \le \dfrac{2\sqrt{6}}{3} \approx 1.63$

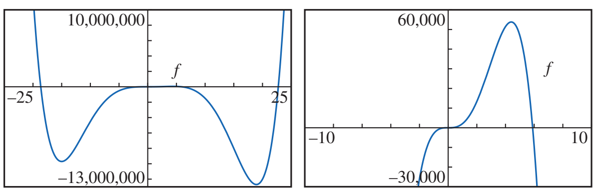
Exercises 4.4 ■ Page 342

1. Increasing: $[0.92, 2.5]$, $[2.58, \infty)$; decreasing: $(-\infty, 0.92]$, $[2.5, 2.58]$; local maximum value: $f(2.5) = 4$; local minimum values: $f(0.92) \approx -5.12$, $f(2.58) \approx 3.998$; concave up:

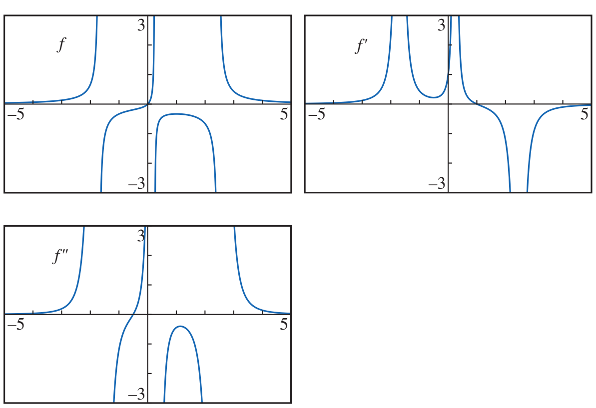
$(-\infty,\ 1.46)$, $(2.54,\ \infty)$; concave down: $(1.46,\ 2.54)$;
IP: $(1.46,\ -1.40)$, $(2.54,\ 3.999)$



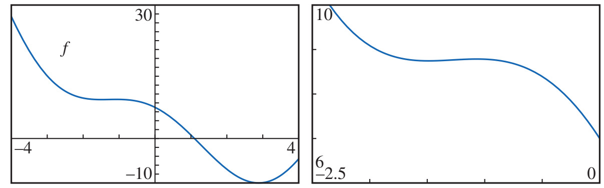
3. Increasing: $[-15,\ 4.40]$, $[18.93,\ \infty)$; decreasing: $(-\infty,\ -15]$, $[4.40,\ 18.93]$; local maximum value: $f(4.40) \approx 53{,}800$; local minimum values: $f(-15) \approx\ -9{,}700{,}000$, $f(18.93) \approx\ -12{,}700{,}000$; concave up: $(-\infty,\ -11.34)$, $(0,\ 2.92)$, $(15.08,\ \infty)$; concave down: $(-11.34,\ 0)$, $(2.92,\ 15.08)$; IP: $(0,\ 0)$, $(-11.34,\ -6{,}250{,}000)$, $(2.92,\ 31{,}800)$, $(15.08,\ -8{,}150{,}000)$



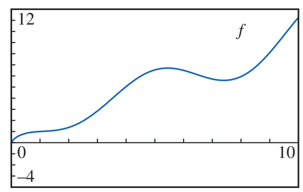
5. Increasing: $(-\infty,\ -1.7)$, $(-1.7,\ 0.24)$, $(0.24,\ 1]$; decreasing: $[1,\ 2.46)$, $(2.46,\ \infty)$; local maximum value: $f(1) = -\dfrac{1}{3}$; concave up: $(-\infty,\ -1.7)$, $(-0.506,\ 0.24)$, $(2.46,\ \infty)$; concave down: $(-1.7,\ -0.506)$, $(0.24,\ 2.46)$; IP: $(-0.506,\ -0.192)$



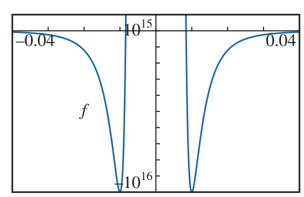
7. Increasing: $[-1.49,\ -1.07]$, $[2.89,\ 4]$; decreasing: $[-4,\ -1.49]$, $[-1.07,\ 2.89]$; local maximum value: $f(-1.07) \approx 8.79$; local minimum values: $f(-1.49) \approx 8.75$, $f(2.89) \approx\ -9.99$; concave up: $(-4,\ -1.28)$, $(1.28,\ 4)$; concave down: $(-1.28,\ 1.28)$; IP: $(-1.28,\ 8.77)$, $(1.28,\ -1.48)$



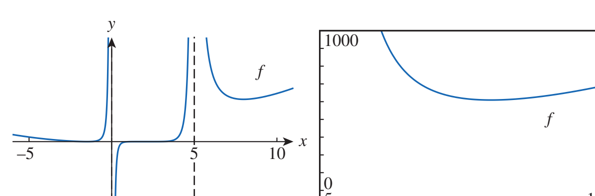
9. Increasing: $[0,\ 5.45]$, $[7.40,\ 10]$;
decreasing: $[5.45,\ 7.40]$;
local maximum value: $f(5.45) \approx 6.70$;
local minimum value: $f(7.40) \approx 5.60$;
concave up: $(1.10,\ 3.54)$, $(6.45,\ 9.52)$;
concave down: $(0,\ 1.10)$, $(3.54,\ 6.45)$, $(9.52,\ 10)$;
IP: $(1.10.1.02)$, $(3.54,\ 4.03)$, $(6.45,\ 6.14)$, $(9.52,\ 9.73)$



11. Increasing: $[-0.01,\ 0)$, $[0.01,\ \infty)$;
decreasing: $(-\infty,\ -0.01]$, $(0,\ 0.01]$;
concave up: $\left(-\dfrac{1}{100}\sqrt[4]{1.8},\ 0\right)$, $\left(0,\ \dfrac{1}{100}\sqrt[4]{1.8}\right)$;
concave down: $\left(-\infty,\ -\dfrac{1}{100}\sqrt[4]{1.8}\right)$, $\left(\dfrac{1}{100}\sqrt[4]{1.8},\ 0\right)$;

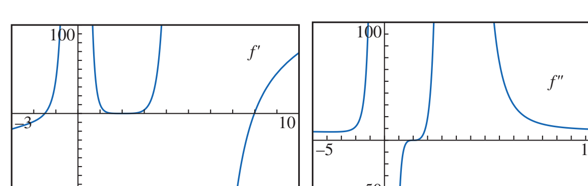


13.

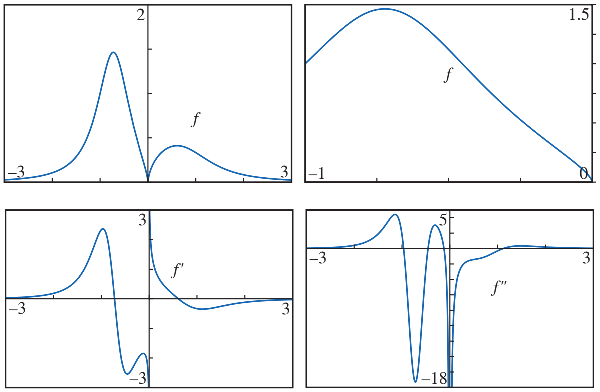


VA: $x = 0$, $x = 5$; HA: none;
local minimum values: $f\left(-\dfrac{3}{2}\right) = 0$, $f(7.98) \approx 609$

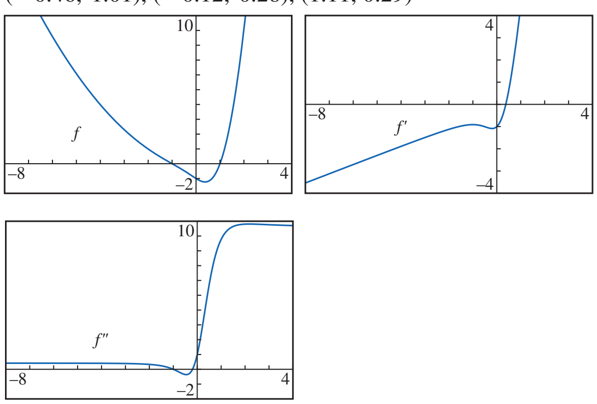
15.



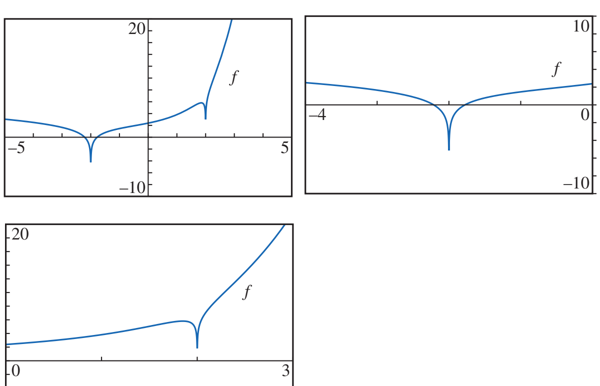
Increasing: $[-1.5, 0)$, $(0, 5)$, $[7.98, \infty)$; decreasing: $(-\infty, -1.5]$, $(5, 7.98]$; concave up: $(-\infty, 0)$, $(2, 5)$, $(5, \infty)$; concave down: $(0, 2)$

17. 

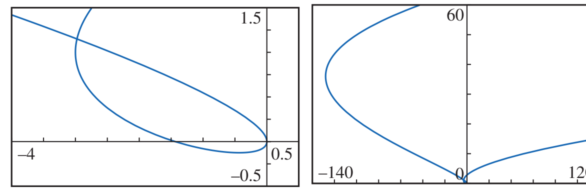
Increasing: $(-\infty, -0.72]$, $[0, 0.61]$; decreasing: $[-0.72, 0]$, $[0.61, \infty)$; local maximum value: $f(-0.72) \approx 1.46$; local minimum value: $f(0.61) \approx 0.41$; concave up: $(-\infty, -0.97)$, $(-0.46, -0.12)$, $(1.11, \infty)$; concave down: $(-0.97, -0.46)$, $(-0.12, 0)$, $(0, 1.11)$; IP: $(-0.97, 1.08)$, $(-0.46, 1.01)$, $(-0.12, 0.28)$, $(1.11, 0.29)$

19. 

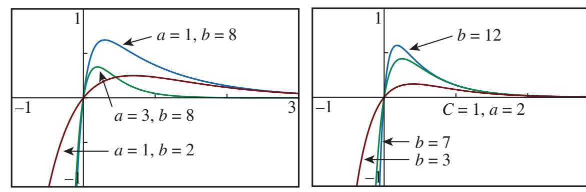
Increasing: $[0.38, \infty)$; decreasing: $(-\infty, 0.38)]$; local minimum value: $f(0.38) = -1.23$; concave up: $(-\infty, -1)$, $(-0.20, \infty)$; concave down: $(-1, -0.20)$; IP: $(-1, 0)$, $(-0.20, -0.79)$

21. 

23. $t = \dfrac{1}{\sqrt{2}}$; $\left(-\dfrac{1}{4}, \dfrac{1}{\sqrt{2}} - \dfrac{1}{2}\ln 2\right)$

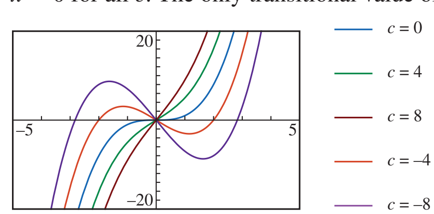
25. 

Horizontal tangent line: $t = \dfrac{1}{4}$, $\left(-\dfrac{111}{256}, -\dfrac{1}{8}\right)$; vertical tangent line: $t = 0$, $(0, 0)$; $t = -4$, $(-128, 36)$; $t = 1$, $(-3, 1)$

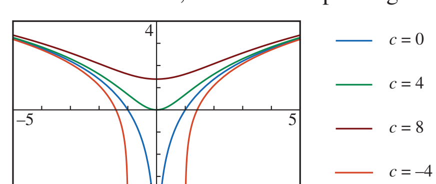
27. 

All graphs pass through the origin, approach the t-axis as t increases and approach $-\infty$ as $t \to -\infty$. As b increases, the slope of the tangent at the origin increases and the local maximum value increases.

29. For $c \geq 0$, f is increasing and has no local maximum or minimum. For $c < 0$, there is a local maximum value and a local minimum value. As c decreases, the local maximum and minimum move further apart. There is an inflection point at $x = 0$ for all c. The only transitional value of c is $c = 0$.

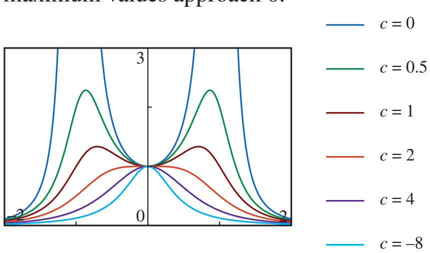


31. For $c < 0$, there are vertical asymptotes at $x = \pm\sqrt{c}$, and as c decreases, the asymptotes get further apart. For $c = 0$, there is a vertical asymptote at $x = 0$. For $c > 0$, there are no asymptotes. There is a local and absolute minimum at $x = 0$. For $c \leq 0$, f is concave down on both intervals on which it is defined. For $c > 0$, there are inflection points at $x = \pm\sqrt{c}$, and as c increases, the inflection points get further apart.



33. f is an even function; for any value of c, $y = 0$ is a horizontal asymptote. For $c > 0$, f has domain $\mathbb{R}$ and $f > 0$. If $c \geq 2$, then $f(0) = 1$ is a maximum. There are two inflection points, which approach the y-axis as $c \to \infty$.

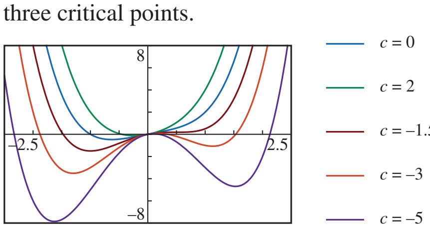
$c = 2$ and $c = 0$ are transitional values of c at which the shape of the curve changes. For $0 < c < 2$, there are three critical points. As c decreases from 2 to 0, the maximum values get larger and larger, and the x-values at which they occur go from 0 to ± 1. There are four inflection points for $0 < c < 2$, and they get farther away from the origin, both vertically and horizontally, as $c \to 0^+$. For $c = 0$, the function is asymptotic to the x-axis and to the lines $x = \pm 1$, approaching $+\infty$ from both sides of each. There are no inflection points.
For $c < 0$, f has four vertical asymptotes. As c decreases, the two exterior asymptotes move away from the origin, while the two interior ones move toward it. There is one local minimum value and two local maximum values. As c decreases, the x-values at which these maxima occur get larger, and the maximum values approach 0.



35. For $c = 0$, there is no inflection point; the curve is concave up everywhere. If c increases, the curve becomes steeper; still no inflection points. If c starts at 0 and decreases, a slight upward bulge appears near $x = 0$, so that there are two inflection points for any $c < 0$.

For $c = 0$, the graph has one critical number, at the absolute minimum near $x = -0.6$. As c increases, the number of critical points does not change. If c decreases from 0, eventually there is another local minimum to the right of the origin, between $x = 1$ and $x = 2$; there is also a maximum near $x = 0$.

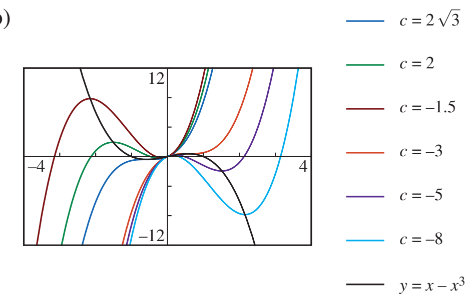
For $c = -1.5$, there appear to be two critical numbers: absolute minimum near $x = -1$, and a horizontal tangent with no extremum near $x = 0.5$. For any c smaller, there are three critical points.



37. (a) Local maximum value at $x = \dfrac{-c - \sqrt{c^2 - 12}}{6}$;

local minimum value at $x = \dfrac{-c + \sqrt{c^2 - 12}}{6}$

(b)



Exercises 4.5 ■ Page 352

1. (a) Indeterminate form of type $\dfrac{0}{0}$

(b) 0

(c) 0

(d) If $f(x) \to 0$ through positive values, then $\lim\limits_{x\to a} \dfrac{p(x)}{f(x)} = \infty$.

If $f(x) \to 0$ through negative values, then $\lim\limits_{x\to a} \dfrac{p(x)}{f(x)} = -\infty$.

(e) Indeterminate form of type $\dfrac{\infty}{\infty}$

3. (a) $-\infty$
(b) Indeterminate form of type $\infty - \infty$
(c) ∞

5. $\dfrac{9}{4}$ **7.** 1 **9.** $\dfrac{1}{6}$

11. 12 **13.** $\dfrac{11}{20}$ **15.** $\dfrac{3}{2}$

17. 2 **19.** 0 **21.** $-\dfrac{1}{2}$

23. 0 **25.** $\ln \dfrac{8}{5}$ **27.** ∞

29. $-\dfrac{1}{2}$ **31.** 0 **33.** $\dfrac{1}{2}(n^2 - m^2)$

35. $\dfrac{1}{3}$ **37.** 0 **39.** 2

41. $\cos a$ **43.** 0 **45.** -1

47. ∞ **49.** $\dfrac{1}{5}$ **51.** 0

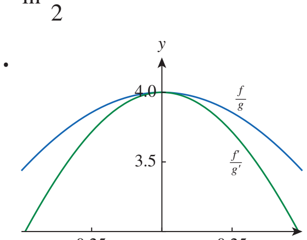
53. 0 **55.** $\ln \dfrac{7}{5}$ **57.** 1

59. e^{ab} **61.** 2 **63.** 1

65. $e^{2/\pi}$ **67.** e^{-8}

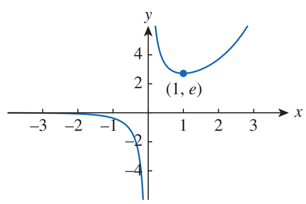
69. $\dfrac{\ln \frac{5}{4}}{\ln \frac{3}{2}} \approx 0.550$

71.

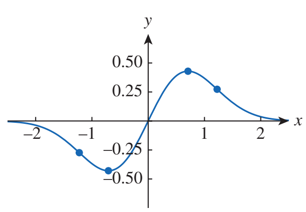


$\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = 4$

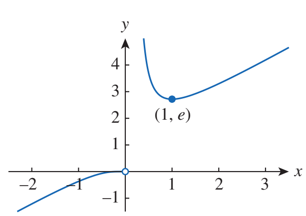
73. HA: $y = 0$; VA: $x = 0$; increasing: $[1, \infty)$; decreasing: $(-\infty, 0)$, $(0, 1]$; local minimum value: $f(1) = e$; concave up: $(0, \infty)$; concave down: $(-\infty, 0)$; No IP



75. HA: $y = 0$; no VA; increasing: $\left[-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right]$; decreasing: $\left(-\infty, -\dfrac{1}{\sqrt{2}}\right]$, $\left[\dfrac{1}{\sqrt{2}}, \infty\right)$; local maximum value: $f\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{1}{\sqrt{2e}}$; local minimum value: $f\left(-\dfrac{1}{\sqrt{2}}\right) = -\dfrac{1}{\sqrt{2e}}$; concave up: $\left(\sqrt{\dfrac{3}{2}}, \infty\right)$, $\left(-\sqrt{\dfrac{3}{2}}, 0\right)$; concave down: $\left(-\infty, -\sqrt{\dfrac{3}{2}}\right)$, $\left(0, \sqrt{\dfrac{3}{2}}\right)$; IP: $(0, 0)$, $\left(\pm\sqrt{\dfrac{3}{2}}, \pm\sqrt{\dfrac{3}{2}}e^{-3/2}\right)$



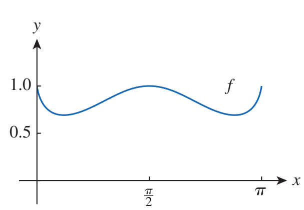
77. (a)



(b) $\lim\limits_{x\to 0^+} f(x) = \infty$, $x = 0$ is a VA; $\lim\limits_{x\to 0^-} f(x) = 0$

(c) Local minimum value: $f(1) = e$; concave up: $(0, \infty)$; concave down: $(-\infty, 0)$; No IP

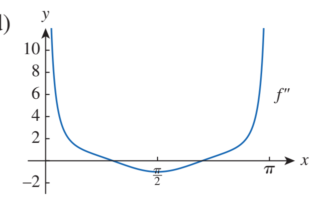
79. (a) f is continuous on intervals of the form $(2n\pi, (2n+1)\pi)$.



(b) $\lim\limits_{x\to 0^+} f(x) = 1$

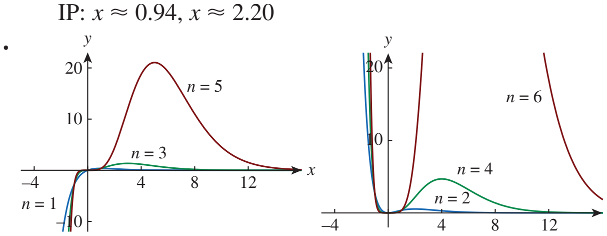
(c) Let $x_1 = \sin^{-1}(e^{-1})$, $x_2 = \dfrac{\pi}{2}$, $x_3 = \pi - \sin^{-1}(e^{-1})$; local maximum value: $(x_2, 1)$; local minimum values: $(x_1, f(x_1)) \approx (0.3767, 0.6922)$, $(x_3, f(x_3)) \approx (2.7649, 0.6922)$

(d)



IP: $x \approx 0.94$, $x \approx 2.20$

81.



All curves pass through the origin and approach $y = 0$ as $x \to \infty$. At $x = 0$: local minimum for n even
At $x = n$: local maximum for all n
As n increases, $(n, f(n))$ gets farther away from the origin.
As n increases, the IP moves farther away from the origin; they are symmetric about the line $x = n$.

83. Original limit or the limit of the reciprocal; 1

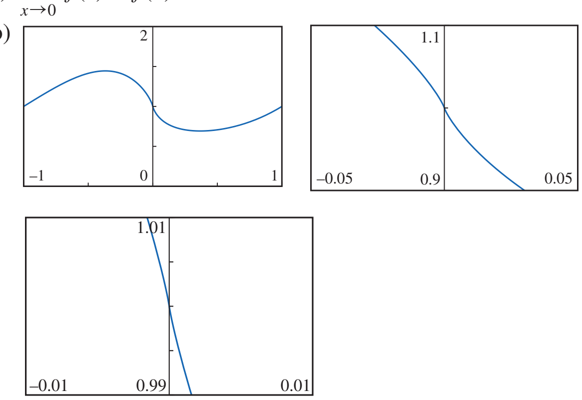
91. $\dfrac{16}{9}a$

93. $\dfrac{1}{2}$

95. 56

99. (a) $\lim\limits_{x\to 0} f(x) = f(0) = 1$

(b)



The graphs suggest that f is differentiable at $x = 0$.

(c) The curve has a vertical tangent at (0, 1) and is not differentiable there. This fact is not seen in the graphs because $\ln|x| \to -\infty$ very slowly as $x \to 0$.

Exercises 4.6 ■ Page 365

1. (a) 11 or 12
(b) $x = y = 11.5$

3. 10 and 10

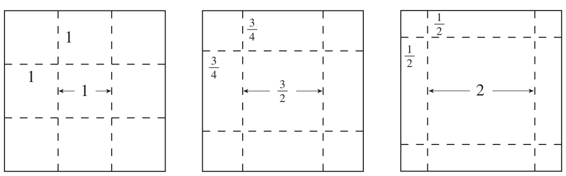
5. $x = y = 25$

7. (C)

9. 60

11. $I = 2$

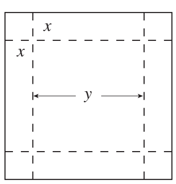
13. (a)



Volume = 1 Volume = 1.6875 Volume = 2

Maximum volume of at least 2 ft^3

(b)



(c) $V = xy^2$

(d) $y + 2x = 3$

(e) $V(x) = x(3 - 2x)^2$

(f) Maximum is $V\left(\frac{1}{2}\right) = 2\ \text{ft}^3$

15. 1000 ft by 1500 ft: middle fence parallel to the short side

17. $40 \times 40 \times 20$

19. $C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{9/2}} \approx 163.54$

21. $\frac{250}{3} \times 125$, area $= 125\left(\frac{250}{3}\right) = 10{,}416.7\ \text{ft}^2$

25. $\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$

27. (2.651, 0.4157)

29. $\sqrt{2}r \times \sqrt{2}r$

31. $\frac{L}{4} \times \frac{\sqrt{3}}{4}L$

33. (a) $y = -2kx + 12 + k^2$

(b) y-intercept: $12 + k^2$; x-intercept: $\frac{k^2 + 12}{2k}$

(c) $A(k) = \frac{(12 + k^2)^2}{4k}$

(d) $k = 2\sqrt{3}$, $A(2\sqrt{3}) = 24\sqrt{3} \approx 41.569$

35. $\sqrt{2}a$

37. $V = \frac{4\pi r^3}{3/\sqrt{3}}$

39. $S = \pi r^2(1 + \sqrt{5})$

41. $\frac{60}{4 + \pi} \times \frac{30}{4 + \pi}$

43. $2\sqrt{30} \times \frac{90}{\sqrt{30}}$

45. $x =$ the length of wire used for the square, maximum: $x = 0$; minimum: $x = \frac{40}{4 + \pi}$

47. $L = 8\frac{\sqrt{1 + 2^{2/3}}}{2^{1/3}} + 4\sqrt{1 + 2^{2/3}} \approx 16.65$ ft

49. $h = 3\sqrt[3]{\frac{6}{\pi}} \approx 3.722$, $r = \frac{3\sqrt{3}}{\sqrt[6]{6\pi^2}} \approx 2.632$

51. $v \approx 53$ mi/h

53. $\frac{E^2}{4r}$

55. (a) $\frac{dS}{d\theta} = \frac{3}{2}s^2 \csc\theta(\csc\theta - \sqrt{3}\cot\theta)$

(b) $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55°$

(c) $S = 6s\left(h + \frac{1}{2\sqrt{2}}s\right)$

57. Row directly to B

59. $6 - \frac{2}{\sqrt{3}} \approx 4.85$ km east of the refinery

61. $\frac{10\sqrt[3]{3}}{1 + \sqrt[3]{3}} \approx 5.905$ ft from the stronger source

63. $(a^{2/3} + b^{2/3})^{3/2}$

65. $2\sqrt{6}$

67. (b) (i) $216{,}000 + 40{,}000\sqrt{10} \approx 342, 491; 342{,}49; 389{:}74$
(ii) 400 **(iii)** 320

69. (a) $p(x) = 19 - \frac{x}{3000}$

(b) \$9.50

71. (a) $p(x) = -\frac{1}{8}x + 500$ where $x \geq 12000$

(b) \$250

(c) \$310

77. $L(x) = x + \sqrt{x^2 - 10x + 29} + \sqrt{x^2 - 10x + 34}$; minimum value: $L(3.593) = 9.352$ m

81. $\theta = \tan^{-1}(\sqrt[3]{1.5}) \Rightarrow L = 9\csc\theta + 6\sec\theta \approx 21.07$ ft

83. $\theta = \frac{\pi}{3}$

85. $x = \sqrt{d(h + d)}$

87. (a) 5.1 km from B

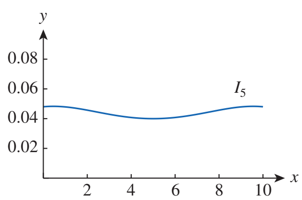
(b) W/L large: fly to a point C closer to B and to D to minimize energy used flying over water; W/L small: fly to a point C that is closer to D than to B to minimize the distance of the flight; x km from B: $\dfrac{W}{L} = \dfrac{\sqrt{25 + x^2}}{x}$

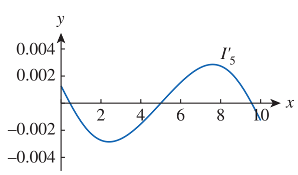
(c) $\dfrac{W}{L} = \dfrac{\sqrt{25 + 13^2}}{13} \approx 1.07$; no value of W/L for which the bird should fly directly to B

(d) $\dfrac{\sqrt{41}}{4} \approx 1.6$

89. (a) $I(x) = \dfrac{k}{x^2 + d^2} + \dfrac{k}{x^2 - 20x + 100 + d^2}$

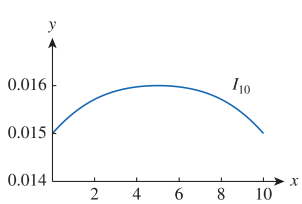
(b) Let $I_5(x) = \dfrac{1}{x^2 + 25} + \dfrac{1}{x^2 - 20x + 125}$

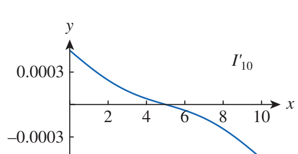




The graphs suggest that $I_5(x)$ has a minimum at $x = 5$ m.

(c) Let $I_{10}(x) = \dfrac{1}{x^2 + 100} + \dfrac{1}{x^2 - 20x + 200}$





The graphs suggest that $I_{10}(x)$ has a minimum at $x = 0$ and $x = 10$.

(d) $d = 5\sqrt{2}$

Exercises 4.7 ■ Page 377

1. (a) $x_2 \approx 7.6$; $x_3 \approx 7.1$

(b) No; tangent line is nearly horizontal.

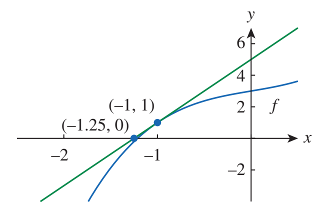
3. $x_2 = \dfrac{9}{2}$

5. $x_1 = a, b, c$ will work; $x_1 = d$ will not work.

7. $x_3 = -2.7186$

9. $x_3 = -1.1529$

11. $x_2 = -1.25$



13. 1.82056420

15. 2.94283096

17. (a) $f(2) = -14 < 0 < 29 = f(3)$, by the IVT there is a number c in (2, 3) such that $f(c) = 0$.

(b) 2.630020

19. 0.876726

21. 3.692586

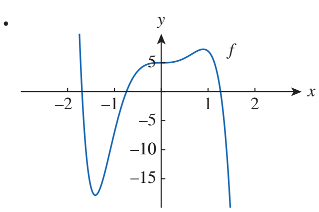
23. $-3.637958, -1.862365, 0.889470$

25. $-0.549700, 2.629658$

27. 0.865474

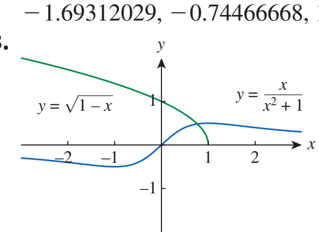
29. 2.792248, 6.988241, 8.623612, 13.622751, 14.602375

31.



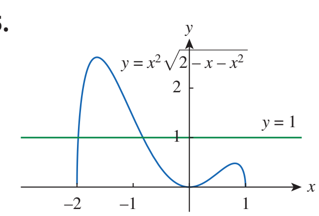
$-1.69312029, -0.74466668, 1.2587094$

33.



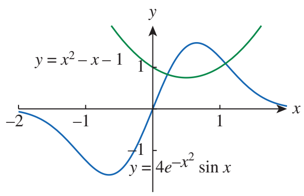
0.76682579

35.



$-1.97806681, -0.82646233$

37.



$0.21916368, 1.08422462$

39. (b) 31.622777

41. Tangent line is horizontal.

43. Each successive approximation is twice as large as the previous one in absolute value.

45. (a) $-1.293227, -0.441731, 0.507854$
(b) -2.0212

47. $(1.519855, 2.306964)$

49. $(0.410245, 0.347810)$

51. 0.76286%, or 9.55% per year compounded monthly

Exercises 4.8 ■ Page 385

1. $F(x) = 2x^2 + 7x + C$

3. $F(x) = \frac{1}{2}x^4 - \frac{2}{9}x^3 + \frac{5}{2}x^2 + C$

5. $F(x) = 4x^3 + 4x^2 + C$

7. $F(x) = 5x^{7/5} + 40x^{1/5} + C$

9. $F(x) = \sqrt{2}x + C$

11. $F(x) = 2x^{3/2} - \frac{3}{2}x^{4/3} + C$

13. $F(x) = \begin{cases} \frac{1}{5}x - 2\ln|x| + C_1 & \text{if } x < 0 \\ \frac{1}{5}x - 2\ln|x| + C_2 & \text{if } x > 0 \end{cases}$

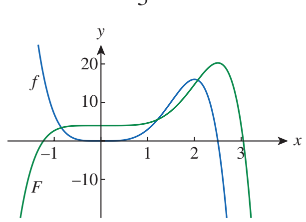
15. $G(t) = 2t^{1/2} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C$

17. $H(\theta) = -2\cos\theta - \tan\theta + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$

19. $F(x) = \frac{2^x}{\ln 2} + \frac{x^3}{2} + C$

21. $F(x) = x^2 + 4x + \frac{1}{x} + C, \quad x > 0$

23. $F(x) = x^5 - \frac{1}{3}x^6 + 4$



25. $f(x) = x^5 - x^4 + x^3 + Cx + D$

27. $f(x) = \frac{1}{3}x^3 + 3e^x + Cx + D$

29. $f(t) = 2t^3 + \cos t + Ct^2 + Dt + E$

31. $f(x) = x + 2x^{3/2} + 5$

33. $f(t) = 4\arctan t - \pi$

35. $f(x) = 3x^{5/3} - 75$

37. $f(x) = 4\sin^{-1}x + 1 - \frac{2\pi}{3}$

39. $f(t) = \begin{cases} \frac{3^t}{\ln 3} - 3\ln(-t) + 1 - \frac{1}{3\ln 3} & \text{if } t < 0 \\ \frac{3^t}{\ln 3} - 3\ln t + 2 - \frac{3}{\ln 3} & \text{if } t > 0 \end{cases}$

41. $f(x) = x^2 - \ln|x|$

43. $f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x - \frac{39}{10}$

45. $f(t) = \frac{1}{12}t^4 - \ln t + \frac{8}{3}t - \frac{11}{3}$

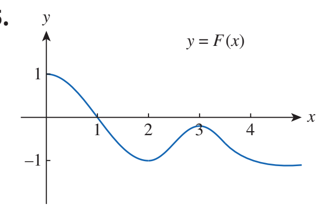
47. $f(x) = e^x + 2\sin x - \frac{2}{\pi}(e^{\pi/2} + 4)x + 2$

49. $f(x) = -\ln x + (\ln 2)x - \ln 2$

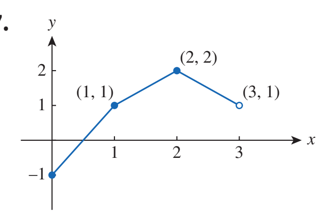
51. $f(1) = 8$

53. b

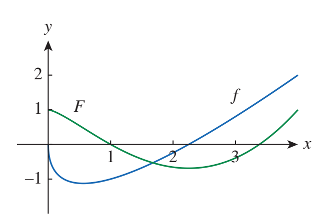
55.



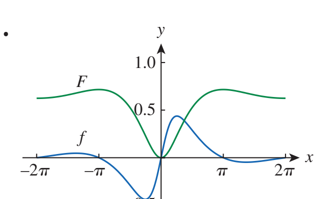
57.



59. $F(x) = x^2 - 2x^{3/2} + 1$



61.



63. $s(t) = -\cos t - \sin t + 1$

65. $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 3$

67. $s(t) = -10 \sin t - 3 \cos t + \frac{6}{\pi}t + 3$

69. (a) $s(t) = 450 - 4.9t^2$

(b) $t_1 = \sqrt{\frac{450}{4.9}} \approx 9.58$ s

(c) $v(t_1) = -9.8\sqrt{\frac{450}{4.9}} \approx -93.9$ m/s

(d) $\frac{5 - \sqrt{8845}}{-9.8} \approx 9.09$ s

73. Yes: $t = 5$ s

75. $2\sqrt{100} = 20$ g

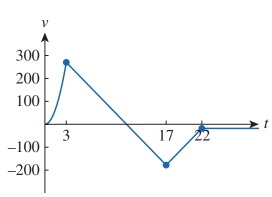
77. $\frac{130}{11} \approx 11.8$ s

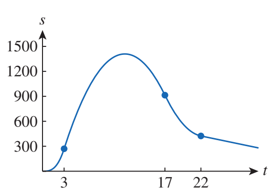
79. $\frac{88}{15} \approx 5.87$ ft/s^2

81. $-\frac{3125}{648} \approx -4.82$ m/s^2

83. (a) $v(t) = \begin{cases} 30t^2 & \text{if } 0 \le t \le 3 \\ -32(t-3) + 270 & \text{if } 3 < t \le 17 \\ 32(t-17) - 178 & \text{if } 17 < t \le 22 \\ -18 & \text{if } t > 22 \end{cases}$

$s(t) = \begin{cases} 10t^3 & \text{if } 0 \le t \le 3 \\ -16(t-3)^2 + 270(t-3) + 270 & \text{if } 3 < t \le 17 \\ 16(t-17)^2 - 178(t-17) + 914 & \text{if } 17 < t \le 22 \\ -18(t-22) + 424 & \text{if } t > 22 \end{cases}$





(b) 11.4375 s; 1409.0625 ft

(c) $\frac{410}{9} \approx 45.6$ s

Chapter 4 Review ■ Page 389

True-False Quiz

1. False **3.** False **5.** True **7.** False **9.** True
11. True **13.** False **15.** True **17.** True **19.** False
21. False

Exercises

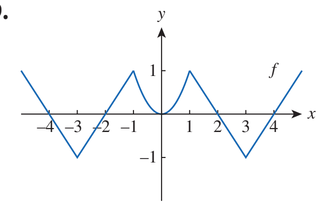
1. Local maximum value: $f(2) = 18$;
local minimum value: $f(4) = 14$;
absolute maximum value: $f(2) = f(5) = 18$;
absolute minimum value: $f(0) = -2$

3. Local minimum value: $f\left(-\frac{1}{3}\right) = -\frac{9}{2}$;
absolute maximum value: $f(2) = \frac{2}{5}$;
absolute minimum value: $f\left(-\frac{1}{3}\right) = -\frac{9}{2}$

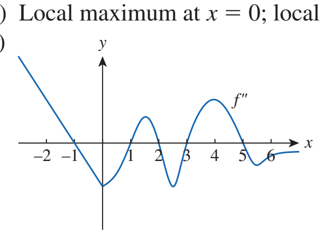
5. Local maximum value: $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3} \approx 2.256$;
local minimum value: $f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} - \sqrt{3} \approx 0.886$;
absolute maximum value: $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3}$;
absolute minimum value: $f(-\pi) = -\pi - 2$

7. Local maximum value: $f(\sqrt{e}) = \frac{1}{2e}$;
absolute maximum value: $f(\sqrt{e}) = \frac{1}{2e}$;
absolute minimum value: $f(1) = 0$

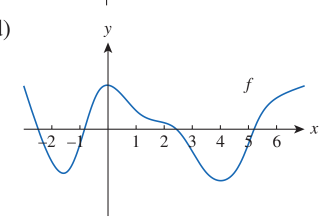
9.



11. (a) Increasing: $[-2, 0], [4, \infty)$;
decreasing: $(-\infty, -2], [0, 4]$
(b) Local maximum at $x = 0$; local minimum at $x = 4$
(c)

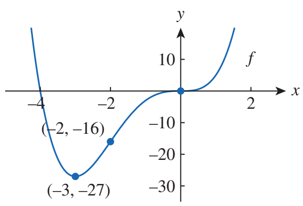


(d)

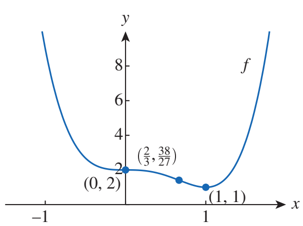


13. (a) None
(b) Increasing: $[-3, \infty)$; decreasing: $(-\infty, -3]$
(c) No local maximum; local minimum value: $f(-3) = -27$
(d) Concave up: $(-\infty, -2)$, $(0, \infty)$;
concave down: $(-2, 0)$; IP $(0, 0)$ $(-2, -16)$

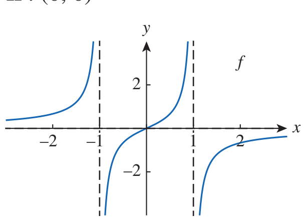
(e)



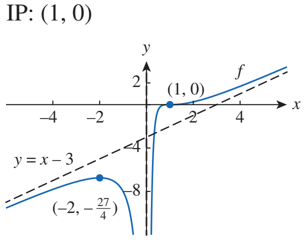
15. (a) None
(b) Increasing: $[1, \infty)$; decreasing: $(-\infty, 1]$
(c) No local maximum; local minimum value: $f(1) = 1$
(d) Concave up: $(-\infty, 0), \left(\frac{2}{3}, \infty\right)$;
concave down: $\left(0, \frac{2}{3}\right)$;
IP: $(0, 2), \left(\frac{2}{3}, \frac{38}{27}\right)$
(e)



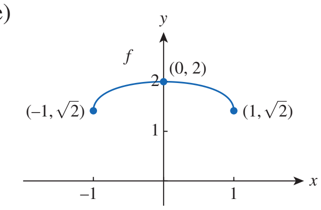
17. (a) VA: $x = 1, x = -1$; HA: $y = 0$
(b) Increasing: $(-\infty, -1), (-1, 1), (1, \infty)$
(c) No local extrema
(d) Concave up: $(-\infty, -1), (0, 1)$;
concave down: $(-1, 0), (1, \infty)$:
IP: $(0, 0)$
(e)



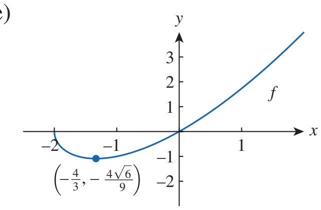
19. (a) VA: $x = 0$; no HA; SA: $y = x - 3$
(b) Increasing: $(-\infty, -2], (0, \infty)$; decreasing: $[-2, 0)$
(c) Local maximum value: $f(-2) = -\frac{27}{4}$
(d) Concave up: $(1, \infty)$;
concave down: $(-\infty, 0), (0, 1)$;
IP: $(1, 0)$
(e)



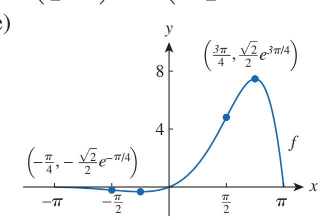
21. (a) VA: none; HA: none
(b) Increasing: $[-1, 0]$; decreasing: $[0, 1]$
(c) Local maximum value: $f(0) = 2$
(d) Concave down: $(-1, 1)$; IP: none
(e)



23. (a) VA: none; HA: none
(b) Increasing: $\left[-\frac{4}{3}, \infty\right)$; decreasing: $\left[-2, -\frac{4}{3}\right]$
(c) Local minimum value: $f\left(-\frac{4}{3}\right) = -\frac{4\sqrt{6}}{9}$
(d) Concave up: $(-2, \infty)$; IP: none
(e)



25. (a) VA: none; HA: none
(b) Increasing: $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$; decreasing: $\left[-\pi, -\frac{\pi}{4}\right]$, $\left[\frac{3\pi}{4}, \pi\right]$
(c) Local maximum value: $f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}e^{3\pi/4}$;
local minimum value: $f\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}e^{-\pi/4}$
(d) Concave up: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; concave down: $\left(-\pi, -\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi\right)$; IP: $\left(-\frac{\pi}{2}, -e^{-\pi/2}\right), \left(\frac{\pi}{2}, e^{\pi/2}\right)$
(e)

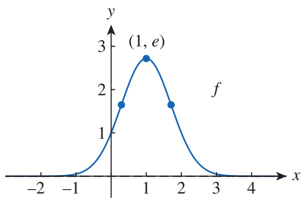


27. (a) VA: none; HA: $y = 0$
(b) Increasing: $(-\infty, 1]$; decreasing: $[1, \infty)$
(c) Local maximum value: $f(1) = e$

(d) Concave up: $\left(-\infty, 1-\dfrac{\sqrt{2}}{2}\right), \left(1+\dfrac{\sqrt{2}}{2}, \infty\right)$; concave down: $\left(1-\dfrac{\sqrt{2}}{2}, 1+\dfrac{\sqrt{2}}{2}\right)$;

IP: $\left(1-\dfrac{\sqrt{2}}{2}, \sqrt{e}\right), \left(1+\dfrac{\sqrt{2}}{2}, \sqrt{e}\right)$

(e)



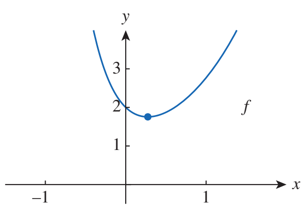
29. (a) VA: none; HA: none

(b) Increasing: $\left[\dfrac{1}{4}\ln 3, \infty\right)$; decreasing: $\left(-\infty, \dfrac{1}{4}\ln 3\right]$

(c) Local minimum value: $f\left(\dfrac{1}{4}\ln 3\right) = 3^{1/4} + 3^{-3/4} \approx 1.75$

(d) Concave up: $(-\infty, \infty)$; IP: none

(e)



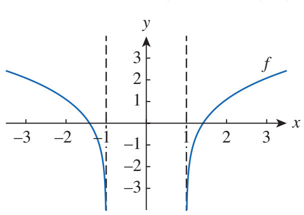
31. (a) VA: $x=-1$, $x=1$; HA: none

(b) Increasing: $(1, \infty)$; decreasing: $(-\infty, -1)$

(c) No local extrema

(d) Concave down: $(-\infty, -1)$, $(1, \infty)$; IP: none

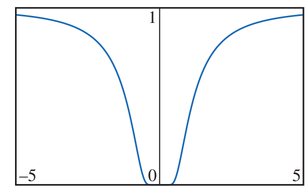
(e)



33. Increasing: $[-\sqrt{3}, 0), (0, \sqrt{3}]$;
decreasing: $(-\infty, -\sqrt{3}], [\sqrt{3}, \infty)$;
local maximum value: $f(\sqrt{3}) = \dfrac{2\sqrt{3}}{9}$;
local minimum value: $f(-\sqrt{3}) = -\dfrac{2\sqrt{3}}{9}$;
concave up: $(-\sqrt{6}, 0), (\sqrt{6}, \infty)$;
concave down: $(-\infty, -\sqrt{6}), (0, \sqrt{6})$;
IP: $\left(\sqrt{6}, \dfrac{5\sqrt{6}}{36}\right), \left(-\sqrt{6}, -\dfrac{5\sqrt{6}}{36}\right)$

35. Increasing: $[-0.23, 0], [1.62, \infty)$;
decreasing: $(-\infty, -0.23], [0, 1.62]$;
local maximum value: $f(0)=2$;
local minimum values: $f(-0.23)=1.96$, $f(1.62)=-19.2$;
concave up: $(-\infty, -0.12), (1.24, \infty)$;
concave down: $(-0.12, 1.24)$;
IP: $(-0.12, 1.98), (1.24, -12.1)$

37. IP: $\left(\pm\sqrt{\dfrac{2}{3}}, e^{-3/2}\right)$



39. Maximum points at $x=-2.96, -0.18, 3.01$;
minimum points at $x=-1.57, 1.57$;
IP: $x=-2.16, -0.75, 0.46, 2.21$

41. For $C > -1$, f is periodic with period 2π and has local maximum at $2n\pi + \dfrac{\pi}{2}$, n is an integer. For $C \le -1$, f has no graph. For $-1 < C \le 1$, f has vertical asymptotes. For $C > 1$, f is continuous on $\mathbb{R}$. As C increases, f moves upward, and its oscillations become less pronounced.

47. $a=-3$, $b=7$

49. 1

51. 4

53. 0

55. $\dfrac{1}{2}$

57. 500, 125

59. (4, 2)

61. $\dfrac{32\pi r^3}{81}$

63. $P=C$

65. $\dfrac{8}{9\pi}$ cm/s

67. $\dfrac{120}{\sqrt{241}} \approx 7.7$ ft/s

69. $L=C$

71. \$11.50

73. 0.268881, 2.770058

75. $f(0.33541803) \approx 1.16718557$

77. $F(x) = -\cos x - \tan x + C$

79. $F(x) = 2x^{1/2} + \dfrac{2}{3}x^{3/2} + C$

81. $f(x) = \dfrac{1}{2}x^2 + 2\sqrt{x} + \dfrac{1}{2}$

83. $f(x) = \dfrac{1}{10}x^5 + \dfrac{1}{4}x^4 - \dfrac{2}{3}x^3 + \dfrac{5}{2}x^2 - \dfrac{251}{60}x + 2$

85. $s(t) = -\sin t - 3\cos t + 3t + 3$

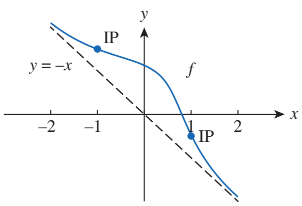
87. No. $v\left(\sqrt{\dfrac{500}{4.9}}\right) \approx -98.995$ m/s

91. (b) $\theta = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$

(c) $R(\theta) = \dfrac{2v^2 \cos\theta \sin(\theta + \alpha)}{g\cos^2\alpha}$

$\theta = \dfrac{\pi}{4} - \dfrac{\alpha}{2}$

93.



95. (a) $V'(t)$ is the rate of change of the volume of the water with respect to time. $H'(t)$ is the rate of change of the height of the water with respect to time. $V'(t)$ and $H'(t)$ are positive.

(b) $V'(t)$ is constant. $V''(t)$ is zero.

(c) $H''(t_1) < 0$: height increasing at a decreasing rate

$H''(t_2) = 0$, $H''(t_3) > 0$

Focus on Problem Solving

5. Absolute maximum value: $f(-5) = e^{45}$; no absolute minimum value

7. $(-2, 4)$, $(2, -4)$

9. 24

11. $-\dfrac{7}{2} < a < -\dfrac{5}{2}$

13. $c > 0$: one inflection point; $c < -e/6$: two inflection points

17. $P\left(\dfrac{1}{2}m, \dfrac{1}{4}m^2\right)$

21. $h = 4r$

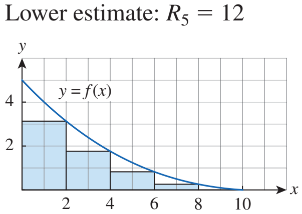
23. $\dfrac{r^2}{R^2 - r^2}$ cm/s

25. $a = \dfrac{1 + \sqrt{5}}{2}$

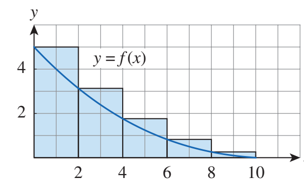
Chapter 5

Exercises 5.1 ■ Page 410

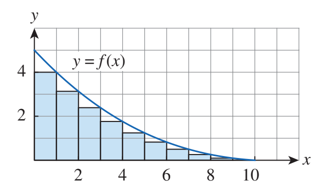
1. (a) Lower estimate: $R_5 = 12$



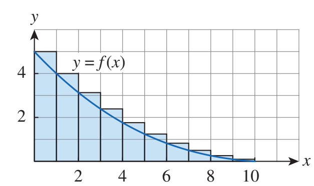
Upper estimate: $L_5 = 22$



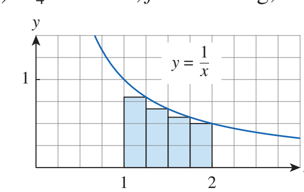
(b) Lower estimate: $R_{10} = 14.4$



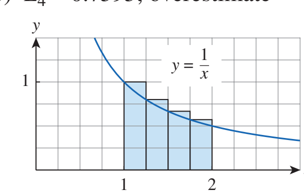
Upper estimate: $L_{10} = 19.4$



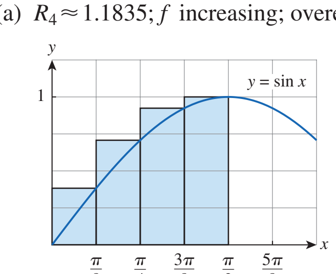
3. (a) $R_4 \approx 0.6345$; f decreasing; underestimate



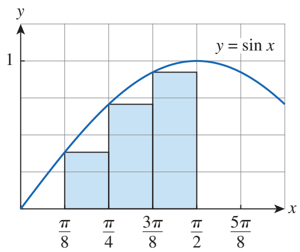
(b) $L_4 \approx 0.7595$; overestimate



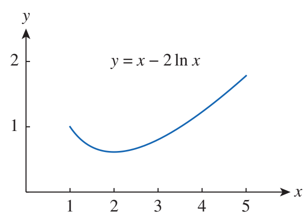
5. (a) $R_4 \approx 1.1835$; f increasing; overestimate



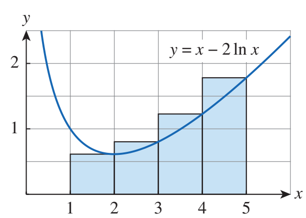
(b) $L_4 \approx 0.7908$; underestimate



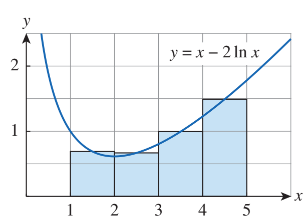
7. (a)



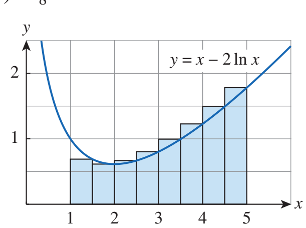
(b) $R_4 \approx 4.425$



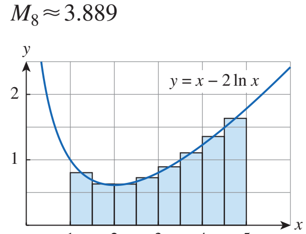
$M_4 \approx 3.843$



(c) $R_8 \approx 4.134$



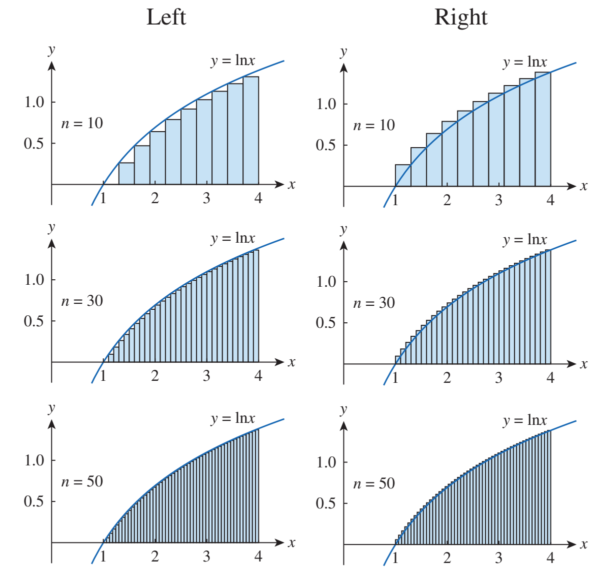
$M_8 \approx 3.889$



9. $R_{10} \approx 0.9194$; $R_{30} \approx 0.9736$; $R_{50} \approx 0.9842$; $R_{100} \approx 0.9921$; exact area $= 1$

11. (a) $L_{10} \approx 2.331629896$; $R_{10} \approx 2.747518204$
$L_{30} \approx 2.475238000$; $R_{30} \approx 2.613867436$
$L_{50} \approx 2.503363648$; $R_{50} \approx 2.586541310$

(b)



(c) f increasing on $[1, 4]$; using $n = 50$, $2.50 < L_{50} < A < R_{50} < 2.59$

13. $f(x_3^*) = 16.0625$

15. $L_6 = 34.7$ ft; $R_6 = 44.8$ ft

17. Lower estimate: $R_5 = 63.2$ L
upper estimate: $L_5 = 70$ L

19. $M_6 \approx 152$ ft

21. $M_6 = 7840$ (infected cells/mL) · days

23. $A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2(1 + 2i/n)}{(1 + 2i/n)^2 + 1} \cdot \frac{2}{n}$

25. $A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i\pi}{2n} \cos\left(\frac{i\pi}{2n}\right) \cdot \frac{\pi}{2n}$

27. Area of the region under the graph of $y = \tan x$ from $x = 0$ to $x = \frac{\pi}{4}$

29. (a) $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n}$

(b) $\frac{1}{4}$

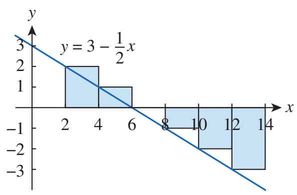
31. $A = e^{-2}(e^2 - 1) \approx 0.8647$

33. 0.4

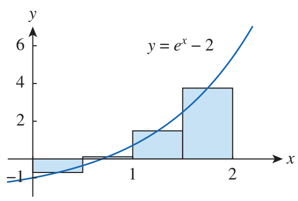
35. $n = 347{,}346$

Exercises 5.2 ■ Page 426

1. $L_6 = -6$; The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis, that is, the net area of the rectangles with respect to the x-axis.



3. $M_4 \approx 2.322986$; The Riemann sum represents the sum of the areas of the three rectangles above the x-axis minus the area of the rectangle below the x-axis, that is, the net area of the rectangles with respect to the x-axis.



5. $R_5 = 6$; $L_5 = 4$; $M_5 = 2$
7. Lower estimate: $L_5 = -64$; upper estimate: $R_5 = 16$
9. $M_4 \approx 124.1644$
11. $M_4 \approx 0.5890$
13. $M_4 \approx 1.6099$
15. $L_{100} \approx 0.89469$; $R_{100} \approx 0.90802$; f is increasing, so L_{100} is an underestimate, R_{100} is an overestimate; therefore $0.8946 < \int_0^2 \frac{x}{x+1}\,dx < 0.9081$

17.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

The values of R_n appear to be approaching 2.

19. $\int_0^1 \frac{e^x}{1+x}\,dx$

21. $\int_2^6 x\ln(1+x^2)\,dx$

23. $\int_2^7 (5x^3-4x)\,dx$

25. $\int_2^4 x^2 dx$

27. $\int_1^3 x\ln x\,dx$

29. -3

31. 0

33. 0

37. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\sqrt{4+\left(1+\frac{2i}{n}\right)^2}\cdot\frac{2}{n}$

39. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{\pi i}{n}\sin\left(\frac{\pi i}{n}\right)\cdot\frac{\pi}{n}$

41. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)^6\cdot\frac{8}{n}\approx 1{,}428{,}553.143$

43. (a) 4 (b) 10 (c) -3 (d) -5

45. $\frac{3}{2}$

47. 12

49. $3+\frac{9}{4}\pi$

51. $\frac{25}{4}$

53. $\frac{1}{2}$

55. 0

57. 3

59. e^5-e^3

61. $\int_{-1}^{5} f(x)\,dx$

63. 122

65. $\int_0^3 f(x)dx < f'(1) < \int_0^8 f(x)dx < \int_4^8 f(x)dx < \int_3^8 f(x)dx$

67. (a) 3 (b) -9 (c) 23 (d) 13 (e) 6 (f) 10

69. 15

75. $0 \le \int_0^1 x^3 dx \le 1$

77. $\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \frac{\pi\sqrt{3}}{12}$

79. $0 \le \int_0^2 xe^{-x}dx \le \frac{2}{e}$

81. $\int_0^1 x^4 dx$

83. $\frac{1}{2}$

Exercises 5.3 ■ Page 439

1. $-\frac{10}{3}$

3. $\frac{21}{5}$

5. -2

7. $5e^{\pi}+1$

9. 36

11. $\frac{55}{63}$

13. $\frac{3}{4}-2\ln 2$

15. $\frac{1}{11}+\frac{9}{\ln 10}$

17. $1+\frac{\pi}{4}$

19. $\dfrac{69}{4}$

21. $\dfrac{\pi}{3}$

23. $\dfrac{\pi}{6}$

25. -3.5

27. $\dfrac{44}{3}$

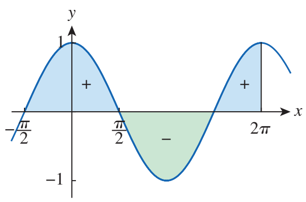
29. $f(x) = \dfrac{1}{x^2}$ is not continuous on $[-1, 3]$; the Evaluation Theorem does not apply.

31. 2

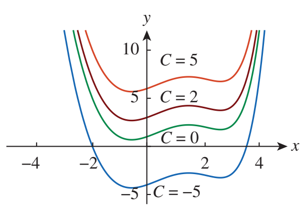
33. ln 4

35. $a \approx -0.87$, $b \approx 0.87$, 1.23

37. $\displaystyle\int_{-\pi/2}^{2\pi} \cos x \, dx = 1$



43. $e^x - \dfrac{2}{3}x^3 + C$



45. $\dfrac{4}{9}x^{9/4} + C$

47. $\dfrac{1}{7}u^7 - \dfrac{1}{3}u^6 - \dfrac{1}{4}u^4 + \dfrac{2}{7}u + C$

49. $\dfrac{2}{7}t^{7/2} + \dfrac{6}{5}t^{5/2} + \dfrac{4}{3}t^{3/2} + C$

51. $\dfrac{x^3}{3} + x + \tan^{-1}x + C$

53. $-\dfrac{1}{r} + 2\ln|r| + r + C$

55. $\tan t + \sec t + C$

57. $2\sin x + C$

59. $\dfrac{x^4}{4} + x^3 + \dfrac{3}{2}x^2 + x + C$

61. $\dfrac{1}{5}$

63. The change in the charge Q from time $t = a$ to $t = b$

65. The net change in the amount of water (in cubic feet) that is in the tank from the third hour to the sixth hour

67. The total bee population after 15 weeks

69. The amount of natural gas produced, in cubic feet, from time $t = 0$ to time $t = 24$ hours

71. Newton-meters, or joules

73. (a) $-\dfrac{3}{2}$ m

(b) $\dfrac{41}{6}$ m

75. (a) $v(t) = \dfrac{1}{2}t^2 + 4t + 5$ m/s

(b) $416\dfrac{2}{3}$ m

77. $46\dfrac{2}{3}$ kg

79. $\dfrac{247}{180}$ miles

81. \$58,000

83. 39.8 ft/s

85. 5443

87. 60,000

91. 3

93. $\dfrac{1}{4}$

Exercises 5.4 ■ Page 451

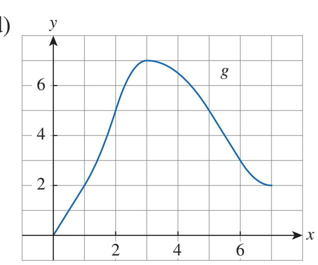
1. One process undoes what the other one does.

3. (a) 0, 2, 5, 7, 3

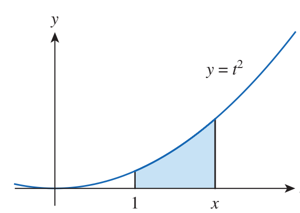
(b) [0, 3]

(c) $x = 3$

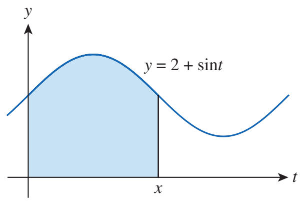
(d)



5. $g'(x) = x^2$



7. $g'(x) = 2 + \sin x$



9. $g'(x) = \ln(1 + x^2)$

11. $h'(u) = \dfrac{\sqrt{u}}{u + 1}$

13. $R'(y) = -y^3 \sin y$

15. $h'(x) = -\dfrac{\arctan(1/x)}{x^2}$

17. $h'(x) = \dfrac{\sqrt{x}}{2(x^2 + 1)}$

19. $g'(x) = \cos^2(x^4) \cdot 4x^3$

21. $g'(x) = -\dfrac{1}{2}\tan\sqrt{x}$

23. $g'(x) = \dfrac{4x}{1 + x^4}$

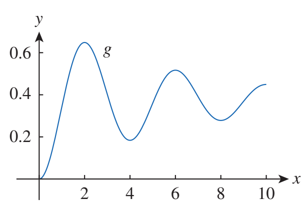
25. $g'(x) = -\sin x(1 + \cos^2 x)^{10} - \cos x(1 + \sin^2 x)^{10}$

27. (a) Local maximum: $x = 2$; 6; local minimum: $x = 4$; 8

(b) Absolute maximum value at $x = 2$

(c) (1, 3), (5, 7), (9, 10)

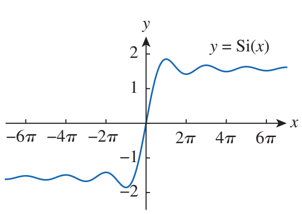
(d)



29. $g''\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{15}}{4}$

31. 2

35. (a)



(b) $x = \pi, -2\pi, 3\pi, -4\pi, 5\pi, -6\pi, \ldots$

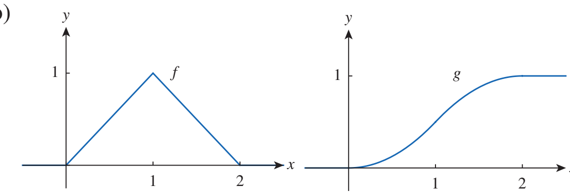
(c) (4.4934, 1.6556)

(d) $y = \pm\dfrac{\pi}{2}$

(e) $x \approx 1.1$

37. (a) $g(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } 0 \le x \le 1 \\ 2x - \frac{1}{2}x^2 - 1 & \text{if } 1 < x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$

(b)

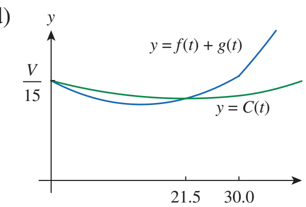


(c) f is not differentiable at $x = 0, 1, 2$. f is differentiable on $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, $(2, \infty)$. g is differentiable on $(-\infty, \infty)$.

39. (b) $T = 30$ months

(c) $C(21.5) \approx 0.05472V$

(d)



Exercises 5.5 ■ Page 463

1. $\dfrac{1}{2}\sin 2x + C$

3. $-\dfrac{1}{2}e^{-x^2} + C$

5. $\dfrac{1}{18(1 - 6t)^3} + C$

7. $-\tan\left(\dfrac{1}{x}\right) + C$

9. $\dfrac{2}{3}(z - 1)^{3/2} + C$

11. $\dfrac{1}{3}e^{x^3} + C$

13. $-\dfrac{2}{3}(1 + \cos t)^{3/2} + C$

15. $\dfrac{1}{2}\tan 2\theta + C$

17. $-\dfrac{1}{5}(4 - y^3)^{5/3} + C$

19. $-\dfrac{1}{5}e^{-5r} + C$

21. $-2\cos\sqrt{x} + C$

23. $\dfrac{1}{3}\ln|z^3 + 1| + C$

25. $\cos(\cos x) + C$

27. $\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C$

29. $\frac{1}{a}\ln|ax+b| + C$

31. $\frac{(\tan^{-1} x)^2}{2} + C$

33. $-e^{\cos t} + C$

35. $\ln|\tan x| + C$

37. $\frac{1}{2}\ln(x^2+4) + C$

39. $-\frac{1}{\pi}\sin\frac{\pi}{x} + C$

41. $\frac{1}{\ln 2}\ln(2^t+3) + C$

43. $2\sqrt{1+\tan t} + C$

45. $-\tan^{-1}(\cos x) + C$

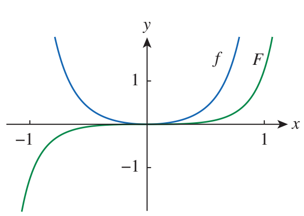
47. $\sin(\ln t) + C$

49. $\frac{1}{2}\tan^{-1}(x^2) + C$

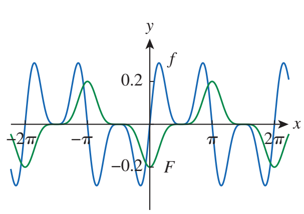
51. $\frac{2}{7}(2+x)^{7/2} - \frac{8}{5}(2+x)^{5/2} + \frac{8}{3}(2+x)^{3/2} + C$

53. $\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C$

55. $\frac{1}{3}\tan^3\theta + C$



57. $-\frac{1}{5}\cos^5 x + C$



59. $\frac{1}{153}(2^{51}+1)$

61. $\frac{1}{5}\ln 16$

63. 0

65. $\frac{4}{3}\sqrt{3}$

67. $2(e^2 - e)$

69. $1 - \cos 1$

71. $\frac{1}{3}a^3$

73. 0

75. $\frac{10}{3}$

77. 0

79. $\frac{T}{\pi}\cos\alpha$

83. 4

85. $\frac{\pi}{8}$

87. 2040 kcal

89. Approximately 11,713 bacteria

91. Approximately 41,667 kg

93. Approximately 4048

95. 5

101. $\frac{\pi^2}{4}$

Exercises 5.6 ■ Page 471

1. $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

3. $\frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C$

5. $-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} + C$

7. $(x^2+2x)\sin x + (2x+2)\cos x - 2\sin x + C$

9. $x\cos^{-1} x - \sqrt{1-x^2} + C$

11. $\frac{1}{5}t^5\ln t - \frac{1}{25}t^5 + C$

13. $-t\cot t + \ln|\sin t| + C$

15. $-\frac{z}{10^z\ln 10} - \frac{1}{10^z(\ln 10)^2} + C$

17. $\frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$

19. $x\tan x - \ln|\sec x| - \frac{1}{2}x^2 + C$

21. $x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$

23. $-6e^{-1} + 3$

25. $\frac{4}{5} - \frac{1}{5}\ln 5$

27. $-\frac{\pi}{4}$

29. $2e^{-1} - 6e^{-5}$

31. $\frac{1}{2}\ln 2 - \frac{1}{2}$

33. $\frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}$

35. $e - 2$

37. $\dfrac{e^t - t - 1}{e^t}$

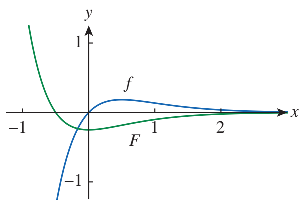
39. 21

41. $2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

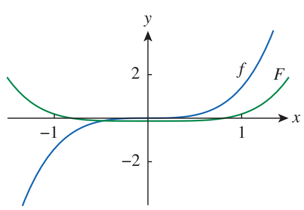
43. $-\dfrac{1}{2} - \dfrac{\pi}{4}$

45. $\dfrac{1}{2}(x^2 - 1)\ln(1 + x) - \dfrac{1}{4}x^2 + \dfrac{x}{2} + C$

47. $-\dfrac{1}{2}xe^{-2x} - \dfrac{1}{4}e^{-2x} + C$



49. $\dfrac{1}{3}x^2(1 + x^2)^{3/2} - \dfrac{2}{15}(1 + x^2)^{5/2} + C$



51. (b) $-\dfrac{1}{4}\cos x \sin^3 x + \dfrac{3}{8}x - \dfrac{3}{16}\sin 2x + C$

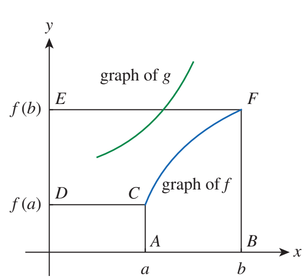
53. (b) $\dfrac{8}{15}$

59. $x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$

61. $2 - e^{-t}(t^2 + 2t + 2)$ meters

63. 2

65. (c) area of $ABFC = bf(b) - af(a) - \displaystyle\int_{f(a)}^{f(b)} g(y)\,dy$



(d) 1

Exercises 5.7 ■ Page 479

1. $\dfrac{1}{5}\cos^5 x - \dfrac{1}{3}\cos^3 x + C$

3. $-\dfrac{11}{384}$

5. π

7. $-\dfrac{2}{3}(\cos\theta)^{3/2} + \dfrac{2}{7}(\cos\theta)^{7/2} + C$

9. $\dfrac{1}{3}\sec^3 x - \sec x + C$

11. $\dfrac{8}{15}$

13. $-\dfrac{\sqrt{9 - x^2}}{x^2} - \sin^{-1}\left(\dfrac{x}{3}\right) + C$

15. $-\dfrac{\sqrt{x^2 + 4}}{4x} + C$

17. $\dfrac{\pi}{24} + \dfrac{\sqrt{3}}{8} - \dfrac{1}{4}$

19. $-\dfrac{\sqrt{4 - x^2}}{4x} + C$

21. $\ln\left|\dfrac{\sqrt{1 + x^2} - 1}{x}\right| + \sqrt{1 + x^2} + C$

23. (a) $\dfrac{2x}{(x + 3)(3x + 1)} = \dfrac{A}{x + 3} + \dfrac{B}{3x + 1}$

(b) $\dfrac{1}{x^3 + 2x^2 + x} = \dfrac{A}{x} + \dfrac{B}{x + 1} + \dfrac{C}{(x + 1)^2}$

25. (a) $\dfrac{x^6}{x^2 - 4} = x^4 + 4x^2 + 16 + \dfrac{A}{x + 2} + \dfrac{B}{x - 2}$

(b) $\dfrac{1}{x^2 + x^4} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx + D}{1 + x^2}$

27. $-3\ln 2 + \ln 3$

29. $\ln|x| - \ln|x + 1| + \ln|x - 1| + C$

31. $\tan^{-1} x + \dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right) + C$

33. $2\ln|x| - \dfrac{1}{2}\ln(x^2 + 3) - \dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{x}{\sqrt{3}} + C$

35. $\tan^{-1} x - \dfrac{1}{2(x^2 + 1)} + C$

37. $\dfrac{1}{2}x^2 - 4x + 16\ln|x + 4| + C$

39. $\dfrac{3}{2} + \ln\dfrac{3}{2}$

41. $\dfrac{3}{2}\ln(\sqrt{x + 3} + 3) + \dfrac{1}{2}\ln|\sqrt{x + 3} - 1| + C$

43. $\dfrac{1}{6}(\sqrt{48} - \sec^{-1} 7)$

45. $\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2x + 1}{\sqrt{3}} + C$

Exercises 5.8 ■ Page 486

1. $\dfrac{1}{2\pi}\tan^2(\pi x) + \dfrac{1}{\pi}\ln|\cos(\pi x)| + C$

3. $-\dfrac{\sqrt{4x^2 + 9}}{9x} + C$

5. $\frac{1}{2}(e^{2x}+1)\arctan(e^x) - \frac{1}{2}e^x + C$

7. $\pi^3 - 6\pi$

9. $-\frac{1}{2}\tan^2\left(\frac{1}{z}\right) - \ln\left|\cos\left(\frac{1}{z}\right)\right| + C$

11. $\frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}}$
$-\frac{1}{12}(6+4y-4y^2)^{3/2} + C$

13. $\frac{1}{9}\sin^3 x[3\ln(\sin x) - 1] + C$

15. $\frac{1}{2\sqrt{3}}\ln\left|\frac{e^x+\sqrt{3}}{e^x-\sqrt{3}}\right| + C$

17. $\frac{1}{5}\ln\left|x^5 + \sqrt{x^{10}-2}\right| + C$

19. $\frac{1}{2}(\ln x)\sqrt{4+(\ln x)^2} + 2\ln\left[\ln x + \sqrt{4+(\ln x)^2}\right] + C$

21. $\sqrt{e^{2x}-1} - \cos^{-1}(e^{-x}) + C$

25. $\frac{2\tan x}{3} + \frac{1}{3}\tan x\sec^2 x + C$

27. $\frac{1}{4}x\sqrt{x^2+4}(x^2+2) - 2\ln(\sqrt{x^2+4}+x) + C$

29. $\frac{1}{15}(2x+1)^{3/2}(3x-1) + C$

31. $-\ln(\cos\theta) - \sec^2\theta + \frac{1}{4}\sec^4\theta + C$

33. (a) $F(x) = -\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C$

Domain of f and F: $(-1, 0) \cup (0, 1)$

(b) Mathematica:

$$F(x) = -\frac{1}{2}\left[\ln(1+\sqrt{1-x^2}) - \ln(1-\sqrt{1-x^2})\right] + C$$

Domain of f and F are the same.

Exercises 5.9 ■ Page 499

1. (a) $L_2 = 6$; $R_2 = 12$; $M_2 = 9.6$
(b) L_2: underestimate; R_2: overestimate; M_2: overestimate
(c) $T_2 = 9$; underestimate
(d) $L_n < T_n < I < M_n < R_n$

3. (a) $T_4 \approx 0.89579$; underestimate
(b) $M_4 \approx 0.908907$; overestimate

$$0.895759 < \int_0^1 \cos(x^2)\,dx < 0.908907$$

5. (a) $M_{10} \approx 0.806598$
(b) $S_{10} \approx 0.804779$

$$\int_0^2 \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln 5 \approx 0.804719$$

$E_M \approx -0.001879$; $E_S \approx -0.000060$

7. (a) $T_8 \approx 2.413790$ (b) $M_8 \approx 2.411543$ (c) $S_8 \approx 2.412232$

9. (a) $T_{10} \approx 0.146879$ (b) $M_{10} \approx 0.147391$
(c) $S_{10} \approx 0.147219$

11. (a) $T_8 \approx 0.451948$ (b) $M_8 \approx 0.451991$ (c) $S_8 \approx 0.451976$

13. (a) $T_8 \approx 4.513618$ (b) $M_8 \approx 4.748256$ (c) $S_8 \approx 4.675111$

15. (a) $T_8 \approx -0.495333$ (b) $M_8 \approx -0.543321$
(c) $S_8 \approx -0.526123$

17. (a) $T_8 \approx 0.902333$; $M_8 \approx 0.905620$
(b) $|E_T| \le \frac{1}{128} = 0.0078125$; $|E_M| \le \frac{1}{256} = 0.00390625$
(c) T_n: $n = 71$; M_n: $n = 50$

19. (a) $T_{10} \approx 1.983524$; $M_{10} \approx 2.008248$; $S_{10} \approx 2.000110$

$$\int_0^\pi \sin x\,dx = 2$$

$E_T \approx 0016476$; $E_M \approx -0.008248$; $E_S \approx -0.000110$

(b) $K = 1$: $|E_T| \le \frac{\pi^3}{1200} \approx 0.025839$;

$|E_M| \le \frac{\pi^3}{2400} \approx 0.012919$; $|E_S| \le \frac{\pi^5}{1{,}800000} \approx 0.000170$

(c) T_n: $n = 509$; M_n: $n = 360$; S_n: $n = 22$

21. (a) 2.8
(b) $M_{10} \approx 7.954926518$
(c) $|E_M| \le \frac{2.8(2\pi - 0)^3}{24 \cdot 10^2} \approx 0.2893916$
(d) $I \approx 7.954926521$
(e) The actual error is much smaller.
(f) $K = 4E \approx 10.9$
(g) $S_{10} \approx 7.953789422$
(h) $|E_S| \le \frac{10.9(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059299814$
(i) The actual error is much smaller.
(j) $n \ge 50$

23.

n	L_n	R_n	T_n	M_n
5	0.742943	1.286599	1.014771	0.992621
10	0.867782	1.139610	1.003696	0.998152
20	0.932967	1.068881	1.000924	0.999538

n	E_L	E_R	E_T	E_M
5	0.257057	−0.286599	−0.014771	0.007379
10	0.132218	−0.139610	−0.003696	0.001848
20	0.067033	−0.068881	−0.000924	0.000462

Observations are the same as after Example 1.

25.

n	T_n	M_n	S_n
6	6.695473	6.252572	6.403292
12	6.474023	6.363008	6.400206

n	E_T	E_M	E_S
6	-0.295473	0.147428	-0.003292
12	-0.074023	0.036992	-0.000206

Observations are the same as after Example 1.

27. (a) 19.8 (b) 20.6 (c) $20.5\overline{3}$

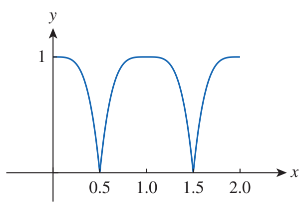
29. $37.7\overline{3}$ ft/s

31. $10{,}177.\overline{3}$ megawatt-hours

33. (a) 23.44 (b) $0.341\overline{3}$

35. 59.4

37.



Exercises 5.10 ■ Page 512

1. (a) Infinite discontinuity: Type 2
(b) Infinite interval: Type 1
(c) Infinite interval: Type 1
(d) Infinite discontinuity: Type 2

3. $A(t) = \frac{1}{2} - \frac{1}{2t^2}$; 0.495; 0.49995; 0.4999995; $\frac{1}{2}$

5. Convergent, 2

7. Divergent

9. Divergent

11. Convergent, $\frac{1}{5}e^{-10}$

13. Divergent

15. Convergent, 1

17. Convergent, $1 - e^{-1}$

19. Divergent

21. Convergent, $\frac{2\pi}{3\sqrt{3}}$

23. Convergent, $-\frac{1}{4}$

25. Divergent

27. Convergent, $-\frac{\pi}{8}$

29. Convergent, 2

31. Divergent

33. Convergent, $\frac{32}{3}$

35. Divergent

37. Convergent, $\frac{9}{2}$

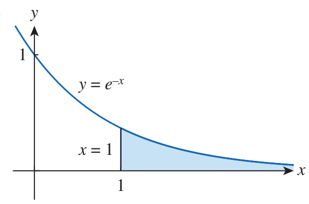
39. Divergent

41. Convergent, $-\frac{1}{4}$

43. Convergent, $-\frac{2}{e}$

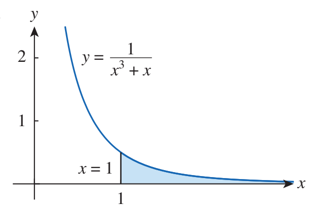
45. Convergent, $-\frac{1}{16}$

47.



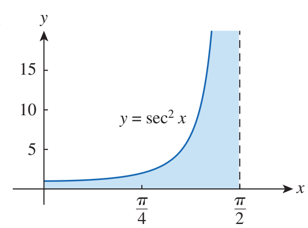
Area $= \frac{1}{e}$

49.



Area $= \frac{1}{2} \ln 2$

51.



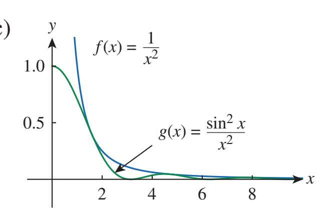
Infinite Area

53. (a)

t	$\int_1^t g(x)\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10,000	0.673407

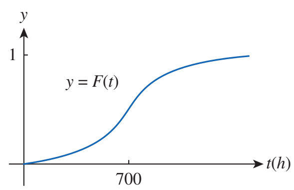
It appears that the integral is convergent.

(c)



Since $\int_1^\infty f(x)\,dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1, t]$, $\int_1^\infty g(x)\,dx$ must be finite, that is, convergent.

55. Convergent
57. Divergent
59. Divergent
61. π
63. $p < 1$, $\dfrac{1}{1-p}$
65. $p > -1$, $-\dfrac{1}{(p+1)^2}$
69. (a)



(b) Rate at which the fraction $F(t)$ of burnt out bulbs increases as t increases

(c) 1

71. $-\dfrac{1}{k} \approx 8264.5$ years
73. $\dfrac{cN}{\lambda(k+\lambda)}$
75. $a > \tan\left(\dfrac{\pi}{2} - 0.001\right) \approx 1000$
79. $C = 1$, $\ln 2$
81. No

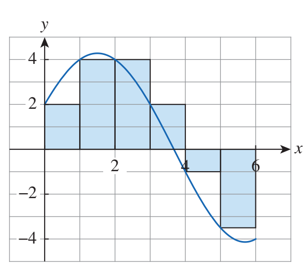
Chapter 5 Review ■ Page 516

True-False Quiz

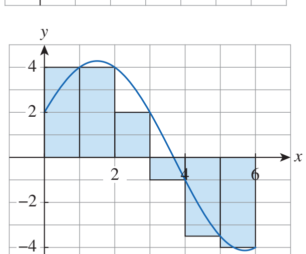
1. True **3.** False **5.** False **7.** True **9.** True
11. True **13.** False **15.** False **17.** False **19.** True
21. False **23.** False **25.** False **27.** False

Exercises

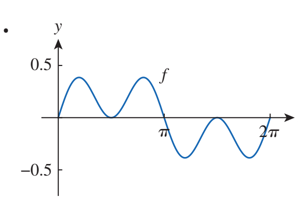
1. (a) $L_6 = 7.5$;



(b) $R_6 = 1.5$;



3. $\dfrac{1}{2} + \dfrac{\pi}{4}$
5. 3
7. c: $f(x)$; b: $f'(x)$; a: $\int_0^x f(t)\,dt$
9. 3, 0, -2
11. $\dfrac{1}{5}T^5 - 4T^2 + 7T$
13. $\dfrac{1}{10}$
15. $\dfrac{49}{15}$
17. $\dfrac{52}{9}$
19. $\dfrac{2}{3\pi}$
21. $\ln 16 - \dfrac{15}{16}$
23. $\dfrac{25}{9} - \dfrac{100}{9}e^{-3}$
25. 0
27. $\arctan e - \dfrac{\pi}{4}$
29. Divergent
31. $-\ln|1 + \cot x| + C|$
33. $-\sin(\cos x) + C$
35. $-\cos(\ln x) + C$
37. $\dfrac{1}{2}\sin^{-1}(x^2) + C$
39. $-\dfrac{1}{2}x^{-2} + \dfrac{1}{2}x^2 + C$
41. $\dfrac{15}{4}$
43. 2
45. $3e^{\sqrt[3]{x}}(x^{2/3} - 2x^{1/3} + 2) + C$
47. $\dfrac{24}{5}$
49. $\dfrac{1}{3}\sqrt{x^2+1}(x^2 - 2) + C$
51.



$\int_0^{2\pi} f(x)\,dx = 0$

53. $F'(x) = -\sqrt{x + \sin x}$
55. $g'(x) = \dfrac{\cos^3 x}{1 + \sin^4 x}$
57. $y' = 3\sin[(3x+1)^4] - 2\sin[(2x)^4]$

59. $-\frac{1}{4}\cot t\csc^3 t - \frac{3}{8}\csc t\cot t + \frac{3}{8}\ln|\csc t - \cot t| + C$

61. $\ln\left|\frac{\sqrt{1+2\sin x}-1}{\sqrt{1+2\sin x}+1}\right| + C$

63. (a) $T_{10} \approx 1.185197$; underestimate
(b) $M_{10} \approx 1.201932$; overestimate
(c) $S_{10} \approx 1.193089$; cannot determine

65. $S_6 \approx 17.739438$

67. (a) $|f^{(4)}(x)| < 3.8$ for $x \in [0, \pi]$
(b) $S_{10} \approx 1.786721$; $|E_S| \le \frac{3.8(\pi-0)^5}{180(10)^4} \approx 0.000646$
(c) $n \ge 30$

69. $4 \le \int_1^3 \sqrt{x^2+3}\,dx \le 4\sqrt{3}$

75. $\frac{1}{36}$

77. Divergent

79. 2

81. Convergent

83. (a) $29.1\overline{6}$ meters (b) 29.5 meters

85. (a) $s(t) = 2t - \frac{4}{3}\cos(3t) + \frac{4}{3}$
(b) $\frac{7}{18}\pi, \frac{11}{18}\pi$

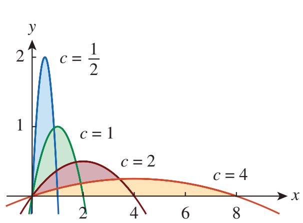
87. The number of barrels of oil consumed from January 1, 2000, through January 1, 2008

89. $\frac{Ce^{-x^2/(4kt)}}{\sqrt{4\pi kt}}$

91. $f(x) = \frac{e^{2x}(1+2x)}{1-e^{-x}}$

Focus on Problem Solving

1. (a)



(b) It appears that the areas are equal.
(c) The vertices of the family of parabolas seem to determine a branch of a hyperbola.
(d) $y = \frac{1}{x}$

3. $f'(x) = x^2\sin(x^2) + 2x\int_0^t \sin(t^2)\,dt$

5. e^{-2}

7. $2k$

11. $[-1, 2]$

13. $\sqrt{1+\sin^4 x}\cos x$

15. $\frac{\pi}{8} - \frac{1}{12}$

17. 0

21. (b) $y = -\sqrt{L^2-x^2} - L\ln\left(\frac{L-\sqrt{l^2-x^2}}{x}\right)$

Chapter 6

Exercises 6.1 ■ Page 533

1. $\frac{45}{4} - \ln 8$

3. $e - \frac{1}{e} + \frac{10}{3}$

5. $e - \frac{1}{e} + \frac{4}{3}$

7. $\frac{9}{2}$

9. $\ln 2 - \frac{1}{2}$

11. $\frac{8}{3}$

13. 72

15. $6\sqrt{3}$

17. $\frac{32}{3}$

19. $\frac{2}{\pi} + \frac{2}{3}$

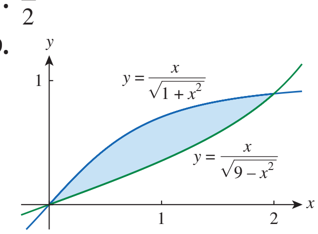
21. $2 - 2\ln 2$

23. $\frac{47}{3} - \frac{9}{2}\sqrt[3]{12}$

25. $\frac{13}{5}$

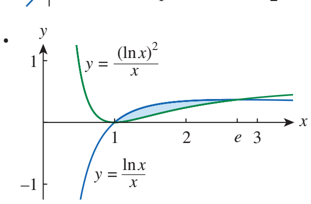
27. $\frac{3}{2}$

29.

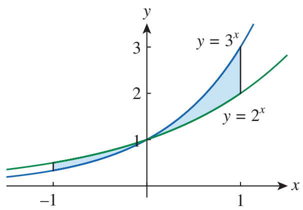


$2\sqrt{5} - 4$

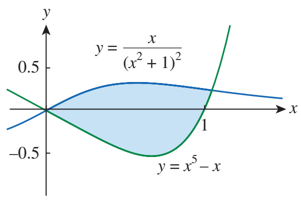
31.



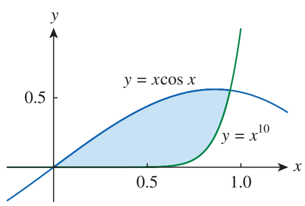
$\frac{1}{6}$

33. 

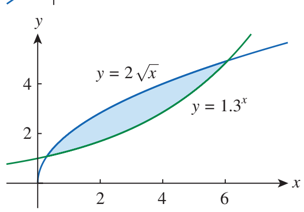
$\dfrac{4}{3 \ln 3} - \dfrac{1}{2 \ln 2}$

35. 

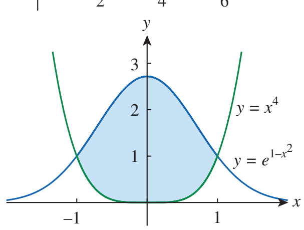
0.5901

37. 

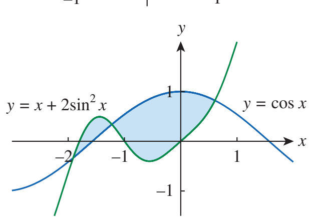
0.3029

39. 

5.1062

41. 

3.6602

43. 

1.7041

45. (a) $y = \dfrac{1}{4}x + 1$ (b) $\dfrac{2}{3}$

47. $117\dfrac{1}{3}$ ft

49. (a) A (b) The distance car A is ahead of car B after 1 minute
(c) A (d) 2.2 mins

51. 4232 cm^2

53. $0.8\overline{6}$; approximately 0.8667 more inches of rain fell at the second location than at the first during the first two hours of the storm.

55. $r\sqrt{R^2 - r^2} + \dfrac{\pi}{2}r^2 - R^2 \arcsin\left(\dfrac{r}{R}\right)$

57. πab

59. $3 - e$

61. $\dfrac{24}{5}\sqrt{3}$

63. $c = \pm 6$

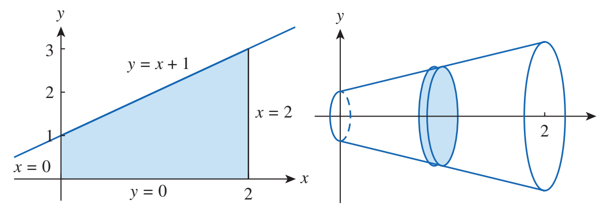
65. $b = 4^{2/3}$

67. $f(t) = 3t^2$

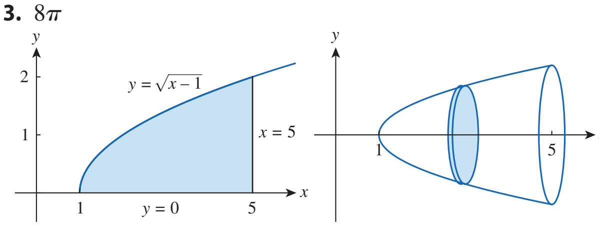
69. $0 < m < 1$; $m - \ln m - 1$

Exercises 6.2 ■ Page 547

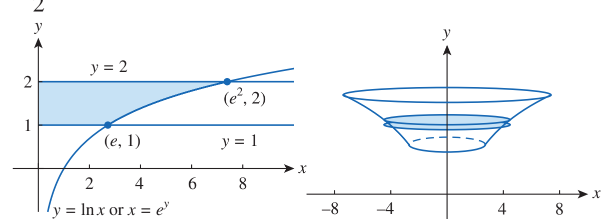
1. $\dfrac{26}{3}\pi$



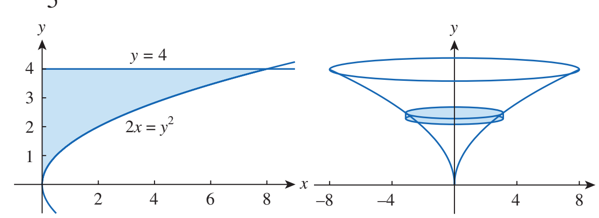
3. 8π



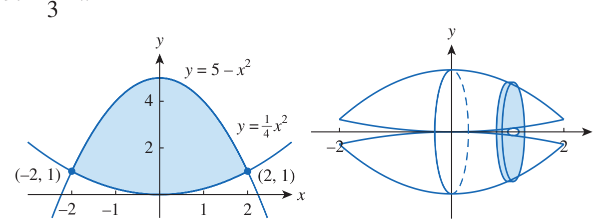
5. $\dfrac{\pi}{2}(e^4 - e^2)$



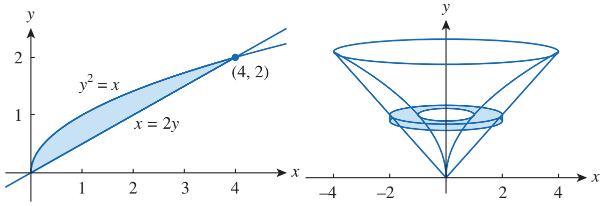
7. $\dfrac{256}{5}\pi$



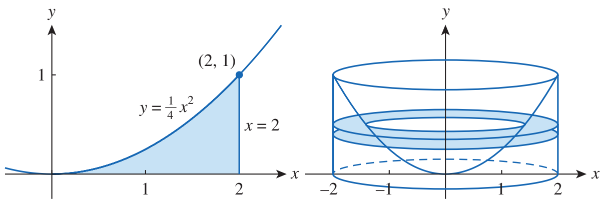
9. $\dfrac{176}{3}\pi$



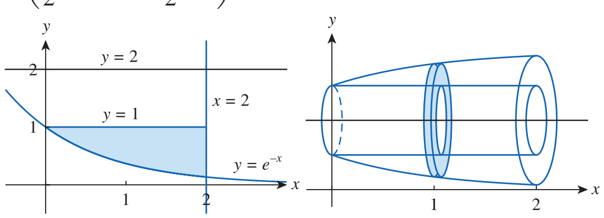
11. $\frac{64}{15}\pi$



13. 2π



15. $\left(\frac{5}{2} + 4e^{-2} - \frac{1}{2}e^{-4}\right)\pi$



17. $\frac{\pi}{2}$

19. $\frac{108}{5}\pi$

21. $\frac{13}{30}\pi$

23. $\left(2\sqrt{2} - \frac{3}{2}\right)\pi$

25. $\frac{2}{3}\pi$

27. $\frac{2}{3}\pi$

29. $\frac{\pi}{9}$

31. $\frac{\pi}{15}$

33. $\frac{2}{9}\pi$

35. $\frac{4}{15}\pi$

37. $V = \pi \int_0^{2\pi} [(4 - \cos x)^2 - (2 + \cos x)^2]\, dx$

39. (a) 3.70110 (b) 6.16850

41. 23.7802

43. 89.023

45. $\frac{11\pi^2}{8}$

47. Volume of the solid obtained by rotating the region $R = \{(x, y) \mid 0 \le x \le \pi,\ 0 \le y \le \sqrt{\sin x}\}$ about the x-axis

49. Volume of the solid obtained by rotating the region $R = \{(x, y) \mid -1 \le y \le 1,\ 0 \le x \le 1 - y^2\}$ about the y-axis

51. Volume of the solid obtained by rotating the region $R = \{(x, y) \mid 1 \le x \le 4,\ 3 - \sqrt{x} \le y \le 3\}$ about the x-axis

53. 1110 cm^3

55. (a) $9\pi^2$

(b) $\frac{9\sqrt{3}}{4}\pi$

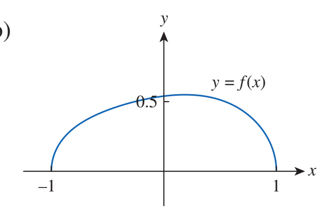
(c) $36\pi + \frac{27}{2}\pi^2$

57. (a) 188.6 units^3

(b) 828.8 units^3

59. (a) $V = \dfrac{4\pi(5a^2 + 18ac + 3[3b^2 + 14bd + 7(c^2 + 5d^2)])}{315}$

(b) $V = \frac{3769\pi}{9375} \approx 1.263$



61. $\frac{1}{3}\pi h(R^2 + Rr + r^2)$

63. $\frac{1}{3}(a^2 + ab + b^2)h$; $a = b$: b^2h; $a = 0$: $\frac{1}{3}b^2h$

65. $\frac{\sqrt{3}}{12}a^2h$

67. $\frac{16}{3}r^3$

69. (a) $\frac{\sqrt{3}}{12}$ (b) $\frac{1}{3}$

71. 2π

73. (a) $V = 2h\int_0^r \sqrt{r^2 - x^2}\, dx$

(b) One quarter of the area of a circle of radius r:

$V = \frac{1}{2}\pi h r^2$

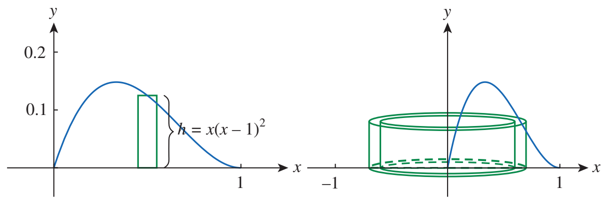
75. $\frac{128}{3\sqrt{3}}$

77. $\frac{16}{3}r^3$

79. $V = 8\int_0^r \sqrt{r^2 - y^2}\sqrt{R^2 - y^2}\, dy$

Exercises 6.3 ■ Page 559

1. $\dfrac{\pi}{15}$



3. $\dfrac{6}{7}\pi$

5. $\pi\left(1 - \dfrac{1}{e}\right)$

7. 8π

9. 4π

11. 192π

13. $\dfrac{16}{3}\pi$

15. $\dfrac{264}{5}\pi$

17. $\dfrac{8}{3}\pi$

19. $\dfrac{13}{3}\pi$

21. (a) $V = 2\pi \displaystyle\int_0^2 x^2 e^{-x} dx$

(b) $V \approx 4.063$

23. (a) $V = 4\pi \displaystyle\int_{-\pi/2}^{\pi/2} (\pi - x) \cos^4 x \, dx$

(b) $V \approx 46.50942$

25. (a) $V = 2\pi \displaystyle\int_0^{\pi} (4 - y)\sqrt{\sin y} \, dy$

(b) $V \approx 36.57476$

27. $V \approx 3.70$

29. Volume of the solid obtained by rotating the region $R = \{(x, y) \mid 0 \le y \le x^4, \ 0 \le x \le 3\}$ about the y-axis

31. Volume of the solid obtained by rotating the region $R = \{(x, y) \mid 0 \le x \le \dfrac{1}{y^2}, \ 1 \le y \le 4\}$ about the line $y = -2$

33. $V \approx 14.450$

35. $V = \dfrac{\pi^3}{32} \approx 0.968946$

37. 8π

39. $4\sqrt{3}\,\pi$

41. $\dfrac{4}{3}\pi$

43. $\dfrac{117}{5}\pi$

45. $a = \dfrac{V}{\pi} + \dfrac{2}{3}$

47. $\dfrac{\pi r^2 h}{3}$

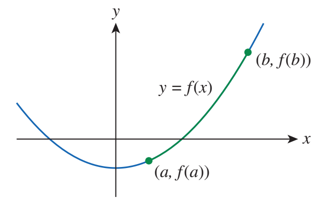
49. $\dfrac{\pi h^3}{6}$

Exercises 6.4 ■ Page 566

1. $4\sqrt{5}$

3. 4π; the unit circle is traversed twice using this parametrization.

5. $f(x) = \pm\dfrac{x^2}{4} + C$; $a = 1$, $b = 4$



7. $L = \displaystyle\int_0^2 \sqrt{1 + e^{-2x}(1 - x)^2} \, dx \approx 2.1024$

9. $L = \displaystyle\int_0^2 \sqrt{1 + (2y - 2)^2} \, dy \approx 2.9579$

11. $L = \displaystyle\int_{-1}^1 \sqrt{1 + 4y^2 e^{2(y^2)}} \, dy \approx 4.2552$

13. $L = \displaystyle\int_0^{2\pi} \sqrt{t^2 + 1} \, dt \approx 21.2563$

15. $\ln(2 + \sqrt{3})$

17. $\dfrac{2}{27}(55\sqrt{55} - 37\sqrt{37})$

19. 2

21. $2(2\sqrt{2} - 1)$

23. $\dfrac{8}{27}\left[10^{3/2} - \left(\dfrac{13}{4}\right)^{3/2}\right]$

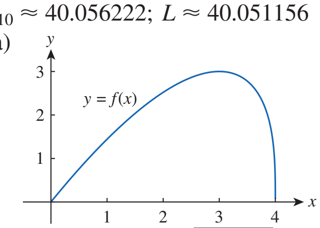
25. 10.0556

27. $S_{10} \approx 5.115840$; $L \approx 5.113568$

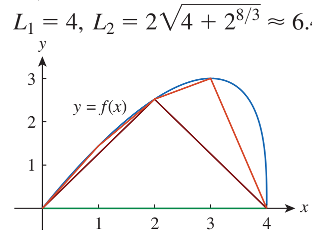
29. $S_{10} \approx 7.11907$; $L \approx 7.118819$

31. $S_{10} \approx 40.056222$; $L \approx 40.051156$

33. (a)



(b) $L_1 = 4$, $L_2 = 2\sqrt{4 + 2^{8/3}} \approx 6.434$, $L_4 \approx 7.497$



(c) $L = \displaystyle\int_0^4 \sqrt{1 + \left(\dfrac{12 - 4x}{3(4 - x)^{2/3}}\right)^2} \, dx$

(d) $L \approx 7.7988$

35. $\dfrac{205}{128} - \dfrac{81}{512}\ln 3$

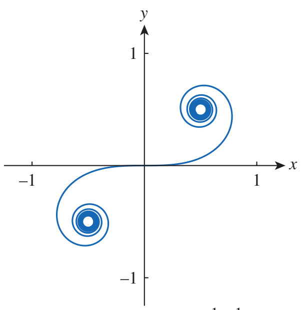
37. $2 - \sqrt{2} + \ln(1 + \sqrt{2}) - \ln\sqrt{3}$

39. $45\sqrt{17} - \dfrac{45}{4}\ln(\sqrt{17} - 4) \approx 209.105$ m

41. $6a$

43. $L = 4\displaystyle\int_0^1 \sqrt{1 + x^{2(2k-1)}(1 - x^{2k})^{1/k-2}}\,dx;\ \lim_{k\to\infty} L_{2k} = 8$

45. (a)



As $t \to \infty$, $(x, y) \to (\frac{1}{2}, \frac{1}{2})$; as $t \to -\infty$, $(x, y) \to (-\frac{1}{2}, -\frac{1}{2})$

(b) t (or $-t$ if $t < 0$)

Exercises 6.5 ■ Page 572

1. 7

3. $\dfrac{4}{3}$

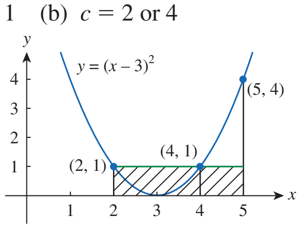
5. $\sqrt{3} - 1$

7. $\dfrac{1}{24}$

9. $\dfrac{2}{5\pi}$

11. (a) 1 (b) $c = 2$ or 4

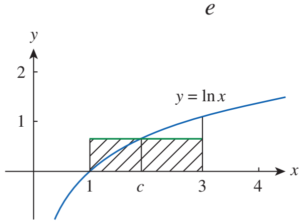
(c)



13. (a) $\dfrac{3}{2}\ln 3 - 1$

(b) $c = e^{(3/2)\ln 3 - 1} = \dfrac{3\sqrt{3}}{e} \approx 1.91$

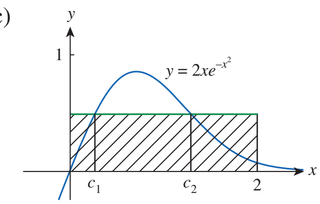
(c)



15. (a) $\dfrac{1}{2}(-e^{-4} + 1)$

(b) $c_1 \approx 0.263$; $c_2 \approx 1.287$

(c)



19. $\dfrac{43}{9}$

21. $\dfrac{9}{8}$

23. $\left(50 + \dfrac{28}{\pi}\right)$°F ≈ 59°F

25. 6 kg/m

27. $\dfrac{5}{4\pi} \approx 0.39789$ L

Exercises 6.6 ■ Page 585

1. 9 ft-lb

3. 180 J

5. $\dfrac{15}{4}$ ft-lb

7. (a) $\dfrac{25}{24} \approx 1.04$ J (b) 10.8 cm

9. $W_2 = 3W_1$

11. (a) 625 ft-lb (b) $\dfrac{1875}{4}$ ft-lb

13. 650,000 ft-lb

15. 3857.28 J

17. 2450 J

19. 1.0584×10^6 J

21. $33{,}000\pi \approx 1.04 \times 10^5$ ft-lb

23. 2.0277 cm

27. (a) $Gm_1m_2\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$ (b) 8.50×10^9 J

29. (a) 187.5 lb/ft^2 (b) 1875 lb (c) 562.5 lb

31. 6.7×10^4 N

33. 9.8×10^3 N

35. 1.2×10^4 lb

37. 5.27×10^5 N

39. (a) 5.63×10^3 lb (b) 5.06×10^4 lb (c) 4.88×10^4 lb (d) 3.03×10^5 lb

41. 2.4892×10^5 N

43. $M_x = 10$; $M_y = 1$; $\left(\dfrac{1}{21}, \dfrac{10}{21}\right)$

45. $\left(0, \dfrac{8}{5}\right)$

47. $\left(\dfrac{1}{e-1}, \dfrac{e+1}{4}\right) \approx (0.58, 0.93)$

49. $M_x = 60$; $M_y = 160$; $\left(\dfrac{8}{3}, 1\right)$

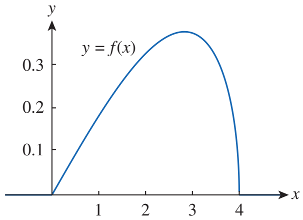
51. (b) $\left(\dfrac{1}{2}, \dfrac{5}{8}\right)$

Exercises 6.7 ■ Page 593

1. \$38,000
3. \$43,866,933.33
5. $450 \ln\left(\frac{45}{8}\right) - 370 \approx \$407.25$
7. \$12,000
9. \$37,753
11. $\frac{2}{3}(16\sqrt{2} - 8) \approx \$9.75$ million
13. $\bar{x} = \frac{(1-k)(b^{2-k} - a^{2-k})}{(2-k)(b^{1-k} - a^{1-k})}$
15. $1.19148 \times 10^{-4}\ \text{cm}^3/\text{s}$
17. $\frac{0.108}{1 - 7e^{-6}} \approx 0.1099$ L/s or 6.594 L/min
19. 0.0962 L/s or 5.77 L/min

Exercises 6.8 ■ Page 602

1. (a) The probability that a randomly selected tire will have a lifetime between 30,000 and 49,000 miles
(b) The probability that a randomly selected tire will have a lifetime of at least 25,000 miles
3. (a) $\int_0^{15} f(t)\,dt$ (b) $\int_{30}^{\infty} f(t)\,dt$
5. (a) $f(x) \geq 0$ for all x; $\int_{-\infty}^{\infty} f(x)\,dx = 1$
(b) $1 - \frac{3}{8}\sqrt{3} \approx 0.3505$
(c)



The probability density function is not symmetric. There is more probability (area under the curve) to the right of 2.
7. (a) $c = \frac{1}{\pi}$ (b) $\frac{1}{2}$
9. (a) $f(x) \geq 0$ for all x; $\int_{-\infty}^{\infty} f(x)\,dx = 1$
(b) 0.16
(c) 0.64
13. (a) 0.2019 (b) 0.5507 (c) 9.7801 min
15. (a) 0.4432 (b) 0.1474
17. (a) 0.0478 (b) 519.74 g
21. $\sigma = \mu$

Chapter 6 Review ■ Page 605

True-False Quiz

1. True **3.** True **5.** True

Exercises

1. $\frac{8}{3}$
3. $\frac{7}{12}$
5. $\frac{4}{3} + \frac{4}{\pi}$
7. 9π
9. (a) 0.3825 (b) 0.8748
11. (a) $\frac{2}{15}\pi$ (b) $\frac{\pi}{6}$ (c) $\frac{8}{15}\pi$
13. $\frac{117}{5}\pi$
15. 256π
17. 0.7834
19. 9.4414
21. The solid obtained by rotating the region $R = \{(x, y) \mid 0 \leq x \leq \frac{\pi}{2},\ 0 \leq y \leq \cos x\}$ about the y-axis
23. The solid obtained by rotating the region $R = \{(x, y) \mid 0 \leq x \leq \pi,\ 0 \leq y \leq 2 - \sin x\}$ about the x-axis
25. The solid obtained by rotating the region $R = \{(x, y) \mid 0 \leq x \leq 1,\ 2 - \sqrt{x} \leq y \leq 2 - x^2\}$ about the x-axis
27. $n = 4$, $\Delta x = 7$, $V \approx \frac{7}{3}\left(\frac{21{,}818}{4\pi}\right) \approx 4051\ \text{cm}^3$
29. $\frac{64}{15}$
31. (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{3}$
33. 1.2968
35. $\frac{124}{5}$
37. 103,500 ft-lb
39. $\frac{512}{15}\delta \approx 2133.3$ lb $(\delta \approx 62.5\ \text{lb/ft}^3)$
41. $\left(2, \frac{2}{3}\right)$
43. $\frac{26}{9}$
45. 0.0704 L/s or 4.225 L/min
47. (a) 0.0214 (b) 0.0062
49. (a) No solution
(b) $\frac{5}{2}$
(c) True for all b

Focus on Problem Solving

1. $f(x) = \sqrt{2x/\pi}$
3. (b) 0.2261
(c) 0.6736 m
(d) (i) $\frac{1}{105\pi} \approx 0.003$ in/s
(ii) $\frac{370\pi}{3} \approx 387\ \text{s} \approx 6.5$ min

7. $y = \dfrac{32}{9}x^2$

9. $h = \dfrac{\sqrt{3}}{2}$

11. $r = \dfrac{h \sin\theta}{\sin\theta + \cos 2\theta}$

13. $\dfrac{5}{6}\pi^3 r^2$

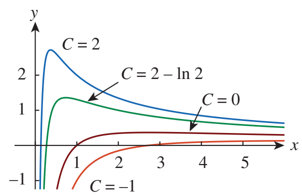
15. $\dfrac{55}{14} \approx 3.93$ cm

Chapter 7

Exercises 7.1 ■ Page 618

5. (a) $k = \pm\dfrac{5}{2}$

7. (b)

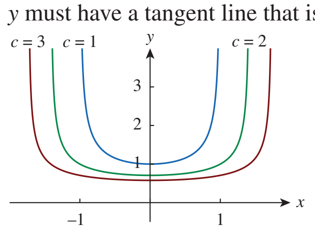


(c) $y = \dfrac{\ln x + 2}{x}$

(d) $y = \dfrac{\ln x + 2 - \ln 2}{x}$

9. (a) If x is close to 0, y' is close to 0. The graph of y must have a tangent line that is nearly horizontal. If x is large, y must have a tangent line that is nearly vertical.

(c)



(d) $y = \left(\dfrac{1}{4} - x^2\right)^{-1/2}$

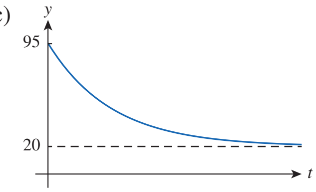
11. (a) $y = 0$, 1, or 5
(b) $(-\infty, 1]$, $[5, \infty)$
(c) $[1, 5]$

13. (C)

15. (a) The coffee cools most quickly as soon as it is removed from the heat source. The rate of cooling decreases toward 0 since the coffee approaches room temperature.

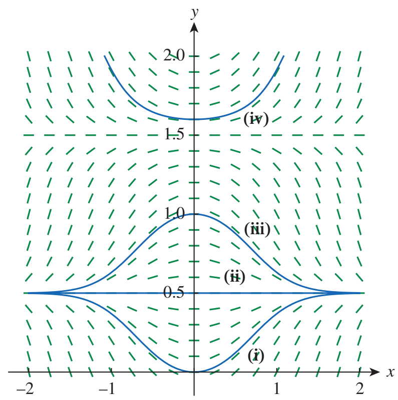
(b) $\dfrac{dy}{dt} = k(y - R)$; k is a proportionality constant, y is the temperature of the coffee, R is the room temperature. Initial condition: $y(0) = 95$. As y approaches R, dy/dt approaches 0.

(c)



Exercises 7.2 ■ Page 627

1. (a)



(b) $y = 0.5$ and $y = 1.5$

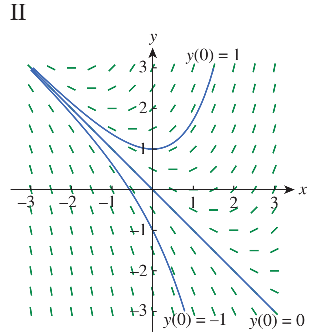
3. (D)

5. (C)

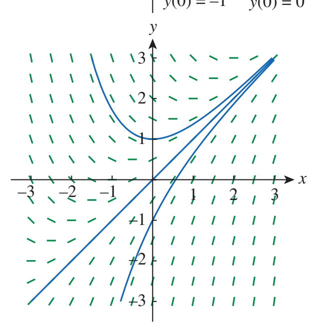
7. I

9. II

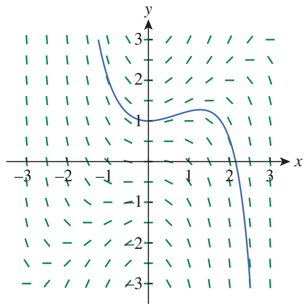
11.



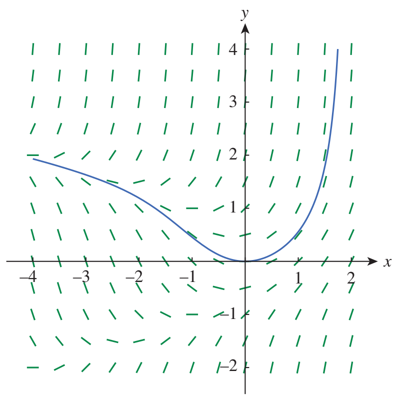
13.



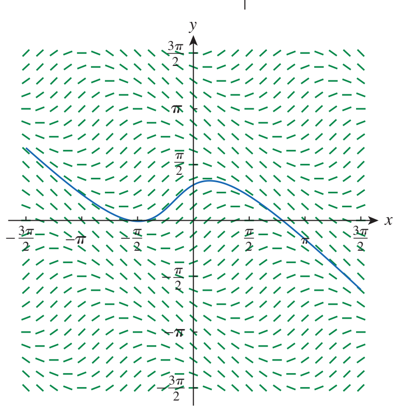
15.



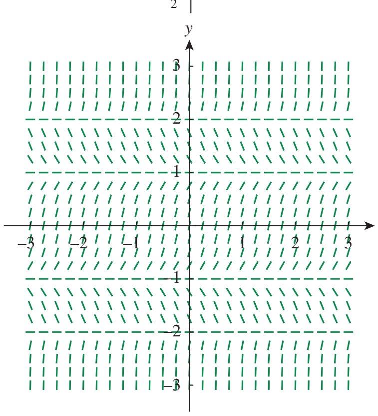
17.



19.



21.



If $y(0) > 2 \;\Rightarrow\; \lim_{x\to\infty} y = \infty$ and $\lim_{x\to-\infty} y = 2$

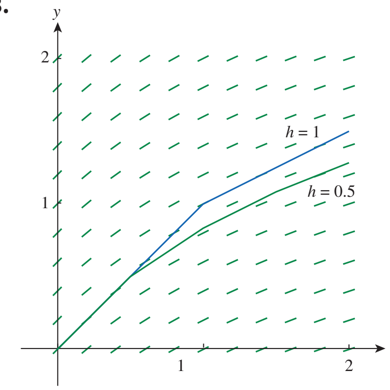
If $1 < y(0) < 2 \;\Rightarrow\; \lim_{x\to\infty} y = 1$ and $\lim_{x\to-\infty} y = 2$

If $-1 < y(0) < 1 \;\Rightarrow\; \lim_{x\to\infty} y = 1$ and $\lim_{x\to-\infty} y = -1$

If $-2 < y(0) < -1 \;\Rightarrow\; \lim_{x\to\infty} y = -2$ and $\lim_{x\to-\infty} y = -1$

If $y(0) < -2 \;\Rightarrow\; \lim_{x\to\infty} y = -2$ and $\lim_{x\to-\infty} y = -\infty$

23.



All estimates are overestimates.

25. $y(1) \approx 1.1949$

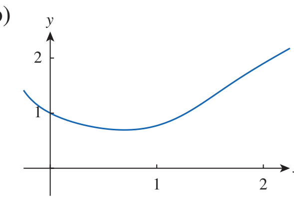
27. $y(0.4) \approx 1.5453$

29. $y(\frac{1}{2}) \approx -2.5$

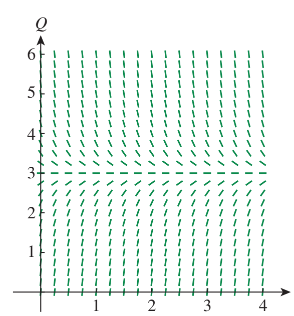
31. (a) $y(0.6) \approx 0.5258$ (b) $y(0.6) \approx 0.5034$

33. (a) $y(2) \approx 1.9000$

(b)



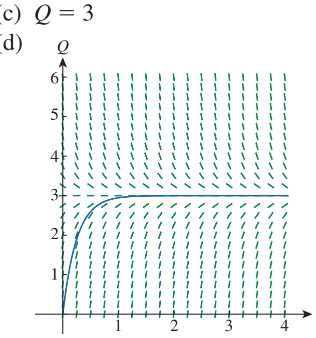
35. (a)



(b) 3

(c) $Q = 3$

(d)



(e) $Q(5) \approx 2.7667$

Exercises 7.3 ■ Page 638

1. $y = \dfrac{-1}{x^3 + C}$; $y = 0$

3. $y = K\sqrt{x^2 + 1}$

5. $y = \pm\sqrt{x^2 + 2\ln|x| + C}$

7. $e^y - y = 2x + \sin x + C$

9. $\theta \sin\theta + \cos\theta = -\dfrac{1}{2}e^{-t^2} + C$

11. $p = Ke^{t^3/3 - t} - 1$

13. $y = -\ln\left(1 - \dfrac{1}{2}x^2\right)$

15. $y = 2e^{x^2 + x} + 1$

17. $u = -\sqrt{t^2 + \tan t + 25}$

19. $\dfrac{1}{2}x^2\ln x - \dfrac{1}{4}x^2 + \dfrac{41}{12} = \dfrac{1}{2}y^2 + \dfrac{1}{3}(3 + y^2)^{3/2}$

21. $y = \dfrac{4a}{\sqrt{3}}\sin x - a$

23. $y = \sqrt{x^2 + 4}$

25. $y = Ke^x - x - 1$

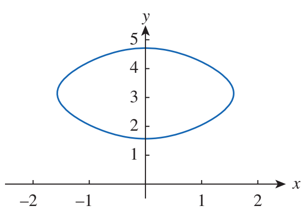
27. (a) $y = \sin(x^2 + C)$ for $-\dfrac{\pi}{2} \le x^2 + C \le \dfrac{\pi}{2}$

(b) $y = \sin x^2$ for $-\sqrt{\pi/2} \le x \le \sqrt{\pi/2}$

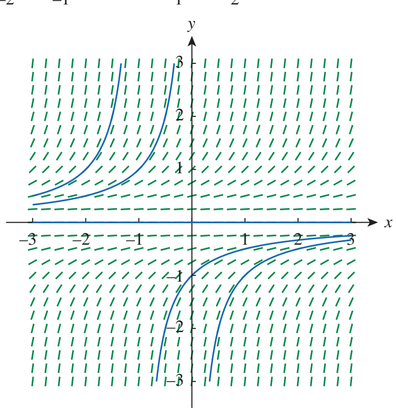
(c) No solution

29. $f(2) \approx 1.742$

31. $\cos y = \cos x - 1$

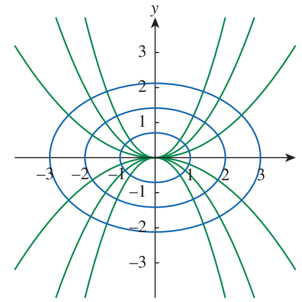


33. (a), (c)

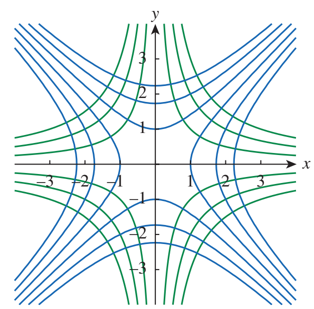


(b) $y = -\dfrac{1}{C + x}$; $y = 0$

35. $y = Cx^2$



37. $x^2 - y^2 = C$



39. $y = 1 + e^{2 - x^2/2}$

41. $y = \left(\dfrac{1}{2}x^2 + 2\right)^2$

43. (a) $y = 3x + 1$; $f(1.5) \approx 5.5$

(b) ∞

(c) $y = 4 - 3e^{-x}$

45. $Q(t) = 3 - 3e^{-4t}$; 3

47. $P(t) = M - Me^{-kt}$; M

49. (a) $\dfrac{d^2y}{dx^2} = \dfrac{-4 - 2y^2}{x^2y^3}$

(b) $y = \dfrac{2}{3}x + \dfrac{7}{3}$; $f(1.2) \approx \dfrac{47}{15} \approx 3.133$

(c) For $1 \le x \le 1.5$, $\dfrac{d^2y}{dx^2} < 0$; curve is concave down, tangent line above curve, overestimate.

(d) $y = \sqrt{4\ln x + 9}$

51. (a) $x(t) = a - \dfrac{4}{(kt + 2/\sqrt{a})^2}$

(b) $t(x) = \dfrac{2}{k\sqrt{a - b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a - b}} - \tan^{-1}\sqrt{\dfrac{b - x}{a - b}}\right)$

53. (a) $\dfrac{dx}{dt} = 0.005(10 - x)$

(b) $x(t) = 10(1 - e^{-0.005t})$

(c) $t = 200\ln 10 \approx 460.517$ days ≈ 1.26 years

55. $p(t) = 0.05 + 0.1e^{-t/90}$

57. (a) $y = \dfrac{130}{3}(1 - e^{-3t/200})$ kg

(b) $\dfrac{130}{3}(1 - e^{-0.9}) \approx 25.7$ kg

59. (a) $v(t) = v_0 e^{-kt/m}$; $s(t) = s_0 + \dfrac{mv_0}{k}(1 - e^{-kt/m})$; $\dfrac{mv_0}{k}$

(b) $v(t) = \dfrac{mv_0}{kv_0 t + m}$; $s(t) = s_0 + \dfrac{m}{k}\ln\left|\dfrac{kv_0 t + m}{m}\right|$; infinitely far

61. (a) $V(t) = be^{Ce^{-at}}$, $C \neq 0$

(b) $V(t) = b^{1 - e^{-at}}$

63. (a) $\dfrac{dA}{dt} - k\sqrt{A}(M - A)$

(b) $A(t) = \left(\dfrac{Ce^{\sqrt{M}kt} - 1}{Ce^{\sqrt{M}kt} + 1}\right)^2$; $C = \dfrac{\sqrt{M} + \sqrt{A_0}}{\sqrt{M} - \sqrt{A_0}}$

Exercises 7.4 ■ Page 650

1. $P(6) = 2e^{0.7944(6)} \approx 234.99$ or about 235 members

3. $P(t) = 3e^{(-0.004\ln 2)t}$

5. (a) $P(t) = 100e^{(\ln 4.2)t} = 100(4.2)^t$

(b) $7408.8 \approx 7409$ bacteria

(c) $(\ln 4.2)\left(100(4.2)^3\right) \approx 10{,}632$ bacteria/h

(d) $t = (\ln 100)/(\ln 4.2) \approx 3.2$ hours

7. (a) $P(t) = 2.810e^{k(t - 1950)}$ where

$$k = \frac{1}{40}\ln\frac{13.018}{2.810} \approx 0.0383287$$

(b) $P(1970) \approx 6.05$; $P(2020) \approx 41.1$

9. (a) $P(t) = 6000e^{t/4}$; $P(5) \approx 20{,}942$ bacteria

(b) $\dfrac{6000(e^2 - 1)}{8} \approx 4791.8$ bacteria/time

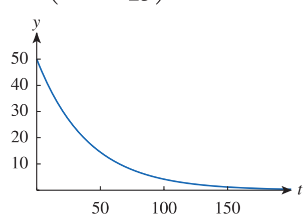
(c) 19167.2

11. (a) $y(t) = 50 \cdot 2^{-t/28}$

(b) $y(40) = 50 \cdot 2^{-40/28} \approx 18.6$ mg

(c) $t = \left(-28\ln\dfrac{1}{25}\right)/\ln 2 \approx 130$ days

(d)



13. (a) $t = -\dfrac{\ln 2}{\ln 0.945} \approx 12.25$ years

(b) $t = -\dfrac{\ln 5}{\ln 0.945} \approx 28.45$ years

15. $y = 5e^{2x}$

17. (a) $T(50) = 20 - \dfrac{20}{3} \approx 13.3°\text{C}$

(b) $t = 25\dfrac{\ln(1/3)}{\ln(2/3)} \approx 67.74$ min

19. (a) $P(3000) = 101.3e^{3\ln(87.14/101.3)} \approx 64.5$ kPa

(b) $P(6187) = 101.3e^{\frac{6187}{1000}\ln\left(\frac{87.14}{101.3}\right)} \approx 39.9$ kPa

21. (a) \$3828.84; \$3840.25; \$3850.08; \$3851.61; \$3852.01; \$3852.08

(b) $\dfrac{dA}{dt} = 0.05A \quad A(0) = 3000$

23. (a) $P(t) = \dfrac{m}{k} + \left(P_0 - \dfrac{m}{k}\right)e^{kt}$

(b) $m < kP_0$

(c) $m = kP_0$; $m > kP_0$

(d) Population was declining

Exercises 7.5 ■ Page 660

1. (a) $M = 1200$, $k = 0.04$

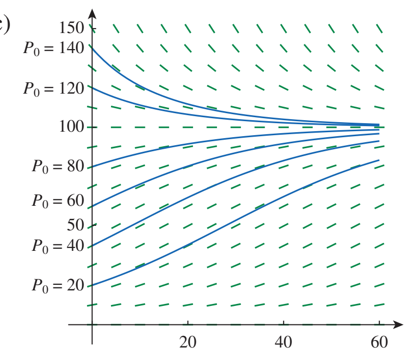
(b) $P(t) = \dfrac{1200}{1 + 19e^{-0.04t}}$

(c) $P(10) = \dfrac{1200}{1 + 19e^{-0.4}} \approx 87$

3. (a) $M = 100$, $k = 0.05$

(b) P near 0 or 100; $P = 50$; $0 < P_0 < 100$; $P_0 > 100$

(c)



(ii) All solutions approach $P = 100$ as t increases. For $0 < P_0 < 100$, solutions increasing. For $P_0 > 100$, solutions decreasing.

(iii) Solutions with $P_0 = 20$ and $P_0 = 40$ have an inflection point at $P = 50$.

(d) $P = 0$ and $P = 100$. Increasing solutions move away from $P = 0$. All nonzero solutions approach $P = 100$ as $t \to \infty$.

5. (a) $M = 400$

(b) $P'(0) = 17.5$

(c) $t = \dfrac{\ln(1/7)}{-0.4} \approx 4.86$ years

7. 9000

9. (a) 3000; carrying capacity, the maximum number of chipmunks the environment is capable of sustaining

(b) 1500

11. (a) 1000; carrying capacity of the model, the entire susceptible population

(b) 500

(c) 400; slope of the curve increasing for $P < 500$

(d) $P(t) = \dfrac{1000}{1 + 4e^{-100{,}000(t - 1)}}$

13. (a) $y(1) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71}} \approx 3.232 \times 10^7$ kg

(b) $t = \dfrac{\ln 3}{0.71} \approx 1.547$ years

15. (a) $\dfrac{dP}{dt} = \dfrac{1}{305}P\left(1 - \dfrac{P}{20}\right)$

(b) 7.91

(c) 15.286; 19.999

17. (a) $\dfrac{dy}{dt} = ky(1 - y)$

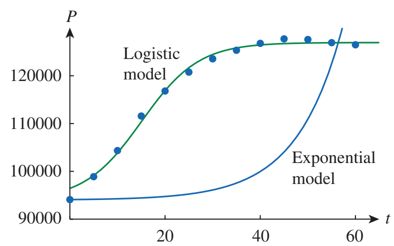
(b) $y = \dfrac{y_0}{y_0 + (1 - y_0)e^{-kt}}$

(c) 7.6 hours

21. Let $t = 0$ correspond to 1960 and subtract 94,000 from the population figures.

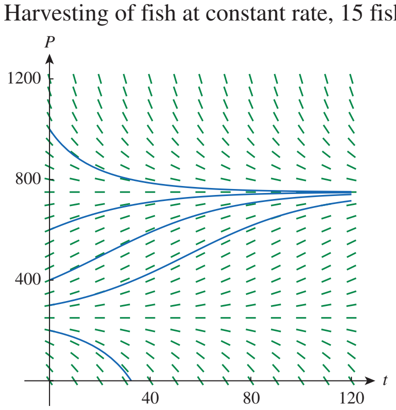
$P_E(t) = 92e^{0.104274t} + 94{,}000$;

$P_L(t) = \dfrac{32941.03}{1 + 12.506e^{-0.167529t}} + 94{,}000$



23. (a) Harvesting of fish at constant rate, 15 fish/week

(b)



(c) $P(t) = 250$ and $P(t) = 750$

(d) $0 < P_0 < 250$: $P(t)$ decreases to 0

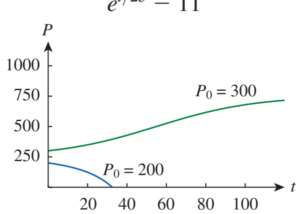
$P_0 = 250$: $P(t)$ constant

$250 < P_0 < 750$: $P(t)$ increases and approaches 750

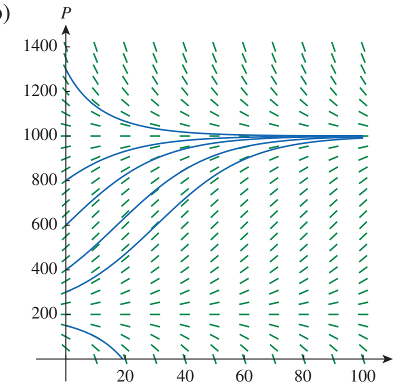
$P_0 = 750$: $P(t)$ constant

$P_0 > 750$: $P(t)$ decreases and approaches 750

(e) $P(t) = \dfrac{250(3e^{t/25} - 11)}{e^{t/25} - 11}$; $P(t) = \dfrac{750(e^{t/25} + 3)}{e^{t/25} + 9}$



25. (b)



$0 < P_0 < 200$: $P(t)$ approaches 0

$P_0 = 200$: $P(t)$ constant

$200 < P_o < 1000$: $P(t)$ increases and approaches 1000

$P_0 = 1000$: $P(t)$ constant

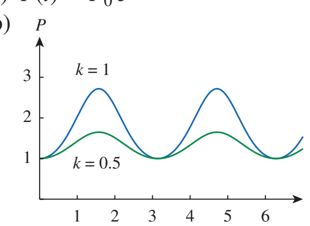
$P_0 > 1000$: $P(t)$ decreases and approaches 1000

Equilibrium solutions: $P(t) = 200$ and $P(t) = 1000$

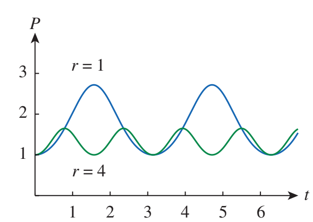
(c) $P(t) = \dfrac{m(M - P_0) + M(P_0 - m)e^{(M-m)(k/M)t}}{M - P_0 + (P_0 - m)e^{(M-m)(k/M)t}}$

27. (a) $P(t) = P_0\, e^{(k/r)[\sin(rt - \phi) + \sin\phi]}$

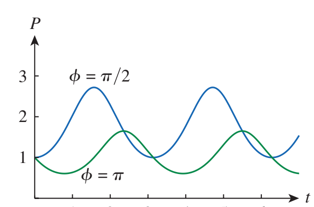
(b)



Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$



Comparing values of r with $P_0 = 1$, $k = 1$, and $\phi = \pi/2$



Comparing values of ϕ with $P_0 = 1$, $k = 1$, and $r = 2$

As k increases, the amplitude increases; as r increases, the amplitude and period decrease, and changes in ϕ produce slight adjustments in the phase shift and amplitude. $P(t)$ oscillates between $P_0 e^{(k/r)[1 + \sin\phi]}$ and $P_0 e^{(k/r)[-1 + \sin\phi]}$. The extreme values are attained when $rt - \phi$ is an odd multiple of $\pi/2$. $\lim\limits_{t\to\infty} P(t)$ does not exist.

Exercises 7.6 ■ Page 668

1. (a) x: predator; y: prey. The growth of the prey population is restricted only by encounters with predators. The predator population increases only by encounters with the prey and not through additional food sources.

(b) y: predator; x: prey. The growth of the prey population is restricted by a carrying capacity of 1000 and by encounters with predators. The predator population increases only by encounters with the prey and not through additional food sources.

3. (a) Competition

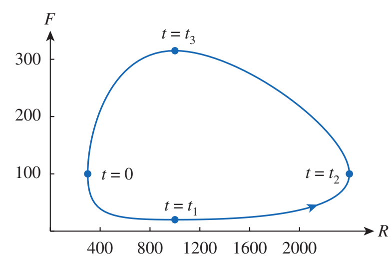
(b) (i) $x = 0$, $y = 0$: if the populations are 0, there is no change.

(ii) $x = 0$, $y = 400$: In the absence of an x-population, the y-population stabilizes at 400.

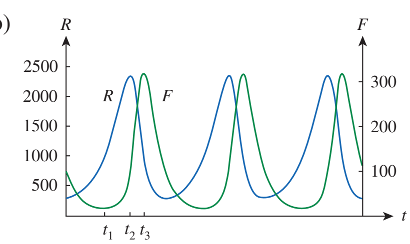
(iii) $x = 125$, $y = 0$: In the absence of a y-population, the x-population stabilizes at 125.

(iv) $x = 50$, $y = 300$: A y-population of 300 is just enough to support a constant x-population of 50.

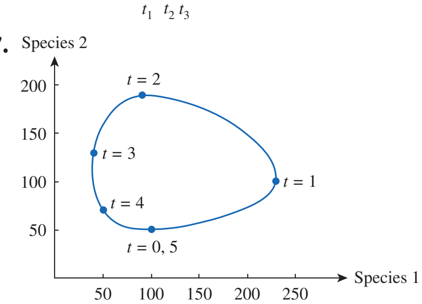
5. (a) The rabbit population starts at about 300, increases to 2400, then decreases back to 300. The fox population starts at 100, decreases to about 20, increases to about 315, decreases to 100, and the cycle starts again.



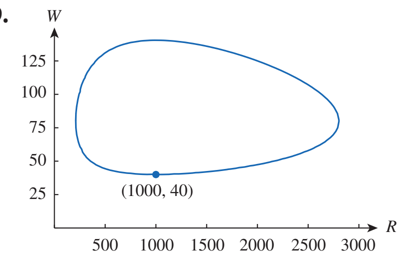
(b)



7.



9.

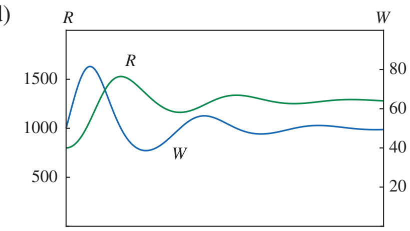


11. (a) Stabilize at 5000

(b) $W = 0$, $R =$: both populations 0
$W = 0$, $R = 5000$: rabbit population stable at 5000
$W = 64$, $R = 1000$: both populations stable

(c) The populations of wolves and rabbits fluctuate around 64 and 1000, respectively, and eventually stabilize at those values.

(d)



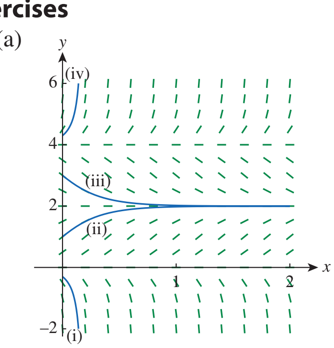
Chapter 7 Review ■ Page 671

True-False Quiz

1. True **3.** False **5.** True

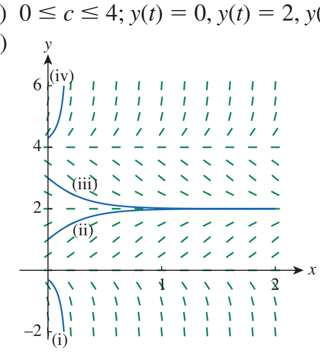
Exercises

1. (a)



(b) $0 \leq c \leq 4$; $y(t) = 0$, $y(t) = 2$, $y(t) = 4$

3. (a)

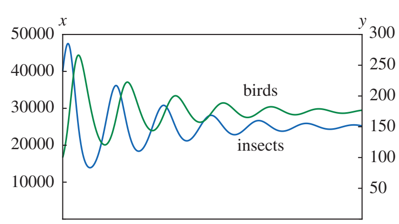


$y(0.3) \approx 0.8$

(b) $y(0.3) \approx 0.757$

(c) $y = x$, $y = -x$; it has a local maximum or minimum

5. $x = -1 + ce^{t - t^2/2}$
7. $r(t) = 5e^{t - t^2}$
9. $y(x) = e^{1 - e^{-x} - xe^{-x}}$
11. (a) $y = \frac{4}{3}x + \frac{2}{3}; f(4.8) \approx \frac{106}{15} \approx 7.067$
(b) $f(4.8) \approx 7.072$
(c) $y = \sqrt{2x^2 + 4}; f(4.8) \approx 7.077$
13. $2y^2 \ln y - y^2 = C - 2x^2$
15. $f(1) \approx 2.5$
17. (A)
19. (a) $100 \cdot 2^{-20/5.24} \approx 7.1$ mg
(b) $t = 5.24 \frac{\ln 100}{\ln 2} \approx 34.8$ years
21. (a) $46\frac{2}{3}^\circ$C
(b) $t = \frac{\ln 1/3}{\ln 4/9} \approx 1.35$ hr
23. (a) $\frac{dL}{dt} = k(L_\infty - L) \quad \Rightarrow \quad L(t) = L_\infty - [L_\infty - L(0)]e^{-kt}$
(b) $L(t) = 53 - 43e^{-0.2t}$
25. $R = A \cdot S^k$ where $A = e^C$ is a positive constant
27. $h + k \ln h = -\frac{R}{V}t + C$
29. (a) Insect population stabilizes at 200,000.
(b) $x = 0$, $y = 0$: both populations zero
$x = 200{,}000$, $y = 0$: no birds, insect population remains at 200,000
$x = 25{,}000$, $y = 175$: both populations stable.
(c) The populations of the birds and insects fluctuate around 175 and 25,000, respectively, and eventually stabilize at those values.
(d)



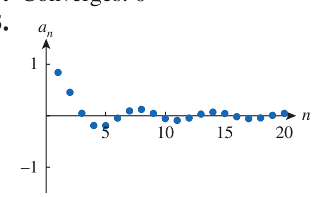
Focus on Problem Solving

1. $f(x) = \pm 10e^x$
5. $y = f(x) = x^{1/n}$
7. 20°C
9. (b) $y = \frac{x^2 - L^2}{4L} - \frac{L}{2} \ln\left(\frac{x}{L}\right)$
(c) As $x \to 0^+$, $y \to \infty$, so the dog never catches the rabbit.
11. (a) 9.8 hours
(b) $2000\pi \approx 6283$ ft^3/h
(c) $t = \frac{813{,}000}{160{,}000} \approx 5.1$ h
13. $x^2 + (y - 6)^2 = 25$; a circle with center (0, 6) and radius 5.

Chapter 8

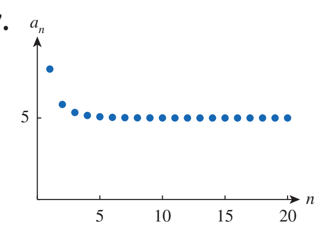
Exercises 8.1 ■ Page 688

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.
(b) The terms a_n approach 8 as n becomes large.
(c) The terms a_n become large as n becomes large.
3. $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}; \frac{1}{2}$
5. $a_n = \frac{1}{2n - 1}$
7. $a_n = -3\left(-\frac{2}{3}\right)^{n-1}$
9. $a_n = 3n + 2$
11. Converges: 5
13. Diverges
15. Converges: 0
17. Converges: 1
19. Converges: 2
21. Diverges
23. Converges: 0
25. Converges: 0
27. Converges: 0
29. Converges: 0
31. Converges: e^2
33. Converges: 0
35. Converges: 1
37. Converges: 0
39. Converges: -2
41. Converges: 0
43. Converges: 0
45.

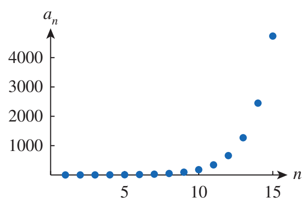


$\lim_{n \to \infty} a_n = 0$

47.



$\lim_{n \to \infty} a_n = 5$

49. 

sequence diverges

51. (a) 1060, 1123.60, 1191.02, 1262.48, 1338.23
(b) Divergent

53. (b) 5734

55. (a) Divergent (b) Convergent

57. (c) $\dfrac{1+\sqrt{5}}{2}$

59. Decreasing; bounded; $0 < a_n \le \dfrac{1}{5}$

61. Not monotonic; not bounded

63. Not monotonic; bounded; $-1 \le \cos n \le 1$

65. Monotonic Sequence Theorem; $5 \le L < 8$

69. (b) $L \approx 0.73909$

73. (b) $1, \dfrac{3}{2}, \dfrac{7}{5}, \dfrac{17}{12}, \dfrac{41}{29}, \dfrac{99}{70}, \dfrac{230}{169}, \dfrac{577}{408}$

Exercises 8.2 ■ Page 700

1. (a) A sequence is an ordered list of numbers; a series is the sum of a list of numbers.
(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

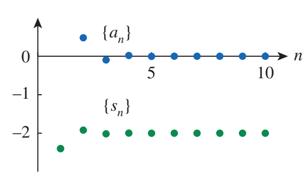
3. 0.5, 0.55, 0.5611, 0.5648, 0.5663, 0.5671, 0.5675, 0.5677, 0.5679, 0.5680; it appears that the series is convergent.

5. 0.8415, 1.7508, 1.8919, 1.1351, 0.1762, −0.1033, 0.5537, 1.5431, 1.9552, 1.4112; it appears that the series is divergent.

7. 2.8854, 5.6161, 8.5015,11.6082, 14.9568, 18.5541, 22.4013, 26.4974, 30.8403, 35.4277; it appears that the series is divergent.

9.

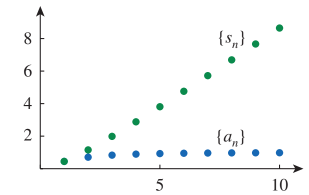
n	s_n
1	−2.40000
2	−1.92000
3	−2.01600
4	−1.99680
5	−2.00064
6	−1.99987
7	−2.00003
8	−1.99999
9	−2.00000
10	−2.00000



The series converges to −2.

11.

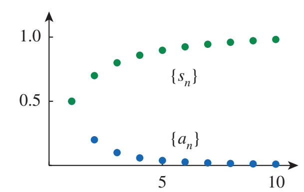
n	s_n
1	0.44721
2	1.15432
3	1.98637
4	2.88080
5	3.80927
6	4.75796
7	5.71948
8	6.68962
9	7.66581
10	8.64639



The series diverges; the terms do not approach 0.

13.

n	s_n
1	0.50000
2	0.70000
3	0.80000
4	0.85882
5	0.89729
6	0.92431
7	0.94431
8	0.95970
9	0.97190
10	0.98180



The series converges; the sum is approximately 1.07667

15. (a) Convergent; $\lim_{n\to\infty} a_n = \dfrac{2}{3}$
(b) Divergent; $\lim_{n\to\infty} a_n \ne 0$

17. Divergent

19. Convergent; $\dfrac{25}{3}$

21. Convergent; $\dfrac{27}{11}$

23. Convergent; $\dfrac{5}{\pi - 1}$

25. Divergent

27. Divergent

29. Convergent; $\dfrac{5}{8}$

31. Divergent

33. Convergent; $\dfrac{7}{3}$

35. Divergent

37. Divergent

39. Convergent; $\dfrac{\sqrt{2}}{\sqrt{2}-1}$

41. Divergent

43. Divergent

45. $s_n = -\ln(n+1)$; divergent

47. $s_n = \dfrac{1}{2} - \dfrac{1}{\sqrt{n+1}}$; convergent; $\dfrac{1}{2}$

49. $s_n = \dfrac{1}{4} - \dfrac{1}{2n} + \dfrac{1}{2n+2}$; convergent; $\dfrac{1}{4}$

51. 16

53. $\dfrac{46}{99}$

55. $\dfrac{838}{333}$

57. $\dfrac{45{,}679}{37{,}000}$

59. $-\dfrac{1}{5} < x < \dfrac{1}{5}$; $\dfrac{-5x}{1+5x}$

61. $-1 < x < 5$; $\dfrac{3}{5-x}$

63. $x < -2$ or $x > 2$; $\dfrac{x}{x-2}$

65. $x < 0$; $\dfrac{1}{1-e^x}$

69. $\dfrac{1}{n^5 - 5n^3 + 4n}$

$$= \frac{1}{24(n-2)} + \frac{1}{24(n+2)} - \frac{1}{6(n-1)} - \frac{1}{6(n+1)} + \frac{1}{4n}$$

$$\lim_{n\to\infty} s_n = \frac{1}{96}$$

71. $a_n = \dfrac{n-2}{2^n}$; $\displaystyle\sum_{n=1}^{\infty} a_n = 3$

73. (a) $\dfrac{N(1-s^{n+1})}{1-s}$; $\dfrac{N}{1-s}$

(b) $N = 1000$

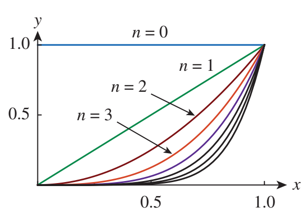
75. (a) $H\left(\dfrac{1+r}{1-r}\right)$ meters

(b) $\sqrt{\dfrac{2H}{g}}\,\dfrac{1+\sqrt{r}}{1-\sqrt{r}}$ seconds

(c) $\sqrt{\dfrac{2H}{g}}\,\dfrac{1+k}{1-k}$ seconds

77. $c = \ln\dfrac{9}{10}$

79.



$$A = \int_0^1 (x^{n-1} - x^n)\,dx = \frac{1}{n(n+1)}$$

Sum of the areas approaches the area of the unit square, 1.

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

81. $b\left(\dfrac{\sin\theta}{1-\sin\theta}\right)$

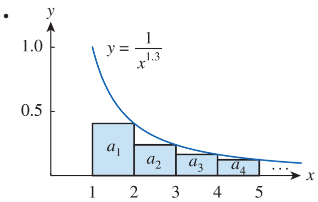
87. The partial sums form an increasing, bounded sequence. By the Monotonic Sequence Theorem, the sequence of partial sums converges, therefore, the series Σa_n converges.

91. (a) $\dfrac{1}{2}, \dfrac{5}{6}, \dfrac{23}{24}, \dfrac{119}{120}$; $s_n = \dfrac{(n+1)! - 1}{(n+1)!}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$

Exercises 8.3 ■ Page 714

1.



$$\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\,dx$$

The integral converges, so the series converges.

3. (a) Σa_n could be convergent or divergent.

(b) Σa_n is convergent.

5. p-series, $p = -b$; geometric series; $b < -1$; $-1 < b < 1$

7. Divergent

9. Convergent

11. Convergent

13. Divergent

15. Convergent

17. Divergent

19. Divergent

21. Convergent

23. Convergent

25. Convergent

27. Convergent

29. Convergent

31. Divergent

33. Convergent

35. Convergent

37. Convergent

39. Convergent

41. Divergent

43. Convergent

45. The function $f(x) = \dfrac{\cos^2 x}{1+x^2}$ is not decreasing on $[1, \infty)$.

47. $p > 1$

49. (a) $s_{10} \approx 1.549768$; at most 0.1

(b) $s \approx 1.64522$

(c) $n > 1000$

51. $s_4 \approx 0.001446$

53. $s_{10} \approx 1.24856$; $R_{10} \le 0.1$

59. Yes; Limit Comparison Test

Exercises 8.4 ■ Page 724

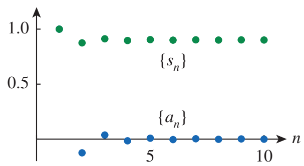
1. (a) An alternating series is a series whose terms are alternatively positive and negative.

(b) $0 < b_{n+1} < b_n$ for all n and $\displaystyle\lim_{n\to\infty} b_n = 0$

(c) $|R_n| \le b_{n+1}$

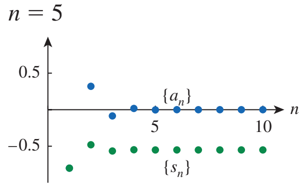
3. Convergent
5. Convergent
7. Convergent
9. Convergent
11. Convergent
13. Divergent
15.

n	a_n	s_n
1	1	1
2	-0.125	0.875
3	0.037037	0.912037
4	-0.015625	0.896412
5	0.008000	0.904412
6	-0.004630	0.899782
7	0.002915	0.902698
8	-0.001953	0.900745
9	0.001371	0.902116
10	-0.001000	0.901116



$|s - s_{10}| \le b_{11} \approx 0.0007513$

17. p not a negative integer
19. $n = 4$
21. $n = 5$
23.



$s \approx -0.55$; $s_7 \approx -0.5507$

25. 0.0676
27. Conditionally convergent
29. Divergent
31. Divergent
33. Absolutely convergent
35. Absolutely convergent
37. Absolutely convergent
39. Absolutely convergent
41. Divergent
43. (A), (D)
45. Absolutely convergent
47. Divergent
49. Divergent

Exercises 8.5 ■ Page 732

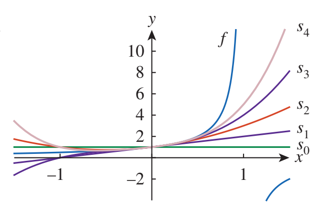
1. A power series centered at a is of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where x is a variable and the c_n's are constants.

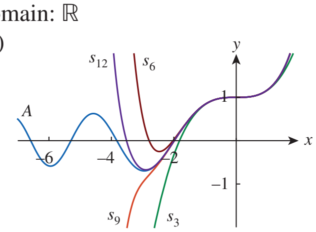
3. $R = 1$; $(-1, 1)$
5. $R = 1$; $[-1, 1)$
7. $R = 0$; $\{0\}$
9. $R = \frac{1}{2}$; $\left(-\frac{1}{2}, \frac{1}{2}\right)$
11. $R = 5$; $(-5, 5]$
13. $R = \infty$; $(-\infty, \infty)$
15. $R = 2$; $(-1, 3]$
17. $R = 8$; $(-14, 2)$
19. $R = \frac{5}{2}$; $[-2, 3)$
21. $R = b$; $(a - b, a + b)$
23. $R = \infty$; $(-\infty, \infty)$
25. $R = 1$; $[-1, 1]$
27. $R = 2$; $(-2, 2)$
29. (a) Given series converges when $x = 1$; $\sum_{n=0}^{\infty} c_n$ is convergent.
(b) Given series diverges when $x = 8$; $\sum_{n=0}^{\infty} c_n 8^n$ is divergent.
(c) Given series converges when $x = -3$; $\sum_{n=0}^{\infty} c_n(-3)^n$ is convergent.
(d) Given series diverges when $x = -9$; $\sum_{n=0}^{\infty} (-1)^n c_n 9^n$ is divergent.

31.



The partial sums appear to converge to f on $(-1, 1)$.

33. (a) Domain: $\mathbb{R}$
(b), (c)



35. $(-1, 1)$; $f(x) = \dfrac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4}$
37. $\sqrt{R}$
39. (a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x - \frac{1}{2}(p+q)}{\frac{1}{2}(q-p)}\right)^n$
(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}\left(\frac{x - \frac{1}{2}(p+q)}{\frac{1}{2}(q-p)}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x-\frac{1}{2}(p+q)}{\frac{1}{2}(q-p)}\right)^n$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{x-\frac{1}{2}(p+q)}{\frac{1}{2}(q-p)}\right)^n$

Exercises 8.6 ■ Page 739

1. 10

3. The functions f and S are the same only on the interval of convergence, $|x|<1$.

5. $f(x)=5\displaystyle\sum_{n=0}^{\infty}4^n x^{2n}$; $\left(-\frac{1}{2},\frac{1}{2}\right)$

7. $f(x)=\dfrac{4}{3}\displaystyle\sum_{n=0}^{\infty}\left(-\frac{2x}{3}\right)^n$; $\left(-\frac{3}{2},\frac{3}{2}\right)$

9. $f(x)=x\displaystyle\sum_{n=0}^{\infty}(-2x^2)^n=\sum_{n=0}^{\infty}(-1)^n 2^n x^{2n+1}$; $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

11. $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{a^{2n+1}}+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{a^{2n+1}}$; $(-a,a)$

13. (a) $f(x)=-\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n}$; $R=1$

(b) $g(x)=x\ln(1-x)=-\displaystyle\sum_{n=1}^{\infty}\frac{x^{n+1}}{n}$

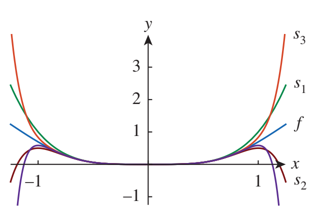
(c) $\ln 2=\displaystyle\sum_{n=1}^{\infty}\frac{1}{n2^n}$

15. $f(x)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+5}}{2n+1}$; $R=1$

17. $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{2^{n+4}}x^{n+3}$; $R=2$

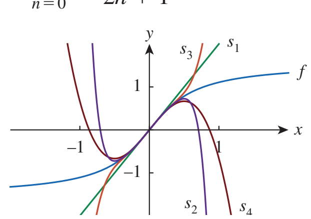
19. $f(x)=\displaystyle\sum_{n=1}^{\infty}n^2x^n$; $R=1$

21. $f(x)=\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{4n}}{n}$



As n increases, s_n approaches f on the interval of convergence, $[-1,1]$.

23. $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n+1}}{2n+1}$



As n increases, s_n approaches f on the interval of convergence, $\left[-\frac{1}{2},\frac{1}{2}\right]$

25. $C+\displaystyle\sum_{n=0}^{\infty}\frac{x^{3n+2}}{3n+2}$; $R=1$

27. $C+\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)^2}$; $R=1$

29. $C+\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{4n^2-1}$; $R=1$

31. $I\approx\dfrac{1}{16}-\dfrac{1}{1536}+\dfrac{1}{61{,}440}-\dfrac{1}{1{,}835{,}008}\approx 0.061865$

33. 0.008969

35. (a) $s(x)=\displaystyle\sum_{n=1}^{\infty}2^{n+1}x^{2n}$

(b) $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

37. $\arctan 0.2\approx 0.19740$

39. (b) 0.920

43. f: $[-1,1]$; f': $[-1,1)$; f'': $(-1,1)$

Exercises 8.7 ■ Page 755

1. $b_8=\dfrac{f^{(8)}(5)}{8!}$

3. $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$; $R=1$

5. $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$; $R=1$

7. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{(2n+1)!}x^{2n+1}$; $R=\infty$

9. $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}x^n$; $R=\infty$

11. $\displaystyle\sum_{n=0}^{\infty}\frac{(\ln 2)^n}{n!}x^n$; $R=\infty$

13. $f(x)=50+105(x-2)+92(x-2)^2+42(x-2)^3+10(x-2)^4+(x-2)^5$; $R=\infty$

15. $f(x)=\ln 2+\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n\,2^n}(x-2)^n$; $R=2$

17. $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{2^n e^6}{n!}(x-3)^n$; $R=\infty$

19. $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}$; $R=\infty$

21. $f(x)=\dfrac{1}{3}+\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot 3^{2n+1}\cdot n!}(x-9)^n$; $R=9$

25. $1-\dfrac{1}{4}x-\displaystyle\sum_{n=2}^{\infty}\frac{3\cdot 7\cdot\cdots\cdot(4n-5)}{4^n n!}x^n$; $R=1$

27. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)x^n}{2^{n+4}}$; $R=2$

29. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{4n+2}$; $R = 1$

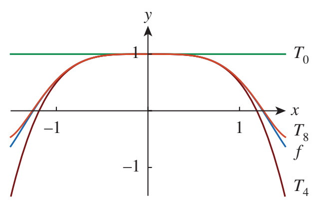
31. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n)!} x^{2n+1}$; $R = \infty$

33. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n}(2n)!} x^{4n+1}$; $R = \infty$

35. $f(x) = \frac{x}{2} + \frac{x}{2} \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^2 \cdot 4^n \cdot n!} x^{2n}$; $R = 2$

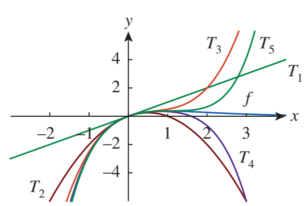
37. $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}$; $R = \infty$

39. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$; $R = \infty$



As n increases, $T_n(x)$ becomes a better approximation to $f(x)$.

41. $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{(n-1)!}$; $R = \infty$



As n increases, $T_n(x)$ becomes a better approximation to $f(x)$.

43. 0.99619

45. (a) $1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} x^{2n}$

(b) $\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n n!} x^{2n+1}$

47. $C + \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{x^{3n+1}}{3n+1}$; $R = 1$

49. $C + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{2n \cdot (2n)!}$; $R = \infty$

51. $C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$; $R = \infty$

53. 0.0059

55. 0.40102

57. $\frac{1}{2}$

59. $\frac{1}{120}$

61. $\frac{3}{5}$

63. $1 - \frac{3}{2}x^2 + \frac{25}{24}x^4 + \cdots$

65. $1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \cdots$

67. $x^2 - \frac{2}{3}x^4 + \frac{23}{45}x^6 + \cdots$

69. e^{-x^4}

71. $\ln(8/5)$

73. $\frac{1}{\sqrt{2}}$

75. $e^3 - 1$

77. (a) $T_3(x) = 2 + 3(x-1) + 2(x-1)^2 + \frac{5}{6}(x-1)^3$

(b) $f(1.5) \approx T_3(1.5) \approx 4.10417$

79. $-\frac{1}{40}$

81. $\frac{7}{3}$

83. (a) $T_2(x) = 3 + 4(x-5) + 10(x-5)^2$

(b) $f(5.4) \approx T_2(5.4) = 6.2$

85. (a) $a_1 = 10$

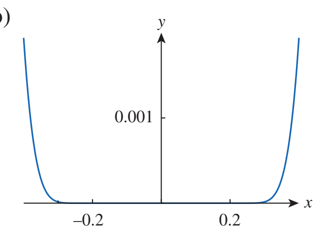
(b) $f''(1) = 22$; less than

(c) $T_4(x) = 4 + 10(x-1) + 11(x-1)^2 + 15(x-1)^3 + 13(x-1)^4$;

$f(2) \approx T_4(2) = 53$

(d) $-4 + 4x - 5(x-1)^2$

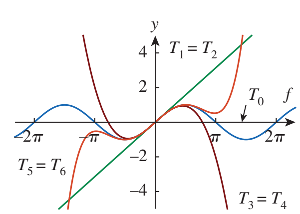
87. (b)



The graph of the function is extremely flat at the origin.

Exercises 8.8 ■ Page 767

1. (a) $T_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

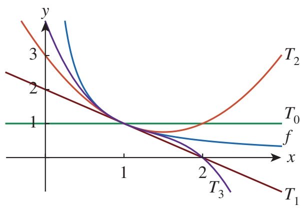


(b)

x	$f(x)$	$T_0(x)$	$T_1(x)$	$T_3(x)$	$T_5(x)$
$\pi/4$	0.7071	0	0.7854	0.7047	0.7071
$\pi/2$	1	0	1.5708	0.9248	1.0045
π	0	0	3.1416	−2.0261	0.5240

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

3. (a) $T_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3$
$= -x^3 + 4x^2 - 6x + 4$

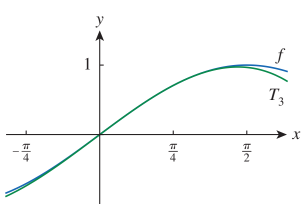


(b)

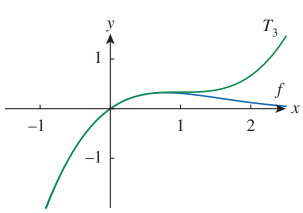
x	$f(x)$	$T_0(x)$	$T_1(x)$	$T_2(x)$	$T_3(x)$
0.9	1.11111	1	1.1	1.11	1.111
1.3	0.76923	1	0.7	0.79	0.763

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

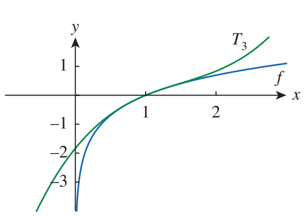
5. $T_3(x) = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right) - \dfrac{1}{4}\left(x - \dfrac{\pi}{6}\right)^2 - \dfrac{\sqrt{3}}{12}\left(x - \dfrac{\pi}{6}\right)^3$



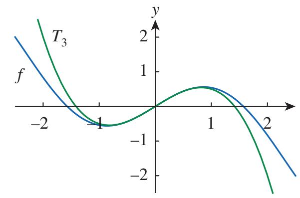
7. $T_3(x) = x - x^2 + \dfrac{x^3}{3}$



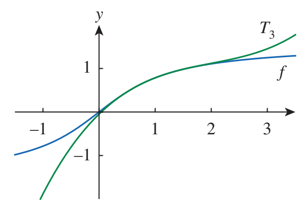
9. $T_3(x) = (x-1) - \dfrac{1}{2}(x-1)^2 + \dfrac{1}{3}(x-1)^3$



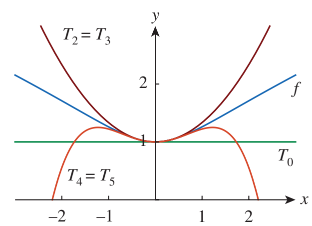
11. $T_3(x) = x - \dfrac{x^3}{2}$



13. $T_3(x) = \dfrac{\pi}{4} + \dfrac{1}{2}(x-1) - \dfrac{1}{4}(x-1)^2 + \dfrac{1}{12}(x-1)^3$



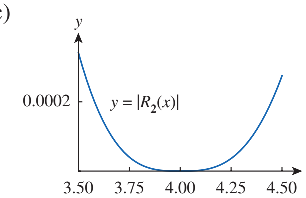
15. $T_5(x) = 1 + \dfrac{1}{3}x^2 - \dfrac{1}{9}x^4$



17. (a) $T_2(x) = \dfrac{1}{2} - \dfrac{1}{16}(x-4) + \dfrac{3}{256}(x-4)^2$

(b) $|R_2(x)| \le \dfrac{15}{6 \cdot 8(3.5)^{7/2}}(0.125) \approx 0.000487$

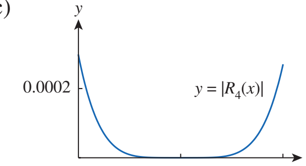
(c)



19. (a) $T_4(x) = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right) - \dfrac{1}{4}\left(x - \dfrac{\pi}{6}\right)^2$
$- \dfrac{\sqrt{3}}{12}\left(x - \dfrac{\pi}{6}\right)^3 + \dfrac{1}{48}\left(x - \dfrac{\pi}{6}\right)^4$

(b) $|R_4(x)| \le \dfrac{1}{5!}\left(\dfrac{\pi}{6}\right)^5 \approx 0.000328$

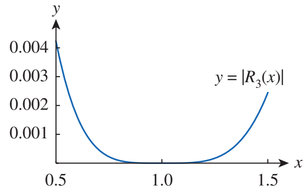
(c)



21. (a) $T_3(x) = \ln 3 + \frac{2}{3}(x-1) - \frac{2}{9}(x-1)^2 + \frac{8}{81}(x-1)^3$

(b) $|R_3(x)| \le \frac{6}{4!} \cdot \frac{1}{16} \approx 0.015625$

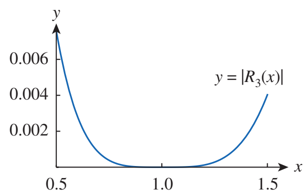
(c)



23. (a) $T_3(x) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$

(b) $|R_3(x)| \le \frac{1}{24} \approx 0.04167$

(c)



25. $\cos\left(\frac{4\pi}{9}\right) \approx 0.17365$

27. $n = 3$

29. $-1.037 < x < 1.037$

31. $-0.86 < x < 0.86$

33. $T_2(t)$ would not be accurate over a full minute since the car could not possibly maintain an acceleration of 2 m/s^2 for that long.

37. (c) Difference between two results: 0.00000000808 km

Chapter 8 Review ■ Page 771

True-False Quiz

1. False **3.** True **5.** False **7.** False **9.** False
11. True **13.** True **15.** False **17.** True **19.** True
21. True

Exercises

1. Convergent; $\frac{1}{2}$

3. Divergent

5. Convergent; 0

7. Convergent; e^{12}

11. Divergent

13. Convergent

15. Divergent

17. Convergent

19. Divergent

21. Convergent

23. $\frac{11}{18}$

25. $\cos\left(\frac{\sqrt{\pi}}{3}\right)$

27. $\frac{416{,}909}{99{,}900}$

29. 0.9721

31. 0.18976224; $R_8 < \frac{1/5^9}{1 - 1/5} = 6.4 \times 10^{-7}$

35. $R = 4$; $[-6, 2)$

37. $R = \frac{1}{2}$; $\left[\frac{5}{2}, \frac{7}{2}\right)$

39.
$$\sin x = \frac{1}{2}\left[1 - \frac{1}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{6}\right)^4 - \cdots\right] + \frac{\sqrt{3}}{2}\left[\left(x - \frac{\pi}{6}\right) - \frac{1}{3!}\left(x - \frac{\pi}{6}\right)^3 + \cdots\right]$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(x - \frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(x - \frac{\pi}{6}\right)^{2n+1}$$

41. $\sum_{n=0}^{\infty}(-1)^n x^{n+2}$; $R = 1$

43. $\ln 4 - \sum_{n=1}^{\infty}\frac{x^n}{n4^n}$; $R = 4$

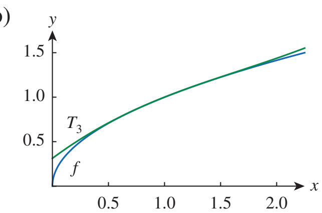
45. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+4}}{(2n+1)!}$; $R = \infty$

47. $\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1 \cdot 5 \cdot 9 \text{ cot} \cdot \cdots \cdot (4n-3)}{2^{6n+1}n!}x^n$; $R = 16$

49. $C + \ln|x| + \sum_{n=1}^{\infty}\frac{x^n}{n \cdot n!}$

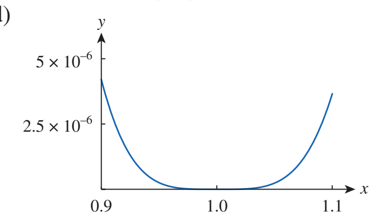
51. (a) $T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$

(b)



(c) $|R_3(x)| \le \frac{15}{16(0.9)^{7/2}4!}(0.1)^4 \approx 6 \times 10^{-6}$

(d)



53. $|R_3(x)| \le \frac{1}{6}$

55. $-\frac{1}{6}$

Focus on Problem Solving

1. 10,897,286,400

3. (b) $x \neq 0$: $-\cot x + \dfrac{1}{x}$; $x = 0$: 0

5. (a) $s_n = 3 \cdot 4^n$; $\ell_n = \left(\dfrac{1}{3}\right)^n$; $p_n = 3 \cdot \left(\dfrac{4}{3}\right)^n$

(c) $\dfrac{2\sqrt{3}}{5}$

11. $(-1, 1)$; $\dfrac{x^3 + 4x^2 + x}{(1 - x)^4}$

15. $\dfrac{1}{1 - 2^{1-p}}$

17. e

19. $\dfrac{\pi}{2\sqrt{3}} - 1$

25. $f_n = \dfrac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$

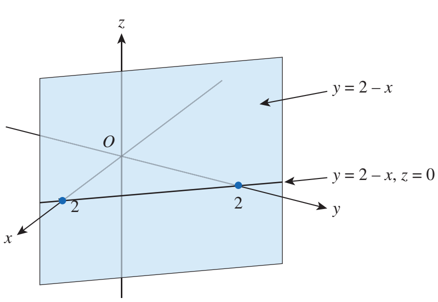
Chapter 9

Exercises 9.1 ■ Page 785

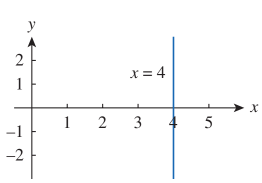
1. $(4, 0, -3)$

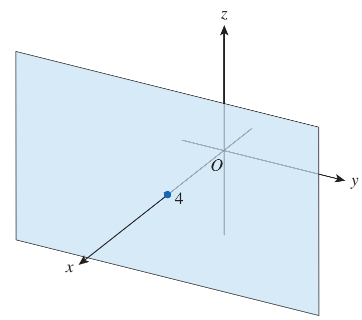
3. Q; R

5. A vertical plane that intersects the xy-plane in the line $y = 2 - x$, $z = 0$

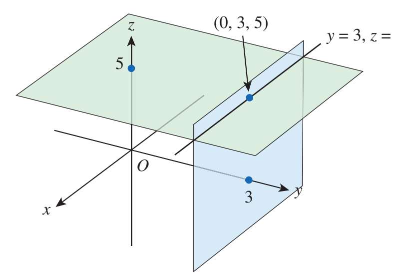


7. (a) $\mathbb{R}^2$: a line parallel to the y-axis;
$\mathbb{R}^3$: a vertical plane parallel to the yz-plane and 4 units in front of it.





(b) $y = 3$: a vertical plane parallel to the xz-plane, 3 units to the right of it; $z = 5$: a horizontal plane parallel to the xy-plane 5 units above it; $y = 3$, $z = 5$: the line of intersection of the planes $y = 3$, $z = 5$.



9. (a) 5 (b) 3 (c) 7 (d) $\sqrt{74} \approx 8.602$ (e) $\sqrt{34} \approx 5.831$
(f) $\sqrt{58} \approx 7.616$

11. $(x - 2)^2 + (y + 6)^2 + (z - 4)^2 = 25$; xy-plane: circle with center $(2, -6, 0)$, radius 3; does not intersect the xz-plane; yz-plane: circle with center $(0, -6, 4)$, radius $\sqrt{21}$

13. $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 25$

15. $(x + 4)^2 + (y - 3)^2 + (z + 1)^2 = 9$; center: $(-4, 3, -1)$; radius: 3

17. $x^2 + (y - 1)^2 + (z - 2)^2 = \dfrac{25}{3}$; center: $(0, 1, 2)$; radius: $\dfrac{5}{\sqrt{3}}$

19. $\left(x + \dfrac{1}{2}\right)^2 + \left(y + \dfrac{11}{7}\right)^2 + z^2 = 1$; center: $\left(-\dfrac{1}{2}, -\dfrac{11}{7}, 0\right)$; radius: 1

21. $(x - 3)^2 + (y - 2)^2 + (z - 7)^2 = 11$

23. $(x - 5)^2 + (y - 4)^2 + (z - 9)^2 = 16$

25. A plane parallel to the xz-plane, 2 units to the left of it

27. A half-space consisting of all points on or in front of the plane $x = -3$.

29. Two horizontal planes; $z = 1$ is parallel to the xy-plane, one unit above it, and $z = -1$ is one unit below it.

31. Circular cylinder with radius 4 whose axis is the x-axis

33. A plane perpendicular to the xz-plane and intersecting the xz-plane in the line $x = z$, $y = 0$

35. Set of points outside the sphere with radius 1 and center $(0, 0, 1)$.

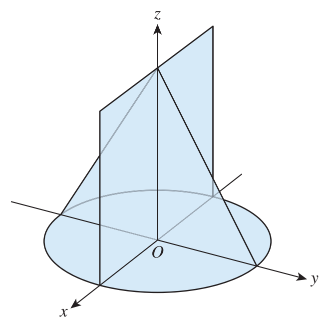
37. Set of points on the parabola $z = x^2$ for any value of y

39. $x^2 + y^2 \leq 4,\ 0 \leq z \leq 8$

41. $x^2 + y^2 + z^2 \leq 4,\ z \geq 0$

45. $\dfrac{11\pi}{12}$

47. Circular base in the xy-plane; vertical cross-section through the center of the base that is parallel to the xz-plane must be a square, and the vertical cross-section parallel to the yz-plane (perpendicular to the square) through the center of the base must be a triangle with two vertices on the circle and the third vertex at the center of the top side of the square.



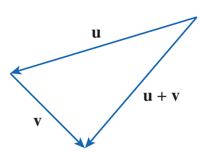
The solid can include any additional points that do not extend beyond these three silhouettes when viewed from directions parallel to the coordinate axes.

Exercises 9.2 ■ Page 795

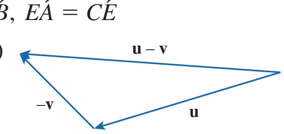
1. (a) Scalar (b) Vector (c) Vector (d) Scalar (e) Scalar (f) Vector

3. $\overrightarrow{AB} = \overrightarrow{DC},\ \overrightarrow{DA} = \overrightarrow{CB},\ \overrightarrow{DE} = \overrightarrow{EB},\ \overrightarrow{EA} = \overrightarrow{CE}$

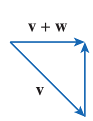
5. (a)



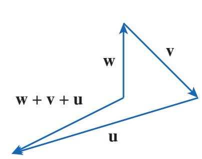
(b)



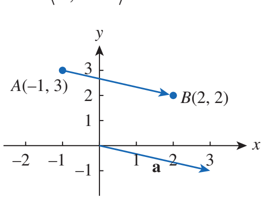
(c)



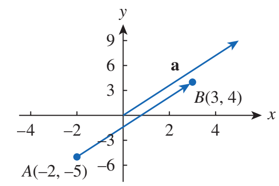
(d)



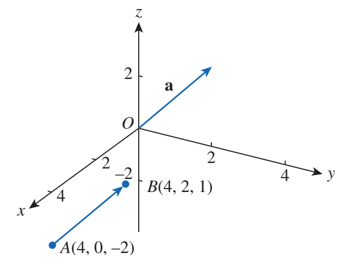
7. $\mathbf{a} = \langle 3, -1 \rangle$



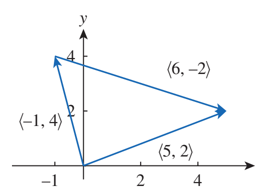
9. $\mathbf{a} = \langle 5, 9 \rangle$



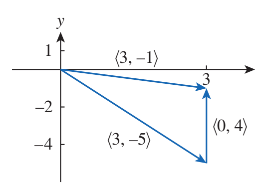
11. $\mathbf{a} = \langle 0, 2, 3 \rangle$



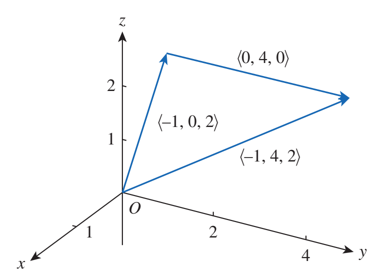
13. $\langle 5, 2 \rangle$



15. $\langle 3, -1 \rangle$



17. $\langle -1, 4, 2 \rangle$



19. $\langle 2, -18 \rangle$; $\langle 1, -42 \rangle$; 13; 10

21. $-\mathbf{i} + \mathbf{j} + 2\,\mathbf{k}$; $-4\,\mathbf{i} + \mathbf{j} + 9\,\mathbf{k}$; $\sqrt{14}$; $\sqrt{82}$

23. $\mathbf{i} + \mathbf{k}$; $5\,\mathbf{i} - 3\,\mathbf{j} + 4\,\mathbf{k}$; $\sqrt{14}$; $\sqrt{70}$

25. $\left\langle \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}} \right\rangle$

27. $\dfrac{8}{9}\,\mathbf{i} - \dfrac{1}{9}\,\mathbf{j} + \dfrac{4}{9}\,\mathbf{k}$

29. $\mathbf{v} = \langle 2, 2\sqrt{3}\rangle$

31. Horizontal component: $60 \cos 40° \approx 45.96$ ft/s; vertical component: $60 \sin 40° \approx 38.57$ ft/s

33. $|\mathbf{F}| = 100\sqrt{7} \approx 264.6$ N; $\theta \approx 139.1°$

35. $\sqrt{493} \approx 22.2$ mi/h; $\theta \approx 98°$; N8°W

37. $\mathbf{T}_1 = -196\,\mathbf{i} + 3.92\,\mathbf{j}$; $\mathbf{T}_2 = 196\,\mathbf{i} + 3.92\,\mathbf{j}$

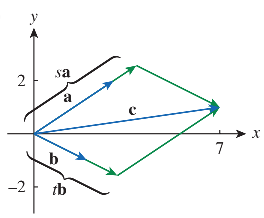
39. $\mathbf{T}_2 \approx -177.39\,\mathbf{i} + 211.41\,\mathbf{j}$, $|\mathbf{T}_2| \approx 275.97$ N
$\mathbf{T}_3 \approx 177.39\,\mathbf{i} + 138.59\,\mathbf{j}$, $|\mathbf{T}_3| \approx 225.11$ N

41. (a) $\theta \approx 0.757$ radians or 43.4° (b) 0.336 h or 20.2 m

43. $\pm\dfrac{1}{\sqrt{17}}(\mathbf{i} + 4\,\mathbf{j})$

45. $\mathbf{0}$

47. (a), (b)



(c) $s \approx 1.3$; $t \approx 1.6$; (d) $s = \dfrac{9}{7}$; $t = \dfrac{11}{7}$

49. $\mathbf{a} = \langle \cos 60°, \cos 72°, \sqrt{1 - (\cos 60°)^2 - (\cos 72°)^2} \approx \langle 0.50, 0.31, 0.81\rangle$

51. Sphere with radius 1 and center (x_0, y_0, z_0)

Exercises 9.3 ■ Page 803

1. (a) No meaning (b) Meaning (c) Meaning (d) Meaning (e) No meaning (f) No meaning

3. $-2000\sqrt{2} \approx -2828.43$

5. -24

7. 81

9. -47

11. $-pq$

13. 1

15. $\dfrac{1}{2}; -\dfrac{1}{2}$

19. Total tolls collected during the day

21. $\dfrac{\pi}{3}$ radians or 60°

23. $\cos^{-1}\left(-\dfrac{1}{2\sqrt{7}}\right) \approx 100.89°$

25. $\cos^{-1}\left(\dfrac{5}{2\sqrt{195}}\right) \approx 79.69°$

27. $\cos^{-1}\left(-\dfrac{2}{3\sqrt{70}}\right) \approx 94.57°$

29. 45°, 45°, 90°

31. $\cos^{-1}\left(-\dfrac{3}{\sqrt{102}}\right) \approx 107.28°$, $\cos^{-1}\left(\dfrac{20}{\sqrt{493}}\right) \approx 25.74°$, 46.98°

33. (a) Orthogonal (b) Orthogonal (c) Neither (d) Neither

35. (a) Orthogonal (b) Orthogonal (c) Neither

37. $b = 0, -2, 2$

39.
$$\mathbf{a} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k} \quad \text{or} \quad \mathbf{a} = -\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

41. $\dfrac{\pi}{4}$ radians or 45°

43. $3, \left\langle \dfrac{9}{5}, -\dfrac{12}{5}\right\rangle$

45. $4, \left\langle -\dfrac{20}{13}, \dfrac{48}{13}\right\rangle$

47. $\dfrac{1}{\sqrt{3}}, \dfrac{1}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

51. $\langle s, t, 3s - 2\sqrt{10}\rangle$, $s, t \in \mathbb{R}$

53. 144 joules

55. $2400 \cos 40° \approx 1838.51$ ft-lb

57. $\dfrac{13}{5}$

59. $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right) \approx 54.74°$

67. (a) The sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of its four sides.

Exercises 9.4 ■ Page 811

1. (a) Meaningful; scalar
(b) No meaning; cross product defined only for two vectors
(c) Meaningful; vector
(d) No meaning; $\mathbf{a} \cdot \mathbf{b}$ is a scalar
(e) No meaning; $(\mathbf{a} \cdot \mathbf{b})$ and $(\mathbf{c} \cdot \mathbf{d})$ are both scalars
(f) Meaningful; scalar

3. $96\sqrt{3}$; into the page

5. $10.8 \sin 80° \approx 10.6$ N · m

7. $\langle 15, -10, -3\rangle$

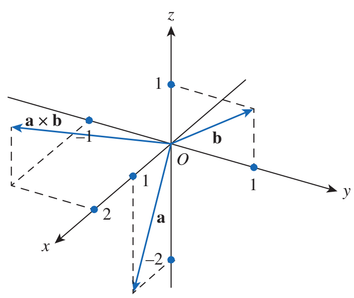
9. $\langle 0, 3, -2\rangle$

11. $-18\,\mathbf{j} - 18\,\mathbf{k}$

13. $\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j} + (-t \sin t - \cos t)\,\mathbf{k}$

15. $(1 - t)\,\mathbf{i} + (t^3 - t^2)\,\mathbf{k}$

17. $2\,\mathbf{i} - \mathbf{j} + \mathbf{k}$



19. $2\,\mathbf{i} + \mathbf{j}$

21. $-2\mathbf{k}$

23. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = 6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}$
$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = 8\,\mathbf{i} + 3\,\mathbf{j} - \mathbf{k}$

25. $\left\langle \dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, -\dfrac{2}{\sqrt{6}} \right\rangle; \left\langle -\dfrac{1}{\sqrt{6}}, -\dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}} \right\rangle$

27. 16

29. $\sqrt{265} \approx 16.279$

31. (a) Any scalar multiple of $\langle -4, 7, -10 \rangle$ (b) $\dfrac{1}{2}\sqrt{165}$

33. (a) Any scalar multiple of $\langle 1, -23, -13 \rangle$ (b) $\dfrac{1}{2}\sqrt{699}$

35. 9

37. 1

39. 16

41. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$

43. $|\mathbf{F}| \approx 417$ N

45. (a) 40 ft-lb (b) 12 ft-lb

47. (a) $\langle a, 2a - 5, a - 1 \rangle$
(b) Inconsistent system: no solution

49. (b) $\dfrac{17}{7}$

Exercises 9.5 ■ Page 823

1. (a) True (b) False (c) True (d) False (e) False
(f) True (g) False (h) True (i) True (j) False (k) True

3. $\mathbf{r} = (2 + 3t)\,\mathbf{i} + (2.4 + 2t)\,\mathbf{j} + (3.5 - t)\,\mathbf{k}$;
$x = 2 + 3t,\ y = 2.4 + 2t,\ z = 3.5 - t$

5. $\mathbf{r} = (1 + t)\,\mathbf{i} + 3t\,\mathbf{j} + (6 + t)\,\mathbf{k}$; $x = 1 + t,\ y = 3t,\ z = 6 + t$

7. $x = 2 + 2t,\ y = 1 + \dfrac{1}{2}t,\ z = -3 - 4t$;
$\dfrac{x - 2}{2} = 2y - 2 = \dfrac{z + 3}{-4}$

9. $x = -8 + 11t,\ y = 1 - 3t,\ z = 4$; $\dfrac{x + 8}{11} = \dfrac{y - 1}{-3},\ z = 4$

11. $x = 1 + t,\ y = -1 + 2t,\ z = 1 + t$; $x - 1 = \dfrac{y + 1}{2} = z - 1$

13. $x = 1 + 5t,\ y = 2t,\ z = -3t$; $\dfrac{x - 1}{5} = \dfrac{y}{2} = \dfrac{z}{-3}$

15. Not perpendicular

17. (a) $\dfrac{x - 1}{-1} = \dfrac{y + 5}{2} = \dfrac{z - 6}{-3}$
(b) xy-plane: $(-1, -1, 0)$; yz-plane: $(0, -3, 3)$;
xz-plane: $\left(-\dfrac{3}{2}, 0, -\dfrac{3}{2}\right)$

19. $\mathbf{r}(t) = (2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}) + t(2\,\mathbf{i} + 7\,\mathbf{j} - 3\,\mathbf{k}),\ 0 \le t \le 1$

21. $x = 10 - 5t,\ y = 3 + 3t,\ z = 1 - 4t,\ 0 \le t \le 1$

23. Parallel

25. Skew

27. $-2x + y + 5z = 1$

29. $y + 2z = -6$

31. $3x - 7z = -9$

33. $6x + 6y + 6z = 11$

35. $x + y + z = 2$

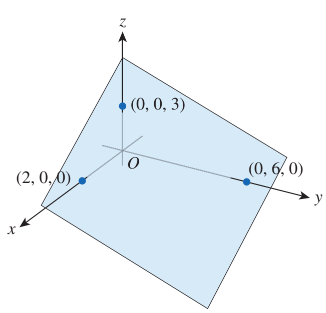
37. $3x + 2y - 5z = 0$

39. $5x - 4y - 9z = 0$

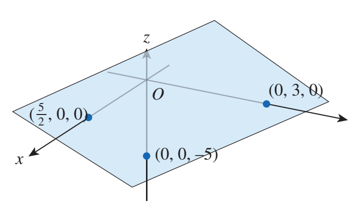
41. $6x - 22y - 29z = -101$

43. $x + y + z = 4$

45. (2, 0, 0), (0, 6, 0), (0, 0, 3)



47. $\left(\dfrac{5}{2}, 0, 0\right)$, (0, 3, 0), (0, 0, −5)



49. (7, −4, 3)

51. Perpendicular

53. Neither; $\theta = \cos^{-1}\left(\dfrac{1}{3}\right) \approx 70.53°$

55. Parallel

57. (a) $x = 1,\ y = -t,\ z = t$

(b) $\theta = \cos^{-1}\left(\dfrac{5}{3\sqrt{3}}\right) \approx 15.79°$

59. $x = 1,\ y - 2 = -z$

61. $4x + y + 2z = 2$

63. $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

65. $x = 3t,\ y = 1 - t,\ z = 2 - 2t$

67. P_1 and P_4 are parallel; P_2 and P_3 are parallel; P_1 and P_4 are identical

69. $\sqrt{\dfrac{61}{14}}$

71. $\dfrac{18}{7}$

73. $\dfrac{5\sqrt{14}}{28}$

77. $\dfrac{1}{\sqrt{6}}$

79. $\dfrac{13}{\sqrt{69}}$

Exercises 9.6 ■ Page 834

1. (a) 25; a 40-knot wind blowing for 15 hours will create waves with estimated heights of 25 feet

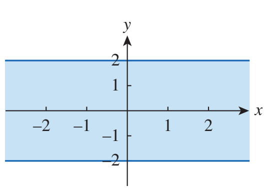
(b) Wave heights produced by 30-knot winds blowing for t hours; the function increases but at a declining rate as t increases

(c) Wave heights produced by winds of speed v blowing for 30 hours; the function appears to increase at an increasing rate

3. (a) 1; 1 (b) $\mathbb{R}^2$ (c) $[-1, 1]$

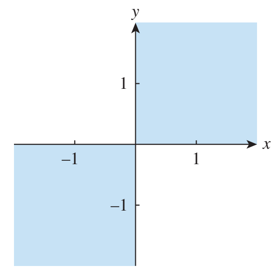
5. (a) $1 + \sqrt{3}$; 1

(b) Domain: $\{(x, y) \mid -2 \le y \le 2\}$

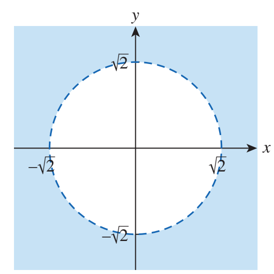


(c) Range: [1, 3]

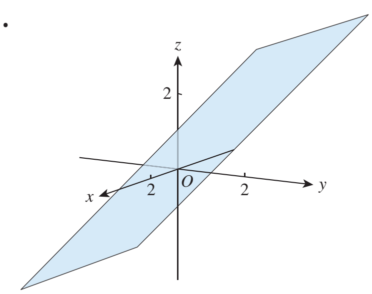
7. $D = \{(x, y) \mid xy \ge 0\}$



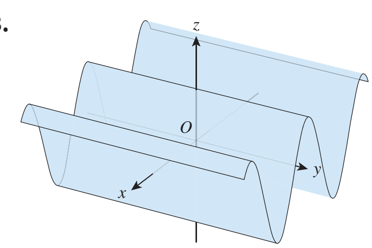
9. $D = \{(x, y) \mid x^2 + y^2 > 2\}$



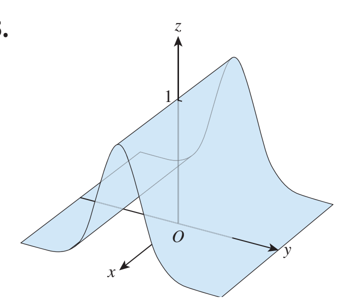
11.



13.

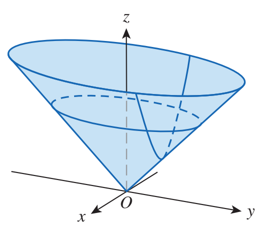


15.

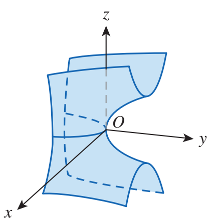


17. (a) VI (b) V (c) I (d) IV (e) II (f) III

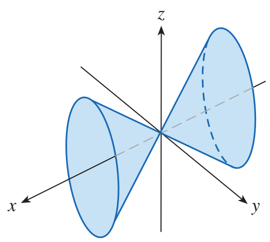
19.



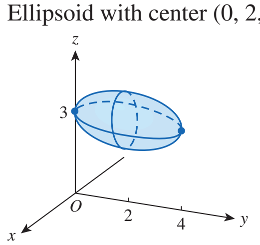
21.



23.



25. Ellipsoid with center (0, 2, 3)



27. (a) Circle or radius 1 centered at the origin

(b) Circular cylinder of radius 1, with axis the z-axis

(c) Circular cylinder of radius 1, with axis the y-axis

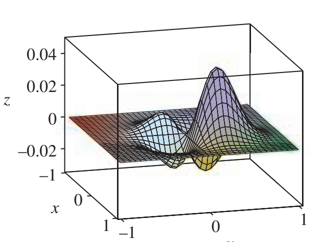
29. (a) $x = k$, $y^2 - z^2 = 1 - k^2$, hyperbola ($k \neq \pm 1$); $y = k$, $x^2 - z^2 = 1 - k^2$, hyperbola ($k \neq \pm 1$); $z = k$, $x^2 + y^2 = 1 + k^2$, circle

(b) The hyperboloid is rotated so that it has axis the y-axis

(c) The hyperboloid is shifted one unit in the negative y-direction

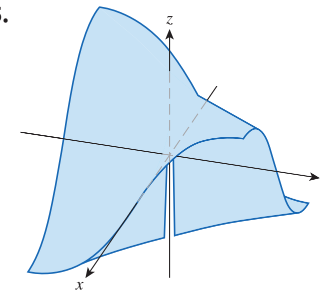
31. III

33.



Maximum value is approximately 0.444. There are two local maximum points and two local minimum points.

35.



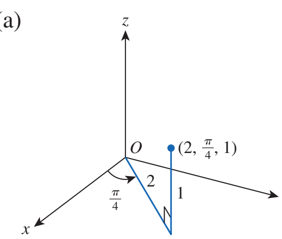
The graph exhibits different limiting values as x and y become large or as (x, y) approaches the origin, depending on the direction being examined.

39. $y^2 + z^2 = 4x^2$; right circular cone with vertex the origin and axis the x-axis

Exercises 9.7 ■ Page 841

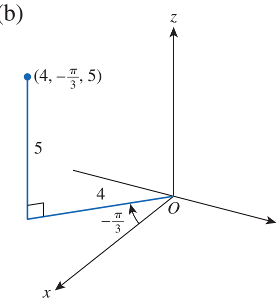
1. See pages 836–838

3. (a)



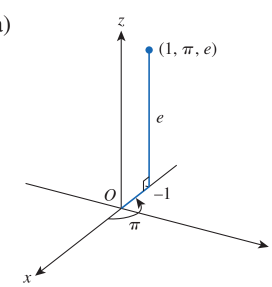
$(\sqrt{2}, \sqrt{2}, 1)$

(b)



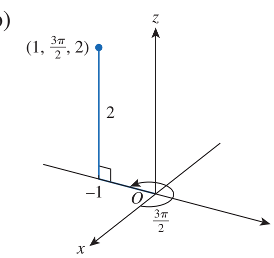
$(2, -2\sqrt{3}, 5)$

5. (a)



$(-1, 0, e)$

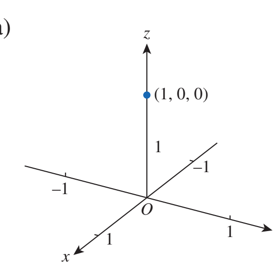
(b)



$(0, -1, 2)$

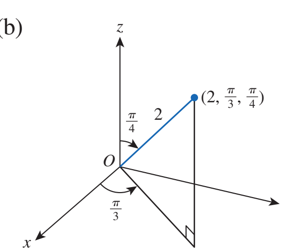
7. (a) $\left(4, \dfrac{\pi}{6}, -1\right)$ (b) $\left(5, \tan^{-1}\left(-\dfrac{3}{4}\right) + 2\pi, 2\right)$

9. (a)



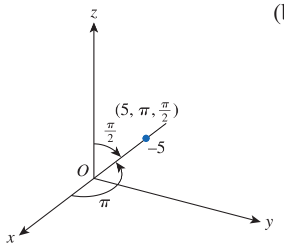
$(0, 0, 1)$

(b)



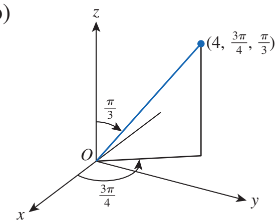
$\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{6}}{2}, \sqrt{2}\right)$

11. (a)



$(-5, 0, 0)$

(b)



$(-\sqrt{6}, \sqrt{6}, 2)$

13. (a) $\left(2, \frac{\pi}{2}, \frac{\pi}{3}\right)$ (b) $\left(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6}\right)$

15. Vertical half-plane including the z-axis

17. Circular paraboloid

19. Ellipsoid

21. Sphere with center the origin and radius 3

23. Sphere centered at $\left(0, \frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$

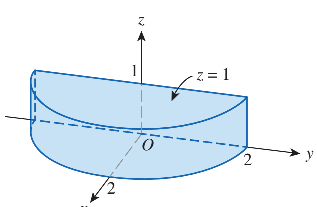
25. (a) $r = 2\sin\theta$

(b) $\rho \sin\phi = 2\sin\theta$

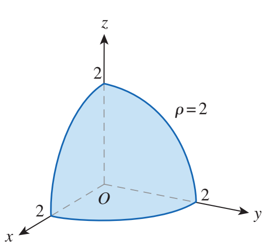
27. (a) $z = 6 - r(3\cos\theta + 2\sin\theta)$

(b) $\rho(3\sin\phi\cos\theta + 2\sin\phi\sin\theta + \cos\phi) = 6$

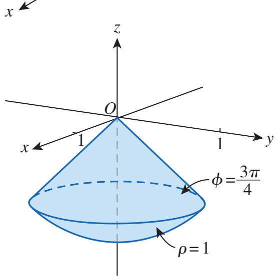
29.



31.



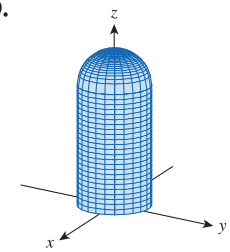
33.



35. Cylindrical coordinates: $6 \le r \le 7$, $0 \le \theta \le 2\pi$, $0 \le z \le 20$.

37. $0 \le \rho \le \cos\phi$, $0 \le \phi \le \frac{\pi}{4}$

39.



41. $s \approx 3960(0.6223) \approx 2464$ mi

Chapter 9 Review ■ Page 843

True-False Quiz

1. False **3.** True **5.** False **7.** False **9.** True
11. True **13.** True **15.** False **17.** False **19.** False
21. False **23.** True

Exercises

1. (a) $(x + 1)^2 + (y - 2)^2 + (z - 1)^2 = 69$

(b) $(y - 2)^2 + (z - 1)^2 = 68$, $x = 0$

(c) Center: $(4, -1, -3)$; radius: $\sqrt{5}$

3. $3\sqrt{2}$; $3\sqrt{2}$; directed out of the page

5. $x = -2, -4$

7. (a) 2 (b) -2 (c) -2 (d) 0

9. $\theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.53°$

11. (a) $\langle 4, -3, 4\rangle$ (b) $\frac{\sqrt{41}}{2}$

13. $F_1 \approx 166$ N; $F_2 \approx 114$ N

15. $x = 4 - 3t$, $y = -1 + 2t$, $z = 2 + 3t$

17. $x = -2 + 2t$, $y = 2 - t$, $z = 4 + 5t$

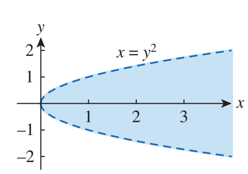
19. $-4x + 3y + z = -14$

21. $x + y + z = 4$

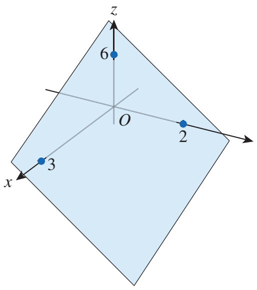
23. Skew

25. (a) $\frac{22}{\sqrt{26}}$ (b) $\frac{3}{\sqrt{2}}$

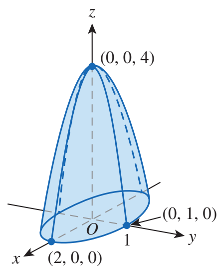
27. $D = \{(x, y) \mid x > y^2\}$



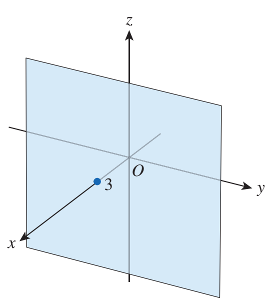
29.



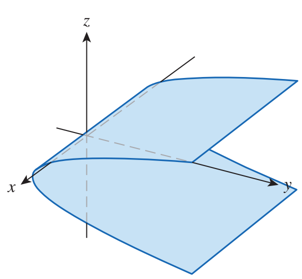
31.



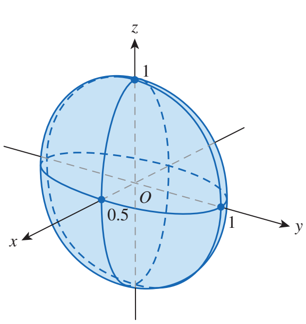
33. Plane parallel to the yz-plane, 3 units in front of it



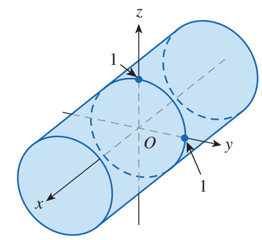
35. Parabolic cylinder whose trace in the xz-plane is the x-axis and which opens to the right



37. Ellipsoid centered at the origin, intercepts $\pm\frac{1}{2}$, ± 1, ± 1



39. Circular cylinder with axis the x-axis



41. $(\sqrt{3},\ 3,\ 2)$; $\left(4,\ \frac{\pi}{3},\ \frac{\pi}{3}\right)$

43. $(2\sqrt{2},\ 2\sqrt{2},\ 4\sqrt{3})$; $\left(4,\ \frac{\pi}{4},\ 4\sqrt{3}\right)$

45. $\rho^2 = 4$

47. $z = 4(x^2 + y^2) = 4r^2$

Focus on Problem Solving

1. $r = \dfrac{2\sqrt{3} - 3}{2}$

3. (a) $\dfrac{x+1}{-2c} = \dfrac{y-c}{c^2-1} = \dfrac{z-c}{c^2+1}$

(b) $x^2 + y^2 = t^2 + 1,\ z = t$

(c) $\dfrac{4\pi}{3}$

5. 20

Chapter 10

Exercises 10.1 ■ Page 856

1. $(-1,\ 2]$

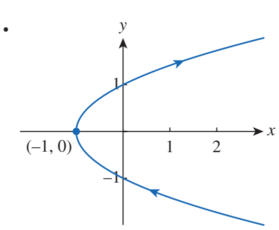
3. $(-3,\ -2) \cup (-2,\ 3)$

5. $(0,\ 2) \cup (2,\ \infty)$

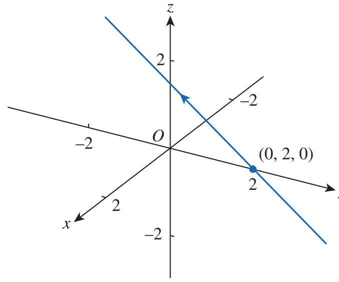
7. $\left\langle 1,\ \frac{1}{2},\ 3 \right\rangle$

9. $\mathbf{i} + 3\mathbf{j} - \pi\mathbf{k}$

11.



13.



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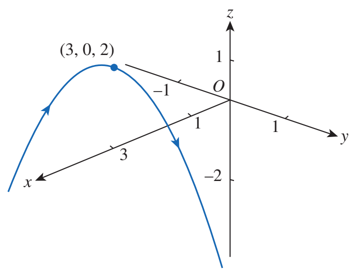
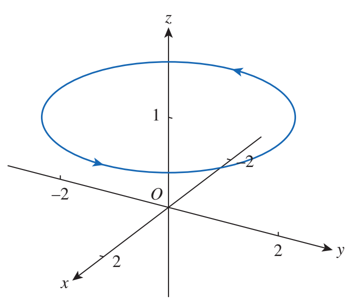


17.



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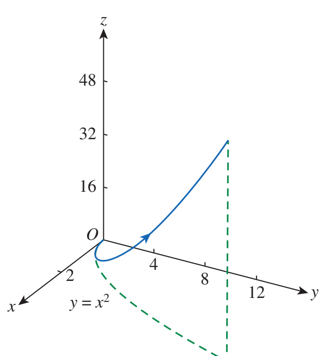

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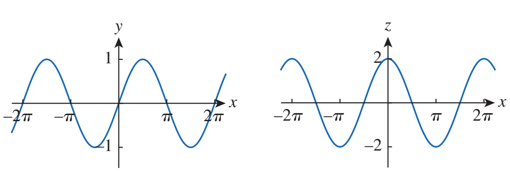
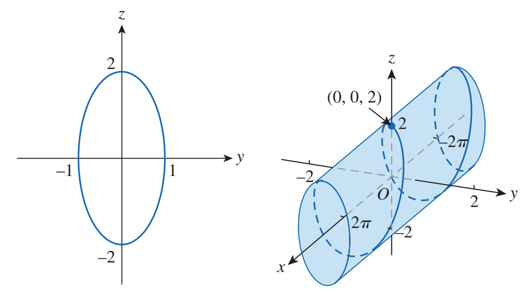




23. $\mathbf{r}(t) = \langle t,\ 2t,\ 3t \rangle,\ 0 \le t \le 1;\ x = t,\ y = 2t,\ z = 3t,\ 0 \le t \le 1$

25. $\mathbf{r}(t) = \langle 1 + 3t,\ -1 + 2t,\ 2 + 5t \rangle,\ 0 \le t \le 1;$
$x = 1 + 3t,\ y = -1 + 2t,\ z = 2 + 5t,\ 0 \le t \le 1$

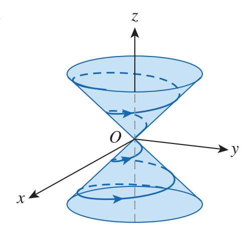
27. $\mathbf{r}(t) = \langle -1 - 2t,\ 2 + 3t,\ -2 + 3t \rangle,\ 0 \le t \le 1;$
$x = -1 - 2t,\ y = 2 + 3t,\ z = -2 + 3t,\ 0 \le t \le 1$

29. II

31. V

33. IV

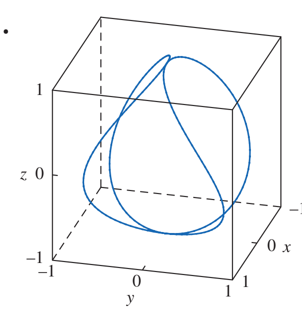
35.



37. $y = e^{x/2};\ z = e^{x};\ z = y^2$

39. $(0, 0, 0),\ (1, 0, 1)$

41.



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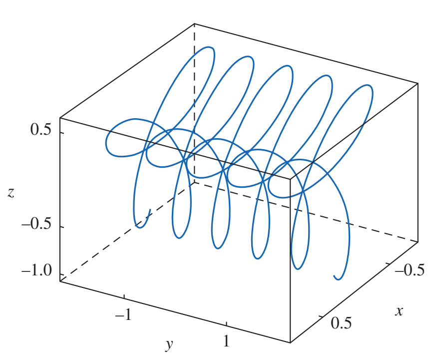

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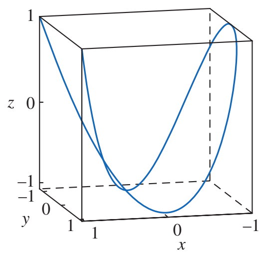

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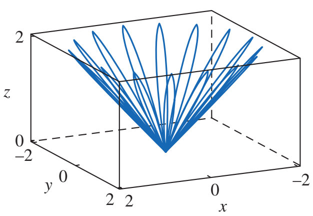


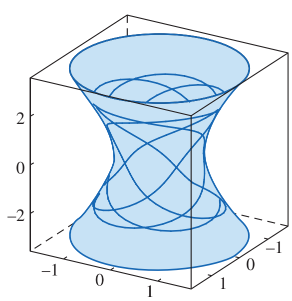
51. $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2 - 1)\,\mathbf{j} + \frac{1}{2}(t^2 + 1)\,\mathbf{k}$

53. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k},\ 0 \le t \le 2\pi$

55. $x = 2\cos t,\ y = 2\sin t,\ z = 4\cos^2 t,\ 0 \le t \le 2\pi$

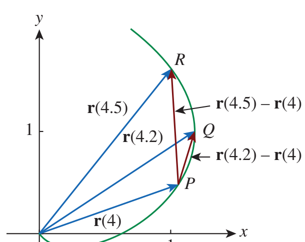
57. yes

59.

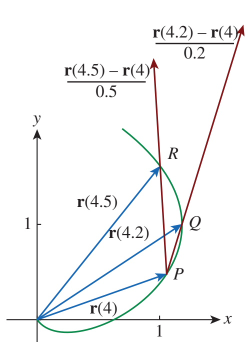


Exercises 10.2 ■ Page 864

1. (a)

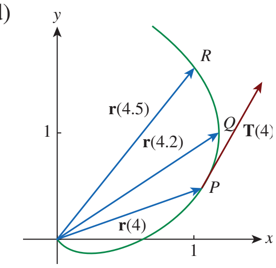


(b)

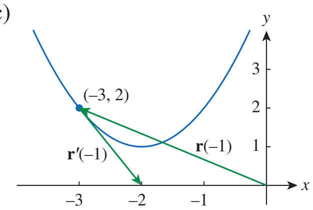


(c) $\mathbf{r}'(4) = \lim_{h\to 0} \dfrac{\mathbf{r}(4+h) - \mathbf{r}(4)}{h}$; $\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$

(d)

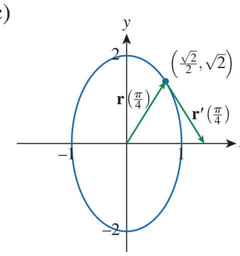


3. (a), (c)



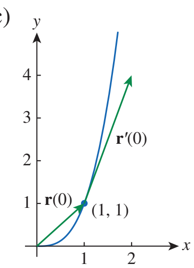
(b) $\mathbf{r}'(t) = \langle 1, 2t\rangle$; $\mathbf{r}'(-1) = \langle 1, -2\rangle$

5. (a), (c)



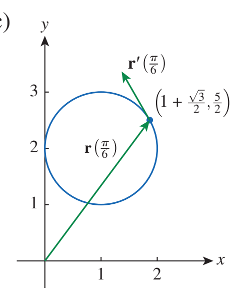
(b) $\mathbf{r}'(t) = \cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$; $\mathbf{r}'\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}\,\mathbf{i} - \sqrt{2}\,\mathbf{j}$

7. (a), (c)



(b) $\mathbf{r}'(t) = e^t\mathbf{i} + 3e^{3t}\mathbf{j}$; $\mathbf{r}'(0) = \mathbf{i} + 3\,\mathbf{j}$

9. (a), (c)



(b) $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j};\ \mathbf{r}'\left(\frac{\pi}{6}\right) = -\frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}$

11. $\mathbf{r}'(t) = \langle -e^{-t},\ 1 - 3t^2,\ 1/t\rangle$

13. $\mathbf{r}'(t) = \langle \sin t + t\cos t,\ 2t\cos t - t^2\sin t,\ 3t^2\rangle$

15. $\mathbf{r}'(t) = 2t\,\mathbf{i} - 2t\sin(t^2)\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k}$

17. $\mathbf{r}'(t) = (t\cos t + \sin t)\,\mathbf{i} + e^t(\cos t - \sin t)\,\mathbf{j} + (\cos^2 t - \sin^2 t)\,\mathbf{k}$

19. $\mathbf{r}'(t) = \mathbf{b} + 2t\,\mathbf{c}$

21. $\mathbf{T}(0) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$

23. $\mathbf{T}(0) = \frac{3}{5}\,\mathbf{j} + \frac{4}{5}\,\mathbf{k}$

25. $\mathbf{T}(1) = \frac{2}{\sqrt{5}}\,\mathbf{j} - \frac{1}{\sqrt{5}}\,\mathbf{k}$

27. $\mathbf{r}'(t) = \langle 1,\ e^t,\ (t+1)e^t\rangle;\ \mathbf{T}(0) = \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle;$

$\mathbf{r}''(t) = \langle 0,\ e^t,\ (t+2)e^t\rangle;\ \mathbf{r}'(t)\cdot\mathbf{r}''(t) = (t^2 + 3t + 3)e^{2t}$

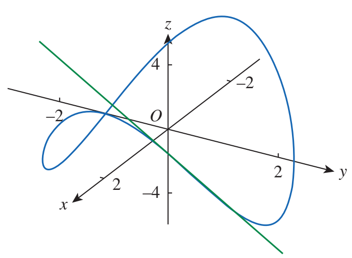
29. $x = 3 + t,\ y = 2t,\ z = 2 + 4t$

31. $x = 1 - t,\ y = t,\ z = 1 - t$

33. $x = -\pi - t,\ y = \pi + t,\ z = -\pi t$

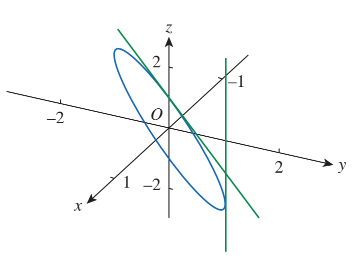
35. $(\sqrt{3},\ 1,\ e^{\pi/6})$

37. $x = \sqrt{3} - t,\ y = 1 + \sqrt{3}t,\ z = 2 - 4\sqrt{3}t$



39. (a) $(1, 2, 1)$

(b)



41. $(1, 0, 4);\ \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74°$

43. $\frac{124}{5}\,\mathbf{i} + \frac{256}{15}\,\mathbf{k}$

45. $\mathbf{i} + \mathbf{j} + \mathbf{k}$

47. $\tan t\,\mathbf{i} + \frac{1}{8}(t^2 + 1)^4\,\mathbf{j} + \left(\frac{1}{3}t^3\ln t - \frac{1}{9}t^3\right)\mathbf{k} + \mathbf{C}$

49. $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} + \left(\frac{2}{3}t^{3/2} - \frac{2}{3}\right)\mathbf{k}$

55. $2t\cos t + 2\sin t - 2\cos t\sin t$

57. $f'(2) = 35$

Exercises 10.3 ■ Page 873

1. $20\sqrt{29}$

3. $e - e^{-1}$

5. $\frac{1}{27}(13^{3/2} - 8)$

7. e^2

9. $L \approx 1.8581$

11. $L \approx 12.0909$

13. 42

15. (a) $s(t) = \sqrt{29}t$

$\mathbf{r}(t(s)) = \frac{2}{\sqrt{29}}s\,\mathbf{i} + \left(1 - \frac{3}{\sqrt{29}}s\right)\mathbf{j} + \left(5 + \frac{4}{\sqrt{29}}s\right)\mathbf{k}$

(b) $\left(\frac{6}{\sqrt{29}},\ 1 - \frac{9}{\sqrt{29}},\ 5 + \frac{12}{\sqrt{29}}\right)$

17. $(3\sin 1,\ 4,\ 3\cos 1)$

19. (a) $\mathbf{T}(t) = \frac{1}{\sqrt{29}}\langle 2\cos t,\ 5,\ -2\sin t\rangle$

$\mathbf{N}(t) = \langle -\sin t,\ 0,\ -\cos t\rangle$

(b) $\frac{2}{29}$

21. (a) $\mathbf{T}(t) = \frac{1}{e^{2t} + 1}\langle \sqrt{2}e^t,\ e^{2t},\ -1\rangle$

$\mathbf{N}(t) = \frac{1}{e^{2t} + 1}\langle 1 - e^{2t},\ \sqrt{2}e^t,\ \sqrt{2}e^t\rangle$

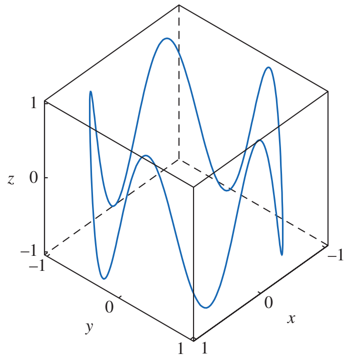
(b) $\frac{\sqrt{2}e^{2t}}{(e^{2t} + 1)^2}$

23. $\frac{6t^2}{(9t^4 + 4t^2)^{3/2}}$

25. $\frac{\sqrt{6}}{2(3t^2 + 1)^2}$

27. $\frac{\sqrt{30}}{18}$

29.



$\kappa(0) = \dfrac{1}{26}$

31. $\kappa(x) = \dfrac{2\sec^2 x\,|\tan x|}{(1 + \sec^4 x)^{3/2}}$

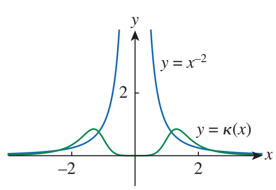
33. $\kappa(x) = \dfrac{|2\ln(x) + 3|}{(1 + (x + 2x\ln(x))^2)^{3/2}}$

35. $\left(-\dfrac{1}{2}\ln 2, \dfrac{1}{\sqrt{2}}\right)$; as $x \to \infty$, $\kappa(x) \to 0$

37. (a) C appears to be changing direction more quickly at P than Q, so we expect the curvature to be greater at P.

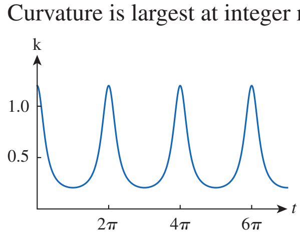
(b) P: $\kappa \approx 1.3$; Q: $\kappa \approx 0.7$

39.



41. a: $y = f(x)$; b: $y = \kappa(x)$

43. Curvature is largest at integer multiples of 2π.



45. $\dfrac{6t^2}{(4t^2 + 9t^4)^{3/2}}$

47. $\dfrac{1}{\sqrt{2}e^t}$

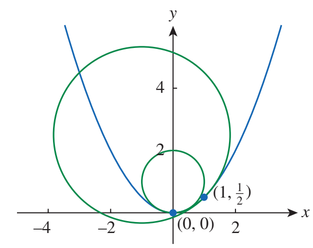
49. $\mathbf{T}(1) = \left\langle \dfrac{2}{3}, \dfrac{2}{3}, \dfrac{1}{3} \right\rangle$; $\mathbf{N}(1) = \left\langle -\dfrac{1}{3}, \dfrac{2}{3}, -\dfrac{2}{3} \right\rangle$;

$\mathbf{B}(1) = \left\langle -\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{2}{3} \right\rangle$

51. $y - 6x = \pi$; $x + 6y = 6\pi$

53. $x + 2y + 2x = 6$; $2x - 2y + z = -3$

55. $x^2 + (y - 1)^2 = 1$; $(x + 1)^2 + \left(y - \dfrac{5}{2}\right)^2 = 8$



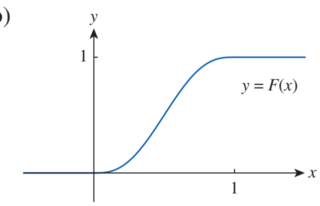
57. No such osculating plane.

59. The curve itself lies in that same plane.

61. $13x - 3y - 4\sqrt{2}z = \sqrt{2}$

69. (a) $P(x) = 6x^5 - 15x^4 + 10x^3$

(b)

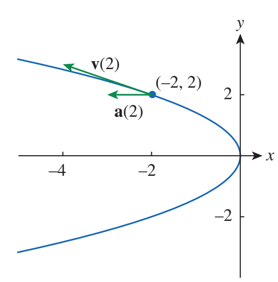


Exercises 10.4 ■ Page 884

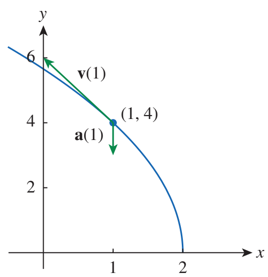
1. (a) $1.8\,\mathbf{i} - 3.8\,\mathbf{j} - 0.7\,\mathbf{k}$; $2.0\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}$; $2.8\,\mathbf{i} + 1.8\,\mathbf{j} - 0.3\,\mathbf{k}$; $2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k}$

(b) $\mathbf{v}(1) \approx 2.4\,\mathbf{i} - 0.8\,\mathbf{j} - 0.5\,\mathbf{k}$; $|\mathbf{v}(1)| \approx 2.58$

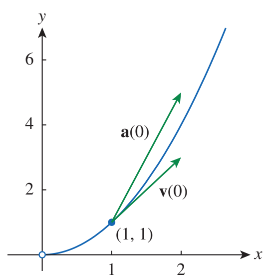
3. $\mathbf{v}(t) = \langle -t, 1 \rangle$; $\mathbf{v}(2) = \langle -2, 1 \rangle$; $\mathbf{a}(t) = \langle -1, 0 \rangle$; $\mathbf{a}(2) = \langle -1, 0 \rangle$; $|\mathbf{v}(t)| = \sqrt{t^2 + 1}$



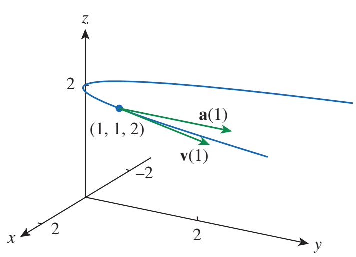
5. $\mathbf{v}(t) = \langle -1, 2/\sqrt{t} \rangle$; $\mathbf{v}(1) = \langle -1, 2 \rangle$; $\mathbf{a}(t) = \langle 0, -1/t^{3/2} \rangle$; $\mathbf{a}(1) = \langle 0, -1 \rangle\, |\mathbf{v}(t)| = \sqrt{1 + 4/t}$



7. $\mathbf{v}(t) = e^t\,\mathbf{i} + 2e^{2t}\,\mathbf{j}$; $\mathbf{v}(0) = \mathbf{i} + 2\,\mathbf{j}$; $\mathbf{a}(t) = e^t\,\mathbf{i} + 4e^{2t}\,\mathbf{j}$;
$\mathbf{a}(0) = \mathbf{i} + 4\,\mathbf{j}$; $|\mathbf{v}(t)| = e^t\sqrt{1 + 4e^{2t}}$



9. $\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j}$; $\mathbf{v}(1) = \mathbf{i} + 2\,\mathbf{j}$; $\mathbf{a}(t) = 2\,\mathbf{j}$;
$\mathbf{a}(1) = 2\,\mathbf{j}$; $|\mathbf{v}(t)| = \sqrt{1 + 4t^2}$



11. $\mathbf{v}(t) = \langle 2t + 1,\ 2t - 1,\ 3t^2 \rangle$; $\mathbf{a}(t) = \langle 2,\ 2,\ 6t \rangle$;
$|\mathbf{v}(t)| = \sqrt{9t^4 + 8t^2 + 2}$

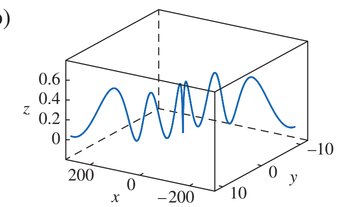
13. $\mathbf{v}(t) = \sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}$;
$\mathbf{a}(t) = e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$;
$|\mathbf{v}(t)| = e^t + e^{-t}$

15. $\mathbf{v}(t) = e^t\langle \cos t - \sin t,\ \sin t + \cos t,\ t + 1 \rangle$;
$\mathbf{a}(t) = e^t\langle -2\sin t,\ 2\cos t,\ t + 2 \rangle$;
$|\mathbf{v}(t)| = e^t\sqrt{t^2 + 2t + 3}$

17. $\mathbf{v}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + \mathbf{k}$;
$\mathbf{r}(t) = \left(\frac{1}{2}t^2 + 1\right)\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k}$

19. $\mathbf{v}(t) = (2t + 1)\,\mathbf{i} + 3t^2\,\mathbf{j} + 4t^3\,\mathbf{k}$;
$\mathbf{r}(t) = (t^2 + t)\,\mathbf{i} + (t^3 + 1)\,\mathbf{j} + (t^4 - 1)\,\mathbf{k}$

21. (a)
$\mathbf{r}(t) = \left(\frac{1}{3}t^3 + t\right)\mathbf{i} + (t - \sin t + 1)\,\mathbf{j} + \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right)\mathbf{k}$
(b)



23. $t = 4$

25. $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k}$; $|\mathbf{v}(t)| = \sqrt{25t^2 + 2}$

27. (a) $100\left(\frac{100\sqrt{3}}{4.9}\right) \approx 3534.8$ m

(b) $100\sqrt{3}\left(\frac{100\sqrt{3}}{9.8}\right) - 4.9\left(\frac{100\sqrt{3}}{9.8}\right)^2 \approx 1530.61$ m

(c) 200 m/s

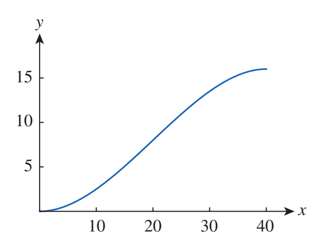
29. $v_0 = \sqrt{90g} \approx 29.7$ m/s

31. $\alpha \approx 10.2°,\ 79.8°$

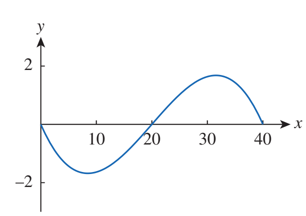
33. $13.0° < \theta < 36.0°,\ 55.4° < \theta < 85.5°$

35. $(250,\ -50,\ 0)$: $|\mathbf{v}(t)| = \sqrt{9300} \approx 96.44$ ft/s

37. (a) 16 m downstream



(b) $\alpha = \sin^{-1}\left(-\frac{2}{5}\right) \approx -23.6°$; head 23.6° south of east (upstream)



39. The path must be contained in a circle that lies in a plane perpendicular to **c**, and the circle is centered on a line through the origin in the direction of **c**.

41. $a_T = \dfrac{18t^3 + 4t}{\sqrt{9t^4 + 4t^2}}$; $a_N = \dfrac{6t^2}{\sqrt{9t^4 + 4t^2}}$

43. $a_T = 0$; $a_N = 1$

45. $a_T \approx 4.5 \text{ cm/s}^2$; $a_N \approx 9.0 \text{ cm/s}^2$

47. $t = 1$

Exercises 10.5 ■ Page 892

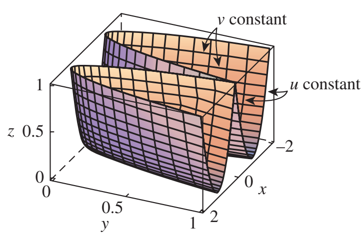
1. P does not lie on the surface; Q does lie on the surface.

3. P does not lie on the surface; Q does lie on the surface.

5. Portion of the elliptical cylinder $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ for $0 \le z \le 2$

7. Portion of the elliptical cylinder $\dfrac{x^2}{9} + z^2 = 1$ for $-1 \le y \le 1$

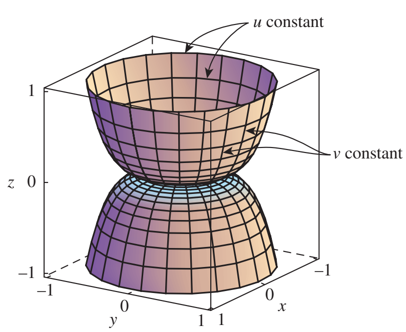
9.



$u = u_0 \quad \Rightarrow \quad y = u_0^2$: grid curves parallel to the xz-plane;

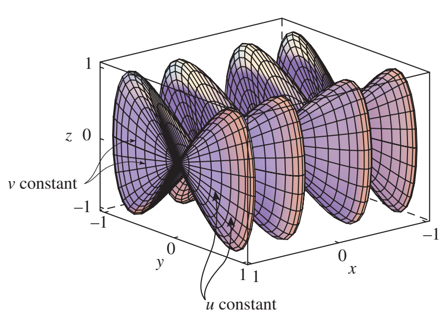
$v = v_0 \quad \Rightarrow \quad z = v_0^2$: grid curves parallel to the xy-plane

11.



$u = u_0 \quad \Rightarrow \quad z = u_0^5$: grid curves are circles parallel to the xy-plane; $v = v_0 \quad \Rightarrow \quad z = u^5$: grid curves lie in vertical planes $y = kx$ through the z-axis

13.



$u = u_0$: grid curves contained in the planes $z = ky$ that pass through the x-axis; $v = v_0$: vertically oriented grid curves shaped like a figure eight

15. IV

17. II

19. III

21. $x = 1 + u + v,\ y = 2 + u - v,\ z = -3 - u + v$

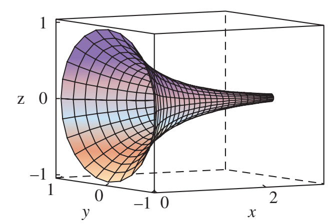
23. $x = x,\ z = z,\ y = \sqrt{1 - x^2 + z^2}$

25. $x = x,\ y = y,\ z = \sqrt{4 - x^2 - y^2}$ where $x^2 + y^2 \le 2$

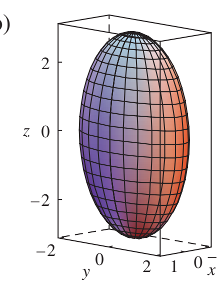
27. $x = x,\ y = 4 \cos\theta,\ z = 4 \sin\theta,\ 0 \le x \le 5,\ 0 \le \theta \le 2\pi$

29. $x = u,\ y = 3 \cos v,\ z = 3 \sin v,\ 0 \le u \le 5,\ \dfrac{\pi}{2} \le v \le \dfrac{3\pi}{2}$

31. $x = x,\ y = e^{-x} \cos\theta,\ z = e^{-x} \sin\theta,\ 0 \le x \le 3,\ 0 \le \theta \le 2\pi$



33. (b)



35. (a) The direction of the spiral is reversed.

(b) The number of coils in the surface doubles within the same parametric domain.

Chapter 10 Review ■ Page 894

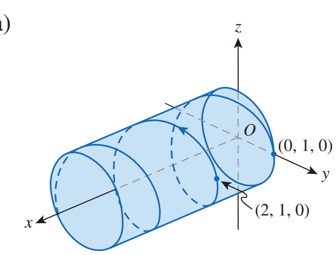
True-False Quiz

1. True **3.** False **5.** False **7.** False **9.** True

11. False **13.** True

Exercises

1. (a)



(b) $\mathbf{r}'(t) = \mathbf{i} - \pi \sin \pi t\, \mathbf{j} + \pi \cos \pi t\, \mathbf{k}$;
$\mathbf{r}''(t) = -\pi^2 \cos \pi t\, \mathbf{j} - \pi^2 \sin \pi t\, \mathbf{k}$

3. $\mathbf{r}(t) = 4 \cos t\, \mathbf{i} + 4 \sin t\, \mathbf{j} + (5 - 4 \cos t)\, \mathbf{k},\ 0 \le t \le 2\pi$

5. $\frac{1}{3}\mathbf{i} - \frac{2}{\pi^2}\mathbf{j} + \frac{2}{\pi}\mathbf{k}$

7. $L \approx 86.631$

9. $\theta = \frac{\pi}{2}$

11. (a) $\mathbf{T}(t) = \frac{1}{\sqrt{13}}\langle 3 \sin t, -3 \cos t, 2\rangle$

(b) $\mathbf{N}(t) = \langle \cos t, \sin t, 0\rangle$

(c) $\mathbf{B}(t) = \frac{1}{\sqrt{13}}\langle -2 \sin t, 2 \cos t, 3\rangle$

(d) $\kappa(t) = \frac{3}{13}\sec t \csc t$

13. $\kappa(1) = \frac{12}{17^{3/2}}$

15. $x - 2y + 2\pi = 0$

17. $\mathbf{v}(t) = (1 + \ln t)\,\mathbf{i} + \mathbf{j} - e^{-t}\,\mathbf{k}$;
$|\mathbf{v}(t)| = \sqrt{2 + 2 \ln t + (\ln t)^2 + e^{-2t}}$; $\mathbf{a}(t) = \frac{1}{t}\,\mathbf{i} + e^{-t}\,\mathbf{k}$

19. $\mathbf{r}(t) = (t^3 + t)\,\mathbf{i} + (t^4 - t)\,\mathbf{j} + (3t - t^3)\,\mathbf{k}$

21. $\alpha = \sin^{-1}\sqrt{0.3675} \approx 37.32°$; $d = \frac{40^2 \sin 2\alpha}{9.8} \approx 157.43$ m

23. $x = 2 \sin \phi \cos \theta,\ y = 2 \sin \phi \sin \theta,\ z = 2 \cos \phi,\ 1 \le \theta \le 2\pi,$
$\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$

25. $\kappa = \pi|t|$

27. (b) $P(x) = 3x^5 - 8x^4 + 6x^3$; this cannot be done with a polynomial of degree 4.

Focus on Problem Solving

1. (a) $\mathbf{v}(t) = -\omega R \sin \omega t\, \mathbf{i} + \omega R \cos \omega t\, \mathbf{j}$

(c) $\mathbf{a}(t) = -\omega^2 \mathbf{r}$

3. (a) $\alpha = \frac{\pi}{2}$; height $= \frac{v_0^2}{2g}$

5. (a) $2\sqrt{7/g}$ ft to the right of the table top;
$|\mathbf{v}| = \sqrt{4 + 7g} \approx 15.1$ ft/s

(b) $\theta = \operatorname{arccot}\frac{\sqrt{7g}}{2} \approx 7.61°$

(c) $2\sqrt{7/g} + 1.197 \approx 2.13$ ft to the right of the table top.

7. $\alpha \approx 0.9855$ or $\alpha \approx 56.47°$

9. $(a_2 - b_3)(x - c_1) + (a_3b_1 - a_1b_3)(y - c_2)$
$+ (a_1b_2 - a_2b_1)(z - c_3) = 0$

Chapter 11

Exercises 11.1 ■ Page 908

1. (a) $f(30, -20) = -53$; if the wind speed is 30 mph and the temperature is -20°F, then the air would feel equivalent to approximately -53°F without wind.

(b) When the temperature is 0°, what wind speed will give a wind-chill index of -31°F? 50 mph

(c) When the wind speed is 45 mph, what temperature will give a wind-chill of -37°F? -5°F

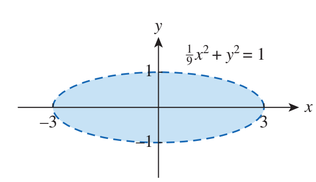
(d) This function gives the wind-chill index for different temperatures when the wind speed is 10 mph. The function decreases as temperature decreases.

(e) This function gives the wind-chill index for different wind speeds when the temperature is -15°F. The function decreases as wind speed increases, and appears to approach a constant value.

3. (b) True

5. (a) $f(160, 70) \approx 20.5$: the surface area of a person who weighs 160 pounds and is 70 in tall is approximately 20.5 square feet.

7. $\{(x, y) \mid 1/9x^2 + y^2 < 1\}$ range: $(-\infty, \ln 9]$



9. (a) $f(1, 1, 1) = 3$

(b) $\{(x, y, z) \mid x^2 + y^2 + z^2 < 4,\ x \ge 0,\ y \ge 0,\ z \ge 0\}$: the portion of the interior of a sphere of radius 2, centered at the origin, that is in the first octant.

11. (a) $C = f(x, y, z) = 8000 + 2.5x + 4y + 4.5z$ dollars

(b) 53,500: it costs \$53,500 to make 3000 small boxes, 5000 medium boxes, and 4000 large boxes.

(c) x, y, and z must be a positive integer or zero.

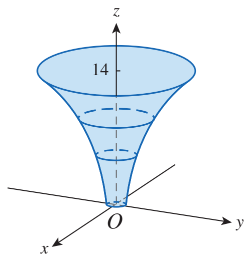
13. (a) B: 30.215; C: 30.03; N: 29.99; P: 29.95

(b) Portland: isobars are closer together.

15. D: 72; K: 62; S: 27

17. A: terrain is quite steep; B: much less steep, perhaps almost flat.

19.



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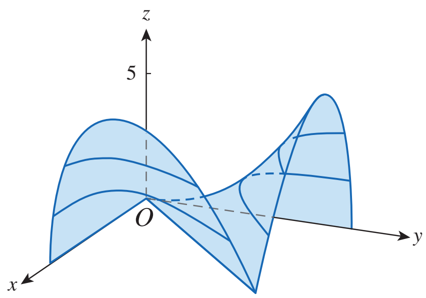

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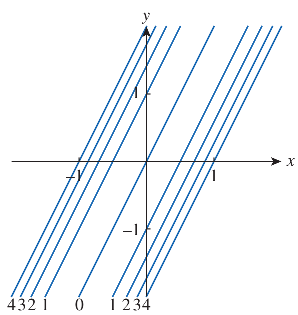

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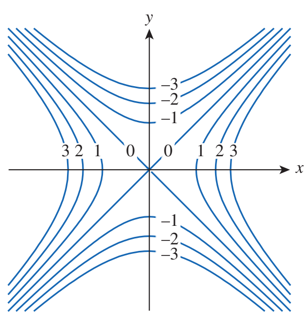

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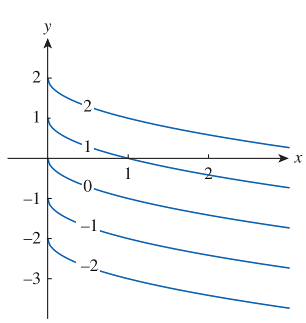

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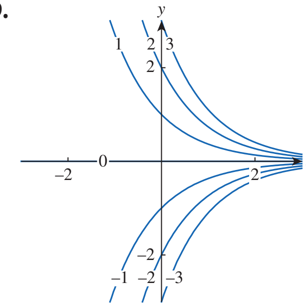

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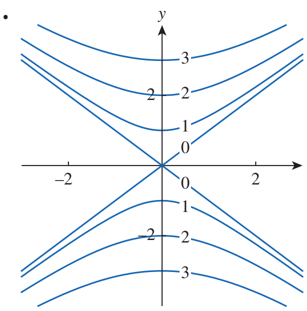

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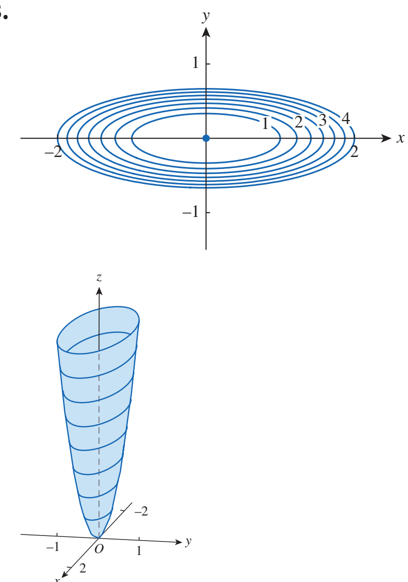

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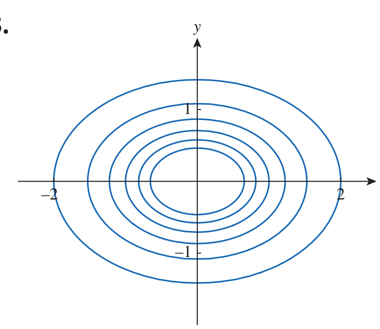

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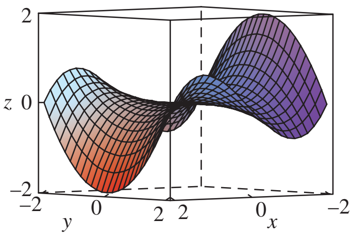

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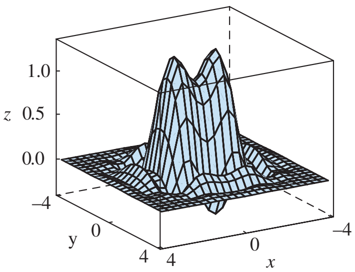


41. (a) C (b) II

43. (a) F (b) I

45. (a) B (b) VI

47. Family of parallel planes with normal vector $\langle 1, 3, 5\rangle$

49. $k > 0$: family of circular cylinders with axis the x-axis and radius $\sqrt{k}$: $k = 0$: x-axis; $k < 0$: no level surfaces

51. (a) Graph of f shifted upward 2 units

(b) Graph of f stretched vertically by a factor of 2

(c) Graph of f reflected about the xy-plane

(d) Graph of f reflected about the xy-plane and then shifted upward 2 units

53. If $c = 0$, the graph is a cylindrical surface. For $c > 0$, the level curves are ellipses. The graph curves upward as we leave the origin, and the steepness increases as c increases. For $c < 0$, the level curves are hyperbolas. The graph curves upward in the y-direction and downward approaching the xy-plane, in the x-direction giving saddle-shaped appearance near $(0, 0, 1)$.

55. The surface is an elliptic paraboloid for $0 < c < 2$, a parabolic cylinder for $c = 2$, and a hyperbolic paraboloid for $c > 2$. For $c < 0$, the graphs have the same shape but are reflected in the plane $x = 0$.

57. (b) $y = 0.75136x + 0.01053$

Exercises 11.2 ■ Page 920

1. Nothing; if f is continuous, $f(3, 1) = 6$

3. $-\dfrac{5}{2}$

5. 88

7. Does not exist

9. Does not exist

11. $\dfrac{\pi}{2}$

13. Does not exist

15. Does not exist

17. Does not exist

19. Does not exist

21. 2

23. 1

25. Does not exist

27. Does not exist

29. The graph shows that the function approaches numbers along different lines.

31. $h(x, y) = \dfrac{1 - xy}{1 + x^2y^2} + \ln\left(\dfrac{1 - xy}{1 + x^2y^2}\right)$; $\{(x, y) \mid xy < 1\}$

33. f is discontinuous on the circle $x^2 + y^2 = 1$

35. $\{(x, y) \mid y \leq x + 1\}$

37. $\mathbb{R}^2$

39. $\{(x, y) \mid x \neq 0, y \neq 0\}$

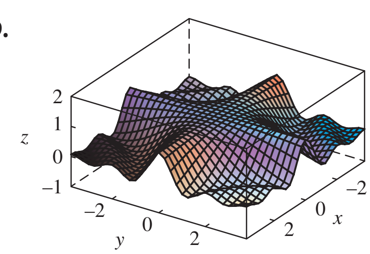
41. $\{(x, y, z) \mid x + y + z \geq 0\}$

43. $\{(x, y) \mid (x, y) \neq (0, 0)\}$

45. 0

47. 0

49.



f is continuous on $\mathbb{R}^2$

Exercises 11.3 ■ Page 932

1. (a) The rate of change of temperature as longitude varies, with latitude and time fixed; the rate of change as only latitude varies; the rate of change as only time varies

(b) Positive, negative, positive

3. (a) $f_v(20, -15) \approx -0.5$: for a wind speed of 20 mph and temperature of $-15°$F, the wind-chill index decreases by $-0.5°$ for each 1 mph increase in wind speed.

$f_T(20, -15) \approx 1.3$: for a wind speed of 20 mph and

temperature of $-15°$F, the wind-chill index increases by $1.3°$ for each degree the temperature increases.

(b) Negative, positive

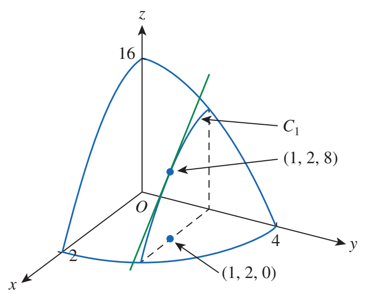
(c) 0

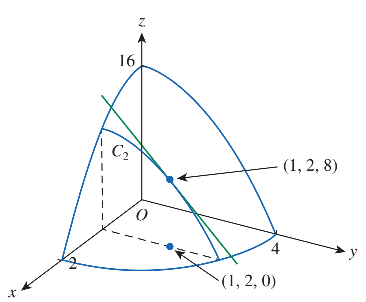
5. (a) Positive (b) Negative

7. (a) Positive (b) Negative

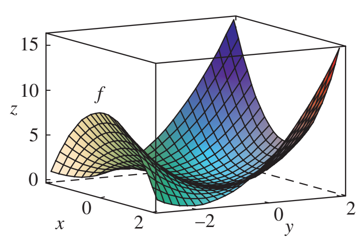
9. $b = f$, $a = f_x$, $c = f_y$

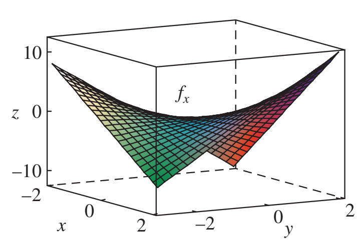
11. $f_x(1, 2) = -8 =$ slope of C_1, $f_y(1, 2) = -4 =$ slope of C_2

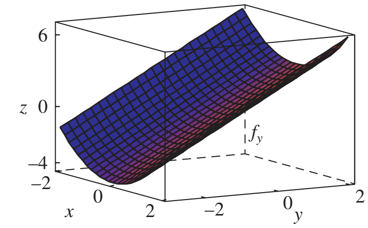




13. $f_x = 2x + 2xy$, $f_y = 2y + x^2$







15. $f_x(x, y) = -3y$, $f_y(x, y) = 5y^4 - 3x$

17. $f_x(x, t) = -\pi e^{-t} \sin \pi x$, $f_t(x, t) = -e^{-t} \cos \pi x$

19. $\dfrac{\partial z}{\partial x} = 20(2x + 3y)^9$, $\dfrac{\partial z}{\partial y} = 30(2x + 3y)^9$

21. $f_x(x, y) = \dfrac{2y}{(x + y)^2}$, $f_y(x, y) = -\dfrac{2x}{(x + y)^2}$

23. $\dfrac{\partial w}{\partial \alpha} = \cos \alpha \cos \beta$, $\dfrac{\partial w}{\partial \beta} = -\sin \alpha \sin \beta$

25. $f_x(x, y) = ye^{xy}(1 + xy)$, $f_y(x, y) = xe^{xy}(1 + xy)$

27. $f_r(r, s) = \dfrac{2r^2}{r^2 + s^2} + \ln(r^2 + s^2)$, $f_s(r, s) = \dfrac{2rs}{r^2 + s^2}$

29. $\dfrac{\partial u}{\partial t} = e^{w/t}\left(1 - \dfrac{w}{t}\right)$, $\dfrac{\partial u}{\partial w} = e^{w/t}$

31. $f_x(x, y) = \cos(y) \cos(x \cos(y))$,
$f_y(x, y) = -x \sin(y) \cos(x \cos(y))$

33. $f_x(x, y) = \cos(x^2)$, $f_y(x, y) = -\cos(y^2)$

35. $f_x(x, y, z) = z - 10xy^3z^4$, $f_y(x, y, z) = -15x^2y^2z^4$,
$f_z(x, y, z) = x - 20x^2y^3z^3$

37. $\dfrac{\partial w}{\partial x} = \dfrac{1}{x + 2y + 3z}$, $\dfrac{\partial w}{\partial y} = \dfrac{2}{x + 2y + 3z}$
$\dfrac{\partial w}{\partial z} = \dfrac{3}{x + 2y + 3x}$

39. $\dfrac{\partial u}{\partial x} = y \sin^{-1}(yz)$, $\dfrac{\partial u}{\partial y} = \dfrac{xyz}{\sqrt{1 - y^2z^2}} + x \sin^{-1}(yz)$,
$\dfrac{\partial u}{\partial z} = \dfrac{xy^2}{\sqrt{1 - y^2z^2}}$

41. $f_x(x, y, z, t) = yz^2 \tan(yt)$,
$f_y(x, y, z, t) = xyz^2 \sec^2(yt) + xz^2\tan(yt)$,
$f_z(x, y, z, t) = 2xyz \tan(yt)$, $f_t(x, y, z, t) = xy^2z^2 \sec^2(yt)$

43. $u_{x_i} = \dfrac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$

45. $\dfrac{1}{5}$

47. $-\dfrac{3}{13}$

49. $\dfrac{1}{4}$

51. 1

53. $f_x(x, y) = y^2 - 3x^2y,\ f_y(x, y) = 2xy - x^3$

55. $\dfrac{\partial z}{\partial x} = \dfrac{3yz - 2x}{2z - 3xy},\ \dfrac{\partial z}{\partial y} = \dfrac{3xz - 2y}{2z - 3xy}$

57. $\dfrac{\partial z}{\partial x} = \dfrac{1 + y^2z^2}{1 + y + y^2z^2},\ \dfrac{\partial z}{\partial y} = -\dfrac{z}{1 + y + y^2z^2}$

59. $\dfrac{\partial z}{\partial x} = \dfrac{yz}{e^z - xy},\ \dfrac{\partial z}{\partial y} = \dfrac{xz}{e^z - xy}$

61. (a) $\dfrac{\partial z}{\partial x} = f'(x),\ \dfrac{\partial z}{\partial y} = g'(y)$

(b) $\dfrac{\partial z}{\partial x} = f'(x + y),\ \dfrac{\partial z}{\partial y} = f'(x + y)$

63. $f_{xx}(x, y) = 6xy^5 + 24x^2y,\ f_{xy}(x, y) = 15x^2y^4 + 8x^3,$

$f_{yx}(x, y) = 15x^2y^4 + 8x^3,\ f_{yy}(x, y) = 20x^3y^3$

65. $f_{xx}(x, y) = -\dfrac{a^2}{(ax + by)^2},\ f_{xy}(x, y) = -\dfrac{ab}{(ax + by)^2},$

$f_{yx}(x, y) = -\dfrac{ab}{(ax + by)^2},\ f_{yy}(x, y) = -\dfrac{b^2}{(ax + by)^2}$

67. $w_{uu} = \dfrac{v^2}{(u^2 + v^2)^{3/2}},\ w_{uv} = -\dfrac{uv}{(u^2 + v^2)^{3/2}},$

$w_{vu} = -\dfrac{uv}{(u^2 + v^2)^{3/2}},\ w_{vv} = \dfrac{u^2}{(u^2 + v^2)^{3/2}}$

69. $z_{xx} = -\dfrac{2x}{(1 + x^2)^2},\ z_{xy} = 0,\ z_{yx} = 0,\ z_{yy} = -\dfrac{2y}{(1 + y^2)^2}$

71. $u_{xy} = u_{yx} = (x^2y + 2x)e^{xy}$

73. $u_y = u_{yx} = 2x\cos(x^2y) - 2x^3y\sin(x^2y)$

75. $f_{xxy} = 12xy,\ f_{yyy} = 72xy$

77. $f_{ttt} = -c^3x^2e^{-ct},\ f_{txx} = -2ce^{-ct}$

79. $f_{xyz} = e^x(\cos(yz) - yz\sin(yz))$

81. $\dfrac{\partial^3 V}{\partial r\,\partial s\,\partial t} = \dfrac{12st^2}{(r + s^2 + t^3)^3}$

83. $\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = ab(b - 1)c(c - 1)(c - 2)x^{a-1}y^{b-2}z^{c-3}$

85. 0

87. (a) Negative (b) Positive (c) Positive
(d) Negative (e) Positive

89. (a) No (b) Yes (c) No (d) Yes (e) Yes

97. (a) $-\dfrac{20}{3}$ (b) $-\dfrac{10}{3}$

99. (a) $\dfrac{\partial T}{\partial P} = \dfrac{V - nb}{nR}$

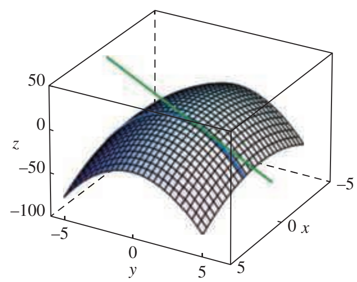
(b) $\dfrac{\partial P}{\partial V} = \dfrac{2n^2a}{V^3} - \dfrac{nRT}{(V - nb)^2}$

101. $-1.34°$F, $-0.43°$F

103. $\dfrac{\partial A}{\partial a} = \dfrac{a}{bc\sin A},\ \dfrac{\partial A}{\partial b} = \dfrac{c\cos A - b}{bc\sin A},$

$\dfrac{\partial A}{\partial c} = \dfrac{b\cos A - c}{bc\sin A}$

105. $x = 1,\ y = 2 + t,\ z = -4 - 8t$



107. (a) $\dfrac{\partial S}{\partial w}(160, 70) \approx 0.0545$: for a person 70 in. tall who weighs 160 pounds, an increase in weight (while height remains constant) causes the surface area to increase at a rate of about 0.0545 square feet per pound.

(b) $\dfrac{\partial S}{\partial h}(160, 70) \approx 0.213$: for a person 70 in. tall who weighs 160 pounds, an increase in height (while weight remains unchanged at 160 pounds) causes the surface area to increase at a rate of about 0.213 square feet per inch of height.

109. $\dfrac{\partial P}{\partial v} = 3Av^2 - \dfrac{B(mg/x)^2}{v^2}$: the rate of change of the power needed during flapping mode with respect to the bird's velocity when the mass and fraction of flapping time remain constant $\dfrac{\partial P}{\partial x} = -\dfrac{2Bm^2g^2}{x^3v}$: the rate at which the power changes with respect to the fraction of time spent in flapping mode when the mass and velocity are held constant $\dfrac{\partial P}{\partial m} = \dfrac{2Bmg^2}{x^2v}$: the rate of change of the power with respect to mass when the velocity and fraction of flapping time remain constant

111. $E_m(400, 8) \approx 0.301$: the average energy needed for a lizard to walk or run 1 km increases at a rate of about 0.301 kcal per gram of body mass increase from 400 g if the speed is 8 km/h

$E_v(400, 8) \approx -4.89$: the average energy needed by a lizard with body mass 400 g decreases at a rate of about 4.89 kcal per km/h when the speed increases from 8 km/h

113. 1

Exercises 11.4 ■ Page 945

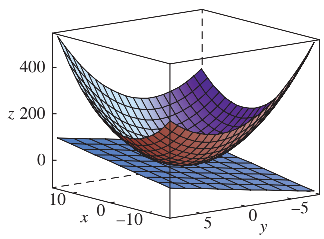
1. $z = -7x - 6y + 5$

3. $x + y - 2z = 0$

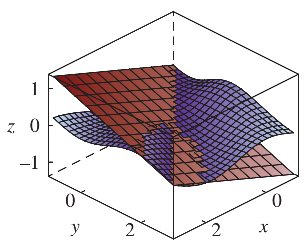
5. $z = x - y + 1$

7. $z = y$

9. Tangent plane: $z = 3x + 7y - 5$



11. Tangent plane: $z = \dfrac{1}{2}x + y - \dfrac{3}{2} + \dfrac{\pi}{4}$

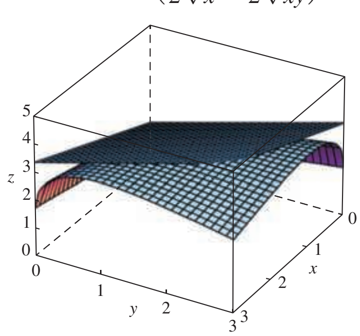


13. $f_x = -\dfrac{1}{10}ye^{-xy/10}(\sqrt{x} + \sqrt{y} + \sqrt{xy})$

$$+ e^{-xy/10}\left(\frac{1}{2\sqrt{x}} + \frac{y}{2\sqrt{xy}}\right)$$

$$f_y = -\frac{1}{10}ye^{-xy/10}(\sqrt{x} + \sqrt{y} + \sqrt{xy})$$

$$+ e^{-xy/10}\left(\frac{1}{2\sqrt{x}} + \frac{x}{2\sqrt{xy}}\right)$$



15. $3x + 4y - 6$

17. $\dfrac{1}{4}x + y + \dfrac{5}{4}$

19. $\dfrac{1}{3}x + y$

25. $f(x, y) \approx 2 - 3y - 1$, $f(6.9, 2.06) \approx -0.28$

27. $f(x, y) \approx 28 + 1.15(v - 40) + 0.45(t - 20)$, $f(43, 24) \approx 33.25$

29. $f(v, T) \approx -35 - 0.5(v - 20) + 1.3(T + 10)$, $f(22, -13) \approx 39.9$

31. $du = e^{-t}\cos(s + 2t)\, ds + e^{-t}[2\cos(s + 2t) - \sin(s + 2t)]\, dt$

33. $dT = -\dfrac{v^2w}{(1 + uvw)^2}\, du + \dfrac{1}{(1 + uvw)^2}\, dv - \dfrac{uv^2}{(1 + uvw)^2}\, dw$

35. $dw = (xz + 1)ye^{xz}\, dx + xe^{xz}\, dy + x^2ye^{xz}\, dz$

37. $du = \dfrac{3}{\sqrt{3x^2 + y^4}}dx + \dfrac{2y^3}{\sqrt{3x^2 + y^4}}dy$

39. $\Delta z = -0.7189$, $dz = -0.73$

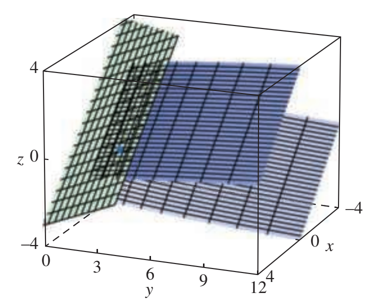
41. $152\ \text{cm}^2$

43. 0.303

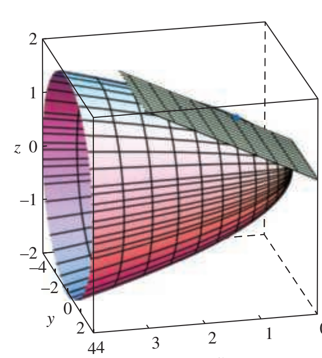
45. -8.83 kPa

47. $-0.165mg$, decreases

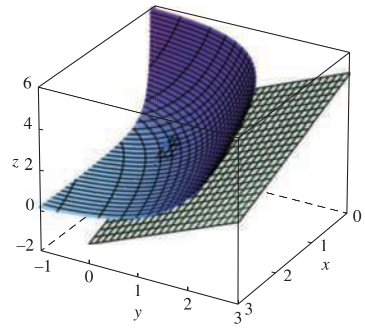
49. $3x - y + 3z = 3$



51. $-x + 2z = 1$



53. $x - y + z = 2$



55. $\epsilon_1 = \Delta x,\ \epsilon_2 = \Delta y$

Exercises 11.5 ■ Page 955

1. $\dfrac{dz}{dt} = 2t(y^3 - 2xy + 3xy^2 - x^2)$

3. $\dfrac{dz}{dt} = \dfrac{3\pi}{(x+2y)^2}\,(ye^{\pi t} + xe^{-\pi t})$

5. $\dfrac{dz}{dt} = \dfrac{1}{\sqrt{1+x^2+y^2}}\left(\dfrac{x}{t} - y\sin t\right)$

7. $\dfrac{dw}{dt} = e^{y/z}\left(2t - \dfrac{x}{z} - \dfrac{2xy}{z^2}\right)$

9. $\dfrac{\partial z}{\partial s} = 5(x-y)^4(2st - t^2);\ \dfrac{\partial z}{\partial t} = 5(x-y)^4(s^2 - 2st)$

11. $\dfrac{\partial z}{\partial s} = \dfrac{2s+2r}{\sqrt{1-(x-y)^2}};\ \dfrac{\partial z}{\partial t} = \dfrac{2s+2t}{\sqrt{1-(x-y)^2}};$

13. $\dfrac{\partial z}{\partial s} = t^2\cos\theta\cos\phi - 2st\sin\theta\sin\phi;$

$\dfrac{\partial z}{\partial t} = 2st\cos\theta\cos\phi - s^2\sin\theta\sin\phi$

15. $\dfrac{\partial z}{\partial s} = \left(yt\sqrt{x} + \dfrac{t}{2\sqrt{x}} + 2x^{3/2}s\right)e^{xy};$

$\dfrac{\partial z}{\partial t} = \left(ys\sqrt{x} + \dfrac{s}{2\sqrt{x}} - 2x^{3/2}t\right)e^{xy}$

17. $\dfrac{\partial z}{\partial s} = \dfrac{2v-3u}{v^2}\sec^2\!\left(\dfrac{u}{v}\right);\ \dfrac{\partial z}{\partial t} = \dfrac{2u+3v}{v^2}\sec^2\!\left(\dfrac{u}{v}\right)$

19. 42

21. $R_s(1, 2) = 32;\ R_t(1, 2) = -39$

23. $g_r(1, 2) = -24;\ g_s(1, 2) = 28$

25. $\dfrac{\partial w}{\partial x} = \dfrac{\partial w}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial w}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial w}{\partial t}\dfrac{\partial t}{\partial x};$

$\dfrac{\partial w}{\partial y} = \dfrac{\partial w}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial w}{\partial s}\dfrac{\partial s}{\partial y} + \dfrac{\partial w}{\partial t}\dfrac{\partial t}{\partial y}$

27. $\dfrac{dt}{dp} = \dfrac{\partial t}{\partial u}\dfrac{\partial u}{\partial p} + \dfrac{\partial t}{\partial v}\dfrac{\partial v}{\partial p} + \dfrac{\partial t}{\partial w}\dfrac{\partial w}{\partial p};$

$\dfrac{dt}{dq} = \dfrac{\partial t}{\partial u}\dfrac{\partial u}{\partial q} + \dfrac{\partial t}{\partial v}\dfrac{\partial v}{\partial q} + \dfrac{\partial t}{\partial w}\dfrac{\partial w}{\partial q};$

$\dfrac{dt}{dr} = \dfrac{\partial t}{\partial u}\dfrac{\partial u}{\partial r} + \dfrac{\partial t}{\partial v}\dfrac{\partial v}{\partial r} + \dfrac{\partial t}{\partial w}\dfrac{\partial w}{\partial r};$

$\dfrac{dt}{ds} = \dfrac{\partial t}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial t}{\partial v}\dfrac{\partial v}{\partial s} + \dfrac{\partial t}{\partial w}\dfrac{\partial w}{\partial s}$

29. $\dfrac{\partial u}{\partial x} = \dfrac{4}{\sqrt{10}},\ \dfrac{\partial u}{\partial y} = \dfrac{3}{\sqrt{10}},\ \dfrac{\partial u}{\partial t} = \dfrac{2}{\sqrt{10}}$

31. $\dfrac{\partial R}{\partial x} = \dfrac{9}{7},\ \dfrac{\partial R}{\partial y} = \dfrac{9}{7}$

33. $\dfrac{\partial u}{\partial p} = 36,\ \dfrac{\partial u}{\partial r} = 24,\ \dfrac{\partial u}{\partial \theta} = 30$

35. $\dfrac{dy}{dx} = \dfrac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}$

37. $\dfrac{dy}{dx} = \dfrac{1 + y - e^y\cos x}{e^y\sin x - x}$

39. $\dfrac{\partial z}{\partial x} = -\dfrac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)},\ \dfrac{\partial z}{\partial y} = -\dfrac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}$

41. $\dfrac{\partial z}{\partial x} = \dfrac{\ln y}{2z - y},\ \dfrac{\partial z}{\partial y} = \dfrac{x + yz}{2yz - y^2}$

43. $\dfrac{\partial z}{\partial x} = \dfrac{1}{y(x+z) - 1},\ \dfrac{\partial z}{\partial y} = -\dfrac{z(x+z)}{y(x+z) - 1}$

45. (a) Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.

(b) -1.1 units/year

47. 8160π in.^{3}/s

49. -0.000031 A/s

51. $\dfrac{dP}{dt} \approx -0.596$, production at that time is decreasing at a rate of about \$596,000 per year

53. 4.65 Hz/s

61. (a) $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$

(b) $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x}r\sin\theta + \dfrac{\partial z}{\partial y}r\cos\theta$

(c) $\dfrac{\partial^2 z}{\partial r\,\partial\theta} = \cos\theta\dfrac{\partial z}{\partial y} - \sin\theta\dfrac{\partial z}{\partial x}$

$+ r\cos\theta\sin\theta\left(\dfrac{\partial^2 z}{\partial y^2} - \dfrac{\partial^2 z}{\partial x^2}\right)$

$+ r(\cos^2\theta - \sin^2\theta)\dfrac{\partial^2 z}{\partial y\partial x}$

63. (b) $\dfrac{\partial^2 z}{\partial s \partial t} = \dfrac{\partial^2 z}{\partial x^2}\dfrac{\partial x}{\partial s}\dfrac{\partial x}{\partial t} + \dfrac{\partial^2 z}{\partial x \partial y}\left(\dfrac{\partial y}{\partial s}\dfrac{\partial x}{\partial t} + \dfrac{\partial y}{\partial t}\dfrac{\partial x}{\partial s}\right)$
$+ \dfrac{\partial z}{\partial x}\dfrac{\partial^2 x}{\partial s\,\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial^2 y}{\partial s\,\partial t} + \dfrac{\partial^2 z}{\partial y^2}\dfrac{\partial y}{\partial s}\dfrac{\partial y}{\partial t}$

Exercises 11.6 ■ Page 969

1. 0.022 millibars/mile

3. -100 ft/mi

5. $6\sqrt{2}$

7. $2\sqrt{3}$

9. (a) $\nabla f(x, y) = 2\cos(2x + 3y)\,\mathbf{i} + 3\cos(2x + 3y)\,\mathbf{j}$
(b) $\nabla f(-6, 4) = 2\mathbf{i} + 3\mathbf{j}$
(c) $D_{\mathbf{u}} f(-6, 4) = \sqrt{3} - \dfrac{3}{2}$

11. (a) $\nabla f(x, y) = 2x \ln y\,\mathbf{i} + (x^2/y)\,\mathbf{j}$
(b) $\nabla f(3, 1) = 9\mathbf{j}$
(c) $D_{\mathbf{u}} f(3, 1) = \dfrac{108}{13}$

13. (a) $\nabla f(x, y, z) = \langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz} \rangle$
(b) $\nabla f(3, 0, 2) = \langle 1, 12, 0 \rangle$
(c) $D_{\mathbf{u}} f(3, 0, 2) = -\dfrac{22}{3}$

15. $\dfrac{23}{10}$

17. $-\dfrac{4\sqrt{10}}{5}$

19. $\dfrac{4}{\sqrt{30}}$

21. $\dfrac{9}{2\sqrt{5}}$

23. $D_{\mathbf{u}} f(2, 2) \approx -3$

25. $\dfrac{22}{\sqrt{30}}$

27. $\sqrt{17}$, $\langle 4, 1 \rangle$

29. $\sqrt{2}$, $\langle 1, 1 \rangle$

31. $\sqrt{6}$, $\langle -1, -1, -2 \rangle$

33. (b) $\langle -12, 92 \rangle$

35. All points on the line $y = x + 1$

37. (a) $-\dfrac{40}{3\sqrt{3}}$

39. (a) $D_{\mathbf{u}} V(3, 4, 5) = \dfrac{32}{\sqrt{3}}$
(b) $\nabla V(3, 4, 5) = \langle 38, 6, 12 \rangle$
(c) $|\nabla V(3, 4, 5)| = 2\sqrt{406}$

41. $\dfrac{327}{13}$

45. $\dfrac{774}{25}$

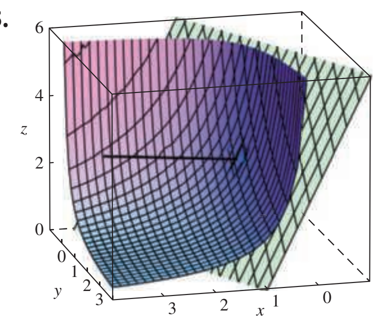
47. (a) $x + y + z = 11$
(b) $x = 3 + t$, $y = 3 + t$, $z = 5 + t$

49. (a) $4x - 5y - z = 4$
(b) $x = 2 + 4t$, $y = 1 - 5t$, $z = -1 - t$

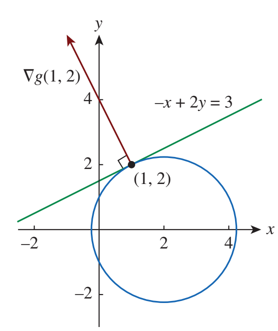
51. (a) $x + y - z = 1$
(b) $x - 1 = y = -z$

53. (a) $x + y + z = 3$
(b) $x = y = z$

55.



57. $\nabla g(1, 2) = \langle -2, 4 \rangle$, $-x + 2y = 3$



59. $\dfrac{x_0}{a^2}x + \dfrac{y_0}{b^2}y - \dfrac{z_0}{c^2}z = 1$

61. $\left(\sqrt{\dfrac{2}{11}}, 2\sqrt{\dfrac{2}{11}}, \dfrac{1}{2}\sqrt{\dfrac{2}{11}}\right), \left(-\sqrt{\dfrac{2}{11}}, -2\sqrt{\dfrac{2}{11}}, -\dfrac{1}{2}\sqrt{\dfrac{2}{11}}\right)$

63. No

67. $\left(-\dfrac{5}{4}, -\dfrac{5}{4}, \dfrac{25}{8}\right)$

71. $x = -1 - 10t$, $y = 1 - 16t$, $z = 2 - 12t$

75. Let $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$.

$$\nabla f = \left\langle \frac{dD_{\mathbf{u}}f - bD_{\mathbf{v}}f}{ad - bc}, \frac{aD_{\mathbf{v}}f - cD_{\mathbf{u}}f}{ad - bc} \right\rangle$$

Exercises 11.7 ■ Page 981

1. (a) f has a local minimum at (1, 1).

(b) f has a saddle point at (1, 1).

3. Local minimum at (1, 1), saddle point at (0, 0)

5. Local minimum: $f\left(\frac{1}{3}, -\frac{2}{3}\right) = -\frac{1}{3}$

7. Local minimum: $f(1, 1) = 0$, $f(-1, -1) = 0$, saddle point at (0, 0)

9. Local maximum: $f(-2, -2) = 4$

11. Local maximum: $f(1, 0) = 3$, $f(-1, 0) = 3$, saddle point at (0, 0)

13. Local maximum: $f(-1, 0) = 5$, local minimum: $f(3, 2) = -31$, saddle points at $(-1, 2)$, (3, 0)

15. Local minimum: $f(1, 1) = 3$

17. Local maximum: $f(1, 1) = f(-1, -1) = e^{-1}$, local minimum: $f(1, -1) = f(-1, 1) = -e^{-1}$, saddle point at (0, 0)

19. Local maximum: $f(0, -2) = 4e^{-2}$, saddle point at (0, 0)

21. Saddle points at $\left(\frac{\pi}{2} + n\pi, 0\right)$

23. Local maximum: $f\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = f\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) = 1$, saddle point at (0, 0)

25. Local maximum: $f(0, y) = 0$, $y < 0$, local minimum: $f(0, y) = 0$, $y > 0$, saddle point at (0, 0)

27. Local maximum: $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2e}$,

local minimum: $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2e}$,

saddle point at (0, 0)

29. Local maximum: $f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{3}{2}$

31. Local minimum: $f(0, -0.794) \approx -1.191$, $f(\pm 1.592, 1.267) \approx -1.310$, saddle points at $(\pm 0.720, 0.259)$, no highest point, lowest points at $(\pm 1.592, 1.267, -1.310)$

33. Local maximum: $f(-1.267, 0) \approx 1.310$, $f(1.629, \pm 1.063) \approx 8.105$, saddle points at $(-0.259, 0)$, (1.526, 0), highest points at $(1.629, \pm 1.063, 8.105)$

35. Local maximum: $f(0.910, 0) \approx 5.731$, local minimum: $f(-0.459, \pm 0.929) \approx 3.868$, $f(3.733, \pm 0.929) \approx -7.077$, saddle points at $(-0.459, 0)$, $(0.910, \pm 0.929)$, (3.733, 0), lowest points at $(3.733, \pm 0.929, -7.077)$

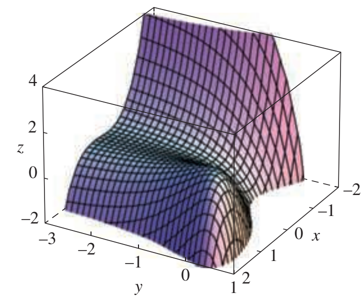
37. Maximum: $f(2, 0) = 9$, minimum: $f(0, 3) = -14$

39. Maximum: $f(\pm 1, 1) = 7$, minimum: $f(0, 0) = 4$

41. Maximum: $f(3, 5) = 19$, minimum: $f(-2, 4) = -12$

43. Maximum: $f(2, 2) = 18$, minimum: $f(-2, -2) = -18$

45. Critical point: (1, 0)



47. $\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)$

49. $(0, \pm 3, 0)$

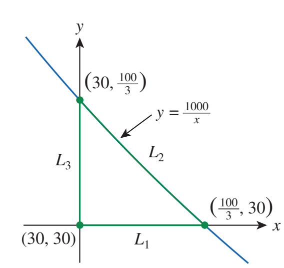
51. $x = y = z = 4$

53. $x = 10$ cm, $y = 10$ cm, $z = 10$ cm

55. Cube with edge length $\frac{8}{\sqrt{6}}$ cm

57. $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$

59. (a) $D = \{(x, y) \mid x \geq 30, 30 \leq y \leq 1000/x\}$



(b) Walls 30 m in length, height $\frac{40}{9}$ m

(c) $x \approx 25.54$ m, $y \approx 20.43$ m, $z \approx 7.67$ m

61. Nitrogen and phosphorus levels 1

65. $\frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1$

Exercises 11.8 ■ Page 991

1. ≈ 59; 30

3. No maximum, minimum: $f(1, 1) = f(-1, -1) = 2$

5. Maximum: $f(2, 3) = 26$, minimum: $f(-2, -3) = -26$

7. Maximum: $f(1, 2) = f(-1, -2) = 2$,
minimum: $f(1, -2) = f(-1, 2) = -2$

9. Maximum: $f(2, 2) = e^4$, no minimum

11. Maximum: $f(2, 0, -1) = 20$,
minimum: $f(-2, 0, 1) = -20$

13. Maximum: $\dfrac{2}{\sqrt{3}}$, minimum: $-\dfrac{2}{\sqrt{3}}$

15. Maximum: $\sqrt{3}$, minimum: 1

17. Maximum: $3 \ln 5$, minimum: $\ln 13$

19. Maximum: $f(1/\sqrt{n}, \ldots, 1/\sqrt{n}) = \sqrt{n}$,
minimum: $f(-1/\sqrt{n}, \ldots, -1/\sqrt{n}) = -\sqrt{n}$

21. Maximum: $f\left(\dfrac{\sqrt{6}}{3}, -\dfrac{\sqrt{6}}{2}, -\dfrac{\sqrt{6}}{6}\right) = 2\sqrt{6}$,
minimum: $f\left(-\dfrac{\sqrt{6}}{3}, \dfrac{\sqrt{6}}{2}, \dfrac{\sqrt{6}}{6}\right) = -2\sqrt{6}$

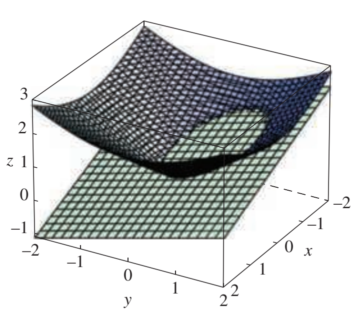
23. Maximum: $f\left(\dfrac{3}{\sqrt{2}}, -\dfrac{3}{\sqrt{2}}\right) = 9 + 12\sqrt{2}$,
minimum: $f(-2, 2) = -8$

25. Maximum: $f\left(\pm\dfrac{1}{\sqrt{2}}, \mp\dfrac{1}{2\sqrt{2}}\right) = e^{1/4} \approx 1.284$,
minimum: $f\left(\pm\dfrac{1}{\sqrt{2}}, \pm\dfrac{1}{2\sqrt{2}}\right) = e^{-1/4} \approx 0.779$

33.–43. See Exercises 46–57 in Section 11.7

45. Maximum: $\dfrac{1}{27}(87{,}500 + 2500\sqrt{10})$,
minimum: $\dfrac{1}{27}(87{,}500 - 2500\sqrt{10})$

47. (a)



(b) Highest: $\left(-\dfrac{4}{3}, 1, \dfrac{5}{3}\right)$, lowest: $\left(\dfrac{4}{13}, -\dfrac{3}{13}, \dfrac{5}{13}\right)$

49. Maximum: ≈ 4.2525, minimum: ≈ -4.7431

51. (a) 1

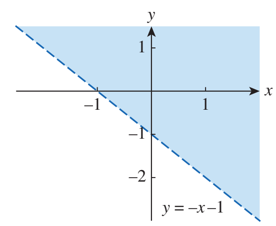
Chapter 11 Review ■ Page 996

True-False Quiz

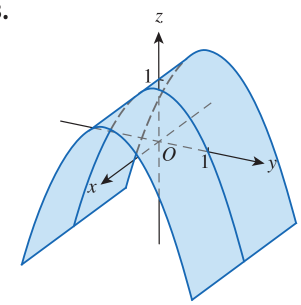
1. True **3.** False **5.** False **7.** True **9.** False **11.** True

Exercises

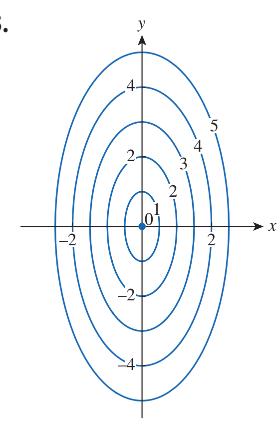
1. $D = \{(x, y) \mid y > -x - 1\}$



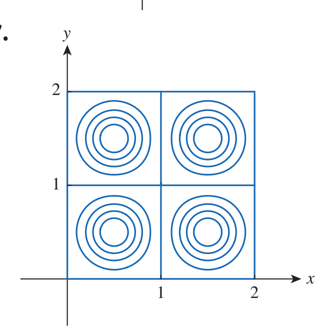
3.



5.



7.



9. (a) 55

(b) Negative, start at (3, 2) and move right, the contours show the surface is descending

(c) $f_y(3, 1) > f_y(1, 3)$, level curves are closer together in the y-direction at (3, 1)

11. Does not exist

13. $T(x, y) \approx 3.5x - 3y + 71$, $T(5, 3.8) \approx 77.1°C$

15. $u_r = -e^{-r}\sin 2\theta$, $u_\theta = 2e^{-r}\cos 2\theta$

17. $w_x = \dfrac{1}{y - z}$, $w_y = -\dfrac{x}{(y - z)^2}$, $w_z = \dfrac{x}{(y - z)^2}$

19. $u_x = -ye^{-xy}\sin\dfrac{y}{z}$, $u_y = e^{-xy}\left(\dfrac{\cos\dfrac{y}{z}}{z} - x\sin\dfrac{y}{z}\right)$,

$u_z = -\dfrac{ye^{-xy}\cos\dfrac{y}{z}}{z^2}$

21. $f_{xx} = 24x$, $f_{yy} = -2x$, $f_{xy} = f_{yx} = -2y$

23. $f_{xx} = -\dfrac{2(x^2 - y^2)}{(x^2 + y^2)^2}$, $f_{yy} = \dfrac{2(x^2 - y^2)}{(x^2 + y^2)^2}$,

$f_{xy} = f_{yx} = -\dfrac{4xy}{(x^2 + y^2)^2}$

25. $f_{xx} = k(k - 1)x^{k-2}y^l z^m$, $f_{yy} = l(l - 1)x^k y^{l-2} z^m$,

$f_{zz} = m(m - 1)x^k y^l z^{m-2}$, $f_{xy} = f_{yx} = klx^{k-1}y^{l-1}z^m$,

$f_{xz} = f_{zx} = kmx^{k-1}y^l z^{m-1}$, $f_{yz} = f_{zy} = lmx^k y^{l-1} z^{m-1}$

29. (a) $z = 8 + 4y + 1$

(b) $x = 1 + 8t$, $y = -2 + 4t$, $z = 1 - t$

31. (a) $2x - 2y - 3z = 3$

(b) $x = 2 + 4t$, $y = -1 - 4t$, $z = 1 - 6t$

33. (a) $4x - y - 2z = 6$

(b) $x = 3 + 8t$, $y = 4 - 2t$, $z = 1 - 4t$

35. $\left(2, \dfrac{1}{2}, -1\right), \left(-2, -\dfrac{1}{2}, 1\right)$

37. $f(x, y, z) \approx 60x + \dfrac{24}{5}y + \dfrac{32}{5}z - 120$, 38.656

39. $\dfrac{du}{dp} = 2xy^3(1 + 6p) + 3x^2y^2(pe^p + e^p) + 4z^3(p\cos p + \sin p)$

41. $\left(\dfrac{\partial z}{\partial s}\right)_{(1,\,2)} = -47$, $\left(\dfrac{\partial z}{\partial t}\right)_{(1,\,2)} = 108$

47. $\nabla f = \langle 2xe^{yz^2}, x^2z^2e^{yz^2}, 2x^2yze^{yz^2}\rangle$

49. $D_{\mathbf{u}}f(-2, 0) = -\dfrac{4}{5}$

51. $\dfrac{\sqrt{145}}{2}$, $\left\langle 4, \dfrac{9}{2}\right\rangle$

53. $-\dfrac{9}{5}$ mph/mile

55. Local minimum: $f(-4, 1) = -11$

57. Local maximum: $f(1, 1) = 1$, saddle points at (0, 0), (3, 0), (0, 3)

59. Maximum: $f(1, 2) = 4$, minimum: $f(2, 4) = -64$

61. Maximum: $f(-1, 0) = 2$, Minimum: $f(1, \pm 1) = -3$, saddle points at $(-1, \pm 1)$, (1, 0)

63. Maximum: $f\left(\pm\sqrt{\dfrac{2}{3}}, \dfrac{1}{\sqrt{3}}\right) = \dfrac{2}{3\sqrt{3}}$,

minimum: $f\left(\pm\sqrt{\dfrac{2}{3}}, -\dfrac{1}{\sqrt{3}}\right) = -\dfrac{2}{3\sqrt{3}}$

65. Maximum: 1, minimum: -1

67. $\left(\pm\dfrac{1}{\sqrt[4]{3}}, \dfrac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right), \left(\pm\dfrac{1}{\sqrt[4]{3}}, -\dfrac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$

69. $\dfrac{2\sqrt{3} - 3}{3}P$, $\dfrac{3 - \sqrt{3}}{6}P$, $(2 - \sqrt{3})P$

Focus on Problem Solving

1. L^2W^2, $\dfrac{1}{4}L^2W^2$

3. (a) $x = w/3$, base $= w/3$

(b) Yes

9. $a = \sqrt{\dfrac{3}{2}}$, $b = \dfrac{3}{\sqrt{2}}$

Chapter 12

Exercises 12.1 ■ Page 1011

1. (a) 288 (b) 144

3. (a) $\dfrac{\pi^2}{4} \approx 4.935$ (b) 0

5. (a) 44 (b) 88

7. (a) 4 (b) -8

9. $U < V < L$

11. (a) 248 (b) 15.5

13. 60

15. 3

17.

n	estimate
1	1.141606
4	1.143191
16	1.143535
64	1.143617
256	1.143637
1024	1.143642

19.

n	estimate
1	36.000000
4	0.399924
16	2.987151
64	3.141532
256	3.141481
1024	3.141461

Exercises 12.2 ■ Page 1018

1. $500y^3, 3x^2$

3. $0, \frac{1}{3}\sin(2\pi x)$

5. 10

7. 2

9. $\frac{261632}{45}$

11. $\frac{21}{2}\ln 2$

13. 0

15. π

17. $\frac{\pi}{8}$

19. $\frac{21}{2}$

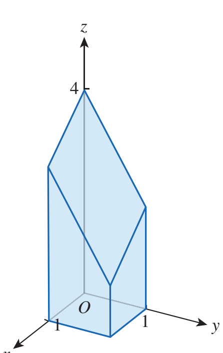
21. $9\ln 2$

23. $\frac{\sqrt{3}-1}{2} - \frac{\pi}{12}$

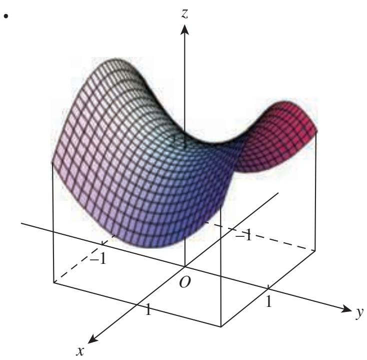
25. $\frac{1}{2}(e^2-3)$

27. $\frac{(\ln 3)^3}{2}$

29.



31.



33. 12

35. $2\pi + 2e - \frac{2}{e}$

37. $\frac{640}{3}$

39. $40 - 8\ln 5$

41. $V \approx 3.0271$

43. $f_{\text{ave}} \approx 3.327$

45. $4\pi^2$

Exercises 12.3 ■ Page 1028

1. 32

3. $\frac{1}{2}(e-1)$

5. $\frac{3}{10}$

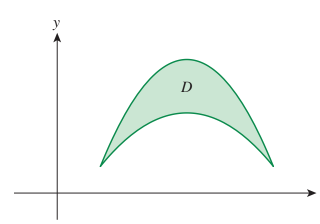
7. $e - 1$

9. $\frac{4}{3}$

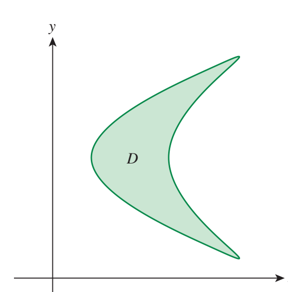
11. π

13. $\frac{1}{2}(1 - e^{-9})$

15. (a)



(b)



17. Type I: $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le x\}$,
Type II: $D = \{(x, y) \mid 0 \le y \le 1, y \le x \le 1\}$, $\frac{1}{3}$

19. $\iint_D y\,dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} y\,dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} y\,dy\,dx$
$= \int_{-1}^2 \int_{y^2}^{y+2} y\,dx\,dy = \frac{9}{4}$

21. $\frac{1}{2}(1 - \cos 1)$

23. $\frac{147}{20}$

25. 0

27. $\frac{7}{18}$

29. $\frac{31}{8}$

31. 6

33. $\frac{128}{15}$

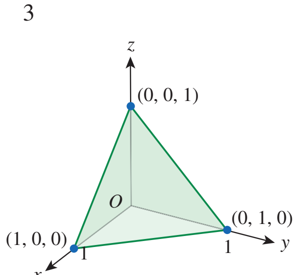
35. $\frac{1}{3}$

37. 0, 1.213; 0.713

39. $\frac{64}{3}$

41. $\frac{5\sqrt{2}}{3}$

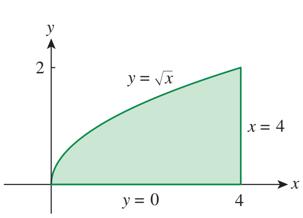
43.



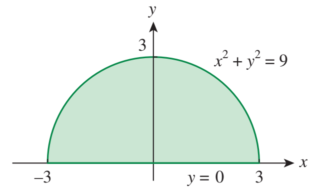
45. $\frac{13{,}984{,}735{,}616}{14{,}549{,}535} \approx 961.181$

47. $\frac{\pi}{2}$

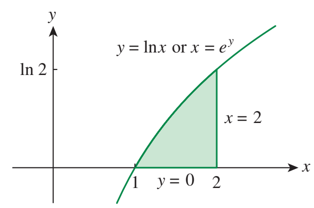
49. $\int_0^2 \int_{y^2}^4 f(x, y)\,dx\,dy$



51. $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} f(x, y)\,dy\,dx$



53. $\int_0^{\ln 2} \int_{e^y}^2 f(x, y)\,dx\,dy$



55. $\frac{e^9 - 1}{6}$

57. $\frac{1}{3} \ln 9$

59. $\frac{1}{3}(2\sqrt{2} - 1)$

61. 1

63. $\frac{\pi}{16} e^{-1/16} \le \iint_Q e^{-(x^2+y^2)^2} dA \le \frac{\pi}{16}$

65. $\frac{3}{4}$

69. 9π

71. $a^2b + \frac{3}{2}ab^2$

73. $\pi a^2 b$

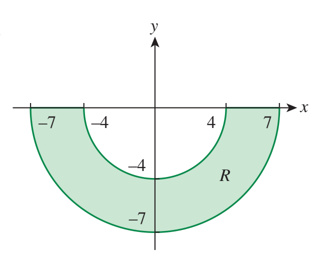
Exercises 12.4 ■ Page 1035

1. $\int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

3. $\int_{-1}^1 \int_0^{(x+1)/2} f(x, y)\,dy\,dx$

5. $\int_{-\pi/2}^{\pi/2} \int_3^6 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

7.



9. 0

11. $\frac{\pi}{2}\sin 9$

13. $\frac{\pi}{2}(1-e^{-4})$

15. $\frac{3}{64}\pi^2$

17. $\frac{\pi}{2}(b^2-a^2)$

19. $\frac{\pi}{4}$

21. $\frac{\pi}{3}+\frac{\sqrt{3}}{2}$

23. 81π

25. $32\sqrt{3}\pi$

27. $\frac{9}{4}\pi$

29. 2π

31. (a) $\frac{4\pi}{3}(r_2^2-r_1^2)^{3/2}$

(b) $\frac{\pi}{6}h^3$

33. $\frac{1}{15}a^5$

35. $\frac{16}{9}$

37. (a) $2\pi(1-Re^{-R}-e^{-R})$ ft^3

(b) $\frac{2(1-Re^{-R}-e^{-R})}{R^2}$ ft^3 (per hour per square foot)

39. $\frac{2}{3}a$

Exercises 12.5 ■ Page 1046

1. $\frac{64}{3}$C

3. $\frac{4}{3},\left(\frac{4}{3},\ 0\right)$

5. $6,\left(\frac{3}{4},\frac{3}{2}\right)$

7. $\frac{1}{4}(e^2-1),\left(\frac{e^2+1}{2(e^2-1)},\frac{4(e^3-1)}{9(e^2-1)}\right)$

9. $\frac{L}{4},\left(\frac{L}{2},\frac{16}{9\pi}\right)$

11. $\left(\frac{3}{8},\frac{3\pi}{16}\right)$

13. $\left(0,\frac{45}{14\pi}\right)$

15. $\left(\frac{2}{5}a,\frac{2}{5}a\right)$ if vertex is $(0,0)$ and sides are along positive axes

17. $\frac{1}{16}(e^4-1),\frac{1}{8}(e^2-1),\frac{1}{16}(e^4+2e^2-3)$

19. $\frac{7}{180}ka^6,\frac{7}{180}ka^6,\frac{7}{90}ka^6$

21. $m=\frac{\pi^2}{8},\ (\bar{x},\bar{y})=\left(\frac{2\pi}{3}-\frac{1}{\pi},\frac{16}{9\pi}\right), I_x=\frac{3\pi^2}{64},$

$I_y=\frac{\pi^2}{16}(\pi^2-3), I_0=\frac{\pi^2}{64}(4\pi^2-9)$

23. (a) $\frac{1}{2}$

(b) 0.375

(c) $\frac{5}{48}\approx 0.1042$

25. (b) (i) $e^{-0.2}\approx 0.8187$

(ii) $1+e^{-1.8}-e^{-0.8}-e^{-1}\approx 0.3481$

(c) 2, 5

27. (a) ≈ 0.500 (b) ≈ 0.632

29. (a) $k\iint_D\left[1-\frac{1}{20}\sqrt{(x-x_0)^2+(y-y_0)^2}\right]dA$

where D is the disk with radius 10 mi centered at the center of the city

(b) $\frac{200}{3}\pi k\approx 209k,\ 200k\left(\frac{\pi}{2}-\frac{8}{9}\right)\approx 136k$, on the edge

Exercises 12.6 ■ Page 1051

1. $\sqrt{14}\,\pi$

3. $3\sqrt{14}$

5. $\frac{\sqrt{2}}{6}$

7. 4

9. $\frac{2\pi}{3}(2\sqrt{2}-1)$

11. $4\pi b\,(b-\sqrt{b^2-a^2})$

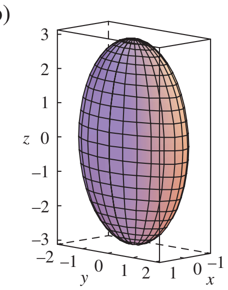
13. 13.9783

15. (a) 24.2055 (b) 24.2476

17. 4.4506

19. $\frac{45}{8}\sqrt{14}+\frac{15}{16}\ln\frac{11\sqrt{5}+3\sqrt{70}}{3\sqrt{5}+\sqrt{70}}$

21. (b)



(c) $\int_0^{2\pi}\int_0^{\pi}\sqrt{36\sin^4 u\cos^2 v + 9\sin^4 u\sin^2 v + 4\cos^2 u\sin^2 u}\,du\,dv$

25. $\frac{98}{3}\pi$

27. 4π

Exercises 12.7 ■ Page 1060

1. $\frac{27}{4}$

3. 1

5. $\frac{1}{3}(e^3 - 1)$

7. $-\frac{1}{3}$

9. $\frac{1}{3}\ln 2$

11. $\frac{3}{20}\sin 1$

13. $\frac{4}{3}$

15. $\frac{3}{28}$

17. $\frac{1}{144}$

19. $\frac{27}{8}$

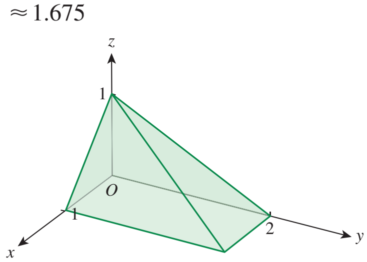
21. 144

23. 128π

25. (a) ≈ 239.64 (b) ≈ 245.91

27. ≈ 1.675

29.



31. $\int_{-2}^{2}\int_0^{4-x^2}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\,dz\,dy\,dx$

$= \int_0^4\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\,dz\,dx\,dy$

$= \int_{-1}^{1}\int_0^{4-4z^2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\,dx\,dy\,dz$

$= \int_0^4\int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\,dx\,dz\,dy$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_0^{4-x^2-4z^2} f(x, y, z)\,dy\,dz\,dx$

$= \int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-z^2}}\int_0^{4-x^2-4z^2} f(x, y, z)\,dy\,dx\,dz$

33. $\int_{-2}^{2}\int_{x^2}^{4}\int_0^{2-y/2} f(x, y, z)\,dz\,dy\,dx$

$= \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{2-y/2} f(x, y, z)\,dz\,dx\,dy$

$= \int_0^4\int_0^{2-y/2}\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\,dx\,dz\,dy$

$= \int_0^2\int_0^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\,dx\,dy\,dz$

$= \int_{-2}^{2}\int_0^{2-x^2/2}\int_{x^2}^{4-2z} f(x, y, z)\,dy\,dz\,dx$

$= \int_0^2\int_{-\sqrt{4-2z}}^{\sqrt{4-2z}}\int_{x^2}^{4-2z} f(x, y, z)\,dy\,dx\,dz$

35. $\int_0^1\int_{\sqrt{x}}^{1}\int_0^{1-y} f(x, y, z)\,dz\,dy\,dx$

$\int_0^1\int_0^{y^2}\int_0^{1-y} f(x, y, z)\,dz\,dx\,dy$

$= \int_0^1\int_0^{1-z}\int_0^{y^2} f(x, y, z)\,dx\,dy\,dz$

$= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x, y, z)\,dx\,dz\,dy$

$= \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x, y, z)\,dy\,dz\,dx$

$= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x, y, z)\,dy\,dx\,dz$

37. $\int_0^1\int_y^1\int_0^y f(x, y, z)\,dz\,dx\,dy$

$\int_0^1\int_0^x\int_0^y f(x, y, z)\,dz\,dy\,dx$

$= \int_0^1\int_0^y\int_y^1 f(x, y, z)\,dx\,dz\,dy$

$= \int_0^1\int_z^1\int_y^1 f(x, y, z)\,dx\,dy\,dz$

$= \int_0^1\int_0^x\int_z^x f(x, y, z)\,dy\,dz\,dx$

$= \int_0^1\int_z^1\int_z^x f(x, y, z)\,dy\,dx\,dz$

39. $\dfrac{79}{30}, \left(\dfrac{358}{553}, \dfrac{33}{79}, \dfrac{571}{553}\right)$

41. $a^5, \left(\dfrac{7}{12}a, \dfrac{7}{12}a, \dfrac{7}{12}a\right)$

43. $I_x = I_y = I_z = \dfrac{2}{3}kL^5$

45. $\dfrac{1}{2}\pi kha^4$

47. (a) $m = \displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} \sqrt{x^2+y^2}\,dz\,dy\,dx$

(b) $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{z} = \frac{1}{m}\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} z\sqrt{x^2+y^2}\,dz\,dy\,dx$$

$$\bar{y} = \frac{1}{m}\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} y\sqrt{x^2+y^2}\,dz\,dy\,dx$$

$$\bar{z} = \frac{1}{m}\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} z\sqrt{x^2+y^2}\,dz\,dy\,dx$$

(c) $I_z = \displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} (x^2+y^2)^{3/2}\,dz\,dy\,dx$

49. (a) $\dfrac{3\pi}{32} + \dfrac{11}{24}$

(b) $\left(\dfrac{28}{9\pi+44}, \dfrac{30\pi+128}{45\pi+220}, \dfrac{45\pi+208}{135\pi+660}\right)$

(c) $\dfrac{68+15\pi}{240}$

51. (a) $\dfrac{1}{8}$ (b) $\dfrac{1}{64}$ (c) $\dfrac{1}{5760}$

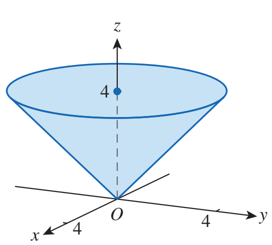
53. $\dfrac{L^3}{8}$

55. (a) The region bounded by the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$

(b) $\dfrac{4\sqrt{6}}{45}\pi$

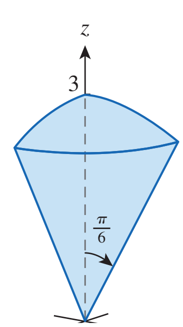
Exercises 12.8 ■ Page 1068

1.



$\dfrac{64\pi}{3}$

3.



$\dfrac{9\pi}{4}(2-\sqrt{3})$

5. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{3}\int_{0}^{2} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$

7. 384π

9. $\pi(e^6 - e - 5)$

11. $\dfrac{2\pi}{5}$

13. (a) 162π (b) $(0, 0, 15)$

15. $m = \dfrac{1}{8}a^2\pi K, (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \dfrac{2}{3}a\right)$

17. $\dfrac{312{,}500}{7}\pi$

19. $\dfrac{15\pi}{16}$

21. $\dfrac{1562}{15}\pi$

23. $\dfrac{\sqrt{3}-1}{3}\pi a^3$

25. (a) 10π (b) $(0, 0, 2.1)$

27. $\left(0, \dfrac{528}{296}, 0\right)$

29. (a) $\left(0, 0, \dfrac{3}{8}a\right)$ (b) $\dfrac{4}{15}Ka^5\pi$

31. $V = \dfrac{1}{3}\pi(2-\sqrt{2}), (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \dfrac{3}{8(2-\sqrt{2})}\right)$

33. $\dfrac{5\pi}{6}$

35. 0

37. $\dfrac{4\sqrt{2}-5}{15}$

39. $\dfrac{136\pi}{99}$

41. (a) $\displaystyle\iiint_C h(P)g(P)\,dV$, where C is the cone

(b) $\dfrac{50}{3}\pi\,(62{,}00)^2\,(12{,}000)^2 \approx 3.1 \times 10^{19}$ ft-lb

Exercises 12.9 ■ Page 1079

1. 16

3. $e^{-u+v}(1 + uv)$

5. $-2e^{2s}$

7. $1 + 8uvw$

9. The parallelogram with vertices $(0, 0), (6, 3), (12, 1), (6, -2)$

11. The region bounded by the line $y = 1$, the y-axis, and $y = \sqrt{x}$

13. $x = \dfrac{1}{3}(u - v), y = \dfrac{1}{3}(u + 2v)$ is one possible transformation, where $S = \{(u, v) \mid -1 \le u \le 1, 1 \le v \le 3\}$

15. $x = u\cos v, y = u\sin v$ is one possible transformation, where $S = \{(u, v) \mid 1 \le u \le \sqrt{2}, 0 \le v \le \pi/2\}$

17. -3

19. 6π

21. $2 \ln 3$

23. (a) $\frac{4}{3}\pi abc$ (b) $1.083 \times 10^{12}\ \text{km}^3$

25. $\frac{8}{5} \ln 8$

27. $\frac{3}{2} \sin 1$

29. $e - e^{-1}$

Chapter 12 Review ■ Page 1081

True-False Quiz

1. True **3.** True **5.** True **7.** True **9.** False

Exercises

1. ≈ 64.0

3. $4e^2 - 4e + 3$

5. $\frac{1}{2} \sin 1$

7. $\frac{2}{3}$

9. $\int_0^{\pi} \int_2^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

11. The region inside the loop of the four-leaved rose $r = \sin 2\theta$ in the first quadrant

13. $\frac{1}{2} \sin 1$

15. $\frac{1}{2}e^6 - \frac{7}{2}$

17. $\frac{1}{4} \ln 2$

19. 8

21. $\frac{81\pi}{5}$

23. $\frac{81}{2}$

25. $\frac{\pi}{96}$

27. $\frac{64}{15}$

29. 176

31. $\frac{2}{3}$

33. $\frac{2}{9} ma^3$

35. (a) $m = \frac{1}{4}$

(b) $(\bar{x}, \bar{y}) = \left(\frac{1}{3}, \frac{8}{15}\right)$

(c) $I_x = \frac{1}{12}, I_y = \frac{1}{24}, I_0 = \frac{1}{8}$

37. (a) $\left(0, 0, \frac{h}{4}\right)$ (b) $\frac{\pi a^4 h}{10}$

39. $\ln(\sqrt{2} + \sqrt{3}) + \frac{\sqrt{2}}{3} \approx 1.6176$

41. $\frac{486}{5}$

43. ≈ 00512

45. (a) $\frac{1}{15}$ (b) $\frac{1}{3}$ (c) $\frac{1}{45}$

47. $\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$

49. $-\ln 2$

51. 0

Focus on Problem Solving

1. 30

3. $\frac{1}{2} \sin 1$

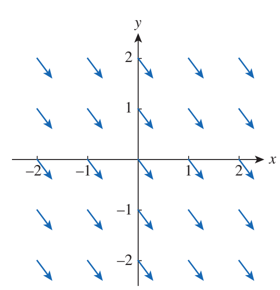
7. (b) 0.90

11. $abc\pi\left(\frac{2}{3} - \frac{8}{9\sqrt{3}}\right)$

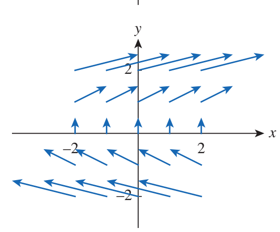
Chapter 13

Exercises 13.1 ■ Page 1094

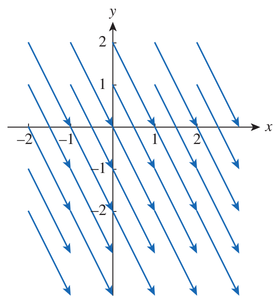
1.



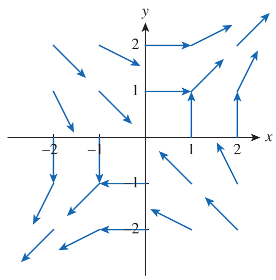
3.



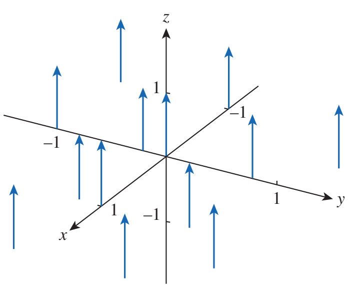
5.



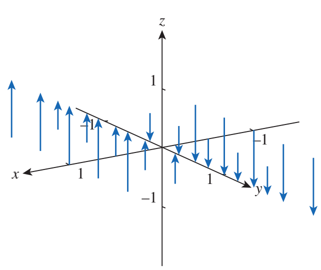
7.



9.



11.



13. II

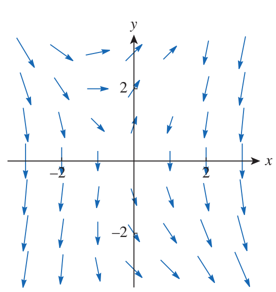
15. I

17. III

19. IV

21. III

23.



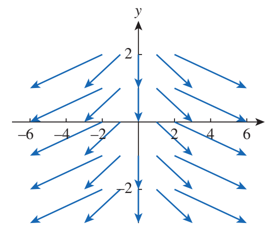
$\mathbf{F}(x, y) = \mathbf{0}$ along the line $y = 2x$

25. $\nabla f(x, y) = (xy + 1)e^{xy}\,\mathbf{i} + x^2e^{xy}\,\mathbf{j}$

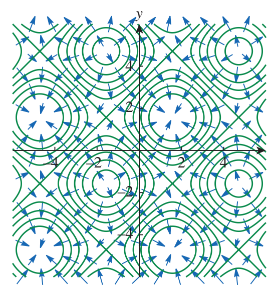
27. $\nabla f(x, y) = y^2 \cos(xy)\,\mathbf{i} + [xy \cos(xy) + \sin(xy)]\,\mathbf{j}$

29. $\nabla f(x, y, z) = \ln(y - 2z)\,\mathbf{i} + \dfrac{x}{y - 2z}\,\mathbf{j} - \dfrac{2x}{y - 2z}\,\mathbf{k}$

31. $\nabla f(x, y) = 2x\,\mathbf{i} - \mathbf{j}$

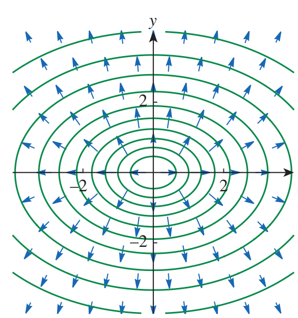


33.



The gradient vectors are perpendicular to the level curves. The gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

35.



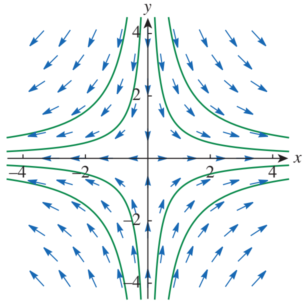
The gradient vectors are perpendicular to the level curves. The gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

37. III

39. II

41. (2.04; 1.03)

43. (a)



The flow lines have equations $y = C/x$.

(b) $y = 1/x,\ x > 0$

Exercises 13.2 ■ Page 1106

1. $\dfrac{1}{54}(145\sqrt{145} - 1)$

3. $\dfrac{2 \cdot 4^6}{5} = 1638.4$

5. $\dfrac{243}{8}$

7. $\dfrac{17}{3}$

9. $\sqrt{5}\pi$

11. $\dfrac{\sqrt{14}}{12}(e^6 - 1)$

13. $\dfrac{1}{5}$

15. $\dfrac{97}{3}$

17. 2

19. $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds$ is positive.

$\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds$ to be negative.

21. $\dfrac{17}{15}$

23. π

25. ≈ -0.1363

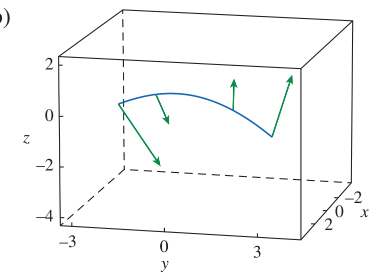
27. ≈ -15.0074

29. ≈ -1.7260

31. 0

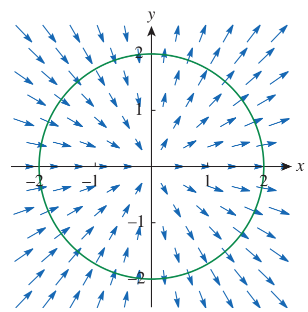
33. (a) -2

(b)



35. (a) 0

(b)



37. $m = \dfrac{1}{2}ka^3,\ (\bar{x}, \bar{y}) = \left(\frac{2}{3}a, \frac{2}{3}a\right)$

39. $m = \sqrt{2}\left(\frac{8}{3}\pi^3 + 2\pi\right),\ (\bar{x}, \bar{y}, \bar{z}) = \left(\dfrac{3\pi(2\pi^2 + 1)}{4\pi^3 + 3}, 0, 0\right)$

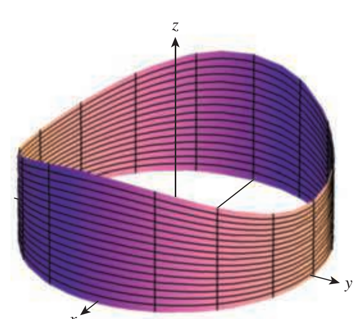
41.
$I_x = 4\sqrt{13}\pi k(1 + 6\pi^2),\ I_y = 4\sqrt{13}\pi k(1 + 6\pi^2),\ I_z = 8\pi\sqrt{13}k$

43. $\dfrac{1}{2}e^2 - \dfrac{1}{2}e + \dfrac{7}{3}$

45. $K\left(\dfrac{1}{2} - \dfrac{1}{\sqrt{30}}\right)$

47. $90\left(185 - \dfrac{9}{2}\right) \approx 1.62 \times 10^4$ ft-lb

49. $1.6\pi \approx 5.03$ L of paint



Exercises 13.3 ■ Page 1117

1. 40

3. $f(x, y) = x^2 - 3xy + 2y^2 - 8y + K$

5. $f(x, y) = ye^x + e^y + K$

7. **F** is not conservative

9. $f(x, y) = ye^x + x \sin y + K$

11. $f(x, y) = x \ln y + x^2y^3 + K$

13. (a) **F** is conservative; the line integral is independent of path; all three curves have the same initial and terminal points.
(b) 16

15. (a) $f(x, y) = \dfrac{1}{2}x^2y^2$ (b) 2

17. (a) $f(x, y) = xe^{xy}$ (b) -1

19. (a) $f(x, y, z) = x^2z + xy^2 + z^3$ (b) 7

21. (a) $f(x, y, z) = xe^y + ze^z$ (b) $2e$

23. $\dfrac{2}{e}$

25. It doesn't matter which curve is chosen.

27. $\dfrac{31}{4}$

29. 2

31. Conservative

33. (a) $r(t) = \pi t\,\mathbf{i} + \pi t\,\mathbf{j},\ 0 \le t \le 1$
(b) $r(t) = \dfrac{\pi}{2}t\,\mathbf{i},\ 0 \le t \le 1$

35. $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial R}{\partial x}$; **F** is not conservative; line integral not independent of path

37. (a) Open (b) Not connected (c) Not simply connected

39. (a) Open (b) Connected (c) Not simply connected

41. (a) $c\left(\dfrac{1}{d_1} - \dfrac{1}{d_2}\right)$
(b) $\approx 1.77 \times 10^{32}$ J
(c) ≈ 1400 J

Exercises 13.4 ■ Page 1126

1. 8π

3. $\dfrac{2}{3}$

5. 12

7. $\dfrac{1}{3}$

9. 0

11. 0

13. -16

15. $-\pi$

17. $-8e + 48e^{-1}$

19. $-\dfrac{1}{12}$

21. 3π

23. (c) $\dfrac{9}{2}$

25. $(\bar{x}, \bar{y}) = \left(\dfrac{4a}{3\pi}, \dfrac{4a}{3\pi}\right)$ if the region is the portion of the disk $x^2 + y^2 = a^2$ in the first quadrant

29. 0

Exercises 13.5 ■ Page 1134

1. (a) $-x^2\,\mathbf{i} + 3xy\,\mathbf{j} - xz\,\mathbf{k}$
(b) yz

3. (a) $ze^x\,\mathbf{i} + (xye^z - yze^x)\,\mathbf{j} - xe^z\,\mathbf{k}$
(b) $y(e^z + e^x)$

5. (a) 0
(b) $\dfrac{2}{\sqrt{x^2 + y^2 + z^2}}$

7. (a) $\left\langle \dfrac{1}{y}, -\dfrac{1}{x}, \dfrac{1}{x} \right\rangle$
(b) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$

9. (a) $-\dfrac{\sqrt{z}}{(1 + y)^2}\,\mathbf{i} - \dfrac{\sqrt{x}}{(1 + z)^2}\,\mathbf{j} - \dfrac{\sqrt{y}}{(1 + x)^2}\,\mathbf{k}$
(b) $-\dfrac{1}{2\sqrt{x}(1 + z)} + \dfrac{1}{2\sqrt{y}(1 + x)} + \dfrac{1}{2\sqrt{z}(1 + y)}$

11. (a) Negative (b) curl $\mathbf{F} = \mathbf{0}$

13. (a) 0
(b) curl **F** points in the negative a-direction

15. (a) Meaningless; f is a scalar field
(b) Vector field
(c) Scalar field
(d) Vector field
(e) Meaningless; **F** is not a scalar field
(f) Vector field
(g) Scalar field
(h) Meaningless; f is a scalar field
(i) Vector field

(j) Meaningless; div **F** is a scalar field

(k) Meaningless; div **F** is a scalar field

(l) Scalar field

17. **F** is not conservative

19. $f(x, y, z) = xe^z + y + K$

21. $f(x, y, z) = \sin xy + \cos z + K$

23. No

Exercises 13.6 ■ Page 1146

1. 49.09

3. 900π

5. $11\sqrt{14}$

7. $\frac{2}{3}(2\sqrt{2} - 1)$

9. $171\sqrt{14}$

11. $\dfrac{\sqrt{3}}{24}$

13. $\dfrac{364\sqrt{2}}{3}\pi$

15. $\dfrac{\pi}{60}(391\sqrt{17} + 1)$

17. 16π

19. 12

21. $\dfrac{713}{180}$

23. $-\dfrac{1}{6}$

25. $-\dfrac{4}{3}\pi$

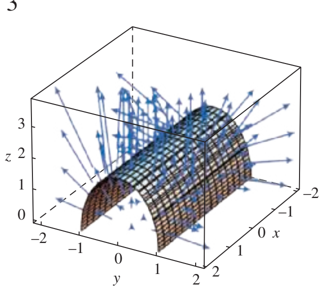
27. 0

29. 48

31. $2\pi + \dfrac{8}{3}$

33. $-\dfrac{151}{33} - \dfrac{1}{220}\sqrt{3}\pi + \dfrac{1977}{176}\ln 7 - \dfrac{9891}{880}\ln 3 + \dfrac{3}{440}\sqrt{3}\tan^{-1}\dfrac{5}{\sqrt{3}}$

35. $\frac{1}{3}(16\pi + 80e^{2/5} - 80e^{-2/5})$



37. $\displaystyle\iint_D \left(P - Q\frac{\partial k}{\partial y} - R\frac{\partial k}{\partial z}\right) dA$

39. $108\sqrt{2}\pi$

41. (a) $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{9}{2})$

(b) $\dfrac{140}{3}\pi k$

43. 0 kg/s

45. $24\,\epsilon_0$

47. $4\pi Kc$

Exercises 13.7 ■ Page 1153

3. 0

5. 0

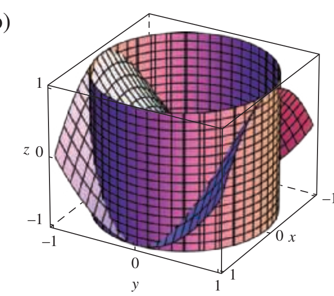
7. 0

9. $2e - 4$

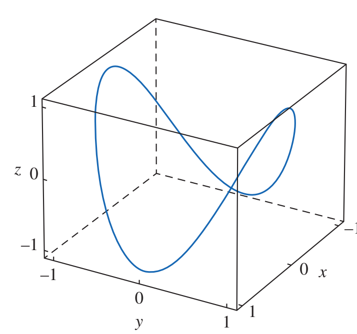
11. 9π

13. (a) π

(b)



(c) $x = \cos t,\ y = \sin t,\ z = \sin^2 t - \cos^2 t,\ 0 \le t \le 2\pi$



15. 8π

19. π

Exercises 13.8 ■ Page 1160

5. $\dfrac{9}{2}$

7. $\dfrac{9\pi}{2}$

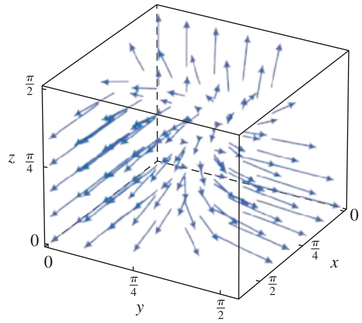
9. 0

11. $\dfrac{32\pi}{3}$

13. 0

15. $4\pi R^5$

17. $\dfrac{19}{64}\pi^2$

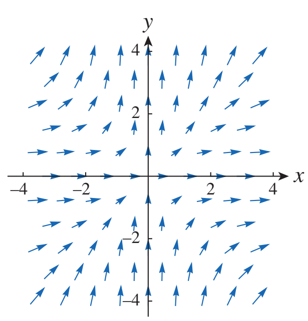


19. $\dfrac{3\pi}{2}$

21. (a) P_1 is a source.

P_2 is a sink.

23.



div $\mathbf{F} > 0$ for $y > -x$ and div $\mathbf{F} < 0$ for $y < -x$

25. $\dfrac{4}{3}\pi$

Chapter 13 Review ■ Page 1163

True-False Quiz

1. False **3.** True **5.** False **7.** False **9.** True
11. True **13.** False

Exercises

1. (a) Negative (b) Positive

3. $6\sqrt{10}$

5. $\dfrac{4}{15}$

7. $\dfrac{110}{3}$

9. $\dfrac{11}{12} - \dfrac{4}{e}$

11. $f(x, y) = e^y + xe^{xy} + K$

13. 0

17. -8π

25. $\dfrac{1}{60}\pi(391\sqrt{17} + 1)$

27. $-\dfrac{64}{3}\pi$

31. $-\dfrac{1}{2}$

35. -4

37. 21

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